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Endogenous interest rate dynamics in asset markets

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Abstract

Starting from a general Itô process model with more assets than driving Brownian motions, we study the term structure model endogenously induced by this complete market. In the Markovian diffusion case, we provide the resulting HJM description and point out a link to finite factor models. But the main contribution is the conceptual approach of considering assets and interest rates within one model which is completely specified by the assets alone. This allows endogenous derivations of dynamic relations between assets and interest rates from global structural assumptions (homogeneity and some spherical symmetry) on the market. Related issues in financial market modelling have been studied by E. Platen.

0 Introduction

Any complete market determines a unique term structure model because the payoffs of all zero coupon bonds are uniquely priced. This is a simple observation, but to the best of our knowledge, it seems not to have been exploited systematically so far. As an immediate consequence, one has in any complete market endogenous relations between interest rates and the underlying assets. If the latter are stocks, these relations are particularly important because they reveal dynamic interactions between interest rates and stocks. The present paper explores this circle of ideas.

The starting point for our approach is a multidimensional Itô process model where all assets can be instantaneously risky and with more assets (n) than driving Brownian motions (m). We think this is a plausible hypothesis for real markets since these contain basic products like stocks and bonds, but also many derivatives, as liquidly traded instruments. The main issue is then how absence of arbitrage enforces relations between model coefficients and thus provides a model which is typically complete because $n > m$. The first half of this idea is basically taken from Jamshidian [Ja], but worked out in a direction more suitable for our goals. For such an arbitrage-free and complete market, we study in detail the endogenously induced term structure model and derive from structural assumptions on the market a number of relations between the dynamics of interest rates and assets. One key point is that all this is done *within* our model and thus in clear contrast to an econometric approach. As a benefit, we obtain new possibilities of formulating and testing hypotheses about financial markets.

Of course, this paper is only a first step and leaves many open problems. Empirical analyses of our assumptions and conclusions remain to be done, and we do not touch upon practical issues like calibration, implementation, or the choice of assets for hedging purposes. We focus exclusively on the conceptual aspect of treating assets and interest rates within one model, with a few applications to show its potential for endogenous derivation of dynamic relations. Hence the entire analysis is more on the theoretical side.

Like pricing by absence of arbitrage, our approach has a partial equilibrium flavour because we take the inital asset model as given and try to extract information about the resulting term structure model. The recent work by Platen ([P99a] – [P00]) goes one step further in that it asks about the origin of the asset model as well. Platen starts not from some asset dynamics, but from basic principles which he uses to derive a model from assumptions on the factors in his setting; see for instance [P99a], [P00], and [P99b] for a recent application to interest rate modelling. Thus while we study the internal structure of a given model, one of Platen's goals is to first provide a conceptual basis for the model itself. Hence our analysis here could form a natural second step to the outcome of Platen's investigations. But we emphasize that neither of the two approaches is a substitute for the other and that both directions can be developed further in future work.

An outline of the paper is as follows. Section 1 introduces our basic setup, characterizes no-arbitrage conditions by restrictions on the model coefficients and describes the endogenous term structure model via its short rate under the risk-neutral measure. All this is done in a general Itô process setting. Section 2 specializes to Markovian diffusions and derives the HJM description of forward rates in our model. Section 3 very briefly makes a link to finite factor models before we present in section 4 a major application of our approach. In an autonomous Markovian diffusion setting, we first study the short rate dynamics under the assumption of homogeneous market coefficients. We then show how a spherical symmetry condition on volatilities determines a unique natural asset index whose use as numeraire gives the asset model a particularly simple structure. Combining these results gives under an additional condition some very interesting relations between the dynamics of assets and interest rates. The final section 5 illustrates by examples the flexibility of our approach if one wants to end up with a particular term structure model.

1 The general setup

Our goal in this section is to derive an endogenous term structure model from a finite system of risky assets under some natural economic assumptions. Parts of this development are very similar to the results in Jamshidian [Ja]; see later for more details. But since both the setup and the results are different, we have decided to start from scratch to keep the paper self-contained.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T_\infty < \infty}$ generated by an $I\!\!R^m$-valued Brownian motion W and augmented by the P-nullsets. The filtration is thus restricted to a fixed finite interval. Our starting point is an n-dimensional process $X = (X_t)_{0 \le t \le T_\infty}$ given by the SDE

$$\frac{dX_t^i}{X_t^i} = dR_t^i = \mu_t^i \, dt + \sum_{j=1}^m \sigma_t^{ij} \, dW_t^j, \qquad X_0^i = x_0^i > 0, \quad i = 1, \dots, n. \tag{1.1}$$

The coefficient processes μ and σ are assumed $I\!\!F$-predictable and such that $\int_0^{T_\infty} |\mu_u^i| \, du < \infty$ P-a.s. and the $I\!\!R^m$-valued process $\sigma^i = (\sigma^{ij})_{j=1,\dots,m}$ is W-integrable for each i. This ensures in particular that (1.1) has a unique strong solution with values in $I\!\!R^n_{++} := \{ x \in I\!\!R^n \,|\, x^i > 0 \}$ for $i = 1, \dots, n$. We think of X as describing the price evolution of n risky assets traded in a financial market where prices are denominated in some fixed currency unit. Note that we do not assume the existence of a locally riskless asset; this means that all the processes σ^i can be nonzero. We assume $n > m$ so that we have later enough assets to obtain a complete market.

A *self-financing trading strategy (SFTS)* in the above market is an $I\!\!R^n$-valued $I\!\!F$-predictable X-integrable process φ such that the value process $V(\varphi) := \sum_{i=1}^n \varphi^i X^i =: \varphi^\top X$ satisfies the self-financing condition

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \varphi_u^\top dX_u, \qquad 0 \le t \le T_\infty,$$

and a solvency constraint $V(\varphi) \ge -c$ for some constant $c \ge 0$. A *tradable numeraire* is the value process $N^\varphi = V(\varphi)$ of an SFTS φ with $V_0(\varphi) = 1$ and $V(\varphi) > 0$; hence

$$N_t^\varphi = \varphi_t^\top X_t = 1 + \int_0^t \varphi_u^\top dX_u, \qquad 0 \le t \le T_\infty. \tag{1.2}$$

Note that since all X^i are strictly positive, there are many tradable numeraires; in fact, normalizing any convex combination of the X^i with constant coefficients yields a tradable numeraire.

A trading strategy φ describes a dynamic portfolio where one holds φ_t^i units of asset i at each time t. A tradable numeraire is therefore a synthetic basket or weighted index whose value is always positive and can thus be used as a new currency; hence the terminology. A more convenient parametrization of tradable numeraires is obtained by replacing φ_t with

$$a_t^i = \frac{\varphi_t^i X_t^i}{N_t^\varphi}, \qquad i = 1, \ldots, n, \tag{1.3}$$

the fraction of total wealth one has in asset i at time t. With the help of the return process R, the tradable numeraire in (1.2) can be expressed as

$$N_t^\varphi = \mathcal{E}\left(\int a^\top dR\right)_t, \qquad 0 \le t \le T_\infty, \tag{1.4}$$

where the process a is $I\!R^n$-valued, $I\!F$-predictable and R-integrable and satisfies $a_t^\top \mathbf{1} \equiv 1$ with $\mathbf{1} = (1 \ldots 1)^\top \in I\!R^n$. Conversely, it is easily seen that each such a induces via (1.4) and (1.3) a tradable numeraire.

Let us now consider the following possible conditions on our market:

(AA) There exist a tradable numeraire N^φ and a local P-martingale $Z > 0$ with $Z_0 = 1$ such that $Z \frac{X}{N^\varphi}$ is a local P-martingale.

This is an *absence-of-arbitrage* type condition: If we discount X by N^φ, then we can (almost) use Z as a density process for an equivalent local martingale measure for X/N^φ. We say "almost" because the measure with density Z_T with respect to P may have total mass < 1; this happens if Z is a strict local P-martingale. Following Duffie [Du] and Jamshidian [Ja], we call Z/N^φ a *state-price deflator* for X and Z a martingale density for X with respect to the numeraire N^φ.

(DC) There exist $I\!F$-predictable processes α ($I\!R$-valued) and λ ($I\!R^m$-valued) with λ W-integrable, $\int_0^{T_\infty} |\alpha_u|\, du < \infty$ P-a.s. and such that

$$\mu + \alpha \mathbf{1} - \sigma\lambda = 0. \tag{1.5}$$

This is a structural condition on the drift μ of X: For each t, μ_t should be in the span of the vector $\mathbf{1}$ and the range of σ_t.

(R1) $\qquad \mathbf{1} \notin \text{range}(\sigma_t) \qquad P\text{-a.s. for every } t \in [0, T_\infty].$

This is a nondegeneracy assumption on the volatility matrix. In the simplest case where $m = 1$ (one driving Brownian motion) and $n = 2$ (two assets), (R1) stipulates that the volatilities of the two assets be different; this is very natural. It is easy to see that (R1) is equivalent to

(R2) There exists an $I\!R^m$-valued $I\!F$-predictable process a with $a_t^\top \mathbf{1} \neq 0$ and $a_t \in \ker(\sigma_t^\top)$ P-a.s. for every $t \in [0, T_\infty]$.

In fact, (R1) implies (R2) if we take for a_t the projection of the vector $\mathbf{1}$ on $\big(\mathrm{range}(\sigma_t)\big)^\perp = \ker(\sigma_t^\top)$, and if $\mathbf{1} = \sigma_t y$ is in $\mathrm{range}(\sigma_t)$, then $\mathbf{1}^\top a_t = y^\top \sigma_t^\top a_t = 0$ for $a_t \in \ker(\sigma_t^\top)$, in contradiction to (R2).

(FR) $\qquad \mathrm{rank}(\sigma_t) = m \qquad P\text{-a.s. for every } t \in [0, T_\infty].$

Since $n > m$, this simply says that the volatility matrix has always full rank. This is a standard assumption to exclude local redundancies between assets.

Our first result makes clear how the above conditions are related. We shall comment below on the relation to other approaches in the literature.

Theorem 1 *The following assertions hold true:*

1) *(AA) is equivalent to (DC).*

2) *If (R2) holds, there is at most one α satisfying (DC). If we have in addition (DC), there exists a unique tradable numeraire N with finite variation, namely $N = \exp\big(-\int \alpha_u \, du\big)$.*

3) *If (R2) and (FR) hold, then there is also at most one λ satisfying (DC), and each tradable numeraire N^φ admits at most one martingale density for X with respect to N^φ. If we have in addition (DC), then each tradable numeraire N^φ admits a unique martingale density for X with respect to N^φ, and X admits a unique state-price deflator.*

Proof. 1) Suppose (AA) holds and Z/N^φ is a state-price deflator for X. Since $I\!F$ is generated by W, $Z > 0$ must be of the form $Z = \mathcal{E}(\int \beta^\top dW)$ by Itô's representation theorem. Combining this with (1.4) and (1.1) shows that Z/N^φ has the form

$$\frac{Z}{N^\varphi} = \mathcal{E}\left(\int \alpha_u \, du - \int \lambda^\top dW\right) \tag{1.6}$$

with $\alpha = -a^\top \mu + |\sigma^\top a|^2 - a^\top \sigma \beta$ and $\lambda = -\beta + \sigma^\top a$. Because a is integrable with respect to $R = \int \mu_u \, du + \int \sigma \, dW$ and β is W-integrable, we see that λ is W-integrable and $\int_0^{T_\infty} |\alpha_u|\,du < \infty$ P-a.s. Moreover, (1.6) and (1.1) yield

$$\begin{aligned} X^i \frac{Z}{N^\varphi} &= x_0^i\, \mathcal{E}\Big(\int \mu_u^i\,du + \int (\sigma^i)^\top dW\Big)\mathcal{E}\Big(\int \alpha_u\,du - \int \lambda^\top dW\Big) \\ &= x_0^i\, \mathcal{E}\Big(\int (\mu_u^i + \alpha_u)\,du + \int (\sigma^i - \lambda)^\top dW - \int (\sigma_u \lambda_u)^i\,du\Big) \end{aligned}$$

for each i by Yor's formula. Note that $\int (\sigma^i)^\top dW = \big(\int \sigma\,dW\big)^i$ because we view σ^i as a vector in $I\!R^m$. Since $X^i \frac{Z}{N^\varphi}$ is a local P-martingale by (AA) and since $\mathcal{E}(L)$ is a local martingale if and only if L is, it follows that

$$\mu^i + \alpha - (\sigma\lambda)^i = 0 \qquad \text{for } i = 1, \dots, n$$

or equivalently that $\mu \in \operatorname{span}\big(\mathbf{1}, \operatorname{range}(\sigma)\big)$, hence (DC).

Conversely, suppose now that (DC) holds. The constant SFTS $\varphi := (1/X_0^1, 0, \dots, 0)^\top \in I\!R^n$ yields $V(\varphi) = \frac{X^1}{X_0^1} =: N^\varphi$ as tradable numeraire. If we set $U := \int \alpha_u\,du - \int \lambda^\top dW$ and $Z := N^\varphi\,\mathcal{E}(U)$, then $Z > 0$ with $Z_0 = 1$, and (DC) readily implies that both Z and XZ/N^φ are local P-martingales. Hence we get (AA).

2) Suppose (R2) holds. If there are two representations $\mu = -\alpha_1\mathbf{1} + \sigma\lambda_1$ and $\mu = -\alpha\mathbf{1} + \sigma\lambda$ as in (DC), then $(\alpha_1 - \alpha)\mathbf{1} = \sigma(\lambda_1 - \lambda)$. Since $\mathbf{1} \notin \operatorname{range}(\sigma)$ by the equivalence of (R2) and (R1), we get $\alpha_1 = \alpha$ and $\sigma\lambda_1 = \sigma\lambda$, hence uniqueness of α in (DC).

Now assume that we also have (DC). By rescaling if necessary, we obtain from (R2) a predictable process a with $a^\top\mathbf{1} = 1$ and $\sigma^\top a = 0$. Using (DC) then gives $a^\top\mu = -\alpha + a^\top\sigma\lambda = -\alpha$ and so a is R-integrable and $N = \mathcal{E}\big(\int a^\top dR\big)$ is a tradable numeraire with

$$N = \mathcal{E}\Big(\int a_u^\top \mu_u\,du + \int (\sigma^\top a)^\top dW\Big) = \exp\big(-\int \alpha_u\,du\big)\,.$$

Finally, let $\tilde N$ be any other tradable numeraire, induced as in (1.4) by a predictable process $\tilde a$ with $\tilde a^\top\mathbf{1} = 1$. If $\tilde N$ is of finite variation, then $\sigma^\top\tilde a = 0$ and (DC) then gives $\tilde a^\top\mu = -\alpha$, hence $\tilde N = N$.

3) Suppose that (R2) and (FR) hold. In part **2)** we already saw that for any two representations $\mu = -\alpha_1\mathbf{1} + \sigma\lambda_1 = -\alpha\mathbf{1} + \sigma\lambda$ as in (DC), we have $\sigma\lambda_1 = \sigma\lambda$. Hence we get $\lambda_1 = \lambda$ because of (FR).

If Z is any martingale density for X with respect to any tradable numeraire N^φ, we get as in part **1)** the representation (1.6) for the state-price deflator Z/N^φ. The argument in part **1)** also shows that the processes α and λ appearing in (1.6) satisfy (DC) and are therefore unique. Because Z has by part **1)** the form $Z = \mathcal{E}(\int \beta^\top dW)$ with $\beta = -\lambda + \sigma^\top a$, we conclude that Z is uniquely determined by a, hence by φ.

Finally, (DC) implies (AA) by part **1)** and so there exists a tradable numeraire N^φ admitting a (unique) martingale density Z for X with respect to N^φ. Since XZ/N^φ is therefore a local P-martingale, so is $\tilde{Z} := N^{\tilde{\varphi}} Z/N^\varphi$ for any SFTS $\tilde{\varphi}$. If $N^{\tilde{\varphi}}$ is in addition strictly positive with $N_0^{\tilde{\varphi}} = 1$, writing $\tilde{Z}X/N^{\tilde{\varphi}} = ZX/N^\varphi$ shows that $\tilde{Z}$ is a martingale density for X with respect to the tradable numeraire $N^{\tilde{\varphi}}$ and unique by the preceding arguments. Moreover, this also shows that any state-price deflator for X must coincide with Z/N^φ because X is strictly positive. ∎

From an economic perspective, Theorem 1 is easy to understand. Since we have more assets than sources of uncertainty ($n > m$), assuming absence of arbitrage as in (AA) must imply two things: a restriction on the drift μ in relation to the volatilities σ, and essentially the existence of an equivalent local martingale measure for discounted prices. Uniqueness of Z/N^φ basically means that we have a complete market, and we shall exploit this below. But the most important result for us is the existence of α under conditions (AA) and (R2), because this scalar process can be interpreted as a natural *endogenous short rate* in our model. For emphasis, we explicitly restate this below as a separate result.

Remarks. 1) At first glance, our setup looks very similar to the general financial market studied in detail in chapter 1 of Karatzas/Shreve [KS]. But closer inspection shows a number of differences which are due to a marked difference between objectives. We do not assume the existence of a money market account with an instantaneous interest rate, and we are not interested in the issue of complete versus incomplete markets. The latter typically occurs if $n \leq m$. In contrast, our condition $n > m$ means that we have potentially too many assets, and we want to identify restrictions on the parameters to keep the system still arbitrage-free. We then use this to deduce *endogenously* the existence and structure of a short rate process.

2) As mentioned at the beginning, some of the key ideas above are strongly inspired by the work of Jamshidian [Ja] (J for short here). However, there are some subtle technical differences and (more importantly) the goals are quite different.

Let us first explain the technical issues. J introduces the notion of locally arbitrage-free (LAF) prices and studies the restrictions imposed by this condition on the structure of asset price processes. His setup is more general because he works with continuous semimartingales over an arbitrary filtration, but he insists on state-price deflators to be continuous. In our framework with positive price processes and a Brownian filtration, the LAF condition is equivalent to our (AA). As mathematical results, J obtains in Theorems 4.2 and 4.3 necessary and sufficient conditions for an asset model to be LAF, but these conditions all still involve his state-price deflator ξ and are therefore not intrinsic descriptions in terms of the assets alone. In Theorem 4.4, J gives a direct condition on the assets which is similar to our (DC), and proves that it is sufficient for the model to be LAF. This corresponds to the (easy) implication "(DC) $\Longrightarrow$ (AA)" from our Theorem 1, but J gives no converse.

While most of J's results are for continuous semimartingales, he does briefly comment on p.310f. on the case of Itô processes. At first glance, the results there look exactly like our equivalence between (AA) and (DC). However, there is still a subtle difference: We do not *assume* a structure for the state-price deflator as in J's equation (4.11), but we *derive* this by using our assumption of a Brownian filtration.

Much more important than the above technicalities is the difference between basic objectives. J is mainly interested in a particular class of derivatives; he proves that homogeneous pay-offs (e.g., LIBOR derivatives) can always be hedged under fairly general assumptions. In particular, one of his main results is that the existence of a savings account and the uniqueness of the short rate r obtained from (DC) are not required for this. In contrast, uniqueness of r and completeness of the market are crucial for our approach because we want to deduce a term structure model from the asset model. Hence we study the solution of (DC) in detail, we add requirements to guarantee a complete market, and we show how to generate the endogenous savings account by a synthetic basket of assets.

In summary, it seems fair to say that part 1) of our Theorem 1 is contained in Jamshidian [Ja]. Parts 2) and 3) are not, and the goals afterwards are different.

For the remainder of the paper, we now focus on the consequences about interest rates that we can draw from Theorem 1. We first restate that part which forms the basis for our subsequent developments.

Corollary 2 *Suppose that X satisfies (AA) and (R2). Then there exists a uniquely determined short rate process $r = (r_t)_{0 \le t \le T_\infty}$ in the sense that we can construct an SFTS φ whose value process is of the form*

$$dV_t(\varphi) = V_t(\varphi) r_t \, dt, \qquad V_0(\varphi) = 1 \tag{1.7}$$

for an $\mathbb{F}$-predictable scalar process r satisfying $\int_0^{T_\infty} |r_u| \, du < \infty$ P-a.s. If we introduce the tradable numeraire $X^0 := V(\varphi)$ as an additional asset, the augmented market $(X^0 \, X^1 \, \dots \, X^n)^\top$ is still arbitrage-free in the sense that there exists a local P-martingale $Z > 0$ with $Z_0 = 1$ such that $\frac{X}{X^0} Z$ is a local P-martingale. Moreover, r has the form $r = a^\top \mu$ for an $\mathbb{F}$-predictable process a with $a^\top \mathbf{1} = 1$, i.e., the endogenous short rate is a generalized convex combination of the individual drift rates μ^i, $i = 1, \dots, n$. If we have in addition (FR), then

$$\mu = r\mathbf{1} + \sigma\lambda \tag{1.8}$$

for a unique $\mathbb{R}^m$-valued W-integrable predictable process λ.

Proof. Existence and structure of r follow immediately from part **2)** of Theorem 1 if we take $r := -\alpha = a^\top \mu$. The last assertion then follows from part **3)** of Theorem 1. ∎

Corollary 3 *Suppose that X satisfies (AA), (R2) and (FR) and denote by r the endogenous short rate from Corollary 2 and by*

$$\hat{\lambda} := (\sigma^\top \sigma)^{-1} \sigma^\top (\mu - r\mathbf{1}) \tag{1.9}$$

the market price of risk process. If

$$\hat{Z} := \mathcal{E}\left(- \int \hat{\lambda}^\top dW \right) \text{ is a true } P\text{-martingale,} \tag{1.10}$$

then X admits a unique equivalent local martingale measure $\hat{P}$ with respect to the tradable numeraire $X^0 = \exp\left(\int r_s \, ds \right)$, given by $d\hat{P} = \hat{Z}_{T_\infty} \, dP$. Actually, $\hat{P}$ is the minimal local martingale measure for the market $(X^0 \, X^1 \, \dots \, X^n)^\top$.

Proof. According to Corollary 2 and part **3)** of Theorem 1, the only candidate for the density process of an equivalent local martingale measure for X with respect to the numeraire X^0 is $Z = \mathcal{E}\left(\int \beta^\top dW \right)$ with $\beta = -\lambda + \sigma^\top a = -\lambda$, since $\sigma^\top a = 0$ for the particular numeraire X^0. Because (FR) implies that the $(m \times m)$-matrix $\sigma^\top \sigma$ is invertible, (1.8) yields

$$\beta = -\lambda = -(\sigma^\top \sigma)^{-1} \sigma^\top (\mu - r\mathbf{1}) = -\hat{\lambda}, \tag{1.11}$$

hence $Z = \hat{Z}$. Since $\hat{Z}$ is a P-martingale by (1.10), $\hat{P}$ as given above is indeed an equivalent local martingale measure for X/X^0. ■

Remark. Combining (1.8) with (1.11) also yields the relation

$$\mu = r\mathbf{1} + \sigma\hat{\lambda}. \tag{1.12}$$

We shall use this several times below.

Corollary 4 *When X satisfies (AA), (R2) and (FR), the endogenous short rate can be explicitly expressed in terms of μ and σ via*

$$r = \frac{\mathbf{1}^\top \left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mu}{\mathbf{1}^\top \left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mathbf{1}}. \tag{1.13}$$

Proof. The $(m \times m)$-matrix $\sigma^\top\sigma$ is invertible by (FR), and (AA) implies (DC). Moreover, (1.12) and (1.9) give $\mu - r\mathbf{1} = \sigma\hat{\lambda} = \sigma(\sigma^\top\sigma)^{-1}\sigma^\top(\mu - r\mathbf{1})$ or in rewritten form

$$\left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mu = r\left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mathbf{1}. \tag{1.14}$$

Now $y := \left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mathbf{1} \neq 0$ since $\mathbf{1} \notin \mathrm{range}(\sigma)$ by (R1); hence multiplying both sides of (1.14) with $y^\top$ yields (1.13) because $\left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)$ is symmetric and idempotent, being the projection on $\ker(\sigma^\top)$. ■

If we compare the representation (1.13) with Corollary 2 and Theorem 1, we see that r can be written as $r = \hat{a}^\top\mu$ with

$$\hat{a} := \frac{\left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mathbf{1}}{\mathbf{1}^\top\left(I_n - \sigma(\sigma^\top\sigma)^{-1}\sigma^\top\right)\mathbf{1}}. \tag{1.15}$$

Moreover, (1.7) shows that the tradable numeraire $\hat{N} = V(\varphi)$ associated to $\hat{a}$ is the *classical savings account*, i.e., $\hat{N} = \mathcal{E}\left(\int \hat{a}^\top dR\right) = \exp\left(\int r_u\,du\right)$. Hence (1.15) can be viewed as an explicit prescription for synthesizing the endogenously determined savings account from the basic securities $X^1, \dots, X^n$.

In summary, the preceding results show that we can start from a general Itô process model for n risky assets and derive a natural endogenous short rate process from a weak absence-of-arbitrage type condition (AA) in combination with a non-degeneracy assumption (R1). If we add a right-hand regularity assumption (FR) for the volatility matrix and an integrability condition (1.10), we also have a unique equivalent local martingale measure and therefore a complete market. This entails in particular unique prices for zero coupon bonds of all maturities $T \leq T_\infty$ and therefore a unique associated term structure model. We shall call this an *endogenously generated term structure model (EGM)*. For the case where the process X has a Markovian structure, we analyze the corresponding EGM in more detail in the next section.

Remarks. 1) We have said at the beginning that we do not assume the existence of a riskless asset among the basic securities $X^1, \ldots, X^n$. But if one of them, say X^{i_0}, does happen to be riskless in the sense that the vector σ^{i_0} is zero, we are of course back in the classical situation of a standard multidimensional Itô process model as in Karatzas/Shreve [KS] (except that we have $n > m$ instead of $n \leq m$). In fact, the drift condition (DC) then enforces that $\mu^{i_0} = -\alpha = r$ so that we can directly use X^{i_0} itself as savings account.

2) We emphasize that our formulation imposes no assumptions on the nature of the securities whose prices are modelled by X. We can for instance think of $X^1, \ldots, X^n$ as stocks that also induce an interest rate model. But we can also consider a combination of some underlyings, some derivatives and some interest rate products. Provided that the required assumptions are satisfied, our results thus yield a joint model for stocks *and* interest rate derivatives in a unified framework. This is in the spirit of the recently developed *market models* and opens up a range of new possibilities for formulating and testing hypotheses about financial markets.

2 The Markovian case

In this section, we specialize our framework to a Markovian diffusion for X; the corresponding endogenously determined interest rate model then turns out to be a finite factor HJM model. More precisely, we assume that X in (1.1) is given by a Markovian system of SDEs

$$\frac{dX_t^i}{X_t^i} = \mu^i(t, X_t)\, dt + \sum_{j=1}^m \sigma^{ij}(t, X_t)\, dW_t^j, \qquad i = 1, \ldots, n, \tag{2.1}$$

where the functions $\mu : [0, T_\infty] \times I\!R^n_{++} \to I\!R^n$ and $\sigma : [0, T_\infty] \times I\!R^n_{++} \to I\!R^{n\times m}$ are sufficiently smooth and such that there exists a continuous $I\!R^n_{++}$-valued process X

on $[0, T_\infty]$ satisfying (2.1). Moreover, we impose for every $(t, x) \in [0, T_\infty] \times I\!R^n_{++}$ the following conditions:

$$\begin{aligned} \mathbf{1} &\notin \operatorname{range}\big(\sigma(t,x)\big), \\ \mu(t,x) &\in \operatorname{span}\Big(\mathbf{1}, \operatorname{range}\big(\sigma(t,x)\big)\Big), \\ \operatorname{rank}\big(\sigma(t,x)\big) &= m. \end{aligned} \tag{2.2}$$

It is clear that (2.2) implies (DC), (R1) and (FR) for X. In addition, we assume that μ and σ satisfy sufficient conditions for (1.10) to hold, i.e.,

$$\hat{Z} = \mathcal{E}\left(-\int \left((\sigma^\top\sigma)^{-1}\sigma^\top(\mu - r\mathbf{1})\right)^\top (u, X_u)\, dW_u\right) \tag{2.3}$$

should be a true P-martingale, where $r(t,x)$ is given by (1.13) and X in (2.3) is the solution of (2.1). Then Corollary 3 yields a unique equivalent local martingale measure $\hat{P}$ for X with respect to the tradable numeraire $X^0 = \exp\left(\int r_s\, ds\right)$. Hence (2.1) gives a *complete market* and every bounded $\mathcal{F}_{T_\infty}$-measurable random variable H is the final value $V_{T_\infty}(\varphi)$ of an SFTS φ such that $V(\varphi)$ is a $\hat{P}$-martingale on $[0, T_\infty]$; see Theorem 16 of Delbaen/Schachermayer [DS].

Now consider a *zero coupon bond (ZCB)* with face value 1 at maturity time $T \le T_\infty$, viewed as a European option with payoff $H \equiv 1$ at time T. Completeness implies that the bond price $B_{t,T}$ at time $t \le T$ is given by

$$B_{t,T} = \hat{E}\left[\exp\left(-\int_t^T r_u\, du\right)\middle|\mathcal{F}_t\right] = E\left[\frac{\hat{Z}_T}{\hat{Z}_t}\exp\left(-\int_t^T r(u, X_u)\, du\right)\middle|\mathcal{F}_t\right], \tag{2.4}$$

and (2.4), (2.3) and the Markov property of X thus make it clear that ZCB prices are of the form $B_{t,T} = b(t, T, X_t)$ for some measurable function b. Note also that $\hat{P}$ is an equivalent martingale measure for each $B_{\cdot,T}$ with respect to the savings account X^0 as numeraire; in the standard terminology from Musiela/Rutkowski [MR] or Björk [Bj98], this means that $\hat{P}$ is a *risk-neutral measure* for the term structure model induced by X.

If b is sufficiently smooth, we can proceed in the standard way to derive the dynamics of ZCB prices. We first apply Itô's formula to $B_{\cdot,T}/X^0_\cdot = b(\cdot, T, X_\cdot)/X^0_\cdot$ to express this ratio as a stochastic integral with respect to X plus a number of finite variation terms. Because $B_{\cdot,T}/X^0_\cdot$ is a $\hat{P}$-martingale on $[0, T]$ by (2.4), we then conclude that the finite variation terms must vanish and finally obtain from the product rule and (2.1) that

$$dB_{t,T} = r(t, X_t) B_{t,T}\, dt + \sum_{i=1}^n X^i_t \frac{\partial b}{\partial x^i}(t, T, X_t)\big(\sigma^i(t, X_t)\big)^\top d\hat{W}_t, \tag{2.5}$$

where $\hat{W} := W + \int \hat{\lambda}_u \, du$ with $\hat{\lambda}$ from (1.9) is by Girsanov's theorem a $\hat{P}$-Brownian motion. The vanishing of the finite variation terms also implies that b must satisfy (at least along the paths of X) a PDE, and so we look for $b(t,T,x)$ by trying to solve the final value problem

$$0 = \frac{\partial b}{\partial t} + \sum_{i=1}^{n} x^i r \frac{\partial b}{\partial x^i} + \frac{1}{2} \sum_{i,k=1}^{n} x^i x^k (\sigma\sigma^\top)^{ik} \frac{\partial^2 b}{\partial x^i \partial x^k} - rb, \qquad b(T,T,x) \equiv 1 \tag{2.6}$$

on $(0,T) \times \mathbb{R}^n_{++}$, where we have suppressed the arguments (t,T,x) for b and (t,x) for r and $\sigma\sigma^\top$. Under regularity assumptions on μ and σ, (2.6) has a sufficiently smooth solution $b(t,T,x)$, and reversing the above reasoning then shows that $b(t,T,X_t)$ gives the ZCB price $B_{t,T}$.

To obtain an HJM description of our model, we now assume that the bond price $B_{t,T}$, hence also the solution of (2.6), is sufficiently smooth in the maturity parameter T and define the usual instantaneous *forward rates*

$$f_{t,T} := -\frac{\partial(\log B_{t,T})}{\partial T} = F(t,T,X_t) \tag{2.7}$$

with

$$F(t,T,x) := -\frac{\partial\big(\log b(t,T,x)\big)}{\partial T}. \tag{2.8}$$

Applying Itô's formula to $\log B_{t,T}$, using (2.5), differentiating with respect to T and using (2.8) yields under regularity assumptions

$$dF(t,T,X_t) = \sum_{i=1}^{n} X_t^i \frac{\partial F}{\partial x^i} (\sigma^i)^\top d\hat{W}_t + \left(\sum_{i,k=1}^{n} X_t^i X_t^k (\sigma\sigma^\top)^{ik} \frac{\partial F}{\partial x^i} \int_t^T \frac{\partial F(t,\tau,X_t)}{\partial x^k} \, d\tau \right) dt.$$

Hence we recognize via (2.7) an HJM dynamics

$$df_{t,T} = \sigma_{\mathrm{HJM}}^\top(t,T,X_t)\, d\hat{W}_t + \left(\sigma_{\mathrm{HJM}}^\top(t,T,X_t) \int_t^T \sigma_{\mathrm{HJM}}(t,\tau,X_t)\, d\tau \right) dt$$

with an $\mathbb{R}^m$-valued HJM forward rate volatility vector process given by

$$\sigma_{\mathrm{HJM}}(t,\tau,X_t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x^i}(t,\tau,X_t) X_t^i \sigma^i(t,X_t). \tag{2.9}$$

The bond price dynamics (2.5) under the risk-neutral measure $\hat{P}$ then becomes

$$dB_{t,T} = B_{t,T} \left(r(t,X_t)\, dt - \left(\int_t^T \sigma_{\mathrm{HJM}}(t,\tau,X_t)\, d\tau \right)^\top d\hat{W}_t \right)$$

because $\frac{\partial b}{\partial x^i} = -b\frac{\partial F}{\partial x^i}$ from (2.8).

In summary, we have obtained the following result:

Proposition 5 *For the model (2.1) with (2.2) and sufficient regularity and integrability conditions for μ and σ, the induced EGM from Corollary 3 has forward rates of the form $f_{t,T} = F(t,T,X_t)$ with forward rate volatility σ_{HJM} given explicitly by (2.9).*

The main point of the present section is that we obtain from a general multi-asset Markovian diffusion model X an endogenous term structure model whose dynamics can be expressed fairly explicitly in terms of the coefficients μ and σ of X. This is most easily seen if we use the description via the short rate given in (1.13) and the risk-neutral measure determined by (2.3). With more effort, we can also write down the HJM description as in (2.9), but this requires in addition the solution $b(t,T,x)$ of the PDE (2.6) to obtain $F(t,T,x)$ via (2.8) by differentiation. Alternatively, one could describe F directly by integrating (2.8) to get $b(t,T,x)$, plugging this into (2.6), using $F(t,t,x) = r(t,x)$ and differentiating with respect to T. However, it seems more difficult to solve the resulting integro-partial differential equation for F than to go via b and (2.6).

3 Links to finite factor models

Geometric aspects of interest rate theory have attracted much attention recently; see for instance the survey by Björk [Bj00]. One question in this context is the realizability of a given term structure model by means of a finite factor model. In this section, we therefore very briefly explain some links between such models and our approach.

As in Duffie/Kan [DK], a *finite factor model (FFM)* is an HJM term structure model with forward rates of the form $f_{t,T} = F(t,T,Y_t)$, $0 \le t \le T \le T_\infty$, where $F : [0,T_\infty] \times [0,T_\infty] \times I\!R^k \to I\!R$ is sufficiently smooth and the $I\!R^k$-valued continuous process Y solves the SDE

$$dY_t = v(t,Y_t)\,dt + \rho(t,Y_t)\,d\hat{W}_t, \quad Y_0 = y_0 \in I\!R^k \qquad \text{under } \hat{P} \tag{3.1}$$

for sufficiently smooth functions v ($I\!R^k$-valued) and ρ ($I\!R^{k\times m}$-valued) on $[0,T_\infty] \times I\!R^k$. In (3.1), $\hat{W}$ is an m-dimensional Brownian motion with respect to $\hat{P}$, and $\hat{P}$ is assumed to be a risk-neutral measure for the term structure model under consideration.

With this terminology, Proposition 5 says that the interest rate model endogenously derived from a complete multi-asset diffusion model (2.1) is given by a finite factor model where the assets X serve as factors Y. To put it more briefly: *Every sufficiently regular Markovian diffusion EGM is an FFM.*

Not surprisingly, there is also a converse to this result. Suppose we start with an FFM corresponding to (3.1) and take enough ZCBs consistent with the given term structure that they form a complete market. If we regard these ZCBs as assets, the resulting EGM will coincide with the initial FFM, at least up to the first time one of the chosen bonds matures. If there are in addition L ZCBs with prices $B_{t,T_\ell} = b(t, T_\ell, Y_t)$, $\ell = 1, \ldots, L$, such that the map $y \mapsto \big(b(t, T_1, y), \ldots, b(t, T_L, y)\big)$ has a left inverse for any $t \in [0, T_\infty]$, we can express $Y_\cdot$ from the $B_{\cdot, T_\ell}$ and the dynamics of the generating ZCBs is seen to be of diffusion type as in (2.1).

Although it works, the above (re-)construction of an EGM from an FFM is not very satisfactory because the connection between the factors Y and the ZCBs as generating assets X is rather complicated. In particular, X always becomes non-autonomous even if Y is autonomous. Thus we ask for a simpler model whose assets are linked to the factors in a more direct way, and the next result goes in that direction.

Proposition 6 *Consider an FFM $f_{t,T} = F(t, T, Y_t)$ in the risk-neutral measure $\hat{P}$ where Y satisfies (3.1). Assume that $m \leq k$ and* $\operatorname{rank}\big(\rho(t,y)\big) = m$ *for all (t, y). If we also have*

$$r = v^i + \frac{1}{2}(\rho\rho^\top)^{ii}, \quad i = 1, \ldots, k, \tag{3.2}$$

$$\mathbf{1} \notin \operatorname{range}(\rho), \tag{3.3}$$

then the processes $X^i := \exp(Y^i)$, $i = 1, \ldots, k$, form a complete asset model of the form (2.1), and the given FFM is endogenously generated by X.

Proof. Applying Itô's formula to X yields

$$\frac{dX_t^i}{X_t^i} = \left(v^i + \frac{1}{2}(\rho\rho^\top)^{ii}\right) dt + (\rho^i)^\top d\hat{W}_t \qquad \text{under } \hat{P},$$

and so (3.2) and (3.3) imply that (DC) $\big($in the form (1.8)$\big)$ and (R1) are satisfied for X and ρ. The assumption $\operatorname{rank}(\rho) \equiv m$ yields (FR), and so Corollary 3 gives a unique endogenous term structure model. By uniqueness, this EGM must coincide with the given FFM. ■

Remarks. 1) Proposition 6 is useful if the FFM has an autonomous factor process because under (3.2) and (3.3), the generating asset process is then autonomous as well. This improves our first construction of a generating asset market via ZCBs. But note also that using $X = \exp(Y)$ as assets implicitly assumes that the factors Y already have some economic significance.

2) The goals behind an FFM and an EGM are in general quite different. In a factor model, one tries to have Y and F as simple as possible to get explicit results about distributions or prices of some interest rate instruments. In particular, the factors Y and the mapping F need not have any economic significance. In contrast, the starting point of an EGM is a collection of traded assets X, and the mechanism generating interest rates from X is given via the arbitrage-free pricing of ZCBs. In a comparison between the two approaches, the focus is therefore more on theoretical insights than on practical applications.

4 Application: Short rate and index

In this section, we exploit our joint framework for assets and interest rates to derive very appealing relations between the short rate and an asset index under some structural assumptions on the market.

4.1 The short rate under homogeneity

We begin with an autonomous version of the Markovian diffusion setting from section 2 so that X is given by

$$\frac{dX_t^i}{X_t^i} = \mu^i(X_t)\,dt + \sum_{j=1}^{m} \sigma^{ij}(X_t)\,dW_t^j, \qquad i = 1, \dots, n. \tag{4.1}$$

Apart from smoothness to justify the calculations below, we assume enough regularity for the results of section 1 to hold so that X endogenously determines the money market. In addition, we assume in this subsection that

$$\mu \text{ and } \sigma \text{ are homogeneous of degree } 0; \tag{4.2}$$

this means that for $h \in \{\mu, \sigma\}$, we have $h(\alpha x) = h(x)$ for all $\alpha > 0$ and $x \in \mathbb{R}^n_{++}$. Because μ and σ describe via (4.1) the return dynamics in our market, this assumption is economically very natural. It formalizes the idea that prices are always in relative terms so that any simultaneous scaling of all asset prices does not affect the return dynamics. (4.2) implies that the short rate function $r(x)$ from (1.13) is also homogeneous of degree 0, and this is the crucial property that drives our subsequent analysis.

Lemma 7 *If $h : \mathbb{R}^n_{++} \to \mathbb{R}$ is in C^2 and homogeneous of degree 0, then*

$$\sum_{i=1}^n x^i \frac{\partial h}{\partial x^i} = 0, \qquad \frac{\partial h}{\partial x^k} + \sum_{i=1}^n x^i \frac{\partial^2 h}{\partial x^i \partial x^k} = 0, \qquad \sum_{i,k=1}^n x^i x^k \frac{\partial^2 h}{\partial x^i \partial x^k} = 0.$$

Proof. By homogeneity, $\alpha \mapsto h(\alpha x)$ is constant on $(0,\infty)$ for each $x \in \mathbb{R}^n_{++}$. Differentiate to get the first result, differentiate that with respect to x^k to get the second one, and multiply by x^k and sum over k to get the third result by using the first one. ■

Since the short rate in our model is given by $r_t = r(X_t)$, Itô's formula, (1.12) and the first property in Lemma 7 yield the short rate dynamics

$$\begin{aligned} dr(X_t) &= \sum_{i=1}^n \frac{\partial r}{\partial x^i} \mu^i X_t^i \, dt + \frac{1}{2} \sum_{i,k=1}^n \frac{\partial^2 r}{\partial x^i \partial x^k} X_t^i X_t^k (\sigma\sigma^\top)^{ik} \, dt + \sum_{i=1}^n \frac{\partial r}{\partial x^i} X_t^i (\sigma \, dW_t)^i \\ &= \frac{1}{2} \sum_{i,k=1}^n \frac{\partial^2 r}{\partial x^i \partial x^k} X_t^i X_t^k (\sigma\sigma^\top)^{ik} \, dt + \sum_{i=1}^n \frac{\partial r}{\partial x^i} X_t^i (\sigma \, d\hat{W}_t)^i \\ &=: \hat{c}(X_t)\, dt + b^\top(X_t)\, d\hat{W}_t \\ &=: c(X_t)\, dt + b^\top(X_t)\, dW_t, \end{aligned}$$

where

$$b(x) = \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x) \sigma^i(x), \tag{4.3}$$

$$\hat{c}(x) = c(x) - (b^\top \hat{\lambda})(x) = \frac{1}{2} \sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(x) (\sigma\sigma^\top)^{ik}(x) \tag{4.4}$$

and $\hat{\lambda} = (\sigma^\top \sigma)^{-1} \sigma^\top (\mu - r\mathbf{1})$ from (1.9) is the market price of risk.

To exploit homogeneity, we now fix some function $\sigma_0 : \mathbb{R}^n \to \mathbb{R}^m$ and define $\tilde{\sigma}^{ij} := \sigma^{ij} - \sigma_0^j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We plug $\sigma^i = \tilde{\sigma}^i + \sigma_0$ into (4.3) and use Lemma 7 to get

$$b(x) = \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x) \tilde{\sigma}^i(x) \tag{4.5}$$

with the $I\!R^m$-valued functions $\tilde{\sigma}^i := (\tilde{\sigma}^{ij})_{j=1,\dots,m}$. Differentiating with respect to x^k, multiplying by $x^k(\tilde{\sigma}^k)^\top$ and summing over k gives

$$\sum_{k=1}^n x^k(\tilde{\sigma}^k)^\top \frac{\partial b}{\partial x^k} = \sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(\tilde{\sigma}\tilde{\sigma}^\top)^{ik} + \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}|\tilde{\sigma}^i|^2 + \sum_{i,k=1}^n x^i x^k \frac{\partial r}{\partial x^i}(\tilde{\sigma}^k)^\top \frac{\partial \tilde{\sigma}^i}{\partial x^k}. \tag{4.6}$$

On the other hand, we can also plug $\sigma^i = \tilde{\sigma}^i + \sigma_0$ into (4.4) and use the first property in Lemma 7, then the second one and then (4.5) to get

$$\hat{c} = c - b^\top \hat{\lambda} = -b^\top \sigma_0 + \frac{1}{2}\sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(\tilde{\sigma}\tilde{\sigma}^\top)^{ik}. \tag{4.7}$$

Finally, we replace the second order derivatives in (4.7) by using (4.6) and use the first property in Lemma 7 to obtain

$$\begin{aligned} \hat{c} &= c - b^\top \hat{\lambda} \\ &= -b^\top \sigma_0 + \frac{1}{2}\sum_{i=1}^n x^i (\tilde{\sigma}^i)^\top \frac{\partial b}{\partial x^i} - \frac{1}{2}\sum_{i,k=1}^n x^i x^k \frac{\partial r}{\partial x^i}(\tilde{\sigma}^k)^\top \frac{\partial \tilde{\sigma}^i}{\partial x^k} \\ &\quad - \frac{1}{2}\sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}\left(|\tilde{\sigma}^i|^2 - \tilde{\sigma}^2_{\mathrm{av}}\right) \end{aligned} \tag{4.8}$$

with $\tilde{\sigma}^2_{\mathrm{av}} := \frac{1}{n}\sum_{i=1}^n |\tilde{\sigma}^i|^2$. This is a general result on the structure of the short rate dynamics in homogeneous models.

4.2 Asset indices

In the preceding arguments, the function σ_0 has been completely arbitrary. To establish a link between interest rates and assets, we want to choose σ_0 as the volatility of a suitable asset index. Our analysis later again needs the Markovian structure imposed above, but the family of indices we consider can be introduced quite generally and so we return for a while to the general setting of section 1. Let $a = (a_t)_{0 \le t \le T_\infty}$ be an $I\!R^n$-valued $I\!F$-predictable R-integrable process with $a_t^\top \mathbf{1} \equiv 1$ and consider the tradable numeraire $I^a = \mathcal{E}\left(\int a^\top dR\right)$ from (1.4). Then I^a is a tradable asset obtained by starting with a unit initial capital and holding at each time t the fraction a_t^i of total wealth in asset i for $i = 1, \dots, n$. The dynamics of I^a is very simple:

$$\frac{dI_t^a}{I_t^a} = a_t^\top dR_t = \bar{\mu}_t(a)\,dt + \bar{\sigma}_t^\top(a)\,dW_t \tag{4.9}$$

with $\bar{\mu}(a) = a^\top \mu$ and the $I\!R^m$-valued process $\bar{\sigma}(a) = \sigma^\top a$. We call I^a the *index* associated to a. Because X and I^a are both stochastic exponentials, the I^a-discounted assets $\tilde{X}(a) := X/I^a$ are readily seen to follow the SDE

$$\frac{d\tilde{X}^i_t(a)}{\tilde{X}^i_t(a)} = \left(\mu^i_t - \bar{\mu}_t(a) - \bar{\sigma}^\top_t(a)\tilde{\sigma}^i_t(a)\right)\,dt + \left(\tilde{\sigma}^i_t(a)\right)^\top dW_t$$

with $\tilde{\sigma}^{ij}(a) := \sigma^{ij} - \bar{\sigma}^j(a)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Intuitively, $\tilde{X}^i(a)$ describes the multiplicative fluctuations of asset X^i around the index I^a, and $\tilde{\sigma}(a)$ is therefore the matrix of *intrinsic volatilities* (with respect to I^a, to be accurate). Like the exchange prices in Platen [P00], the quantities $\tilde{X}^i(a)$ are ratios of two Itô processes and thus have a specific volatility structure.

Looking at the family of indices I^a brings two benefits. Quantities like $\tilde{\sigma}^i(a) = \sigma^i - \bar{\sigma}(a)$ appear in a natural way as intrinsic volatility vectors, and we get an interpretation of the quantity $b^\top\bar{\sigma}(a)$ that appears in (4.8) for $\sigma_0 = \bar{\sigma}(a)$. In fact, (4.9) shows that the instantaneous covariance between the index return dI^a/I^a and the short rate increment dr is given by

$$d\left\langle \int \frac{dI^a}{I^a}, r \right\rangle_t = b^\top(X_t)\bar{\sigma}_t(a)\,dt. \tag{4.10}$$

This relation plus additional structural assumptions on our market will allow us to obtain from (4.8) a close link between assets and interest rates via an index I^a.

So far, our index I^a has been very general. We could for instance look at the arithmetic average $\bar{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X^i$ obtained by taking $a = X/\sum\limits_{i=1}^{n} X^i$. However, we do not choose this index for several reasons. For one thing, its return dynamics is rather complicated. In a model with constant coefficients, it is also well known that $\bar{X}$ is an average of lognormal processes and thus notoriously hard to handle; recall for instance the difficulties in pricing Asian options. Finally, $\bar{X}$ also seems not very reasonable since it corresponds to holding a fixed *number* of assets over time, irrespective of their relative importance. We thus want to use instead an index where the vector a is constant so that we keep over time fixed *fractions* of our wealth invested in the various assets. (4.9) shows that such an index has very simple return dynamics; its drift and volatility vector are simply one fixed generalized convex combination of the individual drifts μ^i and volatility vectors σ^i, respectively. Another positive feature is that I^a is observable by looking at X only. In fact, we have $\int\limits_0^t a^\top dR_s = a^\top R_t = \sum\limits_{i=1}^{n} a^i\left(R^i_t - \frac{1}{2}\langle R^i\rangle_t\right) + \frac{1}{2}\sum\limits_{i=1}^{n} a^i\langle R^i\rangle_t$ if a is constant, and so

$X^i = X_0^i\,\mathcal{E}(R^i)$ and (1.1) yield

$$I_t^a = \mathcal{E}\left(\int a^\top dR\right)_t = \exp\left(\frac{1}{2}\int_0^t \left(\sum_{i=1}^n a^i \left|\sigma_s^i\right|^2 - \left|a^\top \sigma_s\right|^2\right) ds\right) \prod_{i=1}^n \left(\frac{X_t^i}{X_0^i}\right)^{a^i}.$$

But σ is observable from X via the quadratic variations of X, and hence so is I^a. Finally, a strategy with constant a is also economically well motivated as the solution to the Merton problem under constant relative risk aversion.

To find a suitable index I^a, we could now try to determine some constant a via the last observation. But this would require the specification of preferences in terms of a utility function and in addition detailed knowledge about all market coefficients μ and σ. To avoid such delicate issues, we prefer to impose direct assumptions on the volatility structure; these will turn out to determine I^a uniquely even without full information about a.

Lemma 8 *Assume (DC), (R1) and (FR) so that we essentially have a complete arbitrage-free market. Suppose also that for all t and ω, the n volatility vectors $\sigma_t^i(\omega)$ in $I\!R^m$ lie on some sphere $S_t(\omega)$ with center $C_t(\omega)$, say. Then there exists a unique index I^a whose volatility process $\bar\sigma(a)$ coincides with the process $C = (C_t)_{0\le t\le T_\infty}$ of sphere centers, and I^a is completely determined by the volatility structure σ alone.*

Proof. We first observe that $S_t(\omega)$ and $C_t(\omega)$ are unique. In fact, suppose not. Then we should have $\left|\sigma_t^i(\omega) - C_t^\ell(\omega)\right|^2 = \varrho_\ell^2(t,\omega)$ for $\ell = 1,2$ and $i = 1,\dots,n$, with $C_t^1(\omega) \neq C_t^2(\omega)$. Squaring out and taking differences would then yield

$$2\sigma_t(\omega)\left(C_t^2(\omega) - C_t^1(\omega)\right) = \left(\varrho_1^2(t,\omega) - \varrho_2^2(t,\omega) + \left|C_t^2(\omega)\right|^2 - \left|C_t^1(\omega)\right|^2\right)\mathbf{1}$$

so that $\mathbf{1} \in \operatorname{range}\big(\sigma_t(\omega)\big)$, in contradiction to (R1). Due to (FR), the equation $\sigma_t^\top(\omega)a_t = C_t(\omega)$ has a solution $a_t(\omega) \in I\!R^n$ which can be chosen so that a is $I\!F$-predictable since σ and C are. Moreover, $\dim\left(\ker\left(\sigma_t^\top(\omega)\right)\right) \ge n - m \ge 1$ implies that we can choose a to satisfy the condition $a_t^\top(\omega)\mathbf{1} \equiv 1$ as well. Because each σ^i is W-integrable, so is $\bar\sigma(a) = \sigma^\top a = C$. Moreover, multiplying (1.12) by $a^\top$ yields $\bar\mu(a) = a^\top\mu = ra^\top\mathbf{1} + a^\top\sigma\hat\lambda = r + \bar\sigma^\top(a)\hat\lambda$, and since r and $\hat\lambda$ are unique by Corollary 1.2, we see that $\bar\mu(a)$ is also determined uniquely by σ via $\bar\sigma(a)$ and that a is R-integrable. The index I^a thus satisfies all our assertions. ■

Remarks. 1) Note that we always have uniqueness of the index I^a even though the portfolio weights a need not be unique. We only get uniqueness of a in general if we have $n = m + 1$.

2) Apart from giving a unique a, the case $n = m+1$ is also very pleasant because the volatility vectors $\sigma_t^i(\omega)$ then always lie on a sphere. This is because $m+1$ vectors in $I\!R^m$ lie on a sphere if and only if they are not in some at most $(m-1)$-dimensional hyperplane. However, the latter cannot happen due to (FR) and (R1). In fact, (R1) excludes the case where the vectors lie in some hyperplane not containing the origin and (FR) the case of a hyperplane through the origin. If we think of the number n of assets as fixed, we can thus always ensure that the assumptions of Lemma 8 are satisfied if we increase the number m of driving factors. This is of particular interest if n is small, e.g., if we think of a situation with 3 or 4 representative assets that each summarize one market segment. In that case, $n = m+1$ is a very natural condition and allows us to apply Lemma 8.

For brevity, we call the volatility structure σ *spherical* if the vectors $\sigma_t^i(\omega)$, $i = 1, \ldots, n$, lie on a sphere in $I\!R^m$ for each t, ω. The intrinsic volatilities $\tilde{\sigma}^i(a) = \sigma^i - \bar{\sigma}(a)$ with $\bar{\sigma}(a)$ from Lemma 8 are then unique and obviously all have the same length. In the autonomous Markovian case where $\sigma_t(\omega)$ is of the form $\sigma\big(X_t(\omega)\big)$ for a function $\sigma(x)$, it is clear from the proof of Lemma 8 that a and $\tilde{\sigma}^i(a)$ also only depend on $X_t(\omega)$ and are thus given by functions $\tilde{\sigma}^i$ and a on $I\!R^n$.

Definition. A homogeneous Markovian volatility structure $\sigma(x)$ is called *rigidly spherical* if it is spherical and if the associated intrinsic volatility vectors $\tilde{\sigma}^i(a)$ do not depend on x.

Lemma 9 *Assume (DC), (R1) and (FR) and that we have a homogeneous Markovian model as in (4.1). If σ is rigidly spherical, then a from Lemma 8 can be chosen constant (in x).*

Proof. According to the proof of Lemma 8, a is a solution of the equations $\sigma^\top(x)a = C(x)$ and $a^\top \mathbf{1} = 1$. Using $\sigma^i = \tilde{\sigma}^i(a) + \bar{\sigma}(a) = \tilde{\sigma}^i(a) + C$ shows that this can be rewritten as $\tilde{\sigma}^\top a = 0$ and $a^\top \mathbf{1} = 1$, and so if $\tilde{\sigma}$ does not depend on x, then a can be chosen constant as well. ∎

Remark. Combining Lemma 8 and Lemma 9 shows that a volatility structure is rigidly spherical if and only if there exists some index I^a such that the associated intrinsic volatility vectors $\tilde{\sigma}^i(a)$ do not depend on x and all have the same length.

4.3 Links between assets and interest rates

Let us now return to the homogeneous Markovian setting from (4.1) and assume in addition to (DC), (FR) and (R1) plus (4.2) and enough regularity that the volatility structure is rigidly spherical. Then we have from Lemma 8 a unique associated index I^a whose weights a can even be chosen constant by Lemma 9. For ease of notation, we drop all indices a and write I, $\bar{\mu}$, $\bar{\sigma}$ and $\tilde{\sigma}$ from now on. In geometric terms, all vectors $\tilde{\sigma}^i$ have the same length $\tilde{\sigma}_{\mathrm{av}}$ and form a fixed (not necessarily regular) polygon in $I\!R^m$ with all corners on a sphere and the origin as barycentre. The rigidity of this structure explains our choice of terminology. Financially, this situation means that if we use the index I as numeraire, we see for I-discounted prices X/I simply a multidimensional Black-Scholes model with a constant volatility matrix.

Proposition 10 *Suppose we have a rigidly spherical market so that all $\tilde{\sigma}^i = \sigma^i - \bar{\sigma}$ are constant in x and have the same length. Then (4.8) simplifies to*

$$\hat{c} = c - b^\top \hat{\lambda} = -b^\top \bar{\sigma} + \frac{1}{2}\sum_{i=1}^{n} x^i (\tilde{\sigma}^i)^\top \frac{\partial b}{\partial x^i} = -\frac{d}{dt}\left\langle \int \frac{dI}{I}, r \right\rangle + \frac{1}{2}\sum_{i=1}^{n} x^i (\tilde{\sigma}^i)^\top \frac{\partial b}{\partial x^i}. \tag{4.11}$$

Proof. Our assumptions imply that the last two terms in (4.8) vanish. ■

Now assume in addition that our rigidly spherical market induces a short rate process r of "semi-Vasiček" type in the sense that the short rate volatility b is a constant b_0. (We show below that such models exist.) Then we get

Proposition 11 *Consider a rigidly spherical market so that all $\tilde{\sigma}^i = \sigma^i - \bar{\sigma}$ are constant in x and have the same length. If in addition the endogenous short rate has constant volatility b_0, we have the following relation between the index volatility $\bar{\sigma}$, the market price of risk $\hat{\lambda}$ and the dynamics of the short rate r with drift c and volatility b_0 (under the original measure P):*

$$\frac{d}{dt}\left\langle \int \frac{dI}{I}, r \right\rangle = b_0^\top \bar{\sigma} = -c + b_0^\top \hat{\lambda}. \tag{4.12}$$

Proof. Since b is constant, this follows from (4.11). ■

Remark. Since $\hat{c} = c - b^\top \hat{\lambda}$ by (4.4), we also obtain from Proposition 11 the short rate drift $\hat{c}$ under the risk-neutral measure $\hat{P}$ as

$$\hat{c} = -\frac{d}{dt}\left\langle \int \frac{dI}{I}, r \right\rangle.$$

For well-structured markets, our approach yields with (4.12) a very appealing link between the behaviour of interest rates and the overall asset index. In fact, a first economic interpretation of (4.12) can be given as follows. When the short rate is in a "steady state" in the sense that the (usually mean-reverting) drift is approximately zero, the market price of risk is more or less identified via the index volatility. More precisely: *If the real-world short rate drift c is zero, the market price of risk $\hat{\lambda}$ and the index volatility $\bar{\sigma}$ have the same projections on the short rate volatility b_0.*

Even if c is not zero, we can get more intuition from (4.12) if we assume that the market price of risk and the index volatility are proportional, i.e., $\hat{\lambda} = q\bar{\sigma}$ for some scalar function $q(x)$ on $\mathbb{R}^n$. The proof below shows that this is equivalent to assuming that $\frac{|\bar{\mu}-r|}{|\bar{\sigma}|} = |\hat{\lambda}|$, and we also point out later a connection to the numeraire portfolio. We have not yet fully understood the economic interpretation of this condition, but it yields the following interesting result.

Theorem 12 *Suppose we have a rigidly spherical market so that all $\tilde{\sigma}^i = \sigma^i - \bar{\sigma}$ are constant in x and have the same length. Assume also that the endogenous short rate volatility is a constant b_0 and that the market price of risk $\hat{\lambda}$ is proportional to the index volatility $\bar{\sigma}$. Then we have*

$$\frac{c}{|b_0|} = \rho_{I,r}\left(\frac{\bar{\mu}-r}{|\bar{\sigma}|^2} - 1\right)|\bar{\sigma}|, \tag{4.13}$$

where

$$\rho_{I,r} := \frac{b_0^\top \bar{\sigma}}{|b_0||\bar{\sigma}|} = \frac{\frac{d}{dt}\left\langle \int \frac{dI}{I}, r\right\rangle}{\sqrt{\left(\frac{d}{dt}\left\langle \int \frac{dI}{I}\right\rangle\right)\left(\frac{d}{dt}\langle r\rangle\right)}}$$

is the instantaneous correlation between the asset index and the short rate.

Proof. Multiplying (1.12) by $a^\top$ and using $\hat{\lambda} = q\bar{\sigma}$, we get $\bar{\mu} = r + \bar{\sigma}^\top\hat{\lambda} = r + q|\bar{\sigma}|^2$ and therefore $\hat{\lambda} = \frac{\bar{\mu}-r}{|\bar{\sigma}|^2}\bar{\sigma}$. Plugging this into (4.12) gives (4.13). ∎

One important insight from Theorem 12 is the key role played by the ratio

$$q(x) = \frac{|\hat{\lambda}(x)|}{|\bar{\sigma}(x)|} = \frac{|\bar{\mu}(x) - r(x)|}{|\bar{\sigma}(x)|^2}.$$

In view of the index dynamics in (4.9), q is of course an indicator for the average performance of the asset market. But the surprising aspect is that q also gives via (4.13) information about the relation between the dynamics of assets and interest rates. If for instance q is significantly larger than 1, hence if the assets give large returns on average, then a positive (negative) correlation between the index and the short rate goes with an upward (downward) drifting short rate. In Table 1, all possible combinations are listed.

size of q	sign of $\rho_{I,r}$	sign of c
> 1	$+$	$+$
> 1	$-$	$-$
$= 1$	?	0
< 1	$+$	$-$
< 1	$-$	$+$

Table 1. Effect of q and $\rho_{I,r}$ on c

To round off the discussion, we now show that a rigidly spherical market with constant short rate volatility and in addition $\hat{\lambda}$ proportional to $\bar{\sigma}$ actually exists. We first choose a homogeneous rigidly spherical asset volatility structure and set

$$r(x) := r_0 + \sum_{i=1}^{n} \theta^i \log x^i \tag{4.14}$$

with constants r_0 and θ^i satisfying $\sum_{i=1}^{n} \theta^i = 0$. Then $r(x)$ is also homogeneous, and (4.3) yields $b = \sum_{i=1}^{n} \theta^i \tilde{\sigma}^i$ which is indeed constant in x because $\tilde{\sigma}$ is. Next we choose $\hat{\lambda} := q\bar{\sigma}$ for some homogeneous scalar function $q(x)$ and finally obtain a homogeneous $\mu(x)$ from the drift condition (DC) via (1.12). Note that (4.14) is a perturbation of the constant short rate case $r(x) \equiv r_0$ obtained when all θ^i are equal. This example also illustrates that we usually have enough freedom in the choice of our parameters to produce a model with (for instance) a desired short rate process as output. We take up such issues more systematically below.

Remark. We could also formulate some of our results and assumptions in terms of the *numeraire portfolio* introduced by Long [Lo] and recently studied in detail by Becherer [Be]. In our situation with a continuous price process X, the numeraire portfolio is the tradable numeraire N^* whose value process is $N^* = \exp\left(\int r_u \, du\right) / \hat{Z}$ so that

$$\frac{dN_t^*}{N_t^*} = r_t \, dt + \hat{\lambda}_t^\top d\hat{W}_t.$$

The characterizing property of N^* is that every tradable quantity's price becomes after discounting by N^* a local martingale under the original measure P. If we consider in particular as "asset" the index I from Proposition 11, it is readily seen that I/N^* has volatility vector $\bar{\sigma} - \hat{\lambda}$, and so (4.12) shows that the short rate drift under P is

$$c = -b_0^\top (\bar{\sigma} - \hat{\lambda}) = -\frac{d}{dt}\left\langle \int \frac{d(I/N^*)}{I/N^*}, r \right\rangle$$

under the assumptions of Proposition 11. We can also note that the assumption $\hat{\lambda} = q\bar{\sigma}$ used in Theorem 12 is equivalent to saying that the index I and the numeraire portfolio N^* are perfectly locally correlated. It would be interesting to see an economic interpretation for this.

5 Some examples

Since our asset model is very general, one expects the resulting class of interest rate models to be very large. To illustrate that this is indeed the case, we focus in this section on models with two assets and argue that our approach then produces essentially all known one-factor short rate models by suitable choice of the asset volatility structure. We also give some explicit examples.

As in section 4, we start with an autonomous Markovian SDE (4.1) with $n = 2$ assets and $m = 1$ driving Brownian motion; recall that $n > m$. We assume sufficient smoothness and regularity and also that μ and σ are homogeneous of degree 0. Both are thus functions of the ratio $y = x^1/x^2$ so that our asset model is described by four scalar functions $\mu^1(y), \mu^2(y), \sigma^1(y), \sigma^2(y)$. Three of these can be chosen independently; the drift condition (DC) then determines the fourth one and also fixes the short rate function

$$r = \frac{\mu^1\sigma^2 - \mu^2\sigma^1}{\sigma^2 - \sigma^1}.$$

With $Y = X^1/X^2$, Itô's formula gives the dynamics of the short rate $r_t = r(Y_t)$ under the risk-neutral measure $\hat{P}$ as

$$dr_t = Y_t r'(\sigma^1 - \sigma^2)\, d\hat{W}_t + \left(Y_t r' \sigma^2(\sigma^2 - \sigma^1) + \frac{1}{2} Y_t^2 r'' \left|\sigma^1 - \sigma^2\right|^2 \right) dt. \qquad (5.1)$$

If $r(y)$ is invertible with inverse g, we have $Y_t = g(r_t)$ and (5.1) becomes

$$dr_t = \frac{g(r_t)}{g'(r_t)}(\sigma^1 - \sigma^2)\, d\hat{W}_t + \left(\frac{g(r_t)}{g'(r_t)}\sigma^2(\sigma^2 - \sigma^1) - \frac{1}{2}\frac{g''(r_t)}{g'(r_t)} \left| \frac{g(r_t)}{g'(r_t)}(\sigma^1 - \sigma^2) \right|^2 \right) dt; \qquad (5.2)$$

note that all arguments $g(r_t)$ for the functions σ^1, σ^2 have been suppressed.

(5.2) gives the explicit $\hat{P}$-dynamics of the endogenous short rate induced by a two-asset model. Conversely, we can start with any one-factor short rate model and ask for a two-asset model inducing this short rate. To see that this inverse problem can usually be solved, we observe that (5.2) involves three functions: g, σ^2 and the volatility difference $\delta := \sigma^1 - \sigma^2$. But a specification

$$dr_t = \hat{c}(r_t)\,dt + b(r_t)\,d\hat{W}_t \tag{5.3}$$

of (r_t) under the risk-neutral measure only requires two functions $\hat{c}$ and b, and so have three variables g, σ^2, δ that must satisfy the two conditions of matching the coefficients of $d\hat{W}_t$ and dt in (5.2) and (5.3). This problem is typically solvable, and as one expects, the asset model is not uniquely determined by r. We can for example choose σ^2 or δ arbitrarily, and we next examine these two cases in more detail.

Remark. The freedom of choosing σ^2 or δ is still there even if we prescribe the dynamics of (r_t) under both the risk-neutral measure $\hat{P}$ *and* the objective measure P. In fact, this fixes in addition to $\hat{c}$ and b the objective drift c or (equivalently) the $\hat{P}$-market price of risk $\hat{\lambda}$, but we can always match that via the one remaining free drift parameter μ^1 or μ^2.

Consider first the case of a fixed volatility difference $\delta(y) = \sigma^1(y) - \sigma^2(y)$. Comparing (5.2) and (5.3) yields two differential equations

$$\begin{aligned} b(z) &= \frac{g(z)}{g'(z)}\delta\big(g(z)\big), \\ \hat{c}(z) &= -\sigma^2\big(g(z)\big)\,b(z) - \frac{1}{2}\frac{g''(z)}{g'(z)}|b(z)|^2. \end{aligned} \tag{5.4}$$

Since $g(r_0) = Y_0 = X_0^1/X_0^2$, the first equation gives $g(z)$ via

$$\int_{X_0^1/X_0^2}^{g(z)} \frac{dy}{y\delta(y)} = \int_{r_0}^{z} \frac{1}{b(s)}\,ds, \tag{5.5}$$

and we can then determine $\sigma^2\big(g(z)\big)$ from the second equation in (5.4) as

$$\sigma^2\big(g(z)\big) = -\frac{\hat{c}(z)}{b(z)} - \frac{1}{2}\frac{g''(z)}{g'(z)}b(z). \tag{5.6}$$

Hence we know $Y_t = g(r_t)$ and also $\sigma^2(Y_t)$ and $\sigma^1(Y_t) = \delta(Y_t) + \sigma^2(Y_t)$, and so we have the $\hat{P}$-dynamics of the two assets X^1, X^2 that induce our short rate model (5.3).

Example 13 If $\delta(y) = \delta_0 + \delta_1(y - Y_0)$ is linear, (5.5) yields

$$g(z) = Y_0 \frac{\delta_0 - \delta_1 Y_0}{\delta_0 \exp\left(-(\delta_0 - \delta_1 Y_0) \int\limits_{r_0}^{z} \frac{1}{b(s)}\, ds\right) - \delta_1 Y_0} \tag{5.7}$$

after some computation, and substituting this in (5.6) gives σ^2. Several popular short rate models have a volatility b of the form $b(z) = \sigma_0\, z^\gamma$, and then (5.7) is easily evaluated. In fact, the Ho/Lee, Vasiček and Hull/White models all have $\gamma = 0$, the Cox/Ingersoll/Ross model has $\gamma = 0.5$ and the Black/Karasinski model has $\gamma = 1$.

If we fix instead of δ the volatility $\sigma^2(y)$, we can first solve (5.6) for g with initial condition $g(r_0) = X_0^1/X_0^2$. This gives g up to an integration constant, and so $\delta = bg'/g$ is by (5.4) also determined up to a constant.

Example 14 If the volatility $\sigma^2 \equiv \sigma_0^{(2)}$ is constant, we consider the model

$$\begin{aligned} dX_t^1 &= X_t^1 r(Y_t)\, dt + X_t^1 \sigma^1(Y_t)\, d\hat{W}_t, \\ dX_t^2 &= X_t^2 r(Y_t)\, dt + X_t^2 \sigma_0^{(2)}\, d\hat{W}_t. \end{aligned}$$

Then we can solve (5.6) to obtain

$$g(z) = \frac{X_0^1}{X_0^2} + B \int\limits_{r_0}^{z} dy \exp\left(-2 \int\limits_{1}^{y} \left(\frac{\sigma_0^{(2)}}{b(s)} + \frac{\hat{c}(s)}{|b(s)|^2}\right) ds\right)$$

with a constant $B \neq 0$. This determines $r(y)$ implicitly as the inverse function of $g(z)$, and σ^1 is after some computation seen to be

$$\sigma^1(y) = \sigma_0^{(2)} + b(y) \frac{B \exp\left(-2 \int\limits_{r_0}^{y} \left(\frac{\sigma_0^{(2)}}{b(s)} + \frac{\hat{c}(s)}{|b(s)|^2}\right) ds\right)}{\frac{X_0^1}{X_0^2} + B \int\limits_{r_0}^{y} du \exp\left(-2 \int\limits_{1}^{u} \left(\frac{\sigma_0^{(2)}}{b(s)} + \frac{\hat{c}(s)}{|b(s)|^2}\right) ds\right)}.$$

This ends the example.

To conclude, we now show how to find two-asset models with a fixed and constant volatility difference $\delta(y) \equiv \delta_0$ which endogenously induce the Vasiček or the Cox/Ingersoll/Ross short rate models.

Example 15 In the *Vasiček model*, the short rate is given by the SDE

$$dr_t = (\alpha - \beta r_t)\,dt + \gamma\,d\hat{W}_t.$$

Hence $b(z) \equiv \gamma$ and (5.7) yields

$$g(z) = Y_0 \exp\left(\frac{\delta_0}{\gamma}(z - r_0)\right).$$

Now we use (5.6) with $\hat{c}(z) = \alpha - \beta z$ to obtain

$$\sigma^2\big(g(z)\big) = \frac{\beta z - \alpha}{\gamma} - \frac{1}{2}\delta_0. \tag{5.8}$$

Inverting $g(z)$ yields as inverse function

$$r(y) = r_0 + \frac{\gamma}{\delta_0}\log\frac{y}{Y_0},$$

and using (5.8) gives the asset volatilities as

$$\begin{aligned} \sigma^2(y) &= \frac{\beta}{\gamma}r_0 + \frac{\beta}{\delta_0}\log\frac{y}{Y_0} - \frac{\alpha}{\gamma} - \frac{\delta_0}{2}, \\ \sigma^1(y) &= \frac{\beta}{\gamma}r_0 + \frac{\beta}{\delta_0}\log\frac{y}{Y_0} - \frac{\alpha}{\gamma} + \frac{\delta_0}{2}. \end{aligned}$$

This specifies the volatility structure of a homogeneous two-asset model inducing the Vasiček short rate model.

Example 16 The *Cox/Ingersoll/Ross model* has

$$dr_t = (\alpha - \beta r_t)\,dt + \gamma\sqrt{r_t}\,d\hat{W}_t$$

with $r_0 > 0$ and $\alpha \geq \frac{1}{2}\gamma^2$ to keep $r_t > 0$ for all t. (5.7) with $b(z) = \gamma\sqrt{z}$ gives

$$g(z) = Y_0 \exp\left(\frac{2\delta_0}{\gamma}(\sqrt{z} - \sqrt{r_0})\right),$$

the inverse function is

$$r(y) = \left|\frac{\gamma}{2\delta_0}\log\frac{y}{Y_0} + \sqrt{r_0}\right|^2,$$

and again using (5.6) and the expression for $g(z)$ yields

$$\begin{aligned} \sigma^2(y) &= -\frac{\delta_0}{2} + \frac{\beta}{\gamma}\left(\frac{\gamma}{2\delta_0}\log\frac{y}{Y_0} + \sqrt{r_0}\right) + \frac{\frac{\gamma}{4} - \frac{\alpha}{\gamma}}{\frac{\gamma}{2\delta_0}\log\frac{y}{Y_0} + \sqrt{r_0}}, \\ \sigma^1(y) &= \sigma^2(y) + \delta_0, \end{aligned}$$

ending this example.

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