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Phase-field models with hysteresis in one-dimensional thermo-visco-plasticity

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Abstract

The mathematical modelling of nonlinear thermo-visco-plastic developments and of phase transitions in solids have drawn much attention in past years. On the one hand, one is interested how phase transformations on the micro- and/or mesoscales (for instance, between different geometric configurations of the crystal lattice) influence the global thermo-visco-plastic behaviour; on the other hand, the global evolution of solid-solid phase transformations is strongly affected by the presence of micro- and/or mesoscopic stresses. In such situations, a typical macroscopic phenomenon is the occurrence of hysteresis effects, and it is therefore important to model these effects. This paper is a contribution towards this direction. A new one-dimensional model is considered that incorporates both the occurrence of hysteresis effects and of phase transitions. In this connection, the phase transition is described by the evolution of a *phase-field* (which is usually closely related to an order parameter of the phase transition), while the hysteresis effects are accounted for using the mathematical theory of *hysteresis operators* developed in the past thirty years. The model extends recent works of the first two authors on phase-field models with hysteresis to the case when mechanical effects can no longer be ignored or even prevail. It leads to a strongly nonlinear coupled system of partial differential equations in which hysteresis nonlinearities occur at several places, even under time and space derivatives. We show the thermodynamic consistency of the model, and we prove its well-posedness.

1 Introduction and physical motivation

In this paper, we study initial-boundary value problems for systems of partial differential equations of the form

$$\rho\, u_{tt} \,-\, \mu\, u_{xxt} \,=\, \sigma_x \,+\, f(x,t)\,, \quad \text{a.e. in } \Omega_T\,, \tag{1.1}$$

$$\sigma \,=\, \mathcal{H}_1[u_x,w] \,+\, \theta\,\mathcal{H}_2[u_x,w]\,, \quad \text{a.e. in } \Omega_T\,, \tag{1.2}$$

$$\Big(C_V\,\theta \,+\, \mathcal{F}_1[u_x,w]\Big)_t \,-\, \kappa\,\theta_{xx} \,=\, \mu\, u_{xt}^2 \,+\sigma\, u_{xt} \,+\, g(x,t,\theta)\,, \quad \text{a.e. in } \Omega_T\,, \tag{1.3}$$

$$\nu\, w_t \,=\, -\,\psi\,, \quad \text{a.e. in } \Omega_T\,, \tag{1.4}$$

$$\psi \,=\, \mathcal{H}_3[u_x,w] \,+\, \theta\,\mathcal{H}_4[u_x,w]\,, \quad \text{a.e. in } \Omega_T\,, \tag{1.5}$$

$$u(\cdot,0)=u_0\,, \quad u_t(\cdot,0)=u_1\,, \quad \theta(\cdot,0)=\theta_0\,, \quad w(\cdot,0)=w_0\,, \quad \text{a.e. in } \Omega\,, \tag{1.6}$$

$$u(0,t)=0\,, \quad \mu\, u_{xt}(1,t)+\sigma(1,t)=0\,, \quad \theta_x(0,t)=\theta_x(1,t)=0\,, \quad \text{a.e. in } (0,T)\,, \tag{1.7}$$

where $\Omega := (0,1)$, $T > 0$ denotes some final time, and where $\Omega_t := \Omega \times (0,t)$ for $t \in (0,T]$.

The system (1.1)–(1.7) constitutes a model for the one-dimensional thermomechanical developments in a linearly viscous piece of wire of unit length in which a solid-solid phase transition takes place. In this connection, the unknowns u , θ , σ , w , ψ denote displacement, absolute (Kelvin) temperature, elastoplastic stress, phase variable (usually a so-called *generalized freezing index*, cf. [15]), and the thermodynamic force driving the phase transition, respectively. The positive physical constants ρ, μ, C_V , κ , ν denote mass density, viscosity, specific heat, heat conductivity, and a relaxation coefficient, in that order. For the sake of notational convenience, we will always assume without loss of generality that $\rho = \mu = C_V = \kappa = \nu = 1$. Finally, the expressions $\mathcal{H}_j$, $1 \leq j \leq 4$, and $\mathcal{F}_1$, are nonlinearities of *hysteresis type* (to be specified below).

The equations (1.1), (1.3), (1.4) represent the equation of motion, the balance of internal energy, and the phase evolution equation, in that order (see below); equation (1.2) is the constitutive law relating strain and phase variable to the elastoplastic stress, and (1.5) expresses that the phase variable evolves into the opposite direction of the thermodynamic force driving the phase transition. Besides, the boundary conditions (1.7) indicate that the wire is thermally insulated at both ends, fixed at $x = 0$, and stress-free at $x = 1$.

The motivation to study systems of the above type is twofold. On the one hand, it is well-known that for many materials the macroscopic strain-stress (ε - σ , where $\varepsilon = u_x$ is the linearized strain and u is the displacement) relations measured in uniaxial load-deformation experiments strongly depend on the absolute (Kelvin) temperature θ and, at the same time, exhibit a strong elastoplasticity witnessed by the occurrence of *hysteresis loops* that are *rate-independent*, i. e. independent of the speed with which there are traversed. Due to the hysteresis, which reflects the presence of a *rate-independent memory* in the material, the stress-strain relation can no longer be expressed in terms of a simple single-valued function. Among the materials showing very strong temperature-dependent hysteretic effects are the so-called *shape memory alloys* (see Fig. 1 below and Chapter 5 in [2]); but even quite ordinary steels are well-known to exhibit this kind of behaviour (cf. [21]), although to a smaller extent.

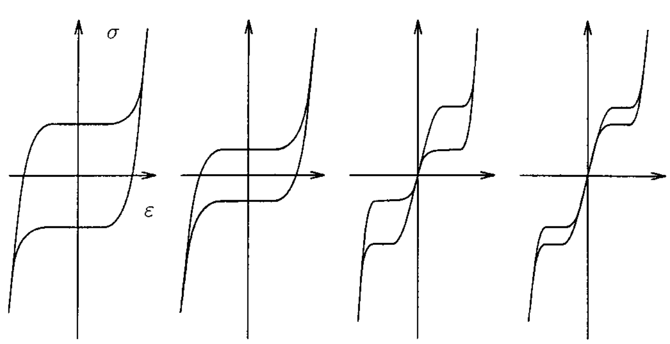

Fig. 1. Schematic load-deformation curves in shape memory alloys, with temperature increasing from left to right.

Usually the occurrence of a hysteresis in the macroscopic stress-strain relations is accompanied (or even triggered) by changes between different configurations of the crystal lattice within the solid. It thus makes sense to complement macroscopic equations of thermoelastoplasticity by field equations accounting for such phase transformations on the micro- and/or mesoscales.

On the other hand, phase transition phenomena are often accompanied by macroscopic hysteresis effects that are caused by thermal and/or mechanical stresses acting on the micro- and/or mesoscales. It then makes sense to complement the field equations describing the macroscopic phase transition by equations modelling such micro- or mesoscopic stresses.

A classical approach to such problems would be the following. One first tries to construct a local *free energy function* of the form

$$F(\varepsilon, w, \theta) = \theta(1 - \log(\theta)) + F_1(\varepsilon, w) + \theta F_2(\varepsilon, w) \tag{1.8}$$

in such a way that the experimentally observed ε - σ and/or w - ψ hysteresis loops are approximately matched using the relations

$$\sigma = \frac{\partial F}{\partial \varepsilon}(\varepsilon, w, \theta), \quad \psi = \frac{\partial F}{\partial w}(\varepsilon, w, \theta), \tag{1.9}$$

then determines the corresponding *internal energy* U and *entropy* S,

$$\begin{aligned} U(\varepsilon, w, \theta) &:= F(\varepsilon, w, \theta) - \theta \frac{\partial F}{\partial \theta}(\varepsilon, w, \theta) = \theta + F_1(\varepsilon, w), \\ S(\varepsilon, w, \theta) &:= -\frac{\partial F}{\partial \theta}(\varepsilon, w, \theta) = \log(\theta) - F_2(\varepsilon, w), \end{aligned} \tag{1.10}$$

and finally inserts these expressions in the governing field equations: equation of motion,

$$u_{tt} - \tilde{\sigma}_x = f, \quad (\tilde{\sigma} = \text{total stress} = \sigma + \text{viscous stress}) \tag{1.11}$$

balance of internal energy,

$$U_t - \theta_{xx} = \tilde{\sigma}\, u_{xt} + g, \quad (U = \text{internal energy}) \tag{1.12}$$

and phase evolution equation (1.4).

We then obtain (1.1), (1.3), (1.4), if we put

$$\begin{aligned} \mathcal{H}_1[\varepsilon, w] &:= \frac{\partial F_1}{\partial \varepsilon}(\varepsilon, w), \quad \mathcal{H}_2[\varepsilon, w] := \frac{\partial F_2}{\partial \varepsilon}(\varepsilon, w), \\ \mathcal{H}_3[\varepsilon, w] &:= \frac{\partial F_1}{\partial w}(\varepsilon, w), \quad \mathcal{H}_4[\varepsilon, w] := \frac{\partial F_2}{\partial w}(\varepsilon, w), \\ \mathcal{F}_1[\varepsilon, w] &:= F_1(\varepsilon, w). \end{aligned} \tag{1.13}$$

In order that a ε - σ (or w - ψ, respectively) hysteresis be modelled by (1.9), $F(\cdot, w, \theta)$ ($F(\varepsilon, \cdot, \theta)$, respectively) needs to be a non-convex function within the range of interesting temperatures.

This approach has advantages: if the nonlinearities involved in (1.1)–(1.7) are smooth functions, then the vast literature on one-dimensional thermoviscoelasticity (we just refer to the fundamental papers [4], [5]) can be applied to derive results concerning well-posedness and asymptotic behaviour. However, while this approach is capable of correctly predicting many of the experimentally observed phenomena, it also has certain disadvantages from the phenomenological (engineering) point of view: the use of a non-convex free energy does not guarantee that a hysteresis actually occurs, and simple functional relations like (1.7) are certainly not able to give a correct account of the inherent memory structures that are responsible for the complicated loopings in the interior of the external hysteresis loops that are observed in experiments.

To avoid these difficulties, the first two authors have recently proposed a different approach using the theory of *hysteresis operators* developed in the past twenty years (let us at least refer to the monographs [8], [20], [22], [2], [9] devoted to this subject). In this approach, we replace the relations (1.9) by the identities (1.2), (1.5), where the expressions $\mathcal{H}_j$, $1 \le j \le 4$ and $\mathcal{F}_1$ are no longer real-valued *functions* but *hysteresis operators* acting between suitable function spaces. This approach has been successfully carried out for the two cases when either we have one-dimensional thermoelastoplastic hysteresis without phase transitions (that is, we have (1.1)–(1.3) with no dependence on w , cf. the papers [10], [11]) or we have a multi-dimensional phase transition without mechanical effects (that is, we have (1.3)–(1.5) with no dependence on u , σ , see [7], [13]–[17]). In this paper, we want to extend some of these results to the fully coupled problem.

In this connection, we also refer the reader to the forthcoming paper [18] where the present authors have investigated a related version of system (1.1)–(1.7): an additional curvature term $\gamma\, u_{xxxx}$, $\gamma > 0$, was added on the left of (1.1), and the boundary conditions for u were of the form $u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0$. We remark at this place that our analysis will not apply to the case when (1.1) is complemented with zero boundary conditions at both ends of the wire; we have to assume a stress-free regime at one of the ends in order to be able to perform a transformation due to [1].

Let us recall some basic facts about the notion of hysteresis operator (for details, we refer to the monographs mentioned above). Let $T > 0$ denote some (final) time. A mapping $\mathcal{H}$ from the set $\mathrm{Map}[0,T] := \{w : [0,T] \to \mathbb{R}\}$ into itself is called a *hysteresis operator* if it is *causal*, that is, if for all $w_1, w_2 \in \mathrm{Map}[0,T]$ and $t \in [0,T]$ we have the implication

$$w_1(\tau) = w_2(\tau) \quad \forall\, \tau \in [0,t] \;\Rightarrow\; \mathcal{H}[w_1](t) = \mathcal{H}[w_2](t)\,,$$

and if it is *rate-independent*, that is, if for every $w \in \mathrm{Map}[0,T]$ and every continuous increasing mapping α of $[0,T]$ onto $[0,T]$ we have

$$\mathcal{H}[w \circ \alpha](t) = \mathcal{H}[w](\alpha(t)) \quad \forall\, t \in [0,T]\,.$$

In the case of partial differential equations, when the input functions not only depend on a time variable $t \in [0,T]$ but also on a space variable $x \in [0,1]$, it is necessary to extend the above notion. In this situation, it is natural to associate with a hysteresis operator $\mathcal{H}$ defined on $\mathrm{Map}\,[0,T]$ in the above sense an operator $\hat{\mathcal{H}}$ acting on $\mathrm{Map}\,([0,1]\times[0,T])$ by simply putting

$$\hat{\mathcal{H}}[w](x,t) \,:=\, \mathcal{H}[w(x,\cdot)](t)\,. \tag{1.14}$$

It is customary to identify the operators $\mathcal{H}$ and $\hat{\mathcal{H}}$. The hysteresis operators appearing in (1.1)–(1.7) have to be understood in this way.

The advantage of this approach is that an operator equation like (1.2), (1.5), is suited much better than a simple relation like (1.9) to keep track of the memory effects imprinted on the material in the past history. In fact, the output at any time $t \in [0,T]$ may depend on the whole evolution of the input in the time interval $[0,t]$. Observe that the rate-independence implies that the hysteresis behaviour cannot be expressed in terms of an integral operator of convolution type, i. e. we are not dealing with a model with fading memory.

Unfortunately, there are also disadvantages: the input-output behaviour of hysteresis operators usually cannot be described explicitly, and they have, as a rule, only very restricted smoothness properties. In fact, nontrivial hysteresis operators are, as a rule, *not differentiable*, but at best only (possibly locally) *Lipschitz continuous* in suitable function spaces; in addition, they carry a *nonlocal memory* with respect to time.

Both non-differentiability and presence of a memory are unpleasant features from the mathematical point of view. For instance, the classical method of deriving higher order a priori estimates for w (namely, differentiation of (1.4) with respect to t and testing with w_t) does not immediately work, since there is no chain rule for the hysteretic nonlinearities; also, we may not simply differentiate (1.2) or (1.5) with respect to x. These facts result in a lack of compactness and thus in difficulties in existence proofs.

However, hysteresis operators usually dissipate energy which typically is proportional to the area of closed traversed loops in the hysteresis diagram. Let us explain this fact for one fundamental hysteresis operator which plays a most prominent rule in the theory, namely the so-called *stop operator* or *Prandtl's normalized elastic-perfectly plastic element.* To this end, let $r>0$ (the yield limit) and $\sigma_r^0 \in [-r,r]$ (the initial stress) be given. For any input function $\varepsilon \in W^{1,1}(0,T)$, we define the output $\sigma_r \in W^{1,1}(0,T)$ as the solution to the variational inequality (the index t denotes time differentiation)

$$\sigma_r(t) \in [-r,r] \quad \forall\, t \in [0,T]\,, \quad \sigma_r(0) \,=\, \sigma_r^0\,, \tag{1.15}$$

$$(\varepsilon_t(t) \,-\, \sigma_{r,t}(t))\;(\sigma_r(t) \,-\, \eta) \,\geq\, 0 \quad \forall\, \eta \in [-r,r]\,, \quad \text{a.e. in } (0,T)\,, \tag{1.16}$$

In Fig. 2, the typical input-output behaviour is depicted.

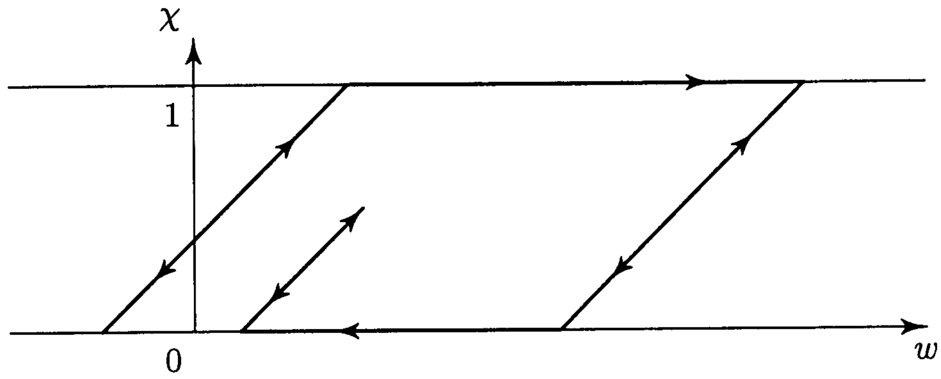

Fig. 2. Prandtl's normalized elastic-perfectly plastic element.

It can easily be proved (see, for instance, [9], where also the multi-dimensional case is treated) that (1.15)–(1.16) admits a unique solution $\sigma_r \in W^{1,1}(0,T)$ for every $\varepsilon \in W^{1,1}(0,T)$ and $\sigma_r^0 \in [-r, r]$. The corresponding solution operator

$$s_r : [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T) : (\sigma_r^0 \,, \varepsilon) \mapsto \sigma_r \,, \tag{1.17}$$

is just the stop operator. It has the well-known property (cf. [2], [9]) that for any $r_1 \,, r_2 \in [0,+\infty)$, $\sigma_{r_j}^0 \in [-r_j, +r_j]$, $j = 1,2$, $t \in [0,T]$, and $\varepsilon_1 \,, \varepsilon_2 \in C[0,T]$, it holds

$$\begin{aligned} &\left| s_{r_1}[\sigma_{r_1}^0, \varepsilon_1](t) - s_{r_2}[\sigma_{r_2}^0, \varepsilon_2](t) \right| \\ \leq\ &|\varepsilon_1(t) - \varepsilon_2(t)| + \max\left\{ |r_1 - r_2| + \max_{0 \le \tau \le t} |\varepsilon_1(\tau) - \varepsilon_2(\tau)| \,, \left| \sigma_{r_1}^0 - \sigma_{r_2}^0 \right| \right\}. \end{aligned} \tag{1.18}$$

Besides, it holds for any $\varepsilon \in W^{1,1}(0,T)$

$$\left\| s_r[\sigma_r^0, \varepsilon] \right\|_\infty \leq r \,, \quad s_r[\sigma_r^0, \varepsilon]_t(t) = \varepsilon_t(t) \quad \text{if} \quad \left| s_r[\sigma_r^0, \varepsilon](t) \right| < r \,, \tag{1.19}$$

$$\left| s_r[\sigma_r^0, \varepsilon]_t \right|^2 = s_r[\sigma_r^0, \varepsilon]_t \, \varepsilon_t \,, \quad \text{a.e. in } (0,T) \,. \tag{1.20}$$

The intrinsic dissipation property of the stop operator results if we insert $\eta = 0$ in (1.16). We then obtain that the energy $\mathcal{P}_r := \frac{1}{2} s_r^2$ of the stop element satisfies the inequality

$$\frac{d}{dt} \mathcal{P}_r[\sigma_r^0, \varepsilon](t) \ \leq\ s_r[\sigma_r^0, \varepsilon](t) \ \varepsilon_t(t) \quad \text{a.e. in } (0,T) \,, \tag{1.21}$$

for all $(\sigma_r^0 \,, \varepsilon) \in [-r, r] \times W^{1,1}(0,T)$, and the difference between the right and the left of (1.21) is the dissipated energy. Equation (1.21) can also be interpreted as a *chain rule inequality* for the energy operator $\mathcal{P}_r$ where the stop operator s_r plays the role of the "derivative" of $\mathcal{P}_r$ with respect to ε (only formally, since $\mathcal{P}_r$ is certainly not differentiable with respect to ε).

Chain rule inequalities of the form (1.21) have proven to be crucial for a successful study of differential equations with hysteresis (for this, see the cited literature). In the case of system (1.1)–(1.7), an appropriate form of such a condition is to postulate the existence of a further hysteresis operator $\mathcal{F}_2$ such that for any $(\varepsilon, w) \in W^{1,1}(0,T) \times W^{1,1}(0,T)$ it holds, for a. e. $t \in (0,T)$,

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_1[\varepsilon,w](t) &\leq \mathcal{H}_1[\varepsilon,w](t)\,\varepsilon_t(t) + \mathcal{H}_3[\varepsilon,w](t)\,w_t(t)\,, \\ \frac{d}{dt}\mathcal{F}_2[\varepsilon,w](t) &\leq \mathcal{H}_2[\varepsilon,w](t)\,\varepsilon_t(t) + \mathcal{H}_4[\varepsilon,w](t)\,w_t(t)\,. \end{aligned} \tag{1.22}$$

We then can associate with system (1.1)–(1.7) free energy, entropy, and internal energy *hysteresis operators* by putting (compare (1.8), (1.10))

$$\begin{aligned} \mathcal{F}[\varepsilon,w,\theta] &:= \theta(1-\log(\theta)) + \mathcal{F}_1[\varepsilon,w] + \theta\,\mathcal{F}_2[\varepsilon,w]\,, \\ \mathcal{S}[\varepsilon,w,\theta] &:= \log(\theta) - \mathcal{F}_2[\varepsilon,w]\,, \\ \mathcal{U}[\varepsilon,w,\theta] &:= \theta + \mathcal{F}_1[\varepsilon,w]\,, \end{aligned} \tag{1.23}$$

where $[\varepsilon,w,\theta] \in \mathrm{Map}[0,T] \times \mathrm{Map}[0,T] \times (0,+\infty)$. Indeed, if we associate σ and ψ as given by (1.2) and (1.5), respectively, with the "derivatives" of $\mathcal{F}$ with respect to ε and w (only formally, as they do not exist), respectively, then we arrive at system (1.1)–(1.5) as field equations. It will turn out later that the validity of (1.22) (rather, of a generalized version thereof, see below) will guarantee the thermodynamic consistency of the model, that is, the temperature stays positive during the evolution, and the *Clausius-Duhem inequality*, which in view of (1.12) can be written in the form

$$\theta\,\frac{d}{dt}\mathcal{S}[\varepsilon,w,\theta] - \frac{d}{dt}\mathcal{U}[\varepsilon,w,\theta] \geq -\tilde{\sigma}\,\varepsilon_t\,, \quad \text{a.e. in } \Omega_T\,, \tag{1.24}$$

where $\tilde{\sigma} = \sigma + \varepsilon_t$ again denotes the total stress, will be satisfied.

The rest of the paper is organized as follows: In section 2, we give a detailed statement of the mathematical problem and of the main mathematical result. Section 3 brings the proof of local existence and global uniqueness, and in the concluding section 4 we prove global existence for system (1.1)–(1.7).

In what follows, the norms of the standard Lebesgue spaces $L^p(\Omega)$, for $1 \leq p \leq \infty$, will be denoted by $\|\cdot\|_p$. Finally, we shall use the usual denotations $W^{m,p}(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}$, $1 \leq p \leq \infty$, for the standard Sobolev spaces.

2 Statement of the problem

We make the following general assumptions on the data of the system.

(H1) $u_0 \in H^2(\Omega)$, $u_1 \in H^1(\Omega)$, $\theta_0 \in H^1(\Omega)$, $w_0 \in H^1(\Omega)$, it holds $\theta_0(x) \geq \delta > 0$ for all $x \in \bar{\Omega}$, and the compatibility condition $u_0(0) = u_1(0) = 0$ is satisfied.

(H2) It holds $f \in H^1(0,T; L^2(\Omega))$.

(H3) We assume that $g : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\exists\, g_0 \in L^\infty(\Omega_T) : \quad \theta \leq 0 \Rightarrow g(x,t,\theta) = g_0(x,t), \tag{2.1}$$

$$\exists\, K_1 > 0 : \quad \left|\frac{\partial g}{\partial \theta}\right| \leq K_1 \quad \text{a.e. in } \Omega \times (0,T) \times \mathbb{R}, \tag{2.2}$$

$$g_0(x,t) \geq 0 \qquad \text{a.e. in } \Omega_T. \tag{2.3}$$

(H4) The operators $\mathcal{H}_j$, $1 \leq j \leq 4$, and $\mathcal{F}_1$ are causal and map $C[0,T] \times C[0,T]$ into $C[0,T]$ and $W^{1,1}(0,T) \times W^{1,1}(0,T)$ into $W^{1,1}(0,T)$. Besides, the following conditions are satisfied:

(i) $\exists\, K_2 > 0 : \forall\, \varepsilon, w \in C[0,T]$ it holds

$$\max_{j \in \{2,4\}} \|\mathcal{H}_j[\varepsilon, w]\|_\infty \leq K_2, \quad \mathcal{F}_1[\varepsilon, w](t) \geq -K_2 \quad \forall\, t \in [0,T]. \tag{2.4}$$

(ii) $\exists\, K_3 > 0 : \forall\, \varepsilon, w \in W^{1,1}(0,T)$ it holds, for a.e. $t \in (0,T)$,

$$\max_{1 \leq j \leq 4} |\mathcal{H}_j[\varepsilon, w]_t(t)| + |\mathcal{F}_1[\varepsilon, w]_t(t)| \leq K_3 \Big(|\varepsilon_t(t)| + |w_t(t)|\Big). \tag{2.5}$$

(iii) $\exists\, K_4 > 0 : \forall\, \varepsilon_1, w_1, \varepsilon_2, w_2 \in C[0,T]$ it holds, for every $t \in [0,T]$,

$$\max_{1 \leq j \leq 4} |\mathcal{H}_j[\varepsilon_1, w_1](t) - \mathcal{H}_j[\varepsilon_2, w_2](t)| \leq K_4 \max_{0 \leq r \leq t} \Big(|\varepsilon_1(r) - \varepsilon_2(r)| + |w_1(r) - w_2(r)|\Big), \tag{2.6}$$

$$|\mathcal{F}_1[\varepsilon_1, w_1](t) - \mathcal{F}_1[\varepsilon_2, w_2](t)| \leq K_4 \Big[|\varepsilon_1(0) - \varepsilon_2(0)| + |w_1(0) - w_2(0)| + \int_0^t \Big(|\varepsilon_{1,t}(r) - \varepsilon_{2,t}(r)| + |w_{1,t}(r) - w_{2,t}(r)|\Big)\, dr\Big]. \tag{2.7}$$

(H5) There exist causal operators $\mathcal{F}_2 : W^{1,1}(0,T) \times W^{1,1}(0,T) \to W^{1,1}(0,T)$, $\mathcal{G} : W^{1,1}(0,T) \to W^{1,1}(0,T)$, and a constant $K_5 > 0$, such that the following conditions are satisfied:

(i) For every $\varepsilon, w \in W^{1,1}(0,T)$ it holds

$$\mathcal{F}_1[\varepsilon, w]_t \leq \varepsilon_t\, \mathcal{H}_1[\varepsilon, w] + \mathcal{G}[w]_t\, \mathcal{H}_3[\varepsilon, w] \quad \text{a.e. in } (0,T), \tag{2.8}$$

$$\mathcal{F}_2[v, w]_t \leq \varepsilon_t\, \mathcal{H}_2[\varepsilon, w] + \mathcal{G}[w]_t\, \mathcal{H}_4[\varepsilon, w] \quad \text{a.e. in } (0,T). \tag{2.9}$$

(ii) For every $w \in W^{1,1}(0,T)$ it holds

$$|\mathcal{G}[w]_t(t)|^2 \;\le\; K_5\, w_t(t)\, \mathcal{G}[w]_t(t) \quad \text{for a.e. } t \in (0,T)\,. \tag{2.10}$$

Remark 1. Owing to **(H4),(iii)** we have, in particular, that for any $\varepsilon\,,\, w \in H^1(0,T)$ and $t \in [0,T]$ it holds

$$\begin{aligned}
&|\mathcal{H}_1[\varepsilon,w](t)| \,+\, |\mathcal{H}_3[\varepsilon,w](t)| \\
\le\;& |\mathcal{H}_1[\varepsilon,w](0)| \,+\, |\mathcal{H}_3[\varepsilon,w](0)| \,+\, 2\,K_4 \max_{0\le r\le t}\left(|\varepsilon(r)-\varepsilon(0)| \,+\, |w(r)-w(0)|\right) \\
\le\;& |\mathcal{H}_1[\varepsilon,w](0)| \,+\, |\mathcal{H}_3[\varepsilon,w](0)| \,+\, 2\,K_4 \int_0^t \Big(|\varepsilon_t(r)| \,+\, |w_t(r)|\Big)\,dr \\
\le\;& |\mathcal{H}_1[\varepsilon,w](0)| \,+\, |\mathcal{H}_3[\varepsilon,w](0)| \,+\, 2\,K_4 \sqrt{t}\,\Big(\int_0^t \Big(|\varepsilon_t(r)|^2 \,+\, |w_t(r)|^2\Big)\,dr\Big)^{1/2}
\end{aligned} \tag{2.11}$$

Besides, a linear growth of $\mathcal{H}_1$ and $\mathcal{H}_3$ with respect to both ε and w is admitted, which, in particular, includes the case of simple linear elasticity. It also follows that for any $\varepsilon, w \in H^1(\Omega; C[0,T])$ it holds, for a.e. $(x,t) \in \Omega_T$,

$$\max_{1\le j\le 4} \left|\Big(\mathcal{H}_j[\varepsilon,w]\Big)_x(x,t)\right| \;\le\; K_4 \max_{0\le r\le t}\Big(|\varepsilon_x(x,r)| \,+\, |w_x(x,r)|\Big)\,. \tag{2.12}$$

Indeed, we only have to apply (2.6) with $\varepsilon_1(x,t) := \varepsilon(x+h,t)$, $\varepsilon_2(x,t) := \varepsilon(x,t)$, $w_1(x,t) := w(x+h,t)$, $w_2(x,t) := w(x,t)$, with some $h > 0$, and then let $h \searrow 0$. Consequently, we may consider first order spatial derivatives of $\mathcal{H}_j[\varepsilon,w]$, and we have $\Big(\mathcal{H}_j[\varepsilon,w]\Big)_x \in L^2(\Omega_T)$, $1 \le j \le 4$.

Remark 2. A typical example where **(H4)**, **(H5)** are fulfilled is given by *Prandtl-Ishlinskii operators* of the form

$$\begin{aligned}
\mathcal{H}_j[\varepsilon,w] &:= \int_0^\infty \varphi_j(r)\, s_r\left[\sigma_r^{0,j}\,,\,\varepsilon\right]\,dr\,, \quad j=1,2\,, \\
\mathcal{H}_j[\varepsilon,w] &:= \int_0^\infty \varphi_j(r)\, s_r\left[\sigma_r^{0,j}\,,\,w\right]\,dr\,, \quad j=3,4\,,
\end{aligned} \tag{2.13}$$

where $\sigma_r^{0,j} \in [-r,+r]$, $1 \le j \le 4$, are given initial values for the operators s_r defined in (1.17), and the weight functions φ_j are non-negative on $[0,+\infty)$ and satisfy

$$\max_{1\le j\le 4} \int_0^\infty (1+r^2)\,\varphi_j(r)\,dr \;<\; +\infty\,. \tag{2.14}$$

Indeed, defining the (energy) operators

$$\begin{aligned}
\mathcal{F}_1[\varepsilon,w] &:= \frac{1}{2}\int_0^\infty \Big(\varphi_1(r)\, s_r^2\left[\sigma_r^{0,1}\,,\,\varepsilon\right] \,+\, \varphi_3(r)\, s_r^2\left[\sigma_r^{0,3}\,,\,w\right]\Big)\,dr \\
\mathcal{F}_2[\varepsilon,w] &:= \frac{1}{2}\int_0^\infty \Big(\varphi_2(r)\, s_r^2\left[\sigma_r^{0,2}\,,\,\varepsilon\right] \,+\, \varphi_4(r)\, s_r^2\left[\sigma_r^{0,4}\,,\,w\right]\Big)\,dr\,,
\end{aligned} \tag{2.15}$$

choosing $\mathcal{G}[w] = w$, and invoking the properties (1.18)–(1.21) of the stop operators s_r , we easily verify the validity of **(H4)**, **(H5)**. Other examples, where the dependence on ε , w is no longer decoupled as in (2.13), can be constructed using multi-dimensional stop operators as basic elements (cf. [15], [16]). For examples where the $\mathcal{H}_j$ are not Prandtl-Ishlinskii operators and $\mathcal{G}$ differs from the identity operator we refer to [14], [15].

We can now formulate the main result of this paper.

Theorem 2.1 *Suppose that the hypotheses* **(H1)** *to* **(H5)** *are satisfied. Then the system* (1.1)–(1.7) *admits a unique strong solution* (u,θ,w) *such that* (1.1)–(1.5) *hold a. e. in* Ω_T *, and such that*

$$u \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^2(\Omega)), \quad w \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)),$$
$$\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)). \tag{2.16}$$

Besides, with the finite norms $\beta_1 := \|u_{xt}\|_{L^1(0,T;L^\infty(\Omega))}$ *and* $\beta_2 := \|w_t\|_{C(\overline{\Omega_T})}$ *it holds*

$$\theta(x,t) \ \geq \ \delta\, e^{-((K_1+K_2\,K_5\,\beta_2)\,t+K_2\,\beta_1)} \quad \text{for all } (x,t) \in \overline{\Omega_T}\,. \tag{2.17}$$

Remark 3. We note at this place that Theorem 2.1 also implies that the second principle of thermodynamics is satisfied for the system (1.1)–(1.7). Indeed, we have $\theta > 0$ a.e. in Ω_T , and the validity of the Clausius-Duhem inequality (1.24) follows from the simple calculation

$$\begin{aligned} & \theta\,\mathcal{S}[\varepsilon,w,\theta]_t \,-\, \mathcal{U}[\varepsilon,w,\theta]_t \,+\, \tilde\sigma\,\varepsilon_t \ = \ -\,\theta\,\mathcal{F}_2[\varepsilon,w]_t \,-\, \mathcal{F}_1[\varepsilon,w]_t \,+\, \sigma\,\varepsilon_t \,+\, \varepsilon_t^2 \\ \geq \ & -\Big(\mathcal{H}_1[\varepsilon,w] \,+\, \theta\,\mathcal{H}_2[\varepsilon,w]\Big)\,\varepsilon_t \,-\, \Big(\mathcal{H}_3[\varepsilon,w] \,+\, \theta\,\mathcal{H}_4[\varepsilon,w]\Big)\,\mathcal{G}[w]_t \,+\, \sigma\,\varepsilon_t \,+\, \varepsilon_t^2 \\ \geq \ & \varepsilon_t^2 \,+\, w_t\,\mathcal{G}[w]_t \ \geq \ 0 \quad \text{a.e. in } \Omega_T\,, \end{aligned} \tag{2.18}$$

where $\mathcal{F}$, $\mathcal{S}$, $\mathcal{U}$ are given by (1.23). We may therefore claim that our system is thermodynamically consistent.

The proof of Theorem 2.1 will be given in the following sections. During its course, we will make repeated use of *Young's inequality*

$$a\,b \ \leq \ \frac{\delta}{2}\,a^2 \,+\, \frac{1}{2\,\delta}\,b^2 \quad \forall\, a,b \in \mathbb{R}\,,\ \delta > 0\,, \tag{2.19}$$

of the elementary inequality

$$|z(t)|^2 \ \leq \ 2\,|z(0)|^2 \,+\, 2\,t\int_0^t z_t^2(r)\,dr \quad \forall\, t \in (0,T) \quad \forall\, z \in H^1(0,T)\,, \tag{2.20}$$

and of the one-dimensional *Gagliardo-Nirenberg inequality*

$$\|w\|_p \;\le\; K_0 \left(\|w\|_q^{1-\omega} \; \|w_x\|_r^{\omega} \;+\; \|w\|_q \right) \quad \forall\, w \in W^{1,r}(\Omega) \cap L^q(\Omega)\,, \tag{2.21}$$

where $K_0 > 0$ is a constant depending only on p, q, r, and where

$$1 \le r \le +\infty\,, \quad 1 \le q \le p \le +\infty\,, \quad \omega \left(\frac{1}{q} - \frac{1}{r} + 1 \right) = \frac{1}{q} - \frac{1}{p}\,. \tag{2.22}$$

3 Local existence

We rewrite the system (1.1)–(1.7), using the transformation due to Andrews [1],

$$u_x \;=\; p + q\,, \quad \text{where } p(x,t) \;:=\; \int_1^x u_t(\xi, t)\, d\xi\,. \tag{3.1}$$

We easily find that (1.1)–(1.6) is equivalent to the system

$$p_t \;-\; p_{xx} \;=\; \sigma \;+\; \int_1^x f(\xi, t)\, d\xi\,, \tag{3.2}$$

$$p(1,t) \;=\; p_x(0,t) \;=\; 0\,, \quad p(x,0) \;=\; \int_1^x u_1(\xi)\, d\xi\,, \tag{3.3}$$

$$\sigma \;=\; \mathcal{H}_1[p+q, w] \;+\; \theta\, \mathcal{H}_2[p+q, w]\,, \tag{3.4}$$

$$q_t \;=\; -\,\sigma \;-\; \int_1^x f(\xi, t)\, d\xi\,, \tag{3.5}$$

$$q(x,0) = u_0'(x) \;-\; \int_1^x u_1(\xi)\, d\xi\,, \tag{3.6}$$

$$\Big(\theta \;+\; \mathcal{F}_1[p+q, w]\Big)_t \;-\; \theta_{xx} \;=\; p_{xx}^2 \;+\; \sigma\, p_{xx} \;+\; g(x,t,\theta)\,, \tag{3.7}$$

$$w_t \;=\; -\,\psi\,, \tag{3.8}$$

$$\psi \;=\; \mathcal{H}_3[p+q, w] \;+\; \theta\, \mathcal{H}_4[p+q, w]\,, \tag{3.9}$$

$$\theta(x,0) = \theta_0(x)\,, \quad w(x,0) = w_0(x)\,, \quad \theta_x(0,t) = \theta_x(1,t) = 0\,. \tag{3.10}$$

Let $V_0 := \{ z \in H^1(\Omega)\,;\, z(1) = 0 \}$, and let V_0^* denote its dual space. We are going to show the following result.

Theorem 3.1 *Suppose that the hypotheses* **(H1)** *to* **(H4)** *are fulfilled. Then there is some* $\hat{\tau} > 0$ *such that the initial-boundary value problem* (3.2)–(3.10) *admits a unique solution quadruple* (p, q, θ, w) *on* $\bar{\Omega} \times [0, \hat{\tau}]$ *satisfying*

$$p \in H^2(0,\hat{\tau}; V_0^*) \,\cap\, H^1(0,\hat{\tau}; H^1(\Omega)) \,\cap\, L^2(0,\hat{\tau}; H^3(\Omega))\,, \tag{3.11}$$

$$q\,,\, w \,\in H^2(0,\hat{\tau}; L^2(\Omega)) \,\cap\, H^1(0,\hat{\tau}; H^1(\Omega)) \,\cap\, C^1([0,\hat{\tau}]; C(\bar{\Omega}))\,, \tag{3.12}$$

$$\theta \in H^1(0,\hat{\tau}; L^2(\Omega)) \,\cap\, L^2(0,\hat{\tau}; H^2(\Omega)) \,\cap\, C(\overline{\Omega_{\hat{\tau}}})\,, \tag{3.13}$$

$$\theta(x,t) \ge \frac{\delta}{2} > 0 \quad \text{for every } (x,t) \in \bar{\Omega} \times [0,\hat{\tau}]\,. \tag{3.14}$$

Proof: We divide the proof of Theorem 3.1 into several steps, each formulated as a separate lemma. The existence part of the proof is based on the following special case of the *Schauder–Tikhonov fixed point principle* (cf., for instance, Theorem 3.6.1 in [6]):

Lemma 3.2 *Let the operator $\mathcal{T}$ map the nonempty, closed, convex, and weakly compact subset $\mathcal{M}$ of the separable Hilbert space X into itself, and suppose that $\mathcal{T}$ is weakly sequentially continuous on $\mathcal{M}$, that is, it holds $\mathcal{T}(v_n) \to \mathcal{T}(v)$ weakly in X whenever $v_n \to v$ weakly in X for some sequence $\{v_n\} \subset \mathcal{M}$. Then $\mathcal{T}$ has a fixed point in $\mathcal{M}$.*

We aim to apply Lemma 3.2 to the following setting. Consider for $\tau \in (0,T]$ the separable Hilbert spaces

$$\begin{aligned} P_\tau &:= H^2(0,\tau;V_0^*) \cap H^1(0,\tau;H^1(\Omega)) \cap L^2(0,\tau;H^3(\Omega)), \\ Q_\tau &:= H^2(0,\tau;L^2(\Omega)) \cap H^1(0,\tau;H^1(\Omega)), \\ Z_\tau &:= H^1(0,\tau;L^2(\Omega)) \cap L^2(0,\tau;H^2(\Omega)), \\ W_\tau &:= H^2(0,\tau;L^2(\Omega)) \cap H^1(0,\tau;H^1(\Omega)), \\ X_\tau &:= P_\tau \times Q_\tau \times Z_\tau \times W_\tau, \end{aligned} \tag{3.15}$$

and introduce the sets

$$\begin{aligned} \mathcal{M}_\tau := \Big\{ &(p,q,\theta,w) \in X_\tau \,;\ (3.3),\ (3.6),\ (3.10),\ \text{hold},\ p_t + q_t = p_{xx} \quad \text{a.e. in } \Omega_\tau, \\ &\int_0^\tau \int_\Omega \left(\theta_t^2 + \theta_{xx}^2\right) dx\,dt + \max_{0\le t\le\tau} \int_\Omega |\theta_x(x,t)|^2\,dx \le M_1, && (3.16) \\ &\max_{(x,t)\in\overline{\Omega_\tau}} |\theta(x,t)| \le M_2, && (3.17) \\ &\max_{0\le t\le\tau} \int_\Omega \left(|q_t(x,t)|^2 + |w_t(x,t)|^2\right) dx \le M_3, && (3.18) \\ &\int_0^\tau \int_\Omega \left(q_{xt}^2 + w_{xt}^2\right) dx\,dt \le M_4, && (3.19) \\ &\int_0^\tau \int_\Omega \left(p_t^2 + p_{xx}^2\right) dt + \max_{0\le t\le\tau} \int_\Omega |p_x(x,t)|^2\,dx \le M_5, && (3.20) \\ &\max_{0\le t\le\tau} \int_\Omega \left(|q_x(x,t)|^2 + |w_x(x,t)|^2\right) dx \le M_6, && (3.21) \\ &\|p_t\|^2_{H^1(0,\tau;V_0^*)} + \int_0^\tau \int_\Omega p_{xt}^2\,dx\,dt + \max_{0\le t\le\tau} \int_\Omega |p_t(x,t)|^2\,dx \le M_7, && (3.22) \\ &\max_{0\le t\le\tau} \int_\Omega |p_{xx}(x,t)|^2\,dx \le M_8, && (3.23) \\ &\int_0^\tau \int_\Omega p_{xxx}^2\,dx\,dt \le M_9, && (3.24) \\ &\min_{(x,t)\in\overline{\Omega_\tau}} \theta(x,t) \ge \frac{\delta}{2} > 0 \Big\}, && (3.25) \end{aligned}$$

where the positive constants M_i, $i = 1, \dots, 9$, will have to be specified later. Obviously, $\mathcal{M}_\tau$ is a nonempty, closed, convex, and bounded (hence weakly compact) subset of the separable Hilbert space X_τ .

Next, we introduce the operator $\mathcal{T}$ on $\mathcal{M}_\tau$ by $\mathcal{T}(\bar{p},\bar{q},\bar{\theta},\bar{w}) := (p,q,\theta,w)$, where for $(\bar{p},\bar{q},\bar{\theta},\bar{w}) \in \mathcal{M}_\tau$ the quadruple (p,q,θ,w) is the unique solution to the linear initial-boundary value problem

$$p_t - p_{xx} = \bar{\sigma} + \int_1^x f(\xi,t)\,d\xi\,, \tag{3.26}$$

$$p(1,t) = p_x(0,t) = 0\,, \quad p(x,0) = \int_1^x u_1(\xi)\,d\xi\,, \tag{3.27}$$

$$\bar{\sigma} = \mathcal{H}_1[\bar{p}+\bar{q},\bar{w}] + \bar{\theta}\,\mathcal{H}_2[\bar{p}+\bar{q},\bar{w}]\,, \tag{3.28}$$

$$q_t = -\bar{\sigma} - \int_1^x f(\xi,t)\,d\xi\,, \tag{3.29}$$

$$q(x,0) = u_0'(x) - \int_1^x u_1(\xi)\,d\xi\,, \tag{3.30}$$

$$\theta_t - \theta_{xx} = -\mathcal{F}_1[\bar{p}+\bar{q},\bar{w}]_t + \bar{p}_{xx}^2 + \bar{\sigma}\,\bar{p}_{xx} + g(x,t,\bar{\theta})\,, \tag{3.31}$$

$$w_t = -\bar{\psi}\,, \tag{3.32}$$

$$\bar{\psi} = \mathcal{H}_3[\bar{p}+\bar{q},\bar{w}] + \bar{\theta}\,\mathcal{H}_4[\bar{p}+\bar{q},\bar{w}]\,, \tag{3.33}$$

$$\theta(x,0) = \theta_0(x)\,, \quad w(x,0) = w_0(x)\,, \quad \theta_x(0,t) = \theta_x(1,t) = 0\,. \tag{3.34}$$

We have the following result.

Lemma 3.3 *There exist $\hat{\tau} \in (0,T]$ and positive constants M_i , $i = 1,\ldots,9$, such that $\mathcal{T}(\mathcal{M}_\tau) \subset \mathcal{M}_\tau$ for any $\tau \in (0,\hat{\tau}]$.*

Proof: Let $\tau \in (0,T]$, be given. Without loss of generality, we may assume that $\tau \le 1$. We have $p_t + q_t = p_{xx}$ a. e. in Ω_τ , and we infer from the general hypotheses that p_t , p_{xx} , q_t , w_t , θ_t , $\theta_{xx} \in L^2(\Omega_\tau)$. Therefore, $\theta \in Z_\tau$. Besides, since $(\bar{p},\bar{q},\bar{\theta},\bar{w}) \in \mathcal{M}_\tau$, it follows from Remark 1. that the right-hand sides of both (3.29) and (3.32) belong to $H^1(\Omega_\tau)$ so that $q \in Q_\tau$ and $w \in W_\tau$.

Next, we consider the parabolic initial-boundary value problem

$$z_t - z_{xx} = v := \bar{\sigma}_t + \int_1^x f_t(\xi,t)\,d\xi\,, \tag{3.35}$$

$$z_x(0,t) = z(1,t) = 0\,, \quad z(x,0) = u_1'(x) + \bar{\sigma}(0) + \int_1^x f(\xi,0)\,d\xi\,. \tag{3.36}$$

Since $z(\cdot,0) \in L^2(\Omega)$, and since the right-hand side v of (3.35) belongs to $L^2(\Omega_\tau)$, it follows from general linear parabolic theory (cf. Lions-Magenes [19]), that (3.35)–(3.36) admits a unique weak solution $z \in L^2(0,\tau;H^1(\Omega)) \cap H^1(0,\tau;V_0^*)$

$\cap C([0,T];\, L^2(\Omega))$, and there is some constant $\hat{C} > 0$, not depending on $\tau \in (0,T]$, such that

$$\|z\|^2_{H^1(0,\tau;V_0^*)} + \int_0^\tau\int_\Omega z_x^2\,dx\,dt + \max_{0\le t\le\tau}\int_\Omega |z(x,t)|^2\,dx$$
$$\le \hat{C}\Big(\int_\Omega |z(x,0)|^2\,dx + \int_0^\tau\int_\Omega v^2\,dx\,dt\Big). \qquad (3.37)$$

Invoking the compatibility condition $u_1(0) = 0$, we easily verify that

$$p(x,t) = \int_1^x u_1(\xi)\,d\xi + \int_0^t z(x,r)\,dr\,, \qquad (3.38)$$

so that $p \in H^2(0,\tau;V_0^*) \cap H^1(0,\tau;H^1(\Omega)) \cap C^1([0,T];\, L^2(\Omega))$, and (3.37) holds for $z = p_t$. Hence, using (3.26), we can conclude that $p_{xxx} \in L^2(\Omega_\tau)$, and also $p_{xx} \in C([0,T];\, L^2(\Omega))$. In conclusion, $p \in P_\tau$, and we have shown that $\mathcal{T}(\mathcal{M}_\tau) \subset X_\tau$.

Now let $(p,q,\theta,w) = \mathcal{T}(\bar{p},\bar{q},\bar{\theta},\bar{w})$ for some $(\bar{p},\bar{q},\bar{\theta},\bar{w}) \in \mathcal{M}_\tau$, where the constants $M_1,\ldots,M_9$ and $\tau \in (0,T]$ are assumed to be fixed. We are going to derive a number of estimates for (p,q,θ,w) in terms of $M_1,\ldots,M_9$ and of the data of the system. In what follows, we denote by C_i , $i \in \mathbb{N}\cup\{0\}$, positive constants which may depend on the given data u_0 , u_1 , θ_0 , w_0 , f , g_0 , and on the constants K_i , $0 \le i \le 4$, but neither on τ nor on $M_1,\ldots,M_9$.

At first, we conclude from (2.11) and from (2.5) that

$$\int_\Omega \Big(|\bar{\sigma}|^2 + |\bar{\psi}|^2\Big)(x,t)\,dx$$
$$\le 2\int_\Omega \Big(\mathcal{H}_1^2[\bar{p}+\bar{q},\bar{w}] + \mathcal{H}_3^2[\bar{p}+\bar{q},\bar{w}] + \bar{\theta}^2\left(\mathcal{H}_2^2[\bar{p}+\bar{q},\bar{w}] + \mathcal{H}_4^2[\bar{p}+\bar{q},\bar{w}]\right)\Big)(x,t)\,dx$$
$$\le 2\,K_2^2\,M_2^2 + C_0\left[1 + t\int_0^t\int_\Omega \Big(\bar{p}_t^2 + \bar{q}_t^2 + \bar{w}_t^2\Big)\,dx\,dr\right]$$
$$\le 2\,K_2^2\,M_2^2 + C_0\Big(1 + t(M_3+M_7)\Big) \quad \text{for all } t\in[0,\tau]\,, \qquad (3.39)$$

$$|\bar{\sigma}_t| + |\bar{\psi}_t| \le 2\,K_3\Big((1+M_2)\left(|\bar{p}_t| + |\bar{q}_t| + |\bar{w}_t|\right) + |\bar{\theta}_t|\Big) \quad \text{a.e. in } \Omega_\tau\,. \qquad (3.40)$$

Besides, it follows from (2.12) that for $1 \le i \le 4$ and a. e. $(x,t) \in \Omega_\tau$ we have

$$\Big|\Big(\mathcal{H}_i[\bar{p}+\bar{q},\bar{w}]\Big)_x(x,t)\Big| \le K_4 \max_{0\le r\le t}\Big(|\bar{p}_x(x,r)| + |\bar{q}_x(x,r)| + |\bar{w}_x(x,r)|\Big)\,. \qquad (3.41)$$

In addition, owing to (2.5),

$$|\mathcal{F}_1[\bar{p}+\bar{q},\bar{w}]_t| \le K_3\left(|\bar{p}_t| + |\bar{q}_t| + |\bar{w}_t|\right) \quad \text{a.e. in } \Omega_\tau\,, \qquad (3.42)$$

as well as, by virtue of **(H3)**,

$$|g(x,t,\bar{\theta}(x,t))| \le g_0(x,t) + K_1\,M_2\,. \qquad (3.43)$$

Now multiply (3.31) first by θ_t , and then by $-\theta_{xx}$, add the resulting equations and integrate over $\Omega \times [0,t]$ for any $t \in (0,\tau]$. Using Young's inequality, and invoking (3.39), (3.42), and (3.43), we find that

$$\begin{aligned} &\int_0^t \int_\Omega \Big(\theta_t^2 \,+\, \theta_{xx}^2\Big)\,dx\,dr \,+\, \int_\Omega |\theta_x(x,t)|^2\,dx \\ \le\;\; & C_1\,\Big[1 + tM_2^2 \,+\, \int_0^t \int_\Omega \Big(\bar{p}_t^2 + \bar{q}_t^2 + \bar{w}_t^2\Big)\,dx\,dr \\ &+ \int_0^t \int_\Omega \bar{p}_{xx}^4\,dx\,dr \,+\, \int_0^t \int_\Omega \bar{\sigma}^2\,\bar{p}_{xx}^2\,dx\,dr\Big]\,. \end{aligned} \tag{3.44}$$

Invoking the Gagliardo-Nirenberg inequality (2.21) for $p = +\infty$, $q = r = 2$, $\omega = 1/2$, we infer that

$$\begin{aligned} \int_0^t \|\bar{p}_{xx}(\cdot,r)\|_\infty^2\,dr \;\;&\le\;\; 2\,K_0^2\,\Big(\int_0^t \int_\Omega \bar{p}_{xx}^2\,dx\,dr \\ &\qquad + \max_{0\le r\le t}\,\|\bar{p}_{xx}(\cdot,r)\|_2 \int_0^t \|\bar{p}_{xxx}(\cdot,r)\|_2\,dr\Big) \\ \le\;\; 2\,K_0^2\,\Big(t\,M_8 \,+\, \sqrt{M_8}\,\sqrt{t}\,\sqrt{M_9}\Big) \;\;&\le\;\; 2\,K_0^2\,\sqrt{t}\,\Big(M_8 \,+\, \sqrt{M_8\,M_9}\Big)\,. \end{aligned} \tag{3.45}$$

It follows that

$$\begin{aligned} \int_0^t \int_\Omega \bar{p}_{xx}^4\,dx\,dr \;\;&\le\;\; \max_{0\le r\le t}\,\|\bar{p}_{xx}(\cdot,r)\|_2^2 \int_0^t \|\bar{p}_{xx}(\cdot,r)\|_\infty^2\,dr \\ &\le\;\; 2\,K_0^2\,\sqrt{t}\,\Big(M_8^2 \,+\, M_8^{3/2}\,\sqrt{M_9}\Big)\,. \end{aligned} \tag{3.46}$$

Besides, by (3.39),

$$\begin{aligned} &\int_0^t \int_\Omega \bar{\sigma}^2\,\bar{p}_{xx}^2\,dx\,dr \;\;\le\;\; \max_{0\le r\le t}\,\|\bar{\sigma}(\cdot,r)\|_2^2 \int_0^t \|\bar{p}_{xx}(\cdot,r)\|_\infty^2\,dr \\ \le\;\; & 2\,K_0^2\,\sqrt{t}\,\Big(M_8 \,+\, \sqrt{M_8\,M_9}\Big)\,\Big(2\,K_2^2\,M_2^2 \,+\, C_0\,(1 \,+\, t\,(M_3 + M_7))\Big)\,. \end{aligned} \tag{3.47}$$

In conclusion, we have shown the estimate

$$\begin{aligned} &\int_0^\tau \int_\Omega \Big(\theta_t^2 \,+\, \theta_{xx}^2\Big)\,dx\,dr \,+\, \max_{0\le t\le\tau} \int_\Omega |\theta_x(x,t)|^2\,dx \\ \le\;\; & C_2\,\Big[1 \,+\, \sqrt{\tau}\,\Big(M_2^2 + M_3 + M_7 + M_8^2 \,+\, M_8^{3/2}\,M_9^{1/2} \\ &\qquad + \Big(M_8 \,+\, M_8^{1/2}\,M_9^{1/2}\Big)\,\Big(1 \,+\, M_2^2 \,+\, M_3 \,+\, M_7\Big)\Big)\Big]\,. \end{aligned} \tag{3.48}$$

Next, we consider (3.29) and (3.32). By the general hypotheses and (3.39), we have

$$\max_{0\le t\le\tau} \int_\Omega \Big(|q_t(x,t)|^2 \,+\, |w_t(x,t)|^2\Big)\,dx \;\le\; C_3\,\Big(1 \,+\, M_2^2 \,+\, t\,(M_3 + M_7)\Big)\,. \tag{3.49}$$

Now differentiate (3.29) and (3.32) with respect to x . Then, by (3.41),

$$|\bar{\sigma}_x(x,t)| + |\bar{\psi}_x(x,t)| \;\le\; 2\,K_2\,|\bar{\theta}_x(x,t)| + 2\,K_4\,(1{+}M_2)\,\max_{0\le r\le t}\Big((|\bar{p}_x| + |\bar{q}_x| + |\bar{w}_x|)(x,r)\Big)\,, \tag{3.50}$$

for a. e. $(x,t) \in \Omega_\tau$. Therefore, using (2.20), we can conclude that

$$\int_0^\tau\!\!\int_\Omega \Big(q_{xt}^2 \,+\, w_{xt}^2\Big)\,dx\,dt \;\;\le\;\; C_4\left(1 \,+\, \int_0^\tau\!\!\int_\Omega \Big(|\bar\sigma_x|^2 \,+\, |\bar\psi_x|^2\Big)\,dx\,dt\right)$$

$$\le\;\; C_5\left[1 \,+\, M_2^2 \,+\, \int_0^\tau\!\!\int_\Omega |\bar\theta_x|^2\,dx\,dt \,+\, (1+M_2^2)\,\tau\int_0^\tau\!\!\int_\Omega\Big(\bar p_{xt}^2 \,+\, \bar q_{xt}^2 \,+\, \bar w_{xt}^2\Big)\,dx\,dt\right]$$

$$\le\;\; C_5\Big(1 \,+\, M_2^2 \,+\, \tau\,(M_1 \,+\, (1+M_2^2)\,(M_4 \,+\, M_7))\Big)\,. \tag{3.51}$$

But then also

$$\max_{0\le t\le\tau}\int_\Omega\Big(|q_x(x,t)|^2+|w_x(x,t)|^2\Big)\,dx \;\le\; C_6\Big(1+M_2^2+\tau\,(M_1+(1{+}M_2^2)\,(M_4+M_7))\Big)\,. \tag{3.52}$$

Next, we consider the linear parabolic system (3.26)–(3.27). Standard parabolic estimates, using the general hypotheses and (3.39), yield that

$$\int_0^\tau\!\!\int_\Omega\Big(p_t^2 \,+\, p_{xx}^2\Big)\,dx\,dt \;\le\; C_7\left(1+\int_0^\tau\!\!\int_\Omega|\bar\sigma|^2\,dx\,dt\right) \;\le\; C_8\Big(1+\tau\,(M_2^2+M_3+M_7)\Big)\,. \tag{3.53}$$

Moreover, since (3.37) is valid for $z = p_t$, we can infer from **(H2)** and from (3.40) that

$$\|p\|^2_{H^2(0,\tau;V_0^*)} \,+\, \int_0^\tau\!\!\int_\Omega p_{xt}^2\,dx\,dt \,+\, \max_{0\le t\le\tau}\int_\Omega|p_t(x,t)|^2\,dx$$

$$\le\;\; C_9\left(1 \,+\, \int_0^\tau\!\!\int_\Omega\Big(\bar\theta_t^2 \,+\, (1+M_2^2)\,\Big(\bar p_t^2 \,+\, \bar q_t^2 \,+\, \bar w_t^2\Big)\Big)\,dx\,dt\right)$$

$$\le\;\; C_9\Big(1 \,+\, M_1 \,+\, \tau\,(1+M_2^2)\,(M_3+M_7)\Big)\,. \tag{3.54}$$

But then we obtain from (3.26), also using (3.28) and **(H2)**, that

$$\max_{0\le t\le\tau}\int_\Omega|p_{xx}(x,t)|^2\,dx \;\;\le\;\; 2\max_{0\le t\le\tau}\int_\Omega|p_t(x,t)|^2\,dx+2\,C_{10}\Big(1+\max_{0\le t\le\tau}\int_\Omega|\bar\sigma(x,t)|^2\,dx\Big)$$

$$\le\;\; C_{11}\Big(1+M_1+M_2^2 \,+\, \tau\,(1+M_2^2)\,(M_3+M_7)\Big)\,. \tag{3.55}$$

Besides, employing (3.50), and (3.54), and arguing as in the derivation of (3.51), we can deduce the estimate

$$\int_0^\tau\!\!\int_\Omega p_{xxx}^2\,dx\,dt \;\;\le\;\; 2\int_0^\tau\!\!\int_\Omega p_{xt}^2\,dx\,dt \,+\, 2\int_0^\tau\!\!\int_\Omega|\bar\sigma_x \,+\, f|^2\,dx\,dt$$

$$\le\;\; C_{12}\Big(1 \,+\, M_1 \,+\, M_2^2 \,+\, \tau\,(M_1 \,+\, (1+M_2^2)\,(M_3+M_4+M_7)\Big)\,. \tag{3.56}$$

Finally, the imbeddings $H^1(0,\tau;\,L^2(\Omega)) \cap L^2(0,\tau;\,H^2(\Omega)) \hookrightarrow C^{\frac12,\frac16}(\overline{\Omega_\tau}) \hookrightarrow C(\overline{\Omega_\tau})$ are continuous (the latter being compact), since Ω is one-dimensional, and there exists a positive constant $\hat C_1$ such that for every $(x,t),(y,s) \in \overline{\Omega_\tau}$ it holds

$$|v(x,t)-v(y,s)| \;\le\; \hat C_1\left(\int_0^\tau\!\!\int_\Omega\Big(v_t^2+v_{xx}^2\Big)\,dx\,dt\right)^{1/2}\Big(|t-s|^{1/6}+|x-y|^{1/2}\Big)\,, \tag{3.57}$$

for every function $v \in H^1(0,\tau;L^2(\Omega)) \cap L^2(0,\tau;H^2(\Omega))$, $0<\tau\le T$. Choosing any constant $C_{13} \ge \|\theta_0\|_\infty$, we therefore obtain that

$$\max_{(x,t)\in\overline{\Omega_\tau}} |\theta(x,t)| \;\le\; C_{13} + \hat{C}_1 \sqrt{\hat{M}_1}\,, \tag{3.58}$$

where $\hat{M}_1$ is equal to the expression on the right-hand side of (3.48).

Now, we can define the constants $M_1 \ldots, M_9$. We make the choices

$$\begin{aligned}
&M_1 := 2\,C_2\,, \quad M_2 := C_{13} + \hat{C}_1\sqrt{M_1}\,, \quad M_3 := 2\,C_3 + C_3\,M_2^2\,,\\
&M_4 := 2\,C_5 + C_5\,M_2^2\,, \quad M_5 := 2\,C_8\,, \quad M_6 := 2\,C_6 + C_6\,M_2^2\,,\\
&M_7 := 2\,C_9 + C_9\,M_1\,, \quad M_8 := 2\,C_{11} + C_{11}\,(M_1+M_2^2)\,,\\
&M_9 := 2\,C_{12} + C_{12}\,(M_1+M_2^2)\,.
\end{aligned} \tag{3.59}$$

It then follows from the estimates (3.48)–(3.49), (3.51)–(3.56), and (3.58), that there exists some $\tau_0 \in (0,T]$ such that the inequalities (3.16)–(3.24) are fulfilled for any $\tau \in (0,\tau_0]$.

To conclude the proof of the lemma, note that by (3.57) we have

$$\max_{x\in\overline{\Omega}} |\theta(x,t)-\theta_0(x)| \;\le\; \hat{C}_1\sqrt{M_1}\,t^{1/6}\,, \tag{3.60}$$

so that, for sufficiently small $\hat{\tau} \in (0,\tau_0]$,

$$\theta(x,t) \;\ge\; \theta_0(x) \,-\, |\theta(x,t)-\theta_0(x)| \;\ge\; \delta \,-\, \hat{C}_1\sqrt{M_1}\,t^{1/6} \;\ge\; \delta/2\,, \tag{3.61}$$

for all $(x,t) \in \Omega \times [0,\hat{\tau}]$. With this, the proof of the lemma is complete. □

Lemma 3.4 *The operator $\mathcal{T}$ is weakly sequentially continuous in $\mathcal{M}_{\hat{\tau}}$.*

Proof: Suppose a sequence $\{(\bar{p}_n,\bar{q}_n,\bar{\theta}_n,\bar{w}_n)\} \subset \mathcal{M}_{\hat{\tau}}$ is given such that

$$(\bar{p}_n,\bar{q}_n,\bar{\theta}_n,\bar{w}_n) \to (\bar{p},\bar{q},\bar{\theta},\bar{w}) \quad \text{weakly in } X_{\hat{\tau}} \quad \text{as } n\to\infty\,. \tag{3.62}$$

Since $\mathcal{M}_{\hat{\tau}}$ is weakly closed, it holds $(\bar{p},\bar{q},\bar{\theta},\bar{w}) \in \mathcal{M}_{\hat{\tau}}$. Now, let

$$(p_n,q_n,\theta_n,w_n) \;:=\; \mathcal{T}(\bar{p}_n,\bar{q}_n\,\bar{\theta}_n,\bar{w}_n)\,, \quad n\in\mathbb{N}\,, \quad (p,q,\theta,w) \;:=\; \mathcal{T}(\bar{p},\bar{q},\bar{\theta},\bar{w})\,. \tag{3.63}$$

We have to show that

$$(p_n,q_n,\theta_n,w_n) \to (p,q,\theta,w) \quad \text{weakly in } X_{\hat{\tau}} \quad \text{as } n\to\infty\,. \tag{3.64}$$

Clearly, as $(p_n,q_n,\theta_n,w_n) \in \mathcal{M}_{\hat{\tau}}$ for all $n\in\mathbb{N}$, we have, on a subsequence which is again indexed by n ,

$$(p_n,q_n,\theta_n,w_n) \to (\hat{p},\hat{q},\hat{\theta},\hat{w}) \quad \text{weakly in } X_{\hat{\tau}} \quad \text{as } n\to\infty\,, \tag{3.65}$$

for some $(\hat{p},\hat{q},\hat{\theta},\hat{w}) \in \mathcal{M}_{\hat{\tau}}$. It remains to show that $(\hat{p},\hat{q},\hat{\theta},\hat{w}) = (p,q,\theta,w)$. The uniqueness of the limit point then entails that (3.65), and thus (3.64), holds for the entire sequence. To this end, note that we have the convergences

$$\begin{aligned} &\bar{\theta}_{n,t} \to \bar{\theta}_t\,, \;\; \bar{\theta}_{n,xx} \to \bar{\theta}_{xx}\,, \;\; \bar{p}_{n,t} \to \bar{p}_t\,, \;\; \bar{p}_{n,xx} \to \bar{p}_{xx}\,, \;\; \bar{p}_{n,xxx} \to \bar{p}_{xxx}\,, \;\; \bar{p}_{n,xt} \to \bar{p}_{xt}\,, \\ &\bar{w}_{n,t} \to \bar{w}_t\,, \;\; \bar{w}_{n,xt} \to \bar{w}_{xt}\,, \;\; \bar{q}_{n,t} \to \bar{q}_t\,, \;\; \bar{q}_{n,xt} \to \bar{q}_{xt}\,, \;\; \text{all weakly in } L^2(\Omega_{\hat{\tau}})\,. \end{aligned} \tag{3.66}$$

By compact imbedding, we may also assume that

$$\bar{\theta}_n \to \bar{\theta}\,, \quad \bar{p}_{n,x} \to \bar{p}_x\,, \quad \text{both uniformly in } \overline{\Omega_{\hat{\tau}}}\,, \tag{3.67}$$

$$\bar{p}_n \to \bar{p}\,, \quad \bar{q}_n \to \bar{q}\,, \quad \bar{w}_n \to \bar{w}\,, \quad \text{all strongly in } L^2(\Omega; C[0,\hat{\tau}])\,. \tag{3.68}$$

But then, owing to **(H4)**, it follows that

$$\bar{\sigma}_n \to \bar{\sigma}\,, \;\; \bar{\psi}_n \to \bar{\psi}\,, \;\; \mathcal{F}_1[\bar{p}_n + \bar{q}_n, \bar{w}_n] \to \mathcal{F}_1[\bar{p} + \bar{q}, \bar{w}]\,, \;\; \text{all strongly in } L^2(\Omega; C[0,\hat{\tau}])\,, \tag{3.69}$$

where $\bar{\sigma}_n$, $\bar{\psi}_n$ have obvious meaning. Also,

$$g(\cdot,\cdot,\bar{\theta}_n) \to g(\cdot,\cdot,\bar{\theta}) \quad \text{uniformly in } \overline{\Omega_{\hat{\tau}}}\,, \tag{3.70}$$

$$\bar{\sigma}_n\, \bar{p}_{n,xx} \to \bar{\sigma}\, \bar{p}_{xx} \quad \text{weakly in } L^2(\Omega_{\hat{\tau}})\,. \tag{3.71}$$

Besides, the sequence $\{\mathcal{F}_1[\bar{p}_n + \bar{q}_n, \bar{w}_n]_t\}$ is bounded in $L^\infty(\Omega_{\hat{\tau}})$, so that we may assume that

$$\mathcal{F}_1[\bar{p}_n + \bar{q}_n, \bar{w}_n]_t \to y \quad \text{weakly-star in } L^\infty(\Omega_{\hat{\tau}}) \tag{3.72}$$

for some $y \in L^\infty(\Omega_{\hat{\tau}})$. But then it follows from (3.69) that $y = \mathcal{F}_1[\bar{p} + \bar{q}, \bar{w}]_t$. Finally, we conclude from (3.66) and (3.67) that

$$\bar{p}_{n,xx}^2 \to \bar{p}_{xx}^2 \quad \text{weakly in } L^2(\Omega_{\hat{\tau}})\,. \tag{3.73}$$

Indeed, we have for any test function $\eta \in C_0^\infty(\Omega_{\hat{\tau}})$ that

$$\begin{aligned} &\lim_{n\to\infty} \int_0^{\hat{\tau}} \int_\Omega \bar{p}_{n,xx}^2\, \eta \, dx\, dt \; = \; - \lim_{n\to\infty} \int_0^{\hat{\tau}} \int_\Omega \Big(\bar{p}_{n,xxx}\, \bar{p}_{n,x}\, \eta \, + \, \bar{p}_{n,xx}\, \bar{p}_{n,x}\, \eta_x \Big)\, dx\, dt \\ = \; &- \int_0^{\hat{\tau}} \int_\Omega \Big(\bar{p}_{xxx}\, \bar{p}_x\, \eta \, + \, \bar{p}_{xx}\, \bar{p}_x\, \eta_x \Big)\, dx\, dt \; = \; \int_0^{\hat{\tau}} \int_\Omega \bar{p}_{xx}^2\, \eta \, dx\, dt\,. \end{aligned} \tag{3.74}$$

Now observe that (3.65) implies, in particular, the convergences

$$\theta_{n,t} \to \hat{\theta}_t\,, \quad \theta_{n,xx} \to \hat{\theta}_{xx}\,, \quad p_{n,t} \to \hat{p}_t\,, \quad p_{n,xx} \to \hat{p}_{xx}\,, \quad q_{n,t} \to \hat{q}_t\,, \quad w_{n,t} \to \hat{w}_t\,, \tag{3.75}$$

all weakly in $L^2(\Omega_{\hat{\tau}})$. Combining all the above convergences, and letting $n \to \infty$, we finally can infer that $(\hat{p},\hat{q},\hat{\theta},\hat{w}) = \mathcal{T}(\bar{p},\bar{q},\bar{\theta},\bar{w})$. This concludes the proof of the lemma.
□

By virtue of Lemma 3.3 and Lemma 3.4, we deduce from Lemma 3.2 that $\mathcal{T}$ has a fixed point in $\mathcal{M}_{\hat{\tau}}$ which then is a solution to the system (3.2)–(3.10). To conclude the proof of Theorem 3.1, we still need to show the uniqueness. We achieve this through the following result, which even shows global uniqueness.

Lemma 3.5 *Let the assumptions of Theorem 3.1 be fulfilled, and let* $\tau \in (0,T]$ *be arbitrary. Then the system* (3.2)–(3.10) *has at most one solution in* X_τ *.*

Proof: Suppose that $(p_i, q_i, \theta_i, w_i) \in X_\tau$, $i = 1,2$, are two solutions to (3.2)–(3.10) on Ω_τ for some $\tau \in (0,T]$. Let $p := p_1 - p_2$, $q := q_1 - q_2$, $\theta := \theta_1 - \theta_2$, $w := w_1 - w_2$, and put $\sigma_i := \mathcal{H}_1[p_i + q_i, w_i] + \theta_i\, \mathcal{H}_2[p_i + q_i, w_i]$, $\psi_i := \mathcal{H}_3[p_i + q_i, w_i] + \theta_i\, \mathcal{H}_4[p_i + q_i, w_i]$, for $i = 1,2$. Then it holds

$$p_t - p_{xx} = \sigma_1 - \sigma_2 \,, \tag{3.76}$$

$$q_t = \sigma_2 - \sigma_1 \,, \tag{3.77}$$

$$\begin{aligned} \theta_t - \theta_{xx} \;=\; & -\mathcal{F}_1[p_1 + q_1, w_1]_t + \mathcal{F}_1[p_2 + q_2, w_2]_t + p_{1,xx}^2 - p_{2,xx}^2 \\ & + \sigma_1\, p_{1,xx} - \sigma_2\, p_{2,xx} + g(x,t,\theta_1) - g(x,t,\theta_2) \,, \end{aligned} \tag{3.78}$$

$$w_t = \psi_1 - \psi_2 \,, \tag{3.79}$$

with corresponding zero initial and boundary conditions. Owing to **(H4)**,**(iii)**, we have for every $(x,t) \in \overline{\Omega_\tau}$

$$\begin{aligned} & \max\{|\sigma_1(x,t) - \sigma_2(x,t)|\,,\, |\psi_1(x,t) - \psi_2(x,t)|\} \\ \le\; & C_1 \Big(|\theta(x,t)| + \max_{0 \le r \le t} (|p(x,r)| + |q(x,r)| + |w(x,r)|) \Big) \\ \le\; & C_1 \Big(|\theta(x,t)| + \int_0^t \Big(|p_t(x,r)| + |q_t(x,r)| + |w_t(x,r)| \Big)\, dr \,, \end{aligned} \tag{3.80}$$

where by C_i , $i \in \mathbb{N}$, we denote positive constants that only depend on the data of the system. Hence, we may multiply (3.76) by p_t , and by $-p_{xx}$, respectively, (3.77) by q_t , and (3.79) by w_t , respectively, add the four resulting equations, integrate over space and time, and apply Young's inequality appropriately, to arrive at the estimate

$$\begin{aligned} \int_0^t \int_\Omega \Big(p_t^2 + p_{xx}^2 + q_t^2 + w_t^2 \Big)\, dx\, ds \;\le\; & C_2 \int_0^t \int_0^s \int_\Omega \Big(p_t^2 + q_t^2 + w_t^2 \Big)\, dx\, dr\, ds \\ & + C_3 \int_0^t \int_\Omega \theta^2\, dx\, ds \,. \end{aligned} \tag{3.81}$$

Next, we integrate (3.78) over $[0,s]$ for some $s > 0$. We obtain

$$\begin{aligned} \theta - \int_0^s \theta_{xx}\, dr \;=\; & -\mathcal{F}_1[p_1 + q_1, w_1] + \mathcal{F}_1[p_2 + q_2, w_2] + \int_0^s \Big(p_{1,xx}^2 - p_{2,xx}^2 \Big)\, dr \\ & + \int_0^s \Big(\sigma_1\, p_{1,xx} - \sigma_2\, p_{2,xx} \Big)\, dr + \int_0^s \Big(g(x,r,\theta_1) - g(x,r,\theta_2) \Big)\, dr \,. \end{aligned} \tag{3.82}$$

Now let $\gamma > 0$ (to be specified later). First, we note that for a. e. $(x, t) \in \Omega_\tau$ it holds

$$|\mathcal{F}_1[p_1 + q_1, w_1](x,t) \;-\; \mathcal{F}_1[p_2 + q_2, w_2](x,t)| \;\le\; C_4 \int_0^t \Big(|p_t| \;+\; |q_t| \;+\; |w_t|\Big)(x,r)\,dr\,, \tag{3.83}$$

so that, using Young's inequality,

$$\begin{aligned} &\int_0^t \int_\Omega |\theta(x,s)| \int_0^s |\mathcal{F}_1[p_1 + q_1, w_1](x,r) \;-\; \mathcal{F}_1[p_2 + q_2, w_2](x,r)|\,dr\,dx\,ds \\ \le\;& \frac{\gamma}{2} \int_0^t \int_\Omega \theta^2\,dx\,ds \;+\; \frac{C_5}{2\,\gamma} \int_0^t \int_0^s \int_\Omega \Big(p_t^2 + q_t^2 + w_t^2\Big)\,dx\,dr\,ds\,. \end{aligned} \tag{3.84}$$

Moreover, owing to **(H3)**, we have

$$\begin{aligned} &\int_0^t \int_\Omega |\theta(x,t)| \int_0^s |g(x,r,\theta_1(x,r)) - g(x,r,\theta_2(x,r)|\,dr\,dx\,ds \\ \le\;& C_6 \int_0^t \int_\Omega |\theta(x,t)| \int_0^s |\theta(x,r)|\,dr\,dx\,ds \\ \le\;& \frac{\gamma}{2} \int_0^t \int_\Omega \theta^2\,dx\,ds \;+\; \frac{C_6}{2\,\gamma} \int_0^t \int_0^s \int_\Omega \theta^2\,dx\,dr\,ds\,. \end{aligned} \tag{3.85}$$

Next, we estimate

$$\begin{aligned} &\int_0^t \int_\Omega |\theta(x,s)| \int_0^s |\sigma_1\,p_{1,xx} \;-\; \sigma_2\,p_{2,xx}|(x,r)\,dr\,dx\,ds \\ \le\;& \int_0^t \int_\Omega |\theta(x,s)| \int_0^s |\sigma_2(x,r)|\,|p_{xx}(x,r)|\,dr\,dx\,ds \\ &+ \int_0^t \int_\Omega |\theta(x,s)| \int_0^s \Big(|p_{1,xx}|\,|\sigma_1 - \sigma_2|\Big)(x,r)\,dr\,dx\,ds \\ :=\;& B_1 \;+\; B_2\,. \end{aligned} \tag{3.86}$$

By the boundedness of σ_2,

$$B_1 \;\le\; \frac{\gamma}{2} \int_0^t \int_\Omega \theta^2\,dx\,ds \;+\; \frac{C_7}{2\,\gamma} \int_0^t \int_0^s \int_\Omega p_{xx}^2\,dx\,dr\,ds\,. \tag{3.87}$$

Next, we employ the Gagliardo-Nirenberg inequality (2.21) with $p = +\infty$, $q = r = 2$, $\omega = 1/2$, to conclude that, for $i = 1, 2$, and every $x \in \bar{\Omega}$,

$$\begin{aligned} &\int_0^s |p_{i,xx}(x,r)|^2\,dr \;\;\le\;\; \int_0^s \|p_{i,xx}(\cdot,r)\|_\infty^2\,dr \\ \le\;& 2\,K_0^2 \int_0^s \Big(\|p_{i,xx}(\cdot,r)\|_2^2 \;+\; \|p_{i,xx}(\cdot,r)\|_2\,\|p_{i,xxx}(\cdot,r)\|_2\Big)\,dr \\ \le\;& C_8 \max_{0 \le r \le s} \|p_{i,xx}(\cdot,r)\|_2^2 \;+\; C_9 \max_{0 \le r \le s} \|p_{i,xx}(\cdot,r)\|_2 \Big(\int_0^s \int_\Omega |p_{i,xxx}|^2\,dx\,dr\Big)^{1/2} \\ \le\;& C_{10}\,. \end{aligned} \tag{3.88}$$

Hence, we have

$$\begin{aligned}\int_0^s \Big(|\sigma_1-\sigma_2|\,|p_{1,xx}|\Big)(x,r)\,dr &\le \Big(\int_0^s |(\sigma_1-\sigma_2)(x,r)|^2\,dr\Big)^{1/2}\Big(\int_0^s |p_{1,xx}(x,r)|^2\,dr\Big)^{1/2}\\ &\le \sqrt{C_{10}}\,\Big(\int_0^s |(\sigma_1-\sigma_2)(x,r)|^2\,dr\Big)^{1/2}, \end{aligned}\tag{3.89}$$

so that, by virtue of (3.80) and of Young's inequality,

$$\begin{aligned} B_2 &\le \frac{\gamma}{2}\int_0^t\int_\Omega \theta^2\,dx\,ds + \frac{C_{11}}{2\gamma}\int_0^t\int_0^s\int_\Omega |\sigma_1-\sigma_2|^2\,dx\,dr\,ds\\ &\le \frac{\gamma}{2}\int_0^t\int_\Omega \theta^2\,dx\,ds + \frac{C_{12}}{2\gamma}\int_0^t\int_0^s\int_\Omega \Big(\theta^2+p_t^2+q_t^2+w_t^2\Big)\,dx\,dr\,ds\,. \end{aligned}\tag{3.90}$$

Finally, using (3.88), we estimate

$$\begin{aligned} &\int_0^t\int_\Omega |\theta(x,s)|\int_0^s \Big(p_{1,xx}^2-p_{2,xx}^2\Big)(x,r)\,dr\,dx\,ds\\ \le\ &\int_0^t\int_\Omega |\theta(x,s)|\int_0^s \Big(|p_{xx}|\,|p_{1,xx}+p_{2,xx}|\Big)(x,r)\,dr\,dx\,ds\\ \le\ &\int_0^t\int_\Omega |\theta(x,s)|\Big(\int_0^s |p_{xx}(x,r)|^2\,dr\Big)^{1/2}\Big(\int_0^s |(p_{1,xx}+p_{2,xx})(x,r)|^2\,dr\Big)^{1/2}dx\,ds\\ \le\ &C_{13}\int_0^t\int_\Omega |\theta(x,s)|\Big(\int_0^s |p_{xx}(x,r)|^2\,dr\Big)^{1/2}dx\,ds\\ \le\ &\frac{\gamma}{2}\int_0^t\int_\Omega \theta^2\,dx\,ds + \frac{C_{14}}{2\gamma}\int_0^t\int_0^s\int_\Omega p_{xx}^2\,dx\,dr\,ds\,. \end{aligned}\tag{3.91}$$

Now, we multiply (3.82) by θ, and integrate over $\Omega\times[0,t]$ for some $t\in[0,\tau]$. Combining the estimates (3.84)–(3.91), and choosing $\gamma>0$ appropriately small, we obtain the inequality

$$\int_0^t\int_\Omega \theta^2\,dx\,ds \le C_{15}\int_0^t\int_0^s\int_\Omega \Big(\theta^2+p_{xx}^2+p_t^2+q_t^2+w_t^2\Big)\,dx\,dr\,ds\,. \tag{3.92}$$

Consequently, combining inequalities (3.81) and (3.92), we have finally shown that

$$\begin{aligned} &\int_0^t\int_\Omega \Big(\theta^2 + p_t^2 + p_{xx}^2 + q_t^2 + w_t^2\Big)\,dx\,ds\\ \le\ &C_{16}\int_0^t\int_0^s\int_\Omega \Big(\theta^2 + p_t^2 + p_{xx}^2 + q_t^2 + w_t^2\Big)\,dx\,dr\,ds\,, \end{aligned}\tag{3.93}$$

whence, by Gronwall's lemma, $p_t = q_t = w_t = \theta = 0$ a.e. in Ω_τ, so that the assertion follows. With this, the proof of Theorem 3.1 is complete. □

4 Global existence

Suppose now that the hypotheses **(H1)** to **(H5)** hold so that (3.2)–(3.10) has a unique solution (p,q,θ,w) on $\Omega_{\hat\tau}$ which satisfies (3.11)–(3.14). Using the compatibility condition $u_0(0)=0$, we then easily verify that (u,θ,w), where

$$u(x,t) = \int_0^x \Big(p(\xi,t)+q(\xi,t)\Big)\,d\xi\,, \tag{4.1}$$

solves (1.1)–(1.7) on $\Omega_{\hat\tau}$ and satisfies (2.16). Now let $\tau \in (0,T]$ be arbitrary such that (u,θ) can be extended to a solution of (1.1)–(1.7) on Ω_τ and satisfies $\theta(x,t) \ge \bar\theta$ for some $\bar\theta > 0$, as well as the smoothness properties (2.16). Owing to the global uniqueness result of Lemma 3.5, this solution is unique. We are now going to derive a number of global a priori estimates. To this end, we denote by C_i, $i \in \mathbb{N}$, positive constants which may depend on the given data of system (1.1)–(1.7), but neither on τ nor on the lower bound $\bar\theta$ for the temperature. For notational convenience, we put $\varepsilon := u_x$.

First estimate.

We multiply (1.1) by u_t, add the result to (1.3), and then integrate over Ω_t, $t \in (0,\tau]$, and by parts. In view of **(H1)**, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \int_\Omega\Big(\theta(x,t)+\mathcal{F}[\varepsilon,w](x,t)\Big)\,dx\\ &\le\; C_1 + \int_0^t\int_\Omega g(x,r,\theta(x,r))\,dx\,dr + \int_0^t\int_\Omega f\,u_t\,dx\,dr\,. \qquad (4.2)\end{aligned}$$

Invoking (2.2), (2.4), **(H2)**, **(H4)**,**(i)**, the positivity of θ, and Gronwall's lemma, we find that

$$\max_{0\le t\le\tau}\Big(\|\theta(\cdot,t)\|_1 + \|u_t(\cdot,t)\|_2\Big) \;\le\; C_2\,. \qquad (4.3)$$

Second estimate.

We multiply (1.3) by $-\theta^{-1}$ and integrate over Ω_t, $t\in(0,\tau]$ (note that θ^{-1} is actually bounded, since $\theta \ge \bar\theta > 0$). It follows

$$\begin{aligned}\int_0^t\int_\Omega\left(\frac{\theta_x^2}{\theta^2}+\frac{\varepsilon_t^2}{\theta}\right)dx\,dr \;&\le\; C_3 + \int_0^t\int_\Omega\frac{1}{\theta}\Big(\mathcal{F}_1[\varepsilon,w]_t - \sigma\,\varepsilon_t - g(x,r,\theta)\Big)\,dx\,dr\\ &\quad + \int_\Omega \log(\theta(x,t))\,dx\,. \qquad (4.4)\end{aligned}$$

In view of (4.3), and of the elementary inequality $\log(\theta)\le\theta$ for $\theta > 0$, the second integral on the right-hand side is bounded. Besides, we obtain from (1.2), (1.4), **(H3)**, **(H5)**, and Young's inequality, that a. e. in Ω_τ it holds

$$\begin{aligned}&\mathcal{F}_1[\varepsilon,w]_t - \sigma\,\varepsilon_t - g(x,r,\theta) \;\le\; \mathcal{H}_3[\varepsilon,w]\,\mathcal{G}[w]_t - \theta\,\mathcal{H}_2[\varepsilon,w]\,\varepsilon_t - g_0(x,r) + K_1\,\theta\\ &\le\; -\,(\theta\,\mathcal{H}_4[\varepsilon,w] + w_t)\,\mathcal{G}[w]_t + K_2\,\theta\,|\varepsilon_t| + K_1\,\theta \;\le\; K_1\,\theta + K_2\,\theta\,|\varepsilon_t| + \frac{K_5}{4}\,K_2^2\,\theta^2\,. \qquad (4.5)\end{aligned}$$

Therefore, using (4.3), we find from Young's inequality that

$$\begin{aligned}\int_0^t\int_\Omega\frac{1}{\theta}\Big(\mathcal{F}_1[\varepsilon,w]_t - \sigma\,\varepsilon_t - g(x,r,\theta)\Big)\,dx\,dr \;&\le\; C_4\Big(1+\int_0^t\int_\Omega|\varepsilon_t|\,dx\,dr\Big)\\ &\le\; C_5 + \frac{1}{2}\int_0^t\int_\Omega\frac{\varepsilon_t^2}{\theta}\,dx\,dr\,. \qquad (4.6)\end{aligned}$$

In conclusion, we have shown the estimate

$$\int_0^\tau\int_\Omega \left(\frac{\theta_x^2}{\theta^2} + \frac{\varepsilon_t^2}{\theta}\right) dx\, dt \ \leq\ C_6\,. \tag{4.7}$$

But then, using (4.3) and Young's inequality,

$$\int_0^\tau\int_\Omega \left|\left(\sqrt{\theta}\right)_x\right| dx\, dt \ =\ \int_0^\tau\int_\Omega \left|\frac{\theta_x}{2\sqrt{\theta}}\right| dx\, dt \ \leq\ C_7 \ +\ \int_0^\tau\int_\Omega \frac{\theta_x^2}{\theta^2}\, dx\, dt \ \leq\ C_8\,, \tag{4.8}$$

whence, using the Gagliardo-Nirenberg inequality (2.21) with $p = +\infty$, $q = 2$, $r = 1$, and $\omega = 1$, and invoking (4.3) once more,

$$\int_0^\tau \|\theta(\cdot,t)\|_\infty\, dt \ \leq\ C_9\,. \tag{4.9}$$

Hence,

$$\int_0^\tau\int_\Omega \theta^2\, dx\, dt \ \leq\ \max_{0\leq t\leq\tau} \|\theta(\cdot,t)\|_1 \int_0^\tau \|\theta(\cdot,t)\|_\infty\, dt \ \leq\ C_{10}\,. \tag{4.10}$$

Third estimate.

We now exploit the decomposition (3.1). We have $u_x = \varepsilon = p + q$, where, owing to (4.3), $\|p\|_{L^\infty(\Omega_\tau)} \leq C_{11}$. Therefore, invoking **(H4)** and (2.11), it holds for every $(x,t) \in \overline{\Omega_\tau}$,

$$|\sigma(x,t)| + |\psi(x,t)| \ \leq\ C_{12} + 2\,K_2\,\theta(x,t) + 2\,K_4 \max_{0\leq r\leq t}\Big(|q(x,r)| + |w(x,r)|\Big). \tag{4.11}$$

Now multiply (3.5) by q , and (3.8) by w , respectively, add the resulting equations, and integrate over $[0,t]$, where $t \in (0,\tau]$. Using Young's inequality, and invoking (4.9), we conclude from (4.11) the estimate

$$\begin{aligned} &\frac{1}{2}\Big(q^2(x,t) + w^2(x,t)\Big) \\ \leq\ & C_{13}\left[1 \,+\, \max_{0\leq r\leq t}\Big(|q(x,r)| + |w(x,r)|\Big)\left(1 \,+\, \int_0^t \Big(|q(x,r)| + |w(x,r)|\Big)\, dr\right)\right] \\ \leq\ & C_{14} \,+\, \frac{1}{4}\max_{0\leq r\leq t}\Big(q^2(x,r) + w^2(x,r)\Big) \,+\, C_{15}\int_0^t \Big(q^2(x,r) + w^2(x,r)\Big)\, dr\,. \end{aligned} \tag{4.12}$$

Taking the maximum with respect to t on both sides, we obtain from Gronwall's lemma that

$$\|q\|_{L^\infty(\Omega_\tau)} \,+\, \|w\|_{L^\infty(\Omega_\tau)} \ \leq\ C_{16}\,, \tag{4.13}$$

whence also

$$\|\varepsilon\|_{L^\infty(\Omega_\tau)} \,+\, \max_{j\in\{1,3\}} \|\mathcal{H}_j[\varepsilon,w]\|_{L^\infty(\Omega_\tau)} \ \leq\ C_{17}\,, \tag{4.14}$$

and we obtain from (3.5) and (3.8), using (4.9)–(4.11), that

$$\|q_t\|_{L^1(0,\tau;L^\infty(\Omega))\,\cap\, L^2(\Omega_\tau)\,\cap\, L^\infty(0,\tau;L^1(\Omega))} \,+\, \|w_t\|_{L^1(0,\tau;L^\infty(\Omega))\,\cap\, L^2(\Omega_\tau)\,\cap\, L^\infty(0,\tau;L^1(\Omega))} \ \leq\ C_{18}\,. \tag{4.15}$$

Moreover, (4.10) and (4.14) imply that the right-hand side of (3.2) is bounded in $L^2(\Omega_\tau)$; hence, using standard parabolic estimates, we can conclude that

$$\|p\|_{H^1(0,\tau;L^2(\Omega))\,\cap\, L^2(0,\tau;H^2(\Omega))\,\cap\, C([0,\tau];H^1(\Omega))} \;\le\; C_{19}\,, \tag{4.16}$$

which yields, in particular, that $u_{xt} = p_{xx}$ is bounded in $L^2(\Omega_\tau)$.

Fourth estimate.

As next step, we perform a classical estimate (cf. [5]), namely, we multiply (1.3) by $-\theta^{-1/3}$ and integrate over Ω_t for $t \in (0,\tau]$. It then follows from (4.5)

$$\begin{aligned}\int_0^t\int_\Omega \left(\theta^{-4/3}\,\theta_x^2 + \theta^{-1/3}\,\varepsilon_t^2\right)\,dx\,dr \;&\le\; C_{20}\left(1 + \int_\Omega \theta^{2/3}(x,t)\,dx + \int_0^t\int_\Omega \theta^{2/3}\,dx\,dr\right)\\ &+ C_{21}\left(\int_0^t\int_\Omega \theta^{2/3}\,|\varepsilon_t|\,dx\,dr \,+\, \int_0^t\int_\Omega \theta^{5/3}\,\,dx\,dr\right). \end{aligned}\tag{4.17}$$

Owing to (4.3) and (4.10), the first, second and fourth integrals on the right of (4.17) are bounded, and the remaining expression is estimated as follows:

$$\begin{aligned}\int_0^t\int_\Omega \theta^{2/3}\,|\varepsilon_t|\,dx\,dr \;&=\; \int_0^t\int_\Omega \theta^{5/6}\,\theta^{-1/6}\,|\varepsilon_t|\,dx\,dr\\ &\le\; \frac{1}{2}\int_0^t\int_\Omega \theta^{-1/3}\,\varepsilon_t^2\,dx\,dr \,+\, \frac{1}{2}\int_0^t\int_\Omega \theta^{5/3}\,dx\,dr\,. \end{aligned}\tag{4.18}$$

Since the second summand on the right of (4.18) is again bounded, we have shown the estimate

$$\int_0^\tau\int_\Omega \left(\theta^{-4/3}\,\theta_x^2 \,+\, \theta^{-1/3}\,\varepsilon_t^2\right)\,dx\,dt \;\le\; C_{22}\,. \tag{4.19}$$

But then $\theta^{1/3}$ is bounded in $L^\infty(0,\tau;L^3(\Omega))\cap L^2(0,\tau;H^1(\Omega))$, and the Gagliardo-Nirenberg inequality (2.21), with $p=+\infty$, $r=2$, $q=3$, and $\omega=2/5$, yields that

$$\int_0^\tau \|\theta(\cdot,t)\|_\infty^{5/3}\,dt \;\le\; C_{23}\,, \tag{4.20}$$

whence, using (4.3) once more,

$$\int_0^\tau\int_\Omega \theta^{8/3}\,dx\,dt \;\le\; C_{24}\,. \tag{4.21}$$

Thus, the right-hand sides of (3.2), (3.5), and (3.8), respectively, are bounded in $L^{8/3}(\Omega_\tau)$, and we can infer from standard parabolic estimates, using $\varepsilon_t = p_{xx}$, that

$$\|p_t\|_{L^{8/3}(\Omega_\tau)} \,+\, \|\varepsilon_t\|_{L^{8/3}(\Omega_\tau)} \,+\, \|q_t\|_{L^{8/3}(\Omega_\tau)} \,+\, \|w_t\|_{L^{8/3}(\Omega_\tau)} \;\le\; C_{25}\,. \tag{4.22}$$

Fifth estimate.

We now turn our attention to (1.3). By virtue of (4.21) and (4.22), and invoking (2.5), we easily verify that the right-hand side of (1.3) is bounded in $L^{4/3}(\Omega_\tau)$. Therefore, multiplying (1.3) by θ, integrating over Ω_τ for $t \in (0,\tau]$, and applying Young's inequality and (4.3), we see that for any $\gamma > 0$ it holds

$$\begin{aligned} \|\theta(\cdot,t)\|_2^2 + \int_0^t \int_\Omega \theta_x^2 \, dx \, dr \;&\leq\; C_{26}\left(1 + \gamma^{-1}\right) + \gamma \int_0^t \int_\Omega \theta^4 \, dx \, dr \\ &\leq\; C_{27}\left(1 + \gamma^{-1} + \gamma \int_0^t \|\theta(\cdot,r)\|_\infty^3 \, dr\right). \qquad (4.23) \end{aligned}$$

Now use (4.3) and the Gagliardo-Nirenberg inequality (2.21) with $p = +\infty$, $q = 1$, $r = 2$, and $\omega = 2/3$, in order to obtain that

$$\int_0^t \|\theta(\cdot,r)\|_\infty^3 \, dr \;\leq\; C_{28}\left(1 + \int_0^t \int_\Omega \theta_x^2 \, dx \, dr\right). \qquad (4.24)$$

Hence, choosing $\gamma > 0$ small enough, we have shown the estimate

$$\|\theta\|_{L^\infty(0,\tau;L^2(\Omega)) \,\cap\, L^2(0,\tau;H^1(\Omega))} \;\leq\; C_{29}\,, \qquad (4.25)$$

whence, using interpolation once more,

$$\|\theta\|_{L^6(\Omega_\tau)} \;\leq\; C_{30}\,. \qquad (4.26)$$

Thus, just as in the derivation of (4.22), we have

$$\|p_t\|_{L^6(\Omega_\tau)} + \|\varepsilon_t\|_{L^6(\Omega_\tau)} + \|q_t\|_{L^6(\Omega_\tau)} + \|w_t\|_{L^6(\Omega_\tau)} \;\leq\; C_{31}\,. \qquad (4.27)$$

But then the right-hand side of (1.3) is bounded in $L^2(\Omega_\tau)$, and standard parabolic estimates, also using (3.57), yield that

$$\|\theta\|_{H^1(0,\tau;L^2(\Omega)) \,\cap\, L^2(0,\tau;H^2(\Omega)) \,\cap\, C([0,\tau];H^1(\Omega)) \,\cap\, C(\overline{\Omega_\tau})} \;\leq\; C_{32}\,. \qquad (4.28)$$

This implies, in particular, that σ_t and ψ_t are bounded in $L^2(\Omega_\tau)$, so that

$$\|q_{tt}\|_{L^2(\Omega_\tau)} + \|w_{tt}\|_{L^2(\Omega_\tau)} \;\leq\; C_{33}\,. \qquad (4.29)$$

Besides, we may argue as in the derivation of (3.37) to conclude that also

$$\|p\|_{H^2(0,\tau;V_0^*) \,\cap\, H^1(0,\tau;H^1(\Omega))} \;\leq\; C_{34}\,. \qquad (4.30)$$

Sixth estimate.

In view of the above estimates, we have, for a. e. $(x,t) \in \Omega_\tau$,

$$\begin{aligned} |\sigma_x(x,t)| + |\psi_x(x,t)| \;&\leq\; C_{35}\left(|\theta_x(x,t)| + \max_{0 \leq r \leq t}\left((|p_x| + |q_x| + |w_x|)(x,r)\right)\right) \\ \leq\; C_{36}&\left(1 + |\theta_x(x,t)| + \int_0^t \Big(|p_{xt}(x,r)| + |q_{xt}(x,r)| + |w_{xt}(x,r)|\Big) \, dr\right). \qquad (4.31) \end{aligned}$$

Hence, differentiating (3.5), and (3.8), respectively, with respect to x , and invoking the estimates (4.28) and (4.30), we easily derive the estimate

$$\|q_{xt}\|_{L^2(\Omega_t)} + \|w_{xt}\|_{L^2(\Omega_t)} \le C_{37} \Big(1 + \int_0^t \Big(\|q_{xt}\|_{L^2(\Omega_r)} + \|w_{xt}\|_{L^2(\Omega_r)}\Big)\, dr\Big), \quad (4.32)$$

whence, using Gronwall's lemma,

$$\|q_{xt}\|_{L^2(\Omega_\tau)} + \|w_{xt}\|_{L^2(\Omega_\tau)} \le C_{38}\,. \quad (4.33)$$

Finally, we conclude from the above estimates that also

$$\|p_{xxx}\|_{L^2(\Omega_\tau)} \le C_{39}\,. \quad (4.34)$$

In conclusion, combining all previously shown estimates, we have shown that

$$\|(p, q, \theta, w)\|_{X_\tau} \le C_{40}\,, \quad (4.35)$$

where X_τ is the space introduced in (3.15).

Conclusion of the proof of Theorem 2.1.

So far we have shown that it holds (4.35) as long as there is some $\bar\theta > 0$ such that $\theta \ge \bar\theta$ on $\overline{\Omega_\tau}$. To conclude the proof of the assertion, we still have to prove the validity of (2.17). To this end, first observe that we have shown above that $\varepsilon_t = p_{xx}$ is bounded in $L^6(\Omega_\tau) \cap L^2(0,\tau; H^1(\Omega))$. In particular, there is some $\beta_1 > 0$, independent of τ , such that

$$\int_0^t \|\varepsilon_t(\cdot, r)\|_\infty\, dr \le \beta_1 \quad \text{for all } t \in (0,\tau]\,. \quad (4.36)$$

Besides, there is some $\beta_2 > 0$, independent of τ , such that

$$\max_{(x,t)\in\overline{\Omega_\tau}} |w_t(x,t)| \le \beta_2\,. \quad (4.37)$$

Now test (1.3) with an arbitrary function $z \in H^1(\Omega_\tau)$ satisfying $z \le 0$ a. e. in Ω_τ . In view of (2.10), (4.5), and (4.37) it follows, for a. e. $t \in (0,T)$,

$$\begin{aligned} &\int_\Omega \Big(z\,\theta_t + z_x\,\theta_x\Big)(x,t)\,dx \le \int_\Omega |z(x,t)| \Big(\big(\mathcal{F}[\varepsilon, w]_t - \sigma\,\varepsilon_t - g(\cdot,\cdot,\theta)\big)(x,t)\Big)\,dx \\ \le\ & \int_\Omega |z(x,t)| \Big(\big(-\theta\,\mathcal{H}_4[\varepsilon, w] - w_t\big)\,\mathcal{G}[w]_t + K_2\,\theta\,|\varepsilon_t| + K_1\,\theta\Big)(x,t)\,dx \\ \le\ & \Big(K_1 + K_2\,K_5\,\beta_2 + K_2\,\|\varepsilon_t(\cdot,t)\|_\infty\Big) \int_\Omega |z(x,t)|\,\theta(x,t)\,dx \\ \le\ & \varphi(t) \int_\Omega |z(x,t)|\,\theta(x,t)\,dx\,, \end{aligned} \quad (4.38)$$

where $\varphi(t) := (K_1 + K_2\,K_5\,\beta_2 + K_2\,\|\varepsilon_t(\cdot,t)\|_\infty)$ is by (4.36) bounded in $L^1(0,\tau)$ by a constant which does not depend on $\tau \in (0,T]$. Now, put

$$z(x,t) := -\Big(\delta \exp\Big(-\int_0^t \varphi(s)\,ds\Big) - \theta(x,t)\Big)^+ \quad \text{for } (x,t) \in \Omega_\tau\,. \quad (4.39)$$

Then it follows from inequality (4.38) that

$$\int_\Omega \left(z \left(z + \delta \exp\Big(-\int_0^t \varphi(s)\,ds\Big)\right)_t + z_x^2 \right)(x,t)\,dx$$
$$\le \; \varphi(t) \int_\Omega |z| \left(|z| + \delta \exp\Big(-\int_0^t \varphi(s)\,ds\Big)\right)(x,t)\,dx\,. \tag{4.40}$$

This yields, in particular,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2(x,t)\,dx + \int_\Omega z_x^2(x,t)\,dx \le \varphi(t)\int_\Omega z^2(x,t)\,dx\,. \tag{4.41}$$

Therefore, by Gronwall's lemma, $z = 0$, and thus,

$$\theta(x,t) \ge \delta\, e^{-((K_1 + K_2 K_5 \beta_2)\, t + K_2 \beta_1)} \quad \text{for all } (x,t) \in \overline{\Omega_\tau}\,. \tag{4.42}$$

Therefore, we can claim that $\tau = T$, and the assertion of Theorem 2.1 is completely proved.

□

Remark 4. It does not present any major difficulties to extend the above proof to the more general case when $\mathcal{H}_3$ and $\mathcal{H}_4$ are *vector* hysteresis operators and, accordingly, (1.4) is a vector differential equation (then, of course, the hypotheses **(H4)** and **(H5)** have to be appropriately modified). Note that this situation has been treated in [18].

Remark 5. It is easy to see that the solution (u, θ, w) depends Lipschitz continuously on the data of the system. Indeed, a closer look at the proof of Lemma 3.5 reveals that $L^2(\Omega)$ - variations of u_0 , u_1 , θ_0 , w_0 and $L^2(\Omega_T)$ - variations of f lead to Lipschitz variations of (p, q, θ, w) in the norm of the space $\big(H^1(0,T;\, L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))\big) \times H^1(0,T;\, L^2(\Omega)) \times L^2(\Omega_T) \times H^1(0,T;\, L^2(\Omega))$. A similar result holds for variations of g . As the line of arguments should be clear, we leave the explicit formulation and the proof of the corresponding result to the reader.

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