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Phase-field systems for multi-dimensional Prandtl-Ishlinskii operators with non-polyhedral characteristics

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Abstract

Hysteresis operators have recently proved to be a powerful tool in modelling phase transition phenomena which are accompanied by the occurrence of hysteresis effects. In a series of papers, the present authors have proposed phase-field models in which hysteresis nonlinearities occur at several places. A very important class of hysteresis operators studied in this connection is formed by the so-called *Prandtl-Ishlinskii* operators. For these operators, the corresponding phase-field systems are in the multi-dimensional case only known to admit unique solutions if the characteristic convex sets defining the operators are polyhedrons. In this paper, we use approximation techniques to extend the known results to multi-dimensional Prandtl-Ishlinskii operators having non-polyhedral convex characteristic sets.

1 Introduction

The theory of hysteresis operators developed in the past twenty years (we refer to the monographs [1], [4], [5], [11], [13] devoted to this subject) has proved to be a powerful tool for solving mathematical problems in various branches of applications such as solid mechanics, material fatigue, ferromagnetism, and many others. Recently, the authors have in a series of papers (cf., [3], [6], [7], [8], [9], [10]) proposed an approach using hysteresis operators to classical *phase-field models* for phase transitions and their generalizations. In particular, in [9] they have studied situations in which *multi-dimensional* hysteresis operators occur. More precisely, they proved existence and uniqueness results for initial-boundary value problems for phase-field systems of the form

$$\mu(\theta)\, w_t \,+\, f_1[w] \,+\, \theta\, f_2[w] = 0\,, \tag{1.1}$$

$$\Big(c_V\,\theta \,+\, F_1[w]\Big)_t \,-\, \kappa\,\Delta\theta = \psi(x,t,\theta)\,, \tag{1.2}$$

for the unknown fields θ (absolute temperature) and $w = (w_1, \ldots, w_M)$ (phase-field variables, $M \in \mathbb{N}$ given), where $f_j = (f_{j,1}, \ldots, f_{j,M})$, $j = 1\,,\,2$. In this connection, $\mu(\theta) > 0$ denotes a relaxation coefficient, and $c_V > 0$ and $\kappa > 0$ represent specific heat and heat conductivity, respectively, both assumed constant throughout this paper. The phase-field variables usually have the meaning of so-called *generalized freezing indices*. For this and more physical background about the system equations (1.1), (1.2), we refer the reader to [9].

Systems of the form (1.1), (1.2) have been studied repeatedly in the literature for the case that μ, $f_{1,k}$, $f_{2,k}$, F_1, ψ are smooth or at least subdifferentiable functions of their respective variables (cf., for instance, the monograph [1]). In contrast to these works, this contribution deals with the case when $f_{1,k}$, $f_{2,k}$, F_1 are no longer *functions*, but *hysteresis operators* acting between suitable function spaces.

Let us recall some basic facts about the notion hysteresis operator (for details, we refer to the monographs mentioned above). Let $T > 0$ denote some (final) time. A mapping f from the set $\mathrm{Map}([0,T];\mathbb{R}^M) := \{w : [0,T] \to \mathbb{R}^M\}$ into itself is called a *hysteresis operator* if it is *causal*, that is, if for all $w_1, w_2 \in \mathrm{Map}([0,T];\mathbb{R}^M)$ and $t \in [0,T]$ we have the implication

$$w_1(\tau) = w_2(\tau) \quad \forall\, \tau \in [0,t] \;\Rightarrow\; f[w_1](t) = f[w_2](t)\,,$$

and if it is *rate-independent*, that is, if for every $w \in \mathrm{Map}([0,T];\mathbb{R}^M)$ and every continuous increasing mapping α of $[0,T]$ onto $[0,T]$ we have

$$f[w \circ \alpha](t) = f[w](\alpha(t)) \quad \forall\, t \in [0,T]\,.$$

In the case of partial differential equations, when the input functions not only depend on a time variable $t \in [0,T]$ but also on a space variable $x \in \Omega \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$, it is necessary to extend the above notion. In this situation, it is natural to associate with a hysteresis operator f defined on $\mathrm{Map}([0,T];\mathbb{R}^M)$ in the above sense the operator $\hat{f}$ acting on $\mathrm{Map}(\Omega \times [0,T];\mathbb{R}^M)$ by simply putting

$$\hat{f}[w](x,t) \;:=\; f[w(x,\cdot)](t)\,. \tag{1.3}$$

Note that for inputs $w \in L^p(\Omega;(C([0,T];\mathbb{R}^M)))$, where $1 \le p \le \infty$, the mapping $x \mapsto \hat{f}[w](x,\cdot)$ is known to be strongly measurable from Ω into $C([0,T];\mathbb{R}^M)$ provided that f is a continuous mapping from $C([0,T];\mathbb{R}^M)$ into $C([0,T];\mathbb{R}^M)$. The latter will be the case for all the operators occurring in this paper. It is customary to denote the extended operator by f, again. The hysteresis operators occurring in (1.1) and (1.2) have to be understood in this sense. In what follows we will not distinguish between the operators f and $\hat{f}$ and denote them both by f. From this simplification no confusion can arise.

Typically, nontrivial hysteresis operators are *not differentiable*, but at best only (possibly locally) *Lipschitz continuous* in suitable function spaces; in addition, they carry a *nonlocal memory* with respect to time in that the output value $f[w](t)$ at any time instant $t \in [0,T]$ depends on the whole input history $w_{|[0,t]}$, and not just on $w(t)$.

Both non-differentiability and presence of a memory are unpleasant features from the mathematical point of view. In particular, the classical method of deriving higher order a priori estimates for w (namely, differentiation of (1.1) with respect to t and testing with w_t) does not work, since there is no chain rule for the hysteretic nonlinearities. This fact results in a lack of compactness and thus in difficulties in existence proofs.

In this paper, we consider operators of the special form

$$f_j[w](t) \ := \ \int_0^\infty \varphi_j(r)\, s_{Z_{j,r}}[w](t)\, dr\,, \tag{1.4}$$

$$F_j[w](t) \ := \ \frac{1}{2}\int_0^\infty \varphi_j(r)\, \left|s_{Z_{j,r}}[w](t)\right|^2\, dr\,, \tag{1.5}$$

for $j = 1, 2$ and $t \in [0, T]$, where w denotes an input function belonging to one of the spaces $W^{1,1}(0, T; \mathbb{R}^M)$ or $C([0, T]; \mathbb{R}^M)$, and where $|\cdot|$ denotes the Euclidean length in $\mathbb{R}^k$, for all $k \in \mathbb{N}$.

Operators of the form (1.4) are called *multi-dimensional Prandtl-Ishlinskii operators*, and the operators (1.5) are their associated *hysteresis potentials* (for these notions, cf. [1], [5]). To interpret the expressions (1.4) and (1.5), suppose that φ_j denote nonnegative and measurable functions on $[0, \infty)$ such that

$$\int_0^\infty \varphi_j(r)\,(1 + r + r^2)\, dr \ < \ +\infty\,, \quad j = 1, 2\,.$$

In addition, let $Z_{j,1}$, $j = 1, 2$, denote some fixed nonempty, closed and bounded convex sets in $\mathbb{R}^M$ such that $0 \in \operatorname{int} Z_{j,1}$. We then set $Z_{j,r} := r\, Z_{j,1} = \{r\, z \in \mathbb{R}^M\,;\ z \in Z_{j,1}\}$ for $r \geq 0$ and $j = 1, 2$, and we denote by $s_{Zj,r}$ the so-called *multidimensional stop operator with characteristic* $Z_{j,r}$.

Let us explain the multi-dimensional stop operator s_Z associated with a general nonempty, closed and convex characteristic set $Z \subset \mathbb{R}^M$ containing the origin (the assumption $0 \in Z$ is immaterial for the definition of s_Z but will be needed later in the paper). To this end, denoting by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^M$, and by the subscript t differentiation with respect to time, we consider the variational inequality

$$\chi(t) \in Z \quad \forall\, t \in [0, T], \quad \langle \chi_t(t) - w_t(t)\,,\, \chi(t) - \varphi\rangle \leq 0 \quad \forall\, \varphi \in Z, \quad \text{for a.e. } t \in (0, T). \tag{1.6}$$

It is a well-known fact (for details, see [5]) that, given any initial datum $\chi^0 \in Z$ and any input function $w \in W^{1,1}(0, T; \mathbb{R}^M)$, there is a unique solution $\chi \in W^{1,1}(0, T; \mathbb{R}^M)$ to (1.6). The related solution operator,

$$s_Z[\chi^0, w] := \chi\,, \tag{1.7}$$

is just the stop operator s_Z.

The dependence on the initial datum χ^0 is very inconvenient from the viewpoint of notational effort. Therefore, we will from now on assume that the initial datum is the same for all stop operators occurring in this paper: we simply require that $\chi^0 = 0$ for all cases. This restriction has no bearing on the generality of the results proved in this paper, while it has the advantage that we can drop the dependence on χ^0, i. e. we can write $s_Z[w]$ instead of $s_Z[\chi^0, w]$. The expressions in (1.4) and (1.5) have to be understood in this sense.

The stop operator admits a simple geometric interpretation: it can easily be shown (see [5], for instance) that the time derivative $s_Z[w]_t(t)$ coincides for almost every $t \in (0,T)$ with the orthogonal projection of $w_t(t)$ onto the tangent cone at Z in the point $w(t)$. The hysteretic input-output behaviour of the stop operator is illustrated in Figure 1 for the simple one-dimensional case when $Z = [0,1]$:

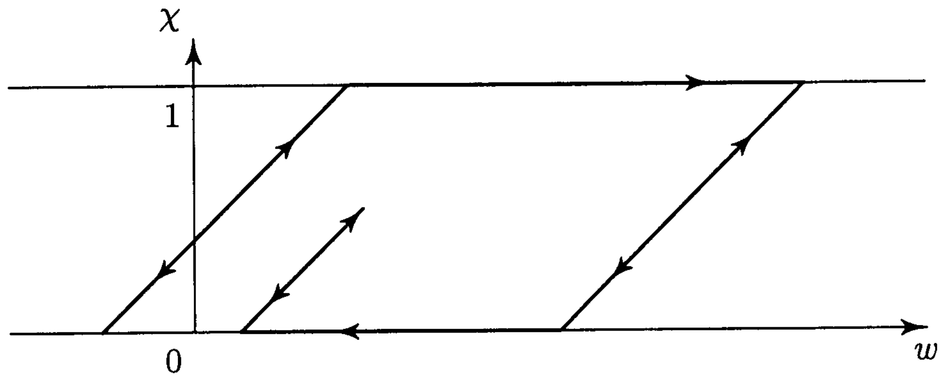

Figure 1 : *A diagram of the stop operator* $s_{[0,1]}$.

Along the upper (lower) threshold line $\chi = 1$ ($\chi = 0$), the process is irreversible and can only move to the right (to the left, respectively), while in between, motions in both directions are admissible. This behaviour is similar to *Prandtl's model of perfect elastoplasticity*, where the horizontal parts of the diagram correspond to plastic yielding and the intermediate lines can be interpreted as linearly elastic trajectories.

The stop operator has an intrinsic *energy dissipation property* which is obtained if we insert $\varphi = 0$ in (1.6). It then follows for a. e. $t \in (0,T)$ that

$$\frac{1}{2}\frac{d}{dt}\,|s_Z[w](t)|^2 \;=\; \langle s_Z[w]_t(t)\,,\, s_Z[w](t)\rangle \;\le\; \langle w_t(t)\,,\, s_Z[w](t)\rangle\;. \tag{1.8}$$

It has been shown in [6], [7], [8], [9] that such energy inequalities play an important role in the analysis of phase-field systems with hysteresis; in particular, they guarantee the thermodynamic consistency of the model. A further property of the stop operator, which follows from its geometric interpretation, is the following: it holds

$$|w_t - s_Z[w]_t| \le |w_t|\,,\quad |s_Z[w]_t| \le |w_t|\,,\quad \langle s_Z[w]_t\,,\, w_t\rangle = |s_Z[w]_t|^2 \ge 0 \quad \text{a.e. in } (0,T). \tag{1.9}$$

We are now in the position to explain the main aim of this paper: in [9] it has been shown that a suitable initial-boundary value problem for the system (1.1), (1.2), with the nonlinearities given by (1.4), (1.5), admits a unique strong solution provided that the characteristic sets $Z_{j,1}$, $j = 1,2$, are polyhedrons containing the origin. In this paper, we aim to extend these results to general *non-polyhedral* characteristics whose interior contain 0. This generalization is of importance in the applications to elastoplasticity: it allows to include the so-called *von Mises yield condition* which was formerly not covered by the theory.

The main difficulty in this generalization lies the continuity properties of the stop operator: if the characteristic Z of the stop operator is a polyhedron containing the origin, then s_Z can be extended to a globally Lipschitz continuous mapping from $C([0,T];\mathbb{R}^M)$ into itself. More precisely (cf. Theorem 4.5 in [5]), there exists some constant $C > 0$, depending on the smallest positive angle between arbitrary facets of the polyhedron that are not parallel, such that for all $w_1, w_2 \in C([0,T];\mathbb{R}^M)$ it holds

$$|s_Z[w_1](t) \; - \; s_Z[w_2](t)| \le C \max_{0\le\tau\le t} |w_1(\tau) - w_2(\tau)| \quad \forall\, t \in [0,T]\,. \tag{1.10}$$

As the Prandtl-Ishlinskii operators defined in (1.4) are superpositions of stop operators, they satisfy an analogous Lipschitz continuity on $C([0,T];\mathbb{R}^M)$ under suitable conditions on the weight functions φ_j. An analogous argument holds for the potential operators defined in (1.5). This special form of Lipschitz continuity was crucial for the proofs given in [9].

For non-polyhedral characteristics the situation is entirely different if $M > 1$: it is well-known (see [5], for instance) that in this case Lipschitz continuity of the form (1.10) cannot be expected to hold for s_Z, in general. Therefore, the chain of arguments employed in [9] does no longer apply. However, s_Z is known to be a continuous mapping from $W^{1,p}(0,T;\mathbb{R}^M)$ into itself for any $p \in [1,+\infty)$ (cf. Theorem 3.12 in [5]); moreover, s_Z can also be extended to a continuous mapping from $C([0,T];\mathbb{R}^M)$ into itself provided that $0 \in \operatorname{int} Z$ (see Lemma 2.4 below).

The idea of proof followed in this paper is to use approximation. First, one approximates the general convex characteristic sets $Z_{j,1}$ from the inside by a sequence of polyhedral sets $Z^n_{j,1}$ in the sense of the Hausdorff distance. Then, we apply the results of [9] to obtain solutions (w_n, θ_n) to the initial-boundary value problem for (1.1), (1.2), when $s_{Z_{j,r}}$ is replaced by $s_{Z^n_{j,r}}$, where $Z^n_{j,r} := r\, Z^n_{j,1}$. Finally, the approximating solutions are shown to converge to a solution of the original problem (which is the most difficult part of the proof, since we do no longer have the Lipschitz continuity (1.10) at our disposal).

The rest of the paper is organized as follows: In Section 2, we give a detailed statement of the mathematical problem and of the main mathematical result, and we collect some properties of hysteresis operators, especially of stop and Prandtl-Ishlinskii operators, which are needed later. Section 3 brings the proof of the main result and some additional remarks.

In what follows, the norms of the standard Lebesgue spaces $L^p(\Omega)$, for $1 \le p \le \infty$, will be denoted by $\|\cdot\|_p$. Finally, we shall use the usual denotations $W^{m,p}(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}$, $1 \le p \le \infty$, for the standard Sobolev spaces. To simplify the notation, we always assume without loss of generality that $C_V = \kappa = 1$.

2 Statement of the Problem

Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded, open domain with Lipschitz boundary $\partial\Omega$. We consider the system (1.1), (1.2), to be satisfied almost everywhere in $\Omega_T := \Omega \times (0,T)$, for some $T > 0$, together with the initial and boundary conditions

$$\theta(\cdot,0) = \theta^0, \; w(\cdot,0) = w^0, \quad \text{a.e. in } \Omega, \quad \frac{\partial\theta}{\partial\mathbf{n}} = 0 \quad \text{a.e. on } \partial\Omega \times (0,T), \tag{2.1}$$

where $\mathbf{n}$ denotes the outer unit normal to $\partial\Omega$.

We make the following hypotheses concerning the data of the system.

(H1) $\quad w^0 \in L^\infty(\Omega)$, $\theta^0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $\exists\, \delta > 0 : \theta^0(x) \geq \delta$ a.e. in Ω. (2.2)

(H2) The function $\mu : (0,\infty) \to (0,\infty)$ is Lipschitz continuous on compact subsets of $(0,\infty)$, and

$$\exists\, \mu_0 > 0 : \quad \mu(\theta) \geq \mu_0 \min\{\theta, 1\} \quad \forall\, \theta > 0. \tag{2.3}$$

(H3) The following conditions hold:

(i) The functions $\varphi_j \in L^1([0,+\infty))$, $j = 1,2$, are non-negative on $[0,+\infty)$ and satisfy

$$\max_{j=1,2} \int_0^\infty \varphi_j(r)\,(1 + r + r^2)\,dr =: \Phi_0 < +\infty. \tag{2.4}$$

(ii) We have $Z_{j,r} := r\,Z_{j,1}$, for $j = 1,2$ and $r \geq 0$, with some bounded, closed and convex sets $Z_{j,1} \subset \mathbb{R}^M$ satisfying $0 \in \operatorname{int} Z_{j,1}$, for $j = 1,2$. Let $K_0 > 0$ and $\rho > 0$ be such that

$$|z| \leq K_0 \quad \forall\, z \in Z_{1,1} \cup Z_{2,1}, \tag{2.5}$$

$$K_\rho(0) := \{x \in \mathbb{R}^M;\, |x| < \rho\} \subset (Z_{1,1} \cap Z_{2,1}). \tag{2.6}$$

(H4) We assume that $\psi : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\exists\, \psi_0 \in L^\infty(\Omega_T) : \quad \theta \leq 0 \Rightarrow \psi(x,t,\theta) = \psi_0(x,t), \tag{2.7}$$

$$\exists\, \Psi_1 > 0 : \quad \left|\frac{\partial\psi}{\partial\theta}\right| \leq \Psi_1 \quad \text{a.e. in } \Omega \times (0,T) \times \mathbb{R}, \tag{2.8}$$

$$\psi_0(x,t) \geq 0 \qquad \text{a.e. in } \Omega_T. \tag{2.9}$$

Remark 2.1 Condition (2.3) is satisfied if $\mu(\theta) = \mu_0\,\theta^\alpha$, $\mu_0 > 0$ fixed, $\alpha \in [0,1]$. For $\alpha = 1$ the system (1.1), (1.2) forms a hysteretic analogue of the *Penrose-Fife* *model* for phase transitions with zero interfacial energy (cf. [12]), while for $\alpha = 0$ the hysteretic analogue of the *Caginalp model* with zero interfacial energy (see [2]) is obtained.

We now formulate the main result of this paper.

Theorem 2.2 *Let the hypotheses* **H1** *to* **H4** *hold. Then there exists a unique solution* (w, θ) *, satisfying the equations* (1.1), (1.2) and $\theta > 0$ *almost everywhere in* Ω_T *, such that*

$$w \in (W^{1,\infty}(0,T;L^\infty(\Omega)))^M \,, \quad \theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(\Omega_T) \,. \tag{2.10}$$

In addition, with $\beta := \Psi_1 + 2\,\Phi_0^2\,K_0^2/\mu_0$ *it holds*

$$\theta(x,t) \geq \delta\, e^{-\beta t} \quad \textit{for a.e.} \quad (x,t) \in \Omega_T \,. \tag{2.11}$$

The proof of Theorem 2.2 will be given in the next section. We now draw a number of conclusions from hypothesis **(H3)** which, on the one hand, will show that all the expressions occurring in (1.1), (1.2) are meaningful and, on the other hand, will be needed in the course of the proof of Theorem 2.2. At first, let us recall the notion of *Hausdorff distance*: If A, B are nonempty, closed subsets of $\mathbb{R}^M$ then their Hausdorff distance $d_H(A,B)$ is defined by

$$d_H(A,B) := \max\{\sup\{\text{dist}(y,A)\,;\, y \in B\}\,,\, \sup\{\text{dist}(x,B)\,;\, x \in A\}\}\,.$$

We have the following result.

Lemma 2.3 *Let* $Z_1\,,\,Z_2 \subset \mathbb{R}^M$ *be nonempty, closed and convex sets such that* $0 \in Z_1 \subset Z_2$ *, and let* $w_1\,,\,w_2 \in W^{1,1}(0,T;\mathbb{R}^M)$ *. Then it holds for every* $t \in [0,T]$

$$\begin{aligned} |\, s_{Z_1}[w_1](t) - s_{Z_2}[w_1](t)\,| &\leq \left(2\,d_H(Z_1\,,\,Z_2)\int_0^t |w_{1,t}(\tau)|\,d\tau\right)^{1/2}, \\ |\, s_{Z_1}[w_1](t) - s_{Z_1}[w_2](t)\,| &\leq \int_0^t |w_{1,t}(\tau) - w_{2,t}(\tau)|\,d\tau\,. \end{aligned} \tag{2.12}$$

Proof: Let $y_j := s_{Z_j}[w_1]$, $j = 1,2$, and let Q be the orthogonal projection onto Z_1. Then, owing to (1.6), we have $\langle y_{1,t} - w_{1,t}\,,\, y_1 - Qy_2\rangle \leq 0$ and $\langle y_{2,t} - w_{1,t}\,,\, y_2 - y_1\rangle \leq 0$, a.e. on $(0,T)$, whence $\langle y_{1,t} - y_{2,t}\,,\, y_1 - y_2\rangle \leq \langle w_{1,t} - y_{1,t}\,,\, y_2 - Qy_2\rangle$. The first inequality in (2.12) now follows from (1.9) and from integration over time. The second inequality is a well-known property of the stop operator. □

The following result, which is a special case of Theorem 3.7 in [5], shows in particular that for $0 \in \operatorname{int} Z$ the stop operator s_Z is $\frac{1}{2}$-Hölder continuous on compact subsets of $C([0,T];\mathbb{R}^M)$.

Lemma 2.4 *Let $\{Z_\varepsilon\}_{0\le\varepsilon\le\bar\varepsilon} \subset \mathbb{R}^M$, $\bar\varepsilon > 0$, be a family of closed, convex and bounded sets satisfying*

$$K_\delta(0) := \{x \in \mathbb{R}^M \ ; \ |x| < \delta\} \subset Z_0 \quad \textit{for some} \quad \delta > 0\,, \tag{2.13}$$

$$0 \in \bigcap_{0\le\varepsilon\le\bar\varepsilon} \operatorname{int} Z_\varepsilon\,, \quad \textit{and} \quad d_H(Z_\varepsilon\,,\, Z_0) \le \frac{\sigma}{2} \quad \textit{for some} \quad 0 < \sigma < \frac{\delta}{2}\,. \tag{2.14}$$

Besides, let $\mathcal{K}$ be a compact subset of $C([0,T];\mathbb{R}^M)$. Then there is a constant $M_0 > 0$ such that for every $\varepsilon_1\,,\, \varepsilon_2 \in [0,\bar\varepsilon]$ and every $w_1\,, w_2 \in \mathcal{K}$ it holds

$$\left\| s_{Z_{\varepsilon_1}}[w_1] - s_{Z_{\varepsilon_2}}[w_2] \right\|_\infty \le 2\,\|w_1 - w_2\|_\infty + M_0\,(\sigma + \|w_1 - w_2\|_\infty)^{1/2}\,. \tag{2.15}$$

Lemma 2.5 *Suppose that* **(H3)** *holds. Then for $j = 1,2$ the following assertions hold true:*

(i) *For any $w \in C([0,T];\mathbb{R}^M)$ and any $r \ge 0$ it holds* $\left\| s_{Z_{j,r}}[w] \right\|_\infty \le r\,K_0\,.$ (2.14)

(ii) $r_1 \ge 0\,, r_2 \ge 0 \quad \Longrightarrow \quad d_H(Z_{j,r_1}\,,\, Z_{j,r_2}) \le |r_1 - r_2|\,K_0\,.$ (2.16)

(iii) *For every $w \in C([0,T];\mathbb{R}^M)$ the mapping $r \mapsto s_{Z_{j,r}}[w]$ is continuous from $[0,+\infty)$ into $C([0,T];\mathbb{R}^M)$.*

(iv) *To any $w \in C([0,T];\mathbb{R}^M)$ put $\hat R(w) := \|w(\cdot) - w(0)\|_\infty/\rho$, with ρ from hypothesis* **(H3)**. *Then for every $r > \hat R(w)$ it holds*

$$s_{Z_{j,r}}[w](t) = w(t) - w(0) \quad \textit{for every}\ t \in [0,T]\,. \tag{2.17}$$

Proof: Let $j \in \{1,2\}$. The assertions **(i)**, **(ii)** follow directly from (2.5) and the definition of the sets $Z_{j,r}$. Then **(iii)** is a direct consequence of Lemma 2.4. To prove **(iv)**, it suffices to assume that $w \in W^{1,1}(0,T;\mathbb{R}^M)$ is given. For $t \in [0,T]$ put $\chi(t) := s_{Z_{j,r}}[w](t)$, $\xi(t) := w(t) - w(0) - \chi(t)$, $\varphi(t) := w(t) - w(0)$. For $r > \hat R(w)$ we obtain from (2.6) that $\varphi(t) \in Z_{j,r}$, and (1.6) with $\varphi(t)$ instead of φ yields that $(|\xi(t)|^2)_t \le 0$ a.e., hence $\xi(t) \equiv 0$. Lemma 2.5 is proved. □

We can now prove a number of important properties of the expressions defined in (1.4) and (1.5).

Lemma 2.6 *Suppose that* **(H3)** *holds. Then the following assertions hold true:*

(i) *The operators* $f_1\,,\,f_2\,,F_1\,,\,F_2$ *given by* (1.4) *and* (1.5), *respectively, are well-defined and continuous mappings from* $C([0,T];\mathbb{R}^M)$ *into itself. Besides, for any* $w\in C([0,T];\mathbb{R}^M)$ *it holds*

$$|f_j[w](t)|\,\le\,\Phi_0\,K_0\,,\quad |F_j[w](t)|\,\le\,\frac{1}{2}\,\Phi_0\,K_0^2\,,\quad j=1,2\,,\quad \forall\,t\in[0,T]\,. \tag{2.18}$$

(ii) *The operators* $F_1\,,\,F_2$ *map* $W^{1,1}(0,T;\mathbb{R}^M)$ *continuously into itself, and for any* $w\in W^{1,1}(0,T;\mathbb{R}^M)$ *it holds*

$$|F_j[w]_t(t)|\,\le\,\Phi_0\,K_0\,|w_t(t)|\,,\quad j=1,2\,,\quad \text{for a.e. }\ t\in(0,T)\,. \tag{2.19}$$

(iii) *There exists some* $\Psi_2>0$ *such that for any* $w_1,w_2\in W^{1,1}(0,T;\mathbb{R}^M)$ *it holds*

$$\max_{j=1,2}\,|F_j[w_1](t)\,-\,F_j[w_2](t)|\,\le\,\Psi_2\int_0^t|w_{1,t}(\tau)\,-\,w_{2,t}(\tau)|\,d\tau\quad\forall\,t\in[0,T]\,. \tag{2.20}$$

An analogous estimate holds for the operators $f_1\,,\,f_2$.

Proof: **(i)**: Let $j\in\{1,2\}$ be fixed, and let $w\in C([0,T];\mathbb{R}^M)$ be given. Then, by virtue of **(H3)**,**(i)** and of Lemma 2.5,(iii), the mappings $r\mapsto\varphi_j(r)\,s_{Z_{j,r}}[w](t)$ and $r\mapsto\varphi_j(r)\,\left|s_{Z_{j,r}}[w]\right|^2$ are measurable on $[0,+\infty)$ for any $t\in[0,T]$. Thus we can conclude from Lemma 2.5,(iv), that

$$f_j[w](t)\;=\;\int_0^{\hat{R}(w)}\varphi_j(r)\,s_{Z_{j,r}}[w](t)\,dr\,+\,(w(t)-w(0))\int_{\hat{R}(w)}^{\infty}\varphi_j(r)\,dr\,. \tag{2.21}$$

Note that both integrals are finite since $\varphi_j\in L^1([0,+\infty))$ and since $\left|s_{Z_{j,r}}[w](t)\right|\le\hat{R}(w)\,K_0$ for $(r,t)\in[0,\hat{R}(w)]\times[0,T]$. Hence, $f_j[w](t)$ is well-defined. Moreover, the uniform continuity of the mapping $(r,t)\mapsto s_{Z_{j,r}}[w](t)$ on the compact set $[0,\hat{R}(w)]\times[0,T]$ yields that $f_j[w]$ is continuous on $[0,T]$. An analogous argument shows that $F_j[w]$ is well-defined and belongs to $C([0,T];\mathbb{R}^M)$. The estimates (2.18) follow directly from (2.4) and (2.5).

It remains to show the continuity of $f_j\,,F_j$ on $C([0,T];\mathbb{R}^M)$. We only argue for f_j; the proof for F_j is similar. Now let $\{w_n\}\subset C([0,T];\mathbb{R}^M)$ be any sequence such that $\lim_{n\to\infty}\|w_n-w\|_\infty=0$. Then, by the continuity of the operators $s_{Z_{j,r}}$ on $C([0,T];\mathbb{R}^M)$, it follows that $\varphi_j(r)\,\|\,s_{Z_{j,r}}[w_n]-s_{Z_{j,r}}[w]\,\|_\infty\to 0$ for almost every $r\in[0,+\infty)$. In view of (2.18), we may apply Lebesgue's theorem to conclude that

$$\|f_j[w_n]-f_j[w]\|_\infty\,\le\,\int_0^\infty\varphi_j(r)\,\|s_{Z_{j,r}}[w_n]-s_{Z_{j,r}}[w]\|_\infty\,dr\,\to 0\quad\text{as }\ n\to\infty\,, \tag{2.22}$$

which shows the continuity.

(ii): Let $j \in \{1,2\}$, and let $w \in W^{1,1}(0,T;\mathbb{R}^M)$. By Theorem 3.12 in [5], $s_{Z_{j,r}}$ is a continuous operator from $W^{1,1}(0,T;\mathbb{R}^M)$ into itself. Therefore, using (1.9) and (2.4), we obtain from Lebesgue's theorem that

$$F_j[w]_t(t) = \int_0^\infty \varphi_j(r) \left\langle s_{Z_{j,r}}[w](t)\,,\, s_{Z_{j,r}}[w]_t(t)\right\rangle dr \quad \text{for a. e. } t \in (0,T)\,, \tag{2.23}$$

and that (2.19) holds. Now suppose that $\{w_n\} \subset W^{1,1}(0,T;\mathbb{R}^M)$ is a sequence such that $\lim_{n\to\infty} \|w_n - w\|_{W^{1,1}(0,T;\mathbb{R}^M)} = 0$. We have to show that $\|F_j[w_n] - F_j[w]\|_{W^{1,1}(0,T;\mathbb{R}^M)} \to 0$. We have:

$$\begin{aligned}
&\int_0^T |F_j[w_n]_t(t) - F_j[w]_t(t)|\, dt \\
\le\; & \int_0^T\!\!\int_0^\infty \varphi_j(r) \left|\left\langle s_{Z_{j,r}}[w_n](t) - s_{Z_{j,r}}[w](t)\,,\, s_{Z_{j,r}}[w_n]_t(t)\right\rangle\right| dr\, dt \\
&+ \int_0^T\!\!\int_0^\infty \varphi_j(r) \left|\left\langle s_{Z_{j,r}}[w](t)\,,\, s_{Z_{j,r}}[w_n]_t(t) - s_{Z_{j,r}}[w]_t(t)\right\rangle\right| dr\, dt \\
\le\; & \|w_{n,t}\|_{W^{1,1}(0,T;\mathbb{R}^M)} \int_0^\infty \varphi_j(r) \left\| s_{Z_{j,r}}[w_n] - s_{Z_{j,r}}[w]\right\|_\infty dr \\
&+ K_0 \int_0^\infty r\,\varphi_j(r) \left\| s_{Z_{j,r}}[w_n] - s_{Z_{j,r}}[w]\right\|_{W^{1,1}(0,T;\mathbb{R}^M)} dr\,.
\end{aligned} \tag{2.24}$$

Now observe that both integrands converge to 0 as $n \to \infty$, for almost every $r \ge 0$. Hence, we may again invoke Lebesgue's theorem to conclude that the right-hand side of (2.24) converges to 0 as $n \to \infty$.

(iii): Let $j \in \{1,2\}$ be fixed, and let $w_1, w_2 \in W^{1,1}(0,T;\mathbb{R}^M)$ be given. Then we conclude from Lemma 2.3 that for all $t \in [0,T]$ it holds

$$\begin{aligned}
|F_j[w_1](t) - F_j[w_2](t)| \;&\le\; K_0 \int_0^\infty r\,\varphi_j(r) \left| s_{Z_{j,r}}[w_1](t) - s_{Z_{j,r}}[w_2](t)\right| dr \\
&\le\; K_0\, \Phi_0 \int_0^t |w_{1,t}(\tau) - w_{2,t}(\tau)|\, d\tau\,,
\end{aligned} \tag{2.25}$$

which proves (2.20). For f_1, f_2 we can argue similarly. □

3 Proof of Theorem 2.2.

We divide the proof into a number of steps.

Step 1: Approximation.

Let $\{x_j^n\}_{n\in\mathbb{N}}$ be a countable dense subset of $Z_{j,1}$ such that $x_j^1 = 0$, $j = 1,2$. We define the nonempty, bounded, closed and convex polyhedrons

$$Z_{j,1}^n := \overline{\operatorname{conv}\{x_j^1, \ldots, x_j^n\}}\,, \quad Z_{j,r}^n := r\, Z_{j,1}^n \quad \text{for } r \ge 0\,, \quad j = 1,2\,, \quad n \in \mathbb{N}\,. \tag{3.1}$$

Then $0 \in Z_{j,r}^n \subset Z_{j,r}^{n+1} \subset Z_{j,r}$, $j = 1, 2$, for all $r \geq 0$ and $n \in \mathbb{N}$. Since $\{x_j^n\}_{n \in \mathbb{N}}$ is dense in $Z_{j,1}$, it holds $K_{\rho/2}(0) \subset Z_{1,1}^n \cap Z_{2,1}^n$ for sufficiently large $n \in \mathbb{N}$, say, for $n \geq N_0$. In what follows, we always tacitly assume that $n \geq N_0$.

We also have that

$$\delta_n := d_H(Z_{1,1}^n, Z_{1,1}) + d_H(Z_{2,1}^n, Z_{2,1}) \to 0 \quad \text{as } n \to \infty . \tag{3.2}$$

Note that $\delta_m \leq \delta_n$ for $m \geq n$, and for $j = 1, 2$ we have

$$d_H(Z_{j,r}^n, Z_{j,r}) = r\, d_H(Z_{j,1}^n, Z_{j,1}) \leq r\, \delta_n \quad \text{for all } n \in \mathbb{N} \quad \text{and } r \geq 0 . \tag{3.3}$$

We now define the hysteresis operators (for $j = 1, 2$ and $n \in \mathbb{N}$)

$$f_j^n[w](t) := \int_0^\infty \varphi_j(r)\, s_{Z_{j,r}^n}[w](t)\, dr , \tag{3.4}$$

$$F_j^n[w](t) := \frac{1}{2} \int_0^\infty \varphi_j(r) \left| s_{Z_{j,r}^n}[w](t) \right|^2 dr , \quad t \in [0, T] , \tag{3.5}$$

and consider the initial-boundary value problem

$$\mu(\theta_n)\, w_{n,t} + f_1^n[w_n] + \theta_n\, f_2^n[w_n] = 0 , \quad \text{a. e. in } \Omega_T , \tag{3.6}$$

$$\theta_{n,t} - \Delta\theta_n = -\, F_1^n[w_n]_t + \psi(x, t, \theta_n) , \quad \text{a. e. in } \Omega_T , \tag{3.7}$$

$$\theta_n(\cdot, 0) = \theta^0 , \quad w_n(\cdot, 0) = w^0 , \quad \text{a. e. in } \Omega , \quad \frac{\partial \theta_n}{\partial \mathbf{n}} = 0 \quad \text{a. e. on } \partial\Omega \times (0, T) . \tag{3.8}$$

Now observe that the inclusion $Z_{j,r}^n \subset Z_{j,r}$ implies that for any $w \in W^{1,1}(0, T; \mathbb{R}^M)$ it holds

$$\left| s_{Z_{j,r}^n}[w](t) \right| \leq r\, K_0 , \quad \forall\, t \in [0, T] , \quad \forall\, r \geq 0 , \quad \forall\, n \in \mathbb{N} . \tag{3.9}$$

Therefore, the estimates (2.18), (2.19) remain valid if we replace the operators f_j , F_j by f_j^n , F_j^n , for all $n \in \mathbb{N}$. But then for every n the system (3.6)–(3.8) satisfies all the assumptions of Theorem 3.3 in [9]: indeed, the choice of the (polyhedral) sets $Z_{j,1}^n$ guarantees the global Lipschitz continuity of the approximating operators f_j^n on the space $C([0, T]; \mathbb{R}^M)$, and the other conditions of Theorem 3.3 in [9] are met if there we put

$$K_2 = K_3 := \Phi_0\, K_0 , \quad K_4 := \Psi_1 , \quad K_5 := 1 , \quad g[w] := w , \quad \Phi_R := \Psi_2 . \tag{3.10}$$

Note that all these constants do not depend on $n \in \mathbb{N}$.

It follows that system (3.6)–(3.8) admits for any $n \in \mathbb{N}$ a unique strong solution (w_n, θ_n) , satisfying the equations (1.1), (1.2) almost everywhere in Ω_T , such that

$$w_n \in (W^{1,\infty}(0, T; L^\infty(\Omega)))^M , \quad \theta_n \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(\Omega_T) , \tag{3.11}$$

and there exist constants $\beta_n > 0$ such that

$$\theta_n(x, t) \geq \delta\, e^{-\beta_n t} \quad \text{for a. e. } (x, t) \in \Omega_T . \tag{3.12}$$

We aim to show that $\{(w_n, \theta_n)\}$ converges to a solution of our original problem.

Step 2: Global estimates for $\{w_n\}$ and $\{\theta_n\}$.

We derive some a priori estimates. In what follows, we denote by $C_i > 0$, $i \in \mathbb{N}$, constants that only depend on the given data of the system, but not on $n \in \mathbb{N}$.

To begin with, we note that a closer inspection of the proof of Theorem 3.3 in [9] reveals that, in the notation used there, it holds

$$\beta_n \le K_4 + \frac{2\, K_2^2\, K_5}{\mu_0} \quad \forall\, n \in \mathbb{N}, \tag{3.13}$$

with the global constants K_i specified above and the constant μ_0 from (2.3), that is, we have $\beta_n \le \beta$. Hence, also using (2.3), we find that for every $n \in \mathbb{N}$ it holds

$$\theta_n(x,t) \ge \delta\, e^{-\beta t}, \quad \mu(\theta_n(x,t)) \ge \hat{\mu} := \mu_0 \min\{\delta\, e^{-\beta T}, 1\} \quad \text{for a.e. } (x,t) \in \Omega_T. \tag{3.14}$$

Next, we show that $\{\theta_n\}$ is bounded in $L^\infty(\Omega_T)$. To this end, we first note that $w_n \in (L^2(\Omega; C[0,T]))^M$, and we can conclude from (2.18) that for a.e. $x \in \Omega$ it holds

$$\max_{j=1,2} \left| f_j^n[w_n](x,t) \right| \le \Phi_0\, K_0 \quad \forall\, t \in [0,T], \tag{3.15}$$

so that

$$\max_{j=1,2} \left| f_j^n[w_n] \right| \le \Phi_0\, K_0 \quad \text{a.e. in } \Omega_T. \tag{3.16}$$

Hence, using (3.6) and (3.14),

$$|w_{n,t}| \le \frac{\Phi_0\, K_0}{\hat{\mu}} (1 + \theta_n) \quad \text{a.e. in } \Omega_T. \tag{3.17}$$

From Lemma 2.6, we can infer that

$$|F_1^n[w_n]_t| \le C_1 (1 + \theta_n) \quad \text{a.e. in } \Omega_T. \tag{3.18}$$

Now, we multiply (3.7) by θ_n^p for $p \ge 1$ and integrate over $\Omega \times [0,t]$. We obtain

$$\begin{aligned} &\frac{1}{p+1} \|\theta_n(t)\|_{p+1}^{p+1} + p \int_0^t \int_\Omega \theta_n^{p-1} |\nabla \theta_n|^2 \, dx\, d\tau \\ \le\ & \frac{1}{p+1} \|\theta_0\|_{p+1}^{p+1} + \int_0^t \int_\Omega \Big(|\psi(x,t,\theta_n)| + |F_1^n[w_n]_t| \Big) \theta_n^p \, dx\, d\tau \\ =:\ & A_1 + A_2. \end{aligned} \tag{3.19}$$

From **(H1)**, we have

$$A_1 \le \frac{1}{p+1} \operatorname{meas}(\Omega)\, \|\theta_0\|_\infty^{p+1} \le \frac{C_2}{p+1}\, C_3^{p+1}. \tag{3.20}$$

Next, we invoke **(H4)** and (3.18) to see that

$$|\psi(x,t,\theta_n)| \,+\, |F_1^n[w_n]_t| \,\le\, C_4\,(1\,+\,\theta_n) \quad \text{a.e. in } \Omega_T\,. \tag{3.21}$$

Hence, using Young's inequality $\theta_n^p \,\le\, \frac{p}{p+1}\,\theta_n^{p+1} \,+\, \frac{1}{p+1}$, we can conclude that

$$A_2 \,\le\, \frac{C_5}{p+1} \,+\, \frac{C_6\,p}{p+1}\int_0^t \|\theta_n(\tau)\|_{p+1}^{p+1}\,d\tau \,+\, C_7\int_0^t \|\theta_n(\tau)\|_{p+1}^{p+1}\,d\tau\,. \tag{3.22}$$

We have thus shown that

$$\|\theta_n(t)\|_{p+1}^{p+1} \,\le\, C_2\,C_3^{p+1} \,+\, C_5 \,+\, ((C_6+C_7)\,p \,+\, C_7)\int_0^t \|\theta_n(\tau)\|_{p+1}^{p+1}\,d\tau\,, \tag{3.23}$$

whence, by virtue of Gronwall's lemma,

$$\|\theta_n(t)\|_{p+1}^{p+1} \,\le\, (C_2\,C_3^{p+1} \,+\, C_5)\exp(((C_6+C_7)\,p \,+\, C_7)\,t)\,, \quad \forall\, t\in[0,T]\,. \tag{3.24}$$

Taking the $(p+1)$-th root on both sides of (3.24), we find that

$$\max_{0\le t\le T}\|\theta_n(t)\|_{p+1} \,\le\, C_8\,, \tag{3.25}$$

and, letting $p\to\infty$,

$$\max_{0\le t\le T}\|\theta_n(t)\|_\infty \,\le\, C_9 \quad \forall\, n\in\mathbb{N}\,. \tag{3.26}$$

Now that an $L^\infty(\Omega_T)$-bound for $\{\theta_n\}$ has been established, we may invoke (3.21) and standard linear parabolic estimates to conclude that also

$$\|\theta_n\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))\cap C([0,T];H^1(\Omega))} \,\le\, C_{10} \quad \forall\, n\in\mathbb{N}\,. \tag{3.27}$$

By (3.17), we also have

$$\|w_n\|_{(W^{1,\infty}(0,T;L^\infty(\Omega)))^M} \,\le\, C_{11} \quad \forall\, n\in\mathbb{N}\,. \tag{3.28}$$

Step 3: Construction of a limit point.

From (3.26) to (3.28), we infer that there exist some pair (w,θ) such that for a subsequence (which is again indexed by n) we have the convergences

$$w_n \to w \qquad \text{weakly-star in } (W^{1,\infty}(0,T;L^\infty(\Omega)))^M\,, \tag{3.29}$$

$$\theta_n \to \theta \qquad \text{weakly in } H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))\,, \text{ and weakly-star in } L^\infty(\Omega_T)\,. \tag{3.30}$$

The compactness of the imbedding $H^1(0,T;L^2(\Omega)) \,\cap\, L^2(0,T;H^2(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$ yields that $\theta(\cdot,0) \,=\, \theta^0$ a.e. in Ω, and the compactness of the imbedding $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))\hookrightarrow L^2(0,T;H^{2-\varepsilon}(\Omega))$ for $0<\varepsilon\le 1/2$ shows that we may assume that

$$\theta_n \to \theta \qquad \text{strongly in } \ L^2(0,T;H^{3/2}(\Omega))\,, \tag{3.31}$$

$$\frac{\partial\theta_n}{\partial\mathbf{n}} \to \frac{\partial\theta}{\partial\mathbf{n}} \qquad \text{strongly in } \ L^2(0,T;L^2(\partial\Omega))\,, \tag{3.32}$$

$$\theta_n \to \theta \qquad \text{pointwise a.e. in } \ \Omega_T\,. \tag{3.33}$$

Hence, we have $\frac{\partial\theta}{\partial\mathbf{n}}=0$ a. e. on $\partial\Omega\times(0,T)$, and, moreover, from (3.14) and (3.26),

$$C_9 \,\ge\, \theta(x,t) \,\ge\, \delta\, e^{-\beta t}\,, \quad \mu(\theta(x,t)) \,\ge\, \hat\mu\,, \quad \text{for a.e. } \ (x,t)\in\Omega_T\,. \tag{3.34}$$

Step 4: $\{w_n\}$ is a Cauchy sequence in $(H^1(0,T;L^2(\Omega)))^M$.

Suppose that $m\ge n\ge N_0$. We put $v^{n,m}:=w_n-w_m$, and $z^{n,m}:=\theta_n-\theta_m$. Subtracting the equations (3.6), (3.7) for m from the corresponding ones for n, we see that

$$\begin{aligned} v_t^{n,m} \;&=\; \frac{1}{\mu(\theta_n)}\Big(f_1^m[w_m]+\theta_m\, f_2^m[w_m]-f_1^n[w_n]-\theta_n\, f_2^n[w_n]\Big)\\ &\quad+\left(\frac{1}{\mu(\theta_m)}-\frac{1}{\mu(\theta_n)}\right)\Big(f_1^m[w_m]+\theta_m\, f_2^m[w_m]\Big),\ \text{a.e. in } \Omega_T, \end{aligned} \tag{3.35}$$

$$z_t^{n,m}-\Delta z^{n,m} \;=\; F_1^m[w_m]_t-F_1^n[w_n]_t+\psi(x,t,\theta_n)-\psi(x,t,\theta_m),\ \text{a.e. in } \Omega_T, \tag{3.36}$$

$$v^{n,m}(\cdot,0)=z^{n,m}(\cdot,0)=0 \quad \text{a.e. in } \Omega, \qquad \frac{\partial z^{n,m}}{\partial\mathbf{n}}=0 \quad \text{a.e. on } \partial\Omega\times(0,T). \tag{3.37}$$

Invoking **(H2)**, (3.14), (3.16), and (3.26), we obtain that

$$|v_t^{n,m}| \,\le\, C_{12}\left(|f_1^n[w_n]-f_1^m[w_m]| \,+\, |f_2^n[w_n]-f_2^m[w_m]| \,+\, |z^{n,m}|\right), \quad \text{a.e. in } \Omega_T\,. \tag{3.38}$$

Now let $j\in\{1,2\}$ be fixed. As $m\ge n$, we have $Z_{j,r}^n\subset Z_{j,r}^m$ for all $r\ge 0$. Therefore, using Lemma 2.3, (2.4), (3.3), and (3.28), we find that a. e. in Ω_T it holds

$$\begin{aligned} \left|f_j^n[w_n]-f_j^m[w_n]\right| \,&\le\, \int_0^\infty \varphi_j(r)\,\left|s_{Z_{j,r}^n}[w_n]-s_{Z_{j,r}^m}[w_n]\right|\,dr\\ \le\, C_{13}\int_0^\infty \varphi_j(r)\left(d_H(Z_{j,r}^n\,,\,Z_{j,r}^m)\right)^{1/2}dr \,&\le\, 2\,C_{13}\,\Phi_0\left(\sqrt{\delta_n}\,+\,\sqrt{\delta_m}\right) \,\le\, C_{14}\sqrt{\delta_n}\,. \end{aligned} \tag{3.39}$$

Moreover, arguing as in the derivation of (2.20) in Lemma 2.6,(iii), we find that for a. e. $(x,t)\in\Omega_T$

$$|f_j^m[w_n](x,t)-f_j^m[w_m](x,t)| \,\le\, C_{15}\int_0^t |w_{n,t}-w_{m,t}|\,(x,\tau)\,d\tau\,. \tag{3.40}$$

Thus, combining (3.38) to (3.40), we arrive at the estimate

$$\int_0^t \int_\Omega |v_t^{n,m}|^2 \, dx \, d\tau \;\le\; C_{16} \Big(\delta_n \,+\, \int_0^t \int_\Omega |z^{n,m}|^2 \, dx \, d\tau \,+\, \int_0^t \int_0^\tau \int_\Omega |v_t^{n,m}|^2 \, dx \, ds \, d\tau \Big) . \tag{3.41}$$

Next, we integrate (3.36) over $[0,\tau]$, multiply by $z^{n,m}$, and integrate the resulting identity over $\Omega \times [0,t]$ and by parts to arrive at the inequality

$$\begin{aligned} &\int_0^t \int_\Omega |z^{n,m}|^2 \, dx \, d\tau \,+\, \frac{1}{2} \int_\Omega \left| \int_0^t \nabla z^{n,m} \, d\tau \right|^2 dx \\ \le\; &\int_0^t \int_\Omega |F_1^m[w_m] - F_1^n[w_n]| \; |z^{n,m}| \, dx \, d\tau \\ &+ \int_0^t \int_\Omega |z^{n,m}(x,\tau)| \int_0^\tau |\psi(x,s,\theta_n(x,s)) - \psi(x,s,\theta_m(x,s))| \, ds \, dx \, d\tau \, . \end{aligned} \tag{3.42}$$

Arguing as above, we see that for a. e. $(x,t) \in \Omega_T$ it holds

$$|F_1^m[w_m](x,t) - F_1^n[w_n](x,t)| \;\le\; C_{17} \Big(\sqrt{\delta_n} \,+\, \int_0^t |v_t^{n,m}(x,\tau)| \, d\tau \Big) . \tag{3.43}$$

Now **(H4)** yields that

$$|\psi(x,t,\theta_n(x,t)) - \psi(x,t,\theta_m(x,t))| \;\le\; \Psi_1 \; |z^{n,m}(x,t)| \, , \quad \text{for a. e. } (x,t) \in \Omega_T \, , \tag{3.44}$$

and it follows from Young's inequality that

$$\begin{aligned} &|z^{n,m}(x,\tau)| \int_0^\tau |\psi(x,s,\theta_n(x,s)) - \psi(x,s,\theta_m(x,s))| \, ds \\ \le\; &\frac{1}{2} \, |z^{n,m}(x,\tau)|^2 \,+\, C_{18} \int_0^\tau |z^{n,m}(x,s)|^2 \, ds \, , \quad \text{a. e. in } \Omega_T \, . \end{aligned} \tag{3.45}$$

Combining (3.41) to (3.45), and applying Young's inequality, we have finally shown that for every $t \in [0,T]$ it holds

$$\int_0^t \int_\Omega \Big(|v_t^{n,m}|^2 \,+\, |z^{n,m}|^2 \Big) \, dx \, d\tau \;\le\; C_{19} \Big(\delta_n \,+\, \int_0^t \int_0^\tau \int_\Omega \Big(|v_t^{n,m}|^2 \,+\, |z^{n,m}|^2 \Big) \, dx \, ds \, d\tau \Big) , \tag{3.46}$$

whence, using Gronwall's lemma,

$$\int_0^t \int_\Omega \Big(|v_t^{n,m}|^2 \,+\, |z^{n,m}|^2 \Big)(x,\tau) \, dx \, d\tau \;\le\; C_{19} \, \delta_n \, e^{C_{19} \, t} \;\to\; 0 \quad \text{as } m,n \to \infty \, . \tag{3.47}$$

From this it follows that $\{w_n\}$ is a Cauchy sequence in $(H^1(0,T;L^2(\Omega)))^M$.

Step 5: (w,θ) is a solution to the original problem.

From Step 4 it follows that $w_n \to w$ strongly in $(H^1(0,T;L^2(\Omega)))^M$, and, in particular, strongly in $(C([0,T];L^2(\Omega)))^M$, and strongly in $(L^2(\Omega;C[0,T]))^M$. Hence,

$w(\cdot,0) = w_0$ a. e. in Ω, and from Lemma 2.6,(i) we infer that

$$\begin{aligned} f_j[w_n] &\to f_j[w], \qquad \text{strongly in } (L^2(\Omega; C[0,T]))^M, \\ F_j[w_n] &\to F_j[w], \qquad \text{strongly in } L^2(\Omega; C[0,T]), \quad j = 1,2. \end{aligned} \tag{3.48}$$

Moreover, since $Z_{j,r}^n \subset Z_{j,r}$ for all $r \geq 0$, $j = 1,2$, we conclude as in the derivation of (3.39) that a. e. in Ω_T it holds

$$\max_{j=1,2} \left(\left| f_j^n[w_n] - f_j[w_n] \right| + \left| F_j^n[w_n] - F_j[w_n] \right| \right) \leq C_{20} \sqrt{\delta_n}. \tag{3.49}$$

Consequently, we have

$$\begin{aligned} f_j^n[w_n] &\to f_j[w], \qquad \text{strongly in } (L^2(\Omega; C[0,T]))^M, \\ F_j^n[w_n] &\to F_j[w], \qquad \text{strongly in } L^2(\Omega; C[0,T]), \quad j = 1,2. \end{aligned} \tag{3.50}$$

Hence, combining the already shown convergences, and letting $n \to \infty$ in (3.6), we have proved that (w, θ) satisfies (1.1) a. e. in Ω_T. To verify that also (1.2) is satisfied, we still have to show that $F_1^n[w_n]_t$ converges in a suitable sense to $F_1[w]_t$. Owing to (3.18) and (3.26), we may assume that $F_1^n[w_n]_t \to y$ weakly-star in $L^\infty(\Omega_T)$ for a suitable $y \in L^\infty(\Omega_T)$. But then (3.50) implies that $y = F_1[w]_t$, which concludes the proof of existence.

Step 6: Proof of uniqueness.

Suppose (w_j, θ_j), $j = 1,2$, are two pairs fulfilling (1.1), (1.2), (2.1) in the sense of the theorem, as well as (2.11). We put $v := w_1 - w_2$ and $z := \theta_1 - \theta_2$. We then have (compare (3.6) to (3.8)):

$$\begin{aligned} v_t \;=\;& \frac{1}{\mu(\theta_1)} \Big(f_1[w_2] + \theta_2 f_2[w_2] - f_1[w_1] - \theta_1 f_2[w_1] \Big) \\ &+ \left(\frac{1}{\mu(\theta_2)} - \frac{1}{\mu(\theta_1)} \right) \Big(f_1[w_2] + \theta_2 f_2[w_2] \Big), \quad \text{a. e. in } \Omega_T, \end{aligned} \tag{3.51}$$

$$z_t - \Delta z \;=\; F_1[w_2]_t - F_1[w_1]_t + \psi(x,t,\theta_1) - \psi(x,t,\theta_2), \quad \text{a. e. in } \Omega_T, \tag{3.52}$$

$$v(\cdot,0) = z(\cdot,0) = 0 \quad \text{a. e. in } \Omega, \quad \frac{\partial z}{\partial \mathbf{n}} = 0 \quad \text{a. e. on } \partial\Omega \times (0,T). \tag{3.53}$$

Performing essentially the same estimates as in the derivation of (3.46) (where here the convex sets $Z_{j,r}$ do not vary so that estimates like (3.39) in terms of δ_n do not occur), we obtain that

$$\int_0^t \int_\Omega \Big(|v_t|^2 + |z|^2 \Big)\, dx\, d\tau \leq C_{21} \int_0^t \int_0^\tau \int_\Omega \Big(|v_t|^2 + |z|^2 \Big)\, dx\, ds\, d\tau, \tag{3.54}$$

and Gronwall's lemma yields $v = z = 0$ a. e. in Ω_T, which proves the uniqueness of the limit. We remark that this entails that all the previously shown convergences in fact hold for the entire sequence $\{(w_n, \theta_n)\}$, and not only for suitable subsequences.

Step 7: Conclusion of the proof.

So far we have shown the existence of a unique solution (w, θ) satisfying (2.11). It remains to show that *any* solution such that $\theta > 0$ a.e. in Ω_T fulfills (2.11). So let us assume that (w, θ) is a solution satisfying (2.10) and $\theta > 0$ a.e. in Ω_T. We have to show the validity of (2.11). To this end, we follow the lines of the proof of Theorem 3.3 in [9] and test (1.2) with an arbitrary function $p \in H^1(\Omega_T)$ satisfying $p \leq 0$ a.e. in Ω_T. Owing to the general assumptions, and due to (1.8), we can infer that for almost every $t \in (0,T)$ it holds

$$\begin{aligned}
&\int_\Omega \Big(p\,\theta_t \,+\, \langle \nabla p\,,\, \nabla\theta\rangle\Big)(x,t)\,dx \\
=\;& \int_\Omega p\,\Big(\psi_0(x,t) \,+\, \psi(x,t,\theta) \,-\, \psi(x,t,0)\Big)\,dx \,+\, \int_\Omega \Big(|p|\,F_1[w]_t\Big)(x,t)\,dx \\
\leq\;& \Psi_1 \int_\Omega \Big(|p|\,\theta\Big)(x,t)\,dx \,+\, \int_\Omega |p|\,\langle w_t\,,\, f_1[w]\rangle\,(x,t)\,dx\,.
\end{aligned} \tag{3.55}$$

We now estimate the term $\langle w_t\,,\, f_1[w]\rangle$. To this end, we consider two different cases.

Case 1: $\theta \geq 1$.

Then, by (2.3), $\mu(\theta) \geq \mu_0 > 0$. It thus follows from (2.18) that

$$\begin{aligned}
&\langle w_t\,,\, f_1[w]\rangle \;\leq\; \Phi_0\,K_0\,|w_t| \;\leq\; \frac{\Phi_0\,K_0}{\mu(\theta)}\,|-f_1[w] \,-\, f_2[w]\,\theta\,| \\
\leq\;& \frac{\Phi_0^2\,K_0^2}{\mu_0}\,(1 \,+\, \theta) \;\leq\; \frac{2\,\Phi_0^2\,K_0^2}{\mu_0}\,\theta\,.
\end{aligned} \tag{3.56}$$

Case 2: $0 < \theta < 1$.

Then (2.3) implies that $\mu(\theta) \geq \mu_0\,\theta$. Hence, using (2.18), and Young's inequality, we find that

$$\begin{aligned}
&\langle w_t\,,\, f_1[w]\rangle \;=\; \langle w_t\,,\, -\mu(\theta)\,w_t \,-\, f_2[w]\,\theta\rangle \\
\leq\;& \frac{\theta^2}{4\,\mu(\theta)}|f_2[w]|^2 \;\leq\; \frac{\Phi_0^2\,K_0^2}{4\,\mu_0}\,\theta\,.
\end{aligned} \tag{3.57}$$

Combining (3.56) and (3.57), we see that in any case

$$\langle w_t\,,\, f_1[w]\rangle \;\leq\; \frac{2\,\Phi_0^2\,K_0^2}{\mu_0}\,\theta\,. \tag{3.58}$$

Combining (3.55) and (3.58), we find that

$$\int_\Omega \Big(p\,\theta_t \,+\, \langle \nabla p\,,\, \nabla\theta\rangle\Big)(x,t)\,dx \;\leq\; \beta \int_\Omega \Big(|p|\,\theta\Big)(x,t)\,dx\,. \tag{3.59}$$

Now, put

$$p(x,t) := -\left(\delta e^{-\beta t} - \theta(x,t)\right)^+ \quad \text{for } (x,t) \in \Omega_T. \tag{3.60}$$

Then it follows from inequality (3.59) that

$$\int_\Omega \left(p\left(p + \delta e^{-\beta t}\right)_t + |\nabla p|^2\right)(x,t)\,dx \le \beta \int_\Omega |p|\left(|p| + \delta e^{-\beta t}\right)(x,t)\,dx. \tag{3.61}$$

This yields, in particular,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega p^2(x,t)\,dx + \int_\Omega |\nabla p|^2(x,t)\,dx \le \beta \int_\Omega p^2(x,t)\,dx. \tag{3.62}$$

Therefore, by Gronwall's lemma, $p = 0$, and thus, $\theta(x,t) \ge \delta e^{-\beta t}$ for a. e. $(x,t) \in \Omega_T$. With this, the assertion of Theorem 2.2 is completely proved. □

Remark 3.1 In Step 7 of the above proof we have used the energy dissipation property $F_j[w]_t \le \langle w_t, f_j[w]\rangle$ of the Prandtl-Ishlinskii operator in order to show (2.11). We note that this property also implies the thermodynamic consistency of the system (1.1), (1.2). For details, we refer the reader to [9].

Remark 3.2 From the above proof, especially from the estimations performed in Step 4, it should be apparent that the (unique) solution (w,θ) depends in a Lipschitz continuous way on the data of the system. Indeed, $L^2(\Omega)$ - variations of the initial values w_0, θ_0, and variations of the convex sets $Z_{j,1}$ with respect to the Hausdorff distance, lead to a corresponding Lipschitz variation of (w,θ) in the norm of $(H^1(0,T;L^2(\Omega)))^M \times L^2(\Omega_T)$. A similar result holds for variations of ψ. As the line of argumentation ought to be clear from the above considerations, we leave the explicit formulation and the proof of the corresponding result to the reader.

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