Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

The moment Lyapunov exponent for conservative systems with small periodic and random perturbations

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submitted: 9th March 2001

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Preprint No. 647
Berlin 2001

2000 *Mathematics Subject Classification.* 60H10, 93E15.

Key words and phrases. Linear stochastic systems with periodic coefficients, stochastic stability, moment Lyapunov exponent, stability index, Hill and Mathieu equations with random excitations.

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Abstract. Much effort has been devoted to the stability analysis of stationary points for linear autonomous systems of stochastic differential equations. Here we introduce the notions of Lyapunov exponent, moment Lyapunov exponent, and stability index for linear nonautonomous systems with periodic coefficients. Most extensively we study these problems for second order conservative systems with small random and periodic escitations. With respect to relations between the intrinsic period of the system and the period of perturbations we consider the incommensurable and commensurable cases. In the first case we obtain an asymptotic expansion of the moment Lyapunov exponent. In the second case we obtain a finite expansion except in situations of resonance. As an application we consider the Hill and Mathieu equations with random excitations.

1. Introduction

Since the development of the tools of stochastic analysis the research of asymptotic properties of random dynamical systems generated by autonomous stochastic differential equations has been very active. Following the pioneering results of Khasminskii [17], [18], a refined theory of stability focused on *pathwise* and *moment Lyapunov exponent* (see [1], [5]) has been developed (Arnold [2]). A more sophisticated notion has been investigated under the names *large deviations* (Baxendale [10]) and *stability index* (Arnold, Khasminskii [4]).

For linear stochastic systems with periodic coefficients not much is known to date. In Section 2 we give an outline of a general theory for such systems using results from [21], [22]. Though there exist some essential differences between systems with constant and periodic coefficients, the main results concerning Lyapunov exponents, moment Lyapunov exponents, and stability indices for stationary points carry over to the periodic case.

In this paper we consider second order conservative systems *with small random and periodic excitations,* of the following form

$$dX^\varepsilon = \omega J X^\varepsilon dt + \varepsilon A(t) X^\varepsilon dt + \sqrt{\varepsilon} \sum_{r=1}^{q} A_r(t) X^\varepsilon \circ dw_r(t), \tag{1.1}$$

where ω is a positive number, J is the matrix of rotation by $-\pi/2$, and $A(t),\ A_r(t),\ 1 \le r \le q$, are l-periodic 2×2-matrices, where $l > 0$.

We recall that in the case of linear autonomous second-order systems there are exact formulas for Lyapunov exponents [18], [15], [16]. But the formulas are mostly rather complicated and systems with small noise are of special interest. This is why many works are devoted to the asymptotics of Lyapunov exponents (see [8], [6], [25], [24], [3], [9], and references therein). In particular in [8], a general expansion has been obtained with evaluation of the n-th remainder term for Lyapunov exponents in the conservative case.

For the moment Lyapunov exponents much less is known (for a general linear autonomous theory see [5]). In [20] deterministic methods for the evaluation of moment Lyapunov exponents for second-order systems are derived. In the case of small noise intensity and small moments some asymptotic expansions have been obtained in [12]. [19] presents asymptotic series expansions of the moment Lyapunov exponents of any order and the stability index in the case of two-dimensional conservative system perturbed by small noise with time independent intensity. These results are applied to the investigation of the stability of orbits in the plane under small diffusion in [22].

In our analysis of (1.1), the main steps are as in [19]. However the periodic case is essentially more complicated in comparison with the autonomous one. First, we have to consider an enlarged homogeneous two-dimensional state-process $(\Phi^\varepsilon, \vartheta)$ defined on a torus, where Φ^ε is the angular part of X^ε on the unit circle, and ϑ uniform motion on the interval $[0, l)$ made into a circle by identifying boundaries. In particular due to this fact, we need to solve in every step of the asymptotic expansion procedure a partial differential equation instead of an ordinary one in the autonomous case. Second, for $\varepsilon = 0$ the considered process on the torus is not always ergodic. It is not ergodic if the intrinsic period $2\pi/\omega$ of the unperturbed system and the period l of the perturbation matrices are *commensurable*, i.e., if there exist integer m and k such that $k\omega l = m\pi$. In the *incommensurable* case, it is possible to obtain an asymptotic expansion of the moment Lyapunov exponent of any order. In the commensurable case, i.e. if $\omega \neq \dfrac{\pi m}{kl}$, $k = 1,\ 2, ..., 2n,\ m = 1,\ 2, ...$, for a fixed integer n, one can obtain a finite expansion with a remainder term of the form $O(\varepsilon^{n+1})$. So, if $\omega \neq \dfrac{\pi m}{2l}$, $m = 1,\ 2, ...$, our approach gives at least the principal term of an expansion with exactness $O(\varepsilon^2)$. The set of excluded frequencies contains the well known ones of the theory of parametric resonance. Both the investigation of resonance cases and a more precise definition of finite expansions are most likely difficult problems which are at present far from being solved.

In Section 2, we briefly recall classical results about moment Lyapunov exponents and stability indices, both for linear and nonlinear systems and give an outline of a general theory for linear stochastic systems with coefficients periodic in t. Section 3 is devoted to establish the setting of the asymptotic analysis for our two-dimensional systems. Sections 4 - 6 are devoted to the incommensurable case. In Section 4 we set up the expansion algorithm for moment Lyapunov exponents of systems including small periodic perturbations. Each step of the algorithm involves in particular the solution of a system of ordinary differential equation for the variable of uniform motion on $[0, l)$. In Section 5 we present the theorem of asymptotic expansion of the moment Lyapunov exponent. The application of the preceding to the asymptotic expansion of stability indices is given in Section 6. In Section 7 we consider the commensurable case. In the final Section 8 some aspects of the Hill and Mathieu oscillators with random excitations are discussed.

2. Moment Lyapunov exponent and stability index for stochastic systems with periodic coefficients. Main theorems

Consider a linear system of stochastic differential equations (SDEs) in the sense of Stratonovich

$$dX = A_0(t)X\,dt + \sum_{r=1}^{q} A_r(t)X \circ dw_r(t). \tag{2.1}$$

Here X takes its values in $\mathbf{R}^d$, $A_r(t)$, $r = 0,\ 1, ..., q$, are $d \times d$ matrices with bounded measurable l-periodic coefficients, $w_r(t)$, $r = 1, ..., q$, are independent standard Wiener processes on a probability space $(\mathbf{\Omega},\ \mathcal{F},\ \mathbf{P})$.

Introduce the cyclic variable ϑ on the circle $\mathbf{S}$ of radius $r = l/2\pi$, the measure $\mu(d\vartheta) = d\vartheta/l$ so that $\mu(\mathbf{S}) = 1$, and the following autonomous version of (2.1) on an enlarged state space

$$dX = A_0(\vartheta)X\,dt + \sum_{r=1}^{q} A_r(\vartheta)X \circ dw_r(t), \qquad d\vartheta = dt, \tag{2.2}$$

which is solved by a homogeneous Markov process (X, ϑ) there.

Let $X(0) = x \neq 0$. Introduce $\Lambda = X\backslash|X|$ and consider the process (ϑ, Λ) which is defined on $\mathbf{D} = \mathbf{S} \times \mathbf{S}^{d-1}$ where $\mathbf{S}^{d-1}$ is the unit sphere in $\mathbf{R}^d$. This process satisfies the Khasminskii type system of SDEs

$$d\Lambda = a_0(\vartheta, \Lambda)dt + \sum_{r=1}^{q} a_r(\vartheta, \Lambda) \circ dw_r(t), \qquad d\vartheta = dt, \tag{2.3}$$

where the d-dimensional vector fields a_r, $r = 0, 1, ..., q$, are defined by

$$a_r(\theta, \lambda) = A_r(\theta)\lambda - (A_r(\theta)\lambda, \lambda)\lambda,\ (\theta, \lambda) \in \mathbf{D}.$$

For $|X(t)|^p$, $-\infty < p < \infty$, we have the following linear equation

$$d|X|^p = p(A_0(\vartheta)\Lambda, \Lambda) \cdot |X|^p dt + p\sum_{r=1}^{q}(A_r(\vartheta)\Lambda, \Lambda)|X|^p \circ dw_r(t). \tag{2.4}$$

Let $X(0) = \lambda \in \mathbf{S}^{d-1}$, $\vartheta(0) = \theta$. The following formula defines a strongly continuous semigroup $T_t(p)$ of positive operators on $\mathbf{C}(\mathbf{D})$ (note that here and in the sequel subscripts to processes indicate the initial state):

$$T_t(p)f(\theta, \lambda) = Ef(\vartheta_\theta(t), \Lambda_{\theta,\lambda}(t))|X_{\theta,\lambda}(t)|^p,\ (\theta, \lambda) \in \mathbf{D},\ f \in \mathbf{C}(\mathbf{D}). \tag{2.5}$$

This fact can be proved by direct checking the definition of a strongly continuous semigroup.

Assume for a moment that X is an autonomous system. Then the following semigroup of positive operators $T_t(p)$ is defined on $\mathbf{C}(\mathbf{S^{d-1}})$:

$$T_t(p)f(\lambda) = Ef(\Lambda_\lambda(t))|X_\lambda(t)|^p,\ \lambda \in \mathbf{S^{d-1}},\ f \in \mathbf{C}(\mathbf{S^{d-1}}). \tag{2.6}$$

It is well known [17], [5], that under some nondegeneracy conditions the process Λ is ergodic and for any $t > 0$, $-\infty < p < \infty$, the operator $T_t(p)$ is compact and irreducible, even strongly positive. We recall that a positive operator Q on $\mathbf{C}(\mathbf{K})$ ($\mathbf{K}$ is a compact set) is called irreducible if $\{0\}$ and $\mathbf{C}(\mathbf{K})$ are the only Q-invariant closed ideals, and Q is called strongly positive if $Qf(x) > 0$, $x \in \mathbf{K}$, for any nontrivial $f \geq 0$. The generalized Perron-Frobenius theorem ensures that for each $p \in \mathbf{R}$ the operator $T_t(p)$ and consequently its generator $L(p)$ has a strictly positive eigenfunction corresponding to the principal eigenvalue $g(p)$, which is real, simple, and strictly dominates the real part of any other point of the spectrum of $L(p)$.

Our next goal is to formulate some basic results concerning stability properties of the system (2.1).

It should be noted that due to the cyclicity of ϑ none of the operators $T_t(p)$ in (2.5), $t > 0$, $-\infty < p < \infty$, is compact. To show this let us consider the set $\mathbf{F}$ of all uniformly bounded functions f depending on the variable θ only. Clearly, $T_t\mathbf{F}$ consists of functions of the form $f(t+\theta)E|X_{\theta,\lambda}(t)|^p$, $f \in \mathbf{F}$, and this set is evidently not compact. Moreover, not every operator $T_t(p)$ in (2.5), $t > 0$, is irreducible in contrast to the operator (2.6). Indeed for θ_0 fixed, let us consider the set $\mathbf{F}$ of all functions f such that $f(\theta_0, \lambda) = 0$ for all $\lambda \in \mathbf{S^{d-1}}$. Clearly, this set is a $T_l(p)$-invariant closed ideal.

But the whole semigroup (2.5) can be irreducible. We recall that a positive semigroup $T_t(p)$ in $\mathbf{C}(\mathbf{D})$ is called irreducible if the operators $T_t(p)$, $t > 0$, have no common invariant closed ideal other than $\{0\}$ and $\mathbf{C}(\mathbf{D})$. A simple sufficient condition of semigroup irreducibility consists in the existence of a set $\mathbf{S}_0$ of Lebesgue measure 0 such that

$$\dim L(a_1(\theta,\lambda), ..., a_q(\theta,\lambda)) = d-1, \ (\theta,\lambda) \in (\mathbf{S} \backslash \mathbf{S}_0) \times \mathbf{S}^{d-1}, \tag{2.7}$$

where L denotes the linear hull spanned by the given vector fields.

It follows from [13], [11] (see also [23]) that the spectrum $\sigma(L(p))$ of the generator $L(p)$ of the positive semigroup (2.5) is not empty and for

$$s(L(p)) := \sup\{\mathrm{Re}\mu : \ \mu \in \sigma(L(p))\} = \max\{\mu \in \mathbf{R} : \ \mu \in \sigma(L(p))\}$$

we have

$$-\infty < s(L(p)) < \infty.$$

Moreover the resolvent $R(\mu, L(p))$ is strongly positive for $\mu > s(L(p))$ because $T_t(p)$ is irreducible, and

$$R(\mu, L(p))f(\theta,\lambda) = \int_0^\infty e^{-\mu t} T_t(p) f(\theta,\lambda) dt. \tag{2.8}$$

It is possible to justify analogously to [21] that the condition (2.7) ensures the compactness of the resolvent for $\mu > s(L(p))$ as well. We note that the properties of irreducibility of the semigroup $T_t(p)$ and of compactness of its resolvent can be fulfilled under some weaker conditions than (2.7). In Section 3 we shall give an alternative condition.

For the completeness of the presentation we prove the following statement (see also [21]).

Theorem 2.1. *Let the following hypothesis be fulfilled:*

$$T_t(p) \text{ is irreducible and } R(\mu, L(p)) \text{ is compact for } \mu > s(L(p)). \tag{2.9}$$

Then there exists a strictly positive eigenfunction h_p of the generator $L(p)$ corresponding to an eigenvalue $g(p)$:

$$L(p)h_p = g(p)h_p, \ h_p > 0. \tag{2.10}$$

The eigenvalue is real and simple, $g(p)$ is bigger or equal to the real part of any other point of the spectrum of $L(p)$. All the points in $\sigma(L(p))$ with real part $g(p)$ are given by $g(p)+i\alpha k,\ k=0,\ \pm 1, ...,$ for some $\alpha \geq 0$, and they are all simple isolated eigenvalues of $L(p)$.

Proof. Let $\mu > s(L(p))$. The relation

$$\sigma(R(\mu, L(p))) = (\mu - \sigma(L(p)))^{-1}$$

implies $(\mu - s(L(p)))^{-1} \in \sigma(R(\mu, L(p)))$ because $s(L(p)) \in \sigma(L(p))$. Since $R(\mu, L(p))$ is strongly positive and compact, $(\mu - s(L(p)))^{-1}$ is a simple isolated eigenvalue of $R(\mu, L(p))$ which exceeds the absolute value of any other eigenvalue of $R(\mu, L(p))$. Moreover there exists a unique $h_p \in \mathbf{C}(\mathbf{D})$ with $h_p > 0$, $||h_p|| = 1$, and a unique positive measure ν_p on $\mathbf{D}$ with $||\nu_p|| = 1$ which are respectively an eigenfunction of the operator $R(\mu, L(p))$ and an eigendistribution of the adjoint operator $R^*(\mu, L(p))$. Denoting $s(L(p))$ by $g(p)$ we get (2.10) and the conjugate equation

$$L^*(p)\nu_p = g(p)\nu_p. \tag{2.11}$$

Further, as $(\mu - s(L(p)))^{-1}$ is a pole of the resolvent of the operator $R(\mu, L(p))$, the number $s(L(p)) = g(p)$ is a pole of $R(\mu, L(p))$ (see [13]). In such a case the generalized Perron-Frobenius theorem [13] (see also [11]) states besides (2.10) and (2.11) that all the points from $\sigma(L(p))$ with real part $g(p)$ are described by $g(p) + i\alpha k,\ k = 0,\ \pm 1, ...,$ for some $\alpha \geq 0$, and they are all simple isolated eigenvalues of $L(p)$. Theorem 2.1 is proved.

As already mentioned, in contrast to (2.6) none of the operators $T_t(p)$ in (2.5), $t > 0$, $-\infty < p < \infty$, is compact, and not each one among them is irreducible. We add that the real part of any point of the spectrum of $L(p)$ different from $g(p)$ is not always strictly less than $g(p)$. However these and some other differences to [5] do not prohibit to carry over the theory of moment Lyapunov exponents to the system (2.1). The basic theorems relating to stability properties of the system (2.1) are analogous to the corresponding ones from [17], [5], and [10] and their proofs will not be included here.

In Theorems 2.2-2.4 $X_{t_0,x_0}(t),\ t \geq t_0$, is the solution of the system (2.6) with $X(t_0) = x_0,\ x_0 \neq 0$. The following theorem is an analogue of the Khasminskii theorem (see also [21], [22]).

Theorem 2.2. *Let the hypothesis (2.9) be fulfilled. Then the process (ϑ, Λ) on $\mathbf{D} = \mathbf{S} \times \mathbf{S}^{d-1}$ is ergodic, there exists an invariant measure $\mu|\mathbf{D}$ and the following P-a.s. limit λ^* exists and does not depend on $(t_0, x_0),\ x_0 \neq 0$:*

$$\lambda^* = \lim_{t\to\infty} \frac{1}{t} \ln |X_{t_0,x_0}(t)| = \lim_{t\to\infty} \frac{1}{t} E \ln |X_{t_0,x_0}(t)| = \int_{\mathbf{D}} Q(\theta, \lambda) d\mu(\theta, \lambda), \tag{2.12}$$

where

$$Q(\theta, \lambda) = (A_0(\theta)\lambda, \lambda) + \frac{1}{2}\sum_{r=1}^{q}((A_r(\theta) + A_r^*(\theta))\lambda, A_r(\theta)\lambda) - \sum_{r=1}^{q}(A_r(\theta)\lambda, \lambda)^2. \tag{2.13}$$

The limit λ^* is *called Lyapunov* exponent of the system (2.1).

The following theorem is an analogue of the Arnold-Oeljeklaus-Pardoux theorem from [5] (see also [21], [22]).

Theorem 2.3. *Let the hypothesis* (2.9) *be fulfilled. Then the limit (which is called the p-th moment Lyapunov exponent for* (2.1))

$$\lim_{t\to\infty} \frac{1}{t} \ln E|X_{t_0,x_0}(t)|^p = g(p) \tag{2.14}$$

exists for any $p \in \mathbf{R}$ *and is independent of* $t_0, x_0,\ x_0 \neq 0$. *The limit* $g(p)$ *coincides with the eigenvalue* $g(p)$ *given by Theorem 2.1 and consequently it satisfies the properties stated there.* $g(p)$ *is a convex analytic function of* $p \in \mathbf{R}$, $g(0) = 0$, $g(p)/p$ *is increasing, and* $g'(0) = \lambda^*$.

If, for example, $\lambda^* < 0$ and $g(p) \to \infty$ as $p \to \infty$ then the equation

$$g(p) = 0 \tag{2.15}$$

has a unique positive root γ^*. It is clear that the trivial solution of (2.1) is p-stable for $0 < p < \gamma^*$ and p-unstable for $p > \gamma^*$. The root γ^* of (2.15) is related to the asymptotic behavior of the probability $P\{\sup_{t\geq t_0} |X_{t_0,x_0}(t)| > \delta\},\ |x_0|/\delta \to 0$, if $\gamma^* > 0$ and of the probability $P\{\inf_{t\geq t_0} |X_{t_0,x_0}(t)| < \delta\},\ |x_0|/\delta \to \infty$, if $\gamma^* < 0$.

The following theorem is an analogue of the Baxendale theorem from [10] (see also [21], [22]).

Theorem 2.4. *Let the hypothesis* (2.9) *be fulfilled. If* $\lambda^* < 0$ *and the equation* (2.15) *has a positive root* $\gamma^* > 0$, *then there exists* $K \geq 1$ *such that for all* $\delta > 0$ *and for all* x_0 *with* $|x_0| < \delta$

$$\frac{1}{K}(|x_0|/\delta)^{\gamma^*} \leq P\{\sup_{t\geq t_0} |X_{t_0,x_0}(t)| > \delta\} \leq K(|x_0|/\delta)^{\gamma^*}. \tag{2.16}$$

If $\lambda^* > 0$ *and the equation* (2.15) *has a negative root* $\gamma^* < 0$, *then there exists* $K \geq 1$ *such that for all* $\delta > 0$ *and for all* x_0 *with* $|x_0| > \delta$

$$\frac{1}{K}(|x_0|/\delta)^{\gamma^*} \leq P\{\inf_{t\geq t_0} |X_{t_0,x_0}(t)| < \delta\} \leq K(|x_0|/\delta)^{\gamma^*}. \tag{2.17}$$

This theorem states that the probability with which a solution of the linear system (2.1) exceeds a threshold is controlled by the number γ^*. Arnold and Khasminskii call this number stability index. Their main result of [4] consists in proving that the estimates (2.16)–(2.17) remain true for a nonlinear system as well.

This result can be carried over to nonlinear nonautonomous systems of the form

$$dY = a_0(t,Y)dt + \sum_{r=1}^{q} a_r(t,Y) \circ dw_r(t). \tag{2.18}$$

In system (2.18) the coefficients $a_r = (a_r^1, ..., a_r^d)^\top,\ r = 0,\ 1, ..., q,$ are l-periodic in t and are equal to zero at $y = 0:\ a_r(t,0) = 0,\ r = 0,\ 1, ..., q.$

The variational system for (2.18) has the form (2.1), where

$$A_r(t) = \left\{ \frac{\partial a_r^i(t,0)}{\partial y^j} \right\},\ i,j = 1, ..., d;\ r = 0,\ 1, ..., q.$$

The following theorem is an analogue of the Arnold-Khasminskii theorem (see also [21], [22]).

Theorem 2.5. *Assume that the linearization* (2.1) *of the system* (2.18) *satisfies hypothesis* (2.9). *Let* λ^* and γ^* *be the Lyapunov exponent and the stability index for* (2.1). *Then*

in case $\lambda^* < 0,\ \gamma^* > 0$ *there exists a sufficiently small* $\delta > 0$ *and positive constants* $c_1,\ c_2$ *such that for all* $|y_0| < \delta$ *the solution* $Y_{t_0,y_0}(t)$ *of* (2.18) *satisfies the inequalities*

$$c_1(|y_0|/\delta)^{\gamma^*} \le P\{\sup_{t\ge t_0} |Y_{t_0,y_0}(t)| > \delta\} \le c_2(|y_0|/\delta)^{\gamma^*}, \tag{2.19}$$

in case $\lambda^* > 0,\ \gamma^* < 0$ *there exists a sufficiently small* $r > 0$, *positive constants* $c_3,\ c_4$, *and a constant* $0 < \alpha < 1$ *such that for any* $\delta \in (0,\ \alpha r)$ *and all* $\delta < |y_0| < \alpha r$ *the solution* $Y_{t_0,y_0}(t)$ of (2.18) *satisfies the inequalities*

$$c_3(|y_0|/\delta)^{\gamma^*} \le P\{\inf_{t_0\le t<\tau} |X_{t_0,x_0}(t)| < \delta\} \le c_4(|y_0|/\delta)^{\gamma^*}. \tag{2.20}$$

Here $\tau := \inf\{t:\ |Y_{t_0,y_0}(t)| > \delta\}$.

3. Perturbed oscillators

Consider the following two-dimensional system

$$dX^\varepsilon = \omega J X^\varepsilon dt + \varepsilon A(t) X^\varepsilon dt + \sqrt{\varepsilon} \sum_{r=1}^{q} A_r(t) X^\varepsilon \circ dw_r(t), \tag{3.1}$$

where $\omega > 0,\ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ A(t),\ A_r(t),\ r = 1, ..., q$, are 2×2-matrices with l-periodic elements, ε is a small parameter. The system (3.1) describes a simple harmonic oscillator perturbed by small fluctuations both deterministic periodic and random diffusive.

Its angular component can be represented in the Khasminskii type form (see [18])

$$d\Phi^\varepsilon = -\omega dt - \varepsilon\bar\Lambda^\top(\Phi^\varepsilon)A(\vartheta)\Lambda(\Phi^\varepsilon)dt - \sqrt{\varepsilon}\sum_{r=1}^{q}\beta_r(\vartheta,\Phi^\varepsilon)\circ dw_r(t), \tag{3.2}$$

$$d\vartheta = dt,$$

where

$$\Lambda(\varphi) = \begin{bmatrix}\cos\varphi \\ \sin\varphi\end{bmatrix},\ \bar\Lambda(\varphi) = \begin{bmatrix}\sin\varphi \\ -\cos\varphi\end{bmatrix},$$

$$\beta_r(\theta,\varphi) = \bar\Lambda^\top(\varphi)A_r(\theta)\Lambda(\varphi).$$

For the p-th power of the radial part $|X^\varepsilon|^p$, $-\infty < p < \infty$, we get the following linear equation

$$d|X^\varepsilon|^p = \varepsilon p\Lambda^\top(\Phi^\varepsilon)A(\vartheta)\Lambda(\Phi^\varepsilon)|X^\varepsilon|^p dt + \sqrt{\varepsilon}p\sum_{r=1}^{q}\alpha_r(\vartheta,\Phi^\varepsilon)|X^\varepsilon|^p\circ dw_r(t), \tag{3.3}$$

where

$$\alpha_r(\theta,\varphi) = \Lambda^\top(\varphi)A_r(\theta)\Lambda(\varphi).$$

Let us write the equations (3.1)-(3.3) in the Ito form. This gives

$$dX^\varepsilon = \omega JX^\varepsilon dt + \varepsilon A(t)X^\varepsilon dt + \frac{1}{2}\varepsilon\sum_{r=1}^{q}A_r^2(t)X^\varepsilon dt + \sqrt{\varepsilon}\sum_{r=1}^{q}A_r(t)X^\varepsilon dw_r(t), \tag{3.4}$$

$$d\Phi^\varepsilon = -\omega dt - \varepsilon\bar\Lambda^\top(\Phi^\varepsilon)A(\vartheta)\Lambda(\Phi^\varepsilon)dt + \frac{1}{2}\varepsilon\sum_{r=1}^{q}\frac{\partial\beta_r}{\partial\varphi}(\vartheta,\Phi^\varepsilon)\beta_r(\vartheta,\Phi^\varepsilon)dt \tag{3.5}$$

$$-\sqrt{\varepsilon}\sum_{r=1}^{q}\beta_r(\vartheta,\Phi^\varepsilon)dw_r(t),$$

$$d\vartheta = dt,$$

$$d|X^\varepsilon|^p = \sqrt{\varepsilon}p\sum_{r=1}^{q}\alpha_r(\vartheta,\Phi^\varepsilon)|X^\varepsilon|^p dw_r(t) \tag{3.6}$$

$$+\varepsilon p[\Lambda^\top(\Phi^\varepsilon)A(\vartheta)\Lambda(\Phi^\varepsilon) - \frac{1}{2}\sum_{r=1}^{q}\frac{\partial\alpha_r}{\partial\varphi}(\vartheta,\Phi^\varepsilon)\beta_r(\vartheta,\Phi^\varepsilon) + \frac{1}{2}p\sum_{r=1}^{q}\alpha_r^2(\vartheta,\Phi^\varepsilon)]|X^\varepsilon|^p dt.$$

The corresponding semigroup is defined by the formula

$$T_t^\varepsilon(p)f(\theta,\varphi) = Ef(\vartheta_\theta(t),\Phi^\varepsilon_{\theta,\varphi}(t))|X^\varepsilon_{\theta,\lambda}(t)|^p,\ (\theta,\varphi)\in\mathbf{D},\ f\in\mathbf{C}(\mathbf{D}), \tag{3.7}$$

where $\mathbf{D}$ is the torus $\{(\theta,\varphi):\ 0\le\theta<l,\ 0\le\varphi<\pi\}$ and $\lambda = \begin{bmatrix}\cos\varphi \\ \sin\varphi\end{bmatrix}$.

The infinitesimal generator $L^\varepsilon(p)$ of the semigroup is given by

$$L^\varepsilon(p) = L_1 + \varepsilon L_2(p), \tag{3.8}$$

where

$$L_1 = -\omega\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta}, \tag{3.9}$$

$$L_2(p) = \frac{1}{2}\sum_{r=1}^{q}\beta_r^2\frac{\partial^2}{\partial\varphi^2} + (-\bar{\Lambda}^\top A\Lambda + \frac{1}{2}\sum_{r=1}^{q}\beta_r\frac{\partial\beta_r}{\partial\varphi} - p\sum_{r=1}^{q}\alpha_r\beta_r)\frac{\partial}{\partial\varphi} \tag{3.10}$$

$$+p(\Lambda^\top A\Lambda - \frac{1}{2}\sum_{r=1}^{q}\beta_r\frac{\partial\alpha_r}{\partial\varphi}) + \frac{1}{2}p^2\sum_{r=1}^{q}\alpha_r^2.$$

The eigenvalue problem (2.10) takes the form

$$L^\varepsilon(p)f^\varepsilon(\theta,\varphi;p) = g^\varepsilon(p)f^\varepsilon(\theta,\varphi;p),\ (\theta,\varphi)\in\mathbf{D}, \tag{3.11}$$

$$f^\varepsilon(0,\varphi;p) = f^\varepsilon(l,\varphi;p),\ \frac{\partial f^\varepsilon(0,\varphi;p)}{\partial\theta} = \frac{\partial f^\varepsilon(l,\varphi;p)}{\partial\theta},$$

$$f^\varepsilon(\theta,0;p) = f^\varepsilon(\theta,\pi;p),\ \frac{\partial f^\varepsilon(\theta,0;p)}{\partial\varphi} = \frac{\partial f^\varepsilon(\theta,\pi;p)}{\partial\varphi},$$

$$f^\varepsilon(0,0;p) = f^\varepsilon(l,0;p) = f^\varepsilon(0,\pi;p) = f^\varepsilon(l,\pi;p) = 1,\ f^\varepsilon(\theta,\varphi;p) > 0.$$

In this section and in Sections 4-6 we shall work under the following incommensurability condition:

$$\frac{\pi}{\omega}\text{ and } l \text{ are incommensurable, i.e. } \frac{\omega l}{\pi} \text{ is irrational.} \tag{3.12}$$

This condition ensures the ergodicity of the process $(\vartheta, \Phi^\varepsilon)$ even if $\varepsilon = 0$.

For two-dimensional systems ($d = 2$) the nondegeneracy condition (2.7) means that for any (θ, λ), θ outside the exceptional set $\mathbf{S}_0$, not all the vectors $a_1(\theta,\lambda), ..., a_q(\theta,\lambda)$ vanish. Since if $\lambda = \begin{bmatrix}\cos\varphi \\ \sin\varphi\end{bmatrix}$, $\beta_r(\theta,\varphi)$ and $a_r(\theta,\lambda)$, are simultaneously equal or nonequal to zero, condition (2.7) is equivalent to

$$\sum_{r=1}^{q}\beta_r^2(\theta,\varphi) \neq 0,\ \text{if } \theta\notin\mathbf{S}_0, \tag{3.13}$$

with a set $\mathbf{S}_0$ of Lebesgue measure zero.

Let us consider the weaker condition

$$\sum_{r=1}^{q}\beta_r^2(\theta,\varphi) \neq 0,\ \text{if } \theta\notin\mathbf{S}_0 \text{ or } \varphi\notin\mathbf{F}_0, \tag{3.14}$$

with a finite set $\mathbf{F}_0 = \{\varphi_1, ..., \varphi_k\} \subset [0,\pi)$. Condition (3.14) allows the vector fields in (2.7) to vanish for finitely many values of φ.

Let us sketch a proof of the fact that condition (3.14) implies hypothesis (2.9). For convenience of notation, let us omit the parameter ε.

Let $P(t,(\theta,\varphi),(d\tilde\theta,d\tilde\varphi))$ be the transition probability function of the Markov process (ϑ,Φ) generated by system (3.5). Since by assumption the diffusion in the first equation of (3.5) vanishes only for finitely many, hence discrete values of φ, or during a time set of measure zero, the transition probability is positive if $t+\theta\in d\tilde\theta$. Moreover, regularity of the span of the vector fields in φ implies that it has the form

$$P(t,(\theta,\varphi),(d\tilde\theta,d\tilde\varphi)) = \delta(t+\theta,d\tilde\theta)p(t,(\theta,\varphi),\tilde\varphi)d\tilde\varphi,$$

where p is a density in $\tilde\varphi$, which is continuous and positive for all $\tilde\varphi$ if $t>0$, and

$$\delta(t+\theta,d\tilde\theta) = \begin{cases} 1, & t+\theta\in d\tilde\theta, \\ 0, & t+\theta\notin d\tilde\theta. \end{cases}$$

For the semigroup we have

$$\begin{aligned} T_t(p)f(\theta,\varphi) &= E[f(\vartheta_\theta(t),\Phi_{\theta,\varphi}(t))E(|X_{\theta,\lambda}(t)|^p/\Phi_{\theta,\varphi}(t))] \\ &= E[f(\vartheta_\theta(t),\Phi_{\theta,\varphi}(t))g(t,\theta,\varphi,\Phi_{\theta,\varphi}(t))] \\ &= \int_0^\pi f(t+\theta,\tilde\varphi)g(t,\theta,\varphi,\tilde\varphi)p(t,(\theta,\varphi),\tilde\varphi)d\tilde\varphi. \end{aligned} \tag{3.15}$$

In (3.15)

$$g(t,\theta,\varphi,\Phi_{\theta,\varphi}(t)) := E(|X_{\theta,\lambda}(t)|^p/\Phi_{\theta,\varphi}(t)).$$

Due to (2.8), we obtain the representation

$$R(\mu,L(p))f(\theta,\varphi) = \int_0^\infty\int_0^\pi e^{-\mu t}f(t+\theta,\tilde\varphi)g(t,\theta,\varphi,\tilde\varphi)p(t,(\theta,\varphi),\tilde\varphi)d\tilde\varphi dt. \tag{3.16}$$

From this representation it follows that the resolvent $R(\mu,L(p))$ is strongly positive and consequently irreducible. Therefore (see [14], [11]) the semigroup $T_t(p)$ is irreducible as well. Moreover, it is not difficult to prove directly that for sufficiently large $\mu>0$ the representation (3.16) implies the compactness of the operator $R(\mu,L(p))$. Due to Hilbert's resolvent equality the resolvent $R(\mu,L(p))$ is compact for any $\mu\in\rho(L(p))$, where $\rho(L(p))$ is the resolvent set of $L(p)$. So, we have justified that condition (3.14) implies hypothesis (2.9). In theorems dealing with the two-dimensional system (3.1) condition (3.14) will therefore be used instead of (2.9).

4. The procedure of asymptotic expansion

Recalling the eigenvalue problem (3.11), let us suppose that $g^\varepsilon(p)$ and $f^\varepsilon(\theta,\varphi;p)$ allow asymptotic expansions

$$g^\varepsilon(p) = g_0(p)+\varepsilon g_1(p)+\ldots+\varepsilon^n g_n(p)+\ldots$$

$$f^\varepsilon(\theta,\varphi;p) = f_0(\theta,\varphi;p)+\varepsilon f_1(\theta,\varphi;p)+\ldots+\varepsilon^n f_n(\theta,\varphi;p)+\ldots$$

$$f_0(0,0;p)=f_0(l,0;p)=f_0(0,\pi;p)=f_0(l,\pi;p)=1,\ f_0(\theta,\varphi;p)>0,$$

$$f_n(0,0;p)=f_n(l,0;p)=f_n(0,\pi;p)=f_n(l,\pi;p)=0,\ n=1,2,\ldots\ .$$

Inserting these expressions into (3.11) and equating coefficients of equal power of ε leads to the following set of equations in which we write L_2 instead of $L_2(p)$ (see [19])

$$L_1 f_0 = g_0 f_0, \tag{4.1}$$
$$L_1 f_1 + L_2 f_0 = g_0 f_1 + g_1 f_0,$$
$$L_1 f_2 + L_2 f_1 = g_0 f_2 + g_1 f_1 + g_2 f_0,$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$L_1 f_n + L_2 f_{n-1} = g_0 f_n + g_1 f_{n-1} + \dots + g_n f_0.$$

Each function f_j is periodic in θ and φ. Consider the first equation of (4.1):

$$-\omega \frac{\partial f_0}{\partial \varphi} + \frac{\partial f_0}{\partial \theta} = g_0 f_0. \tag{4.2}$$

Integrating (4.2) in θ from 0 to l and in φ from 0 to π, and using periodicity and positivity of f_0, we find that $g_0(p) = 0$. Therefore $f_0(\theta, \varphi; p) = F(\omega\theta + \varphi; p)$, $(\theta, \varphi) \in \mathbf{D}$,where F is some function of one variable. Again by periodicity, and since the condition of incommensurability (3.12) is fulfilled, we conclude that $f_0(\theta, \varphi; p) = 1$. Consequently the second equation of (4.1) has the form:

$$L_1 f_1 + c = g_1, \tag{4.3}$$

where c denotes the function periodic in θ and φ given by

$$c(\theta, \varphi; p) = L_2 f_0 = p(\Lambda^\top A \Lambda - \frac{1}{2} \sum_{r=1}^{q} \beta_r \frac{\partial \alpha_r}{\partial \varphi}) + \frac{1}{2} p^2 \sum_{r=1}^{q} \alpha_r^2.$$

Due to periodicity of f_1 in θ and φ, we get

$$\int_0^l \int_0^\pi L_1 f_1 d\varphi d\theta = \int_0^l \int_0^\pi (-\omega \frac{\partial f_1}{\partial \varphi} + \frac{\partial f_1}{\partial \theta}) d\varphi d\theta = 0.$$

Hence

$$g_1(p) = \frac{1}{l\pi} \int_0^l \int_0^\pi c(\theta, \varphi; p) d\varphi d\theta. \tag{4.4}$$

The function $c(\theta, \varphi; p)$ can be written as a partial sum of a Fourier series in φ with coefficients depending on θ

$$c(\theta, \varphi; p) = \frac{1}{2} a_1^0(\theta; p) + a_1^1(\theta; p) \cos 2\varphi + b_1^1(\theta; p) \sin 2\varphi$$
$$+ a_1^2(\theta; p) \cos 4\varphi + b_1^2(\theta; p) \sin 4\varphi.$$

Clearly

$$\int_0^l (g_1(p) - \frac{1}{2} a_1^0(\theta; p)) d\theta = 0. \tag{4.5}$$

The function $f_1(\theta, \varphi; p)$ has to be determined as periodic in θ and φ with the boundary condition $f_1(0, 0; p) = 0$, and also of the form of a partial sum of a Fourier series in φ

$$f_1(\theta, \varphi; p) = \frac{1}{2}A_1^0(\theta; p) + A_1^1(\theta; p)\cos 2\varphi + B_1^1(\theta; p)\sin 2\varphi + A_1^2(\theta; p)\cos 4\varphi + B_1^2(\theta; p)\sin 4\varphi. \tag{4.6}$$

Substituting (4.6) in (4.3), we obtain the system of linear differential equations with constant parameters with respect to the unknown coefficients A_1^0, A_1^1, B_1^1, A_1^2, B_1^2, the dot denoting differentiation with respect to θ:

$$\frac{1}{2}\dot{A}_1^0 = g_1 - \frac{1}{2}a_1^0, \tag{4.7}$$

$$\begin{aligned} \dot{A}_1^1 - 2\omega B_1^1 &= -a_1^1 \\ \dot{B}_1^1 + 2\omega A_1^1 &= -b_1^1, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \dot{A}_1^2 - 4\omega B_1^2 &= -a_1^2 \\ \dot{B}_1^2 + 4\omega A_1^2 &= -b_1^2. \end{aligned} \tag{4.9}$$

This system consists of three subsystems (4.7), (4.8), and (4.9). Our aim is to find l-periodic solutions for these systems. Due to (4.5) the periodic solution of (4.7) has the form

$$A_1^0(\theta; p) = A_1^0(0; p) + \int_0^\theta (2g_1(p) - a_1^0(\vartheta; p))d\vartheta, \tag{4.10}$$

where the constant $A_1^0(0; p)$ will be determined below.

All the solutions of the homogeneous part of (4.8) are π/ω-periodic. But due to the condition of incommensurability, this part has no l-periodic solution. The same statement holds for the homogeneous part of (4.9) with ω replaced by 2ω. Since the right hand sides of the systems are given by l-periodic functions, both (4.8) and (4.9) have unique l-periodic solutions. We do not need the explicit description of these solutions, which are obtained via integrations. Let us finally determine $A_1^0(0; p)$. Using the condition $f_1(0, 0; p) = 0$ and (4.6), we get

$$\frac{1}{2}A_1^0(0; p) + A_1^1(0; p) + A_1^2(0; p) = 0. \tag{4.11}$$

Because $A_1^1(0; p)$ and $A_1^2(0; p)$ are already known, we can solve for $A_1^0(0; p)$. Therefore both $g_1(p)$ and $f_1(\theta, \varphi; p)$ are determined.

In the next step we find g_2 and f_2 from the equations

$$L_1 f_2 = g_2 + g_1 f_1 - L_2 f_1, \;\; f_2(0, 0; p) = 0. \tag{4.12}$$

Again periodicity in θ and φ lead to

$$g_2(p) = \frac{1}{l\pi}\int_0^l \int_0^\pi (L_2 f_1 - g_1 f_1)d\varphi d\theta. \tag{4.13}$$

From the preceding step and the definition of L_2 it is easy to see that the function $g_1 f_1 - L_2 f_1$ can be represented in the form

$$g_1 f_1 - L_2 f_1 = \frac{1}{2} a_2^0(\theta; p) + \sum_{k=1}^{4} (a_2^k(\theta; p) \cos 2k\varphi + b_2^k(\theta; p) \sin 2k\varphi) \tag{4.14}$$

with known coefficients $a_2^0, \ a_2^k, \ b_2^k, \ k = 1, ..., 4$.

An analogous ansatz as before leads us to

$$f_2(\theta, \varphi; p) = \frac{1}{2} A_2^0(\theta; p) + \sum_{k=1}^{4} (A_2^k(\theta; p) \cos 2k\varphi + B_2^k(\theta; p) \sin 2k\varphi) \tag{4.15}$$

and the subsequent calculation of the coefficients $A_2^0, \ A_2^k, \ B_2^k, \ 1 \le k \le 4$. Using the additional boundary condition caused by periodicity

$$\frac{1}{2} A_2^0(0; p) + \sum_{k=1}^{4} A_2^k(0; p) = 0,$$

the function f_2 is uniquely determined.

The recursive step of order n starts with the averaging condition which leads to

$$g_n(p) = \frac{1}{l\pi} \int_0^l \int_0^\pi (L_2 f_{n-1} - g_1 f_{n-1} - ... - g_{n-1} f_1) d\varphi d\theta, \tag{4.16}$$

continues with the observation that $L_2 f_{n-1} - g_1 f_{n-1} - ... - g_{n-1} f_1$ possesses a representation as a trigonometrical polynomial in φ of order $4n$ and consequently motivates the ansatz

$$f_n(\theta, \varphi; p) = \frac{1}{2} A_n^0(\theta; p) + \sum_{k=1}^{2n} (A_n^k(\theta; p) \cos 2k\varphi + B_n^k(\theta; p) \sin 2k\varphi). \tag{4.17}$$

Starting with (4.17) we determine $A_n^0(\theta; p)$ from a one-dimensional linear differential equation, and every pair $A_n^k(\theta; p), \ B_n^k(\theta; p), \quad k = 1, ..., 2n$, via integration from a two-dimensional system of linear differential equations. Analogously to (4.11) the following condition has to be used

$$\frac{1}{2} A_n^0(0; p) + \sum_{k=1}^{2n} (A_n^k(0; p) = 0. \tag{4.18}$$

Thus, the procedure (4.1) is justified and the following theorem holds true.

Theorem 4.1. *Let the condition of incommensurability* (3.12) *be fulfilled. Then the procedure* (4.1) *can be realized in the form* (4.16)-(4.17). *We have* $g_0 = 0, \ f_0 = 1$. *The coefficients* $A_n^0(\theta; p), \ A_n^k(\theta; p), \ B_n^k(\theta; p), \ k = 1, ..., 2n, \ n \in \mathbf{N}$, *are uniquely determined, and can be explicitly given by integrations.*

5. Theorem on asymptotic expansion of moment Lyapunov exponent

Theorem 5.1. *Assume* (3.14) *and the condition of incommensurability* (3.12) *are fulfilled. Let* $g_1(p),\ \dots\ ,\ g_n(p)$ *and* $f_0(\theta,\varphi;p),\ f_1(\theta,\varphi;p),\ \dots,\ f_n(\theta,\varphi;p)$ *be the functions obtained from the algorithm* (4.16)-(4.17) *described in Theorem 4.1. Then for any* $n \in \mathbf{N}$ *we have*

$$g^{\varepsilon}(p) = \varepsilon g_1(p) + \dots + \varepsilon^n g_n(p) + O(\varepsilon^{n+1})\ (\varepsilon \to 0), \tag{5.1}$$

where $O(\varepsilon^{n+1})$ *is bounded uniformly in* p *restricted to compacts of* $\mathbf{R}$.

Proof. Fix $n \in \mathbf{N}$ and introduce

$$g_n^{\varepsilon}(p) = g_0(p) + \varepsilon g_1(p) + \dots + \varepsilon^n g_n(p), \tag{5.2}$$

$$f_n^{\varepsilon}(\theta,\varphi;p) = f_0(\theta,\varphi;p) + \varepsilon f_1(\theta,\varphi;p) + \dots + \varepsilon^n f_n(\theta,\varphi;p). \tag{5.3}$$

The algorithm of Theorem 4.1 immediately gives

$$L^{\varepsilon}(p) f_n^{\varepsilon} = g_n^{\varepsilon}(p) f_n^{\varepsilon} + O(\varepsilon^{n+1}), \tag{5.4}$$

where $O(\varepsilon^{n+1})$ is uniformly bounded in p restricted to compacts in $\mathbf{R}$.

By means of Ito's formula, (5.4) guarantees that

$$\begin{aligned} d f_n^{\varepsilon}(\vartheta_\theta(t), \Phi_{\theta,\varphi}(t);p)|X^{\varepsilon}_{\theta,\lambda}(t)|^p &= g_n^{\varepsilon}(p) f_n^{\varepsilon}(\vartheta_\theta(t), \Phi_{\theta,\varphi}(t);p)|X^{\varepsilon}_{\theta,\lambda}(t)|^p dt \\ &\quad + O(\varepsilon^{n+1})|X^{\varepsilon}_{\theta,\lambda}(t)|^p dt + dM_t, \end{aligned} \tag{5.5}$$

where M_t denotes a martingale part which is not explicited further, and $O(\varepsilon^{n+1})$ is again uniformly bounded in p restricted to compacts in $\mathbf{R}$.

Omitting the subscripts θ, φ, λ, we conclude, taking expectations

$$\frac{d}{dt} E f_n^{\varepsilon} |X^{\varepsilon}|^p = g_n^{\varepsilon}(p) E f_n^{\varepsilon} |X^{\varepsilon}|^p + E[O(\varepsilon^{n+1})|X^{\varepsilon}|^p]. \tag{5.6}$$

As $f_0 = 1$, (5.3) implies that for any compact $\mathbf{K} \subset \mathbf{R}$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$

$$\frac{1}{2} \le f_n^{\varepsilon}(\theta,\varphi;p) \le \frac{3}{2},\ (\theta,\varphi) \in \mathbf{D}, p \in \mathbf{K}. \tag{5.7}$$

It is not difficult to deduce from (5.6) and (5.7) that there exist positive constants K and C such that the following inequalities true for $(\theta,\varphi) \in \mathbf{D}$, $p \in \mathbf{K}$

$$(g_n^{\varepsilon}(p) - K\varepsilon^{n+1}) E f_n^{\varepsilon} |X^{\varepsilon}|^p \le \frac{d}{dt} E f_n^{\varepsilon} |X^{\varepsilon}|^p \le (g_n^{\varepsilon}(p) + K\varepsilon^{n+1}) E f_n^{\varepsilon} |X^{\varepsilon}|^p, \tag{5.8}$$

$$C \exp(g_n^{\varepsilon}(p) - K\varepsilon^{n+1})t \le E f_n^{\varepsilon} |X^{\varepsilon}|^p \le C \exp(g_n^{\varepsilon}(p) + K\varepsilon^{n+1})t. \tag{5.9}$$

Taking the logarithmic time average of (5.9) yields

$$g_n^{\varepsilon}(p) - K\varepsilon^{n+1} \le \frac{\ln E f_n^{\varepsilon} |X^{\varepsilon}|^p}{t} - \frac{\ln C}{t} \le g_n^{\varepsilon}(p) + K\varepsilon^{n+1}. \tag{5.10}$$

Due to (5.7), the following limits exist

$$\lim_{t\to\infty}\frac{\ln Ef_n^\varepsilon|X^\varepsilon|^p}{t}=\lim_{t\to\infty}\frac{\ln E|X^\varepsilon|^p}{t}=g^\varepsilon(p), \tag{5.11}$$

and therefore (5.10) implies

$$g_n^\varepsilon(p)-K\varepsilon^{n+1}\le g^\varepsilon(p)\le g_n^\varepsilon(p)+K\varepsilon^{n+1}.$$

This completes the proof of Theorem 5.1.

6. Asymptotic expansion of the stability index

The stability index γ^ε is defined as the nonzero root of the equation

$$g^\varepsilon(\gamma^\varepsilon)=0. \tag{6.1}$$

An asymptotic expansion for the stability index

$$\gamma^\varepsilon=\gamma_0+\gamma_1\varepsilon+...+\gamma_n\varepsilon^n+... \tag{6.2}$$

is obtained without any essential modification of the arguments given in [19].

Let us insert the expansion into the formal equation

$$g_1(\gamma^\varepsilon)+\varepsilon g_2(\gamma^\varepsilon)+...+\varepsilon^{n-1}g_n(\gamma^\varepsilon)+...=0.$$

Equating coefficients of equal power of ε leads to a hierarchy of equations the first four of which are given by

$$\begin{gathered}g_1(\gamma_0)=0,\\ \dot g_1(\gamma_0)\gamma_1+g_2(\gamma_0)=0,\\ \dot g_1(\gamma_0)\gamma_2+\frac{1}{2}\ddot g_1(\gamma_0)\gamma_1^2+\dot g_2(\gamma_0)\gamma_1+g_3(\gamma_0)=0,\\ \dot g_1(\gamma_0)\gamma_3+\frac{1}{6}\dddot g_1(\gamma_0)\gamma_1^3+\ddot g_1(\gamma_0)\gamma_1\gamma_2+\frac{1}{2}\ddot g_2(\gamma_0)\gamma_1^2+\dot g_2(\gamma_0)\gamma_2+\dot g_3(\gamma_0)\gamma_1+g_4(\gamma_0)=0.\end{gathered} \tag{6.3}$$

Here the dot denotes differentiation with respect to p. In the algorithm of Theorem 4.1, every function $g_k(p)$ is a polynomial in p. If the first equation in (6.3) possesses a root $\gamma_0\neq 0$ and $\dot g_1(\gamma_0)\neq 0$, then the system (6.3) can be solved.

The function $g_1(p)$ is given by formula (4.4), which states that $g_1(p)$ is a quadratic function of the form $g_1(p)=c_0p^2+c_1p$, where

$$c_0=\frac{1}{2l\pi}\int_0^l\int_0^\pi\sum_{r=1}^q\alpha_r^2d\varphi d\theta,\ c_1=\frac{1}{l\pi}\int_0^l\int_0^\pi\sum_{r=1}^q(\Lambda^\top A\Lambda-\frac{1}{2}\sum_{r=1}^q\beta_r\frac{\partial\alpha_r}{\partial\varphi})d\varphi d\theta. \tag{6.4}$$

If $c_0\neq 0,\ c_1\neq 0$, then

$$\gamma_0=-\frac{c_1}{c_0}\neq 0,\ \dot g_1(\gamma_0)=2c_0\gamma_0+c_1=-c_1\neq 0.$$

To be more precise, assume that $-c_1 > 0$. Consider the interval $\mathbf{I} = [-\frac{3c_1}{4c_0}, -\frac{5c_1}{4c_0}]$. The function $g_1(p)$ takes the values $-\frac{3c_1^2}{16c_0}$ and $\frac{5c_1^2}{16c_0}$ at the boundaries of this interval. On $\mathbf{I}$ the function $g^\varepsilon(p)$ has form $g^\varepsilon(p) = \varepsilon g_1(p) + O(\varepsilon^2)$, where $|O(\varepsilon^2)| \le K\varepsilon^2$ and K is independent of $p \in \mathbf{I}$. Therefore, just as $g_1(p)$, the function $g^\varepsilon(p)$ for sufficiently small ε has different signs at the boundaries of $\mathbf{I}$. Since $g^\varepsilon(p)$ is convex, its unique root on $\mathbf{I}$ is given by γ^ε on this interval. Furthermore, we may write with some $\xi \in \mathbf{I}$

$$0 = g^\varepsilon(\gamma^\varepsilon) = \varepsilon g_1(\gamma^\varepsilon) + O(\varepsilon^2) = \varepsilon g_1(\gamma_0) + \varepsilon \dot{g}_1(\xi)(\gamma^\varepsilon - \gamma_0) + O(\varepsilon^2).$$

Since $g_1(\gamma_0) = 0$ and $\dot{g}_1 \ge -c_1/2$ on $\mathbf{I}$, we get the asymptotic relation

$$\gamma^\varepsilon - \gamma_0 = O(\varepsilon).$$

Now consider the function

$$g_n^\varepsilon(p) := \varepsilon g_1(p) + ... + \varepsilon^n g_n(p).$$

For sufficiently small ε this function has different signs at the boundaries of $\mathbf{I}$ and its first derivative on $\mathbf{I}$ is bounded below by $-c_1\varepsilon/4$. Hence similar arguments as above yield the following estimate for the unique root γ_n^ε of the function $g_n^\varepsilon(p)$:

$$\gamma^\varepsilon - \gamma_n^\varepsilon = O(\varepsilon^n).$$

On the other hand, it is not difficult to show that

$$\gamma_0 + \gamma_1\varepsilon + ... + \gamma_{n-1}\varepsilon^{n-1} - \gamma_n^\varepsilon = O(\varepsilon^n).$$

Consequently the following theorem is proved (see [19]).

Theorem 6.1. *Let the conditions of Theorem 5.1 be fulfilled. Assume* $c_0 \ne 0,\ c_1 \ne 0$. *Then the stability index of the system* (3.1) *has the following asymptotic expansion:*

$$\gamma^\varepsilon = \gamma_0 + \gamma_1\varepsilon + ... + \gamma_n\varepsilon^n + O(\varepsilon^{n+1}),$$

where $\gamma_0,\ \gamma_1, ..., \gamma_n$ *can be found recursively from* (6.3).

Remark 6.1 Let us add a few remarks concerning the case $c_0 = 0$. In this case

$$\int_0^l \int_0^\pi \sum_{r=1}^q \alpha_r^2 \, d\varphi d\theta = 0.$$

This implies that there are smooth functions $\nu_i, 1 \le i \le r$ on $[0, l)$ such that for almost all $\theta \in [0, l), 1 \le r \le q$ we have $A_r = \nu_r\, J$ and consequently

$$\alpha_r = 0, \quad \beta_r = \nu_r, \quad 1 \le r \le q.$$

If we denote with $a_{ij}, i, j = 1, 2$, the coefficients of the matrix A and $\nu^2 = \sum_{r=1}^q \nu_r^2$ we may describe the operator $L_2(p)$ by

$$L_2(p) = \frac{1}{2}\nu^2 \frac{\partial^2}{\partial\varphi^2} - \frac{1}{2}\left[a_{12} - a_{21} - (a_{12} + a_{21})\cos 2\varphi + (a_{11} - a_{22})\sin 2\varphi\right]\frac{\partial}{\partial\varphi}$$

$$+\frac{p}{2}\,[a_{11}+a_{22}+(a_{11}-a_{22})\cos 2\varphi+(a_{12}+a_{21})\sin 2\varphi].$$

Setting

$$c=\frac{p}{2}[a_{11}+a_{22}+(a_{11}-a_{22})\cos 2\varphi+(a_{12}+a_{21})\sin 2\varphi],$$

$r=|X^\varepsilon|$, and writing $\overline{f}=\frac{1}{l}\int_0^l f(\theta)d\theta$ for measurable functions f on $[0,l)$, we obtain

$$c_1=\frac{p}{2}[\overline{a_{11}}+\overline{a_{22}}],\qquad dr=\varepsilon\, c\, dt.$$

Now we have to distinguish three possibilities.

(i) In case $\overline{(a_{11}-a_{22})^2}+\overline{(a_{12}+a_{21})^2}=0$, the differential equation for r is deterministic and we have $g(p)=\varepsilon\,\frac{p}{2}\,(\overline{a_{11}}+\overline{a_{22}})$.

(ii) Next assume $\overline{(a_{11}-a_{22})^2}+\overline{(a_{12}+a_{21})^2}\neq 0$ and $\overline{a_{11}}+\overline{a_{22}}\neq 0$. Then $g_1(p)=\frac{p}{2}(\overline{a_{11}}+\overline{a_{22}})$, obviously. To compute $g_2(p)$, note that from the equations (4.6) - (4.9) we have

$$f_1(\theta,\varphi;p)=\frac{1}{2}A_1^0(\theta;p)+A_1^1(\theta;p)\cos 2\varphi+B_1^1(\theta;p)\,\sin 2\varphi$$

with

$$\frac{1}{2}\dot{A}_1^0=\frac{p}{2}\,[\overline{a_{11}}+\overline{a_{22}}-(a_{11}+a_{22})],$$

(6.5) $$\dot{A}_1^1-2aB_1^1=-\frac{p}{2}(a_{11}-a_{22}),$$

(6.6) $$\dot{B}_1^1+2aA_1^1=-\frac{p}{2}(a_{12}+a_{21}).$$

Multiplying (6.5) with A_1^1, (6.6) with B_1^1 and taking averages immediately yields

$$\overline{(a_{11}-a_{22})A_1^1}=-\overline{(a_{12}+a_{21})B_1^1},$$

whence we easily obtain from the formulas for $L_2(p)$ and f_1 the equation $g_2(p)=0$. To compute $g_3(p)$, note that by induction

$$f_2=\frac{1}{2}A_2^0+A_2^1\cos 2\varphi+B_2^1\sin 2\varphi+A_2^2\cos 4\varphi+B_2^2\sin 4\varphi.$$

Hence

$$\begin{aligned}g_3(p)&=\frac{1}{l}\frac{1}{\pi}\int_0^l\int_0^\pi(L_2(p)\,f_2-g_2\,f_2)d\varphi d\theta\\&=(\frac{1}{2}+\frac{p}{4})\overline{A_2^1(a_{11}-a_{22})}+(\frac{1}{2}+\frac{p}{4})\overline{B_2^1(a_{12}+a_{21})}\\&\quad+\frac{p}{4}[\overline{A_2^0(a_{11}+a_{22})}-\overline{A_2^0}\;\overline{(a_{11}+a_{22})}],\end{aligned}$$

with A_2^0, A_2^1, B_2^1 determined by

$$\frac{1}{2}\dot{A}_2^0=(\frac{1}{2}+\frac{p}{4})\,[A_1^1(a_{11}-a_{22})+B_1^1(a_{12}+a_{21})],$$

$$\dot{A}_2^1-2aB_2^1=-2\nu^2A_1^1-B_1^1(a_{12}-a_{21})+\frac{p}{2}A_1^1(a_{11}+a_{22})+\frac{p}{4}A_1^0(a_{11}-a_{22}),$$

$$\dot{B}_2^1+2aA_2^1=-2\nu^2B_1^1-A_1^1(a_{12}-a_{21})+\frac{p}{2}B_1^1(a_{11}+a_{22})+\frac{p}{4}A_1^0(a_{12}+a_{21}).$$

As in [19], the asymptotic expansion of the stability index therefore takes the form

$$\gamma^\varepsilon = \gamma_{-2}\varepsilon^{-2} + \gamma_{-1}\varepsilon^{-1} + \gamma_0 + \gamma_1\varepsilon + \dots$$

with coefficients recursively determined from the expansion of $g^\varepsilon(p)$.

(iii) Finally, in case $\overline{(a_{11} - a_{22})^2} + \overline{(a_{12} + a_{21})^2} \neq 0$ and $\overline{a_{11}} + \overline{a_{22}} = 0$ the analysis of (ii) gives in addition $g_1(p) = 0 = g_2(p)$.

7. The finite expansion of the moment Lyapunov exponent in the commensurable case

In Section 4, it was shown that the algorithm of Theorem 4.1 produces unique solutions. In the commensurable case the property of uniqueness is violated from the very beginning. Let us show this fact. Consider equation (4.2). Again $g_0 = 0$ since we look for periodic and positive f_0. The equation

$$-\omega \frac{\partial f_0}{\partial \varphi} + \frac{\partial f_0}{\partial \theta} = 0 \tag{7.1}$$

has besides $f_0 = 1$ many other solutions. Indeed, let $\omega l/\pi = m/k$ and F be an arbitrary differentiable π/k-periodic positive function. Then $f_0(\theta, \varphi) = F(\omega\theta + \varphi)$ is a solution of (7.1) which is positive, l-periodic in θ, and π-periodic in φ.

Among the solutions of (7.1) we choose $f_0 = 1$ in the following, as in the incommensurable case.

Let us analyze the existence of an expansion. Supposing that $g_0 = 0$, $f_0 = 1$, g_1, $f_1, \dots,$ g_{n-1}, f_{n-1} are found (maybe they are not unique), let us return to the n-th step of the algorithm of Theorem 4.1. The coefficient $g_n(p)$ is found via (4.16). The function $f_n(\theta, \varphi; p)$ is determined by A_n^0, A_n^k, B_n^k, $k = 1, \dots, 2n$. For A_n^0 we obtain a one-dimensional linear differential equation which does not cause any difficulties. For every pair A_n^k, B_n^k we obtain a two-dimensional system of linear differential equations of the form

$$\begin{aligned} \dot{A}_n^k - 2k\omega B_n^k &= -a_n^k \\ \dot{B}_n^k + 2k\omega A_n^k &= -b_n^k, \end{aligned} \tag{7.2}$$

where a_n^k, b_n^k are some l-periodic functions in θ which depend also on the parameter p. In general the functions a_n^k, b_n^k as l-periodic ones have frequencies $\omega_m = 2\pi m/l$, $m = 1, 2, \dots$. If all these frequencies are present, the necessary and sufficient condition for the existence of an l-periodic solution of (7.2) consists in

$$\frac{2\pi m}{l} \neq 2k\omega, \ m = 1, \ 2, \dots . \tag{7.3}$$

Moreover, under (7.3) system (7.2) has only one l-periodic solution.

So, if

$$\omega \neq \frac{\pi m}{kl}, \ k = 1, \dots, 2n, \ m = 1, \ 2, \dots, \tag{7.4}$$

then all the systems (7.2), $k = 1, \dots, 2n$, have a unique l-periodic solution.

Consider now the case when one of the frequencies $\omega_m = 2\pi m/l$ coincides with $2k\omega$ and the right-hand side of (7.2) possesses a harmonic of this frequency, then the system has no l-periodic solution. In this case the n-th step of the algorithm considered is impossible. However if the right-hand side of (7.2) does not possess any nontrivial harmonic of the frequency $2k\omega$, this system does have an l-periodic solution and even an infinite set such solutions. So, even if the condition (7.4) is violated, for a fixed n there is a possibility that each of the systems of (7.2), $k = 1, ..., 2n$, has an l-periodic solution, and the n-th step of the algorithm of Theorem 4.1 is feasible. In case of solvability we find $g_0 = 0,\ f_0 = 1,\ g_1(p),\ f_1(\theta, \varphi; p), ...,\ g_n(p),\ f_n(\theta, \varphi; p)$, maybe not uniquely. As earlier we can introduce $g_n^\varepsilon,\ f_n^\varepsilon$ in accordance with (5.2), (5.3) and obtain equation (5.4). All the arguments in the proof of Theorem 5.1 remain true in the considered situation as well, and consequently we obtain the finite expansion (5.1). As a result the following theorem is obtained.

Theorem 7.1. *Let $g_0 = 0,\ f_0 = 1$, suppose the algorithm of Theorem 4.1 is solvable to the $(n-1)$-st step, and gives $g_1(p),\ f_1(\theta, \varphi; p), ...,\ g_{n-1}(p),\ f_{n-1}(\theta, \varphi; p),\ g_n(p)$. Assume that each of the systems (7.2), $k = 1, ..., 2n$, possesses an l-periodic solution. Then the following finite expansion holds*

$$g^\varepsilon(p) = \varepsilon g_1(p) + ... + \varepsilon^n g_n(p) + O(\varepsilon^{n+1})\ (\varepsilon \to 0),$$

where $O(\varepsilon^{n+1})$ is bounded uniformly in p restricted to compacts of $\mathbf{R}$. *A sufficient condition of solvability for the systems (7.2) is given by (7.4). In particular, if*

$$\omega \neq \frac{\pi m}{2l},\ m = 1,\ 2, ..., \tag{7.5}$$

or if systems (4.8), (4.9) are solvable, then

$$g^\varepsilon(p) = \varepsilon g_1(p) + O(\varepsilon^2). \tag{7.6}$$

Remark 7.1. Clearly $g_1(p), ..., g_n(p)$ are obtained uniquely in spite of the fact that the functions $f_1, ..., f_n$ may not be unique.

Remark 7.2. Let $g_1(p) = c_0 p^2 + c_1 p$ where $c_0 \neq 0,\ c_1 \neq 0$. Assume the algorithm of Theorem 4.1 is solvable to the n-th step. Then the stability index of system (3.1) has the finite expansion

$$\gamma^\varepsilon = \gamma_0 + \gamma_1 \varepsilon + ... + \gamma_{n-1}\varepsilon^{n-1} + O(\varepsilon^n),$$

where $\gamma_0,\ \gamma_1, ..., \gamma_{n-1}$ can be found recursively from (6.3). In particular, if (7.5) is fulfilled or if system (4.8), (4.9) is solvable, then

$$\gamma^\varepsilon = \gamma_0 + O(\varepsilon) = -\frac{c_1}{c_0} + O(\varepsilon). \tag{7.7}$$

Remark 7.3. If in the commensurable case $2\pi m/l$ coincides with $2k\omega$, resonance occurs and the system (7.2) may possess no l-periodic solution. However in the incommensurable case, a similar situation arises. Indeed if $\omega l/\pi$ is close to some rational number m/k, we encounter difficulties which are close to resonance problems, because the periodic solution obtained has a very large amplitude.

Remark 7.4. If the systems (4.8), (4.9) are solvable, and g_1 and f_1 are found, then along with (7.6) it is possible to get a bound for ε such that the sign of $g^\varepsilon(p)$ is conserved. Knowing g_1 and f_1 actually provides

$$L^\varepsilon(p)f_1^\varepsilon(\theta,\varphi;p) = \varepsilon g_1 f_1^\varepsilon + \varepsilon^2(L_2 f_1 - g_1 f_1) = \varepsilon g_1(p) + \varepsilon^2 L_2 f_1(\theta,\varphi;p). \tag{7.8}$$

Let, for instance, $g_1(p) < 0$ for some fixed p. Clearly for all sufficiently small ε the system (3.1) is p-stable. Let us find an upper bound for ε such that the system remains p-stable. For p-stability it is sufficient that $f_1^\varepsilon > 0$ and $L^\varepsilon(p)f_1^\varepsilon < 0$, more precisely

$$1 + \varepsilon f_1(\theta,\varphi;p) > 0,\ g_1(p) + \varepsilon L_2 f_1(\theta,\varphi;p) < 0,\ \text{for all } (\theta,\varphi) \in \mathbf{D}. \tag{7.9}$$

Introduce $m(p) := \min_{\theta,\varphi} f_1(\theta,\varphi;p),\ M(p) := \max_{\theta,\varphi} L_2 f_1(\theta,\varphi;p)$. If $m(p) \geq 0$, then the first part of condition (7.9) is fulfilled for all ε; if $M(p) \leq 0$, then the second part of condition (7.9) is fulfilled for all ε. In these cases the system of two inequalities reduces to a single one. Consider the more complicated case $m(p) < 0$ and $M(p) > 0$. In this case for all ε satisfying the inequality

$$0 < \varepsilon < \min\{-\frac{1}{m(p)}, -\frac{g_1(p)}{M(p)}\}$$

p-stability of (3.1) is ensured.

8. The Hill and Mathieu oscillators with small damping and noise

Let us start by giving a more constructive general formula for $g_1(p)$. Using the expressions for α_r and β_r from Section 3, and denoting the coefficients of A_r by a_r^{ij}, $r = 1, ..., q$, $i, j = 1,\ 2$, one can use (4.4) to evaluate

$$g_1(p) = c_0 p^2 + c_1 p = \frac{1}{8l}p^2 \sum_{r=1}^{q} \int_0^l (\frac{3}{2}(a_r^{11})^2 + \frac{3}{2}(a_r^{22})^2 + a_r^{11}a_r^{22} + \frac{1}{2}(a_r^{12} + a_r^{21})^2)d\theta \tag{8.1}$$

$$+\frac{1}{2l}p\int_0^l (a_{11} + a_{22})d\theta + \frac{1}{8l}p\sum_{r=1}^{q}\int_0^l ((a_r^{12} + a_r^{21})^2 + (a_r^{11} - a_r^{22})^2)d\theta.$$

So, if system (4.8), (4.9) is solvable, then due to Theorem 7.1 $g^\varepsilon(p) = \varepsilon g_1(p) + O(\varepsilon^2)$.

Let us consider the Stratonovich form of the Hill equation with small friction and small noise

$$\ddot{X} = -\omega^2(1 + \varepsilon a(t))X - \varepsilon\alpha\dot{X} + \sqrt{\varepsilon}\beta X \circ \dot{w}_1 + \sqrt{\varepsilon}\gamma\dot{X} \circ \dot{w}_2, \tag{8.2}$$

where $a,\ \alpha,\ \beta,\ \gamma$ are 2π-periodic functions in t, i.e., $l = 2\pi$. If $a(t) = \cos t$ the Hill equation is known as the Mathieu equation. Introduce $X_1 = \dfrac{1}{\omega}X,\ X_2 = \dfrac{1}{\omega^2}\dot{X}$. Then equation (8.2) takes the form of system (3.1):

$$dX_1 = \omega X_2 dt, \tag{8.3}$$

$$dX_2 = -\omega(1 + \varepsilon a(t))X_1 dt - \varepsilon\alpha X_2 dt + \sqrt{\varepsilon}\frac{\beta}{\omega}X_1 \circ dw_1 + \sqrt{\varepsilon}\gamma X_2 \circ dw_2.$$

Our next aim is to get systems (4.7)-(4.9) for (8.3). For the convenience of the reader we give some formulae of routine calculations which are necessary for determining f_1 and $L_2 f_1$.

We have $\Lambda = [\cos\varphi \ \ \sin\varphi]^\top$, $\bar{\Lambda} = [\sin\varphi \ \ -\cos\varphi]^\top$,

$$A(\theta) = \begin{bmatrix} 0 & 0 \\ -\omega a(\theta) & -\alpha \end{bmatrix},\ A_1(\theta) = \begin{bmatrix} 0 & 0 \\ \beta/\omega & 0 \end{bmatrix},\ A_2(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix},$$

$$\alpha_0 := \Lambda^\top A \Lambda = -\frac{1}{2}\omega a(\theta)\sin 2\varphi - \alpha\sin^2\varphi,\ \beta_0 := \bar{\Lambda}^\top A\Lambda = \omega a(\theta)\cos^2\varphi + \frac{\alpha}{2}\sin 2\varphi,$$

$$\alpha_1 = \Lambda^\top A_1\Lambda = \frac{\beta}{2\omega}\sin 2\varphi,\ \beta_1 = \bar{\Lambda}^\top A_1\Lambda = -\frac{\beta}{\omega}\cos^2\varphi,$$

$$\alpha_2 = \Lambda^\top A_2\Lambda = \gamma\sin^2\varphi,\ \beta_2 = \bar{\Lambda}^\top A_2\Lambda = -\frac{\gamma}{2}\sin 2\varphi.$$

Further,

$$\sum_{r=1}^{2}\alpha_r^2 = \frac{\beta^2}{4\omega^2}\sin^2 2\varphi + \gamma^2\sin^4\varphi,\ \sum_{r=1}^{2}\alpha_r\beta_r = -\frac{\beta^2}{2\omega^2}\sin 2\varphi\cos^2\varphi - \frac{\gamma^2}{2}\sin^2\varphi\sin 2\varphi,$$

$$\sum_{r=1}^{2}\beta_r^2 = \frac{\beta^2}{\omega^2}\cos^4\varphi + \frac{\gamma^2}{4}\sin^2 2\varphi,\ \sum_{r=1}^{2}\beta_r\frac{\partial\alpha_r}{\partial\varphi} = -\frac{\beta^2}{\omega^2}\cos^2\varphi\cos 2\varphi - \frac{\gamma^2}{2}\sin^2 2\varphi,$$

$$\sum_{r=1}^{2}\beta_r\frac{\partial\beta_r}{\partial\varphi} = -\frac{\beta^2}{\omega^2}\cos^2\varphi\sin 2\varphi + \frac{\gamma^2}{4}\sin 4\varphi.$$

Therefore

$$c(\theta,\varphi;p) = p(\alpha_0 - \frac{1}{2}\sum_{r=1}^{2}\beta_r\frac{\partial\alpha_r}{\partial\varphi}) + \frac{1}{2}p^2\sum_{r=1}^{2}\alpha_r^2 \tag{8.4}$$

$$= \frac{1}{2}a_1^0(\theta;p) + a_1^1(\theta;p)\cos 2\varphi + b_1^1(\theta;p)\sin 2\varphi + a_1^2(\theta;p)\cos 4\varphi + b_1^2(\theta;p)\sin 4\varphi,$$

where

$$\frac{1}{2}a_1^0 = p(-\frac{\alpha}{2} + \frac{\beta^2}{8\omega^2} + \frac{\gamma^2}{8}) + \frac{p^2}{16}(\frac{\beta^2}{\omega^2} + 3\gamma^2),\ a_1^1 = p(\frac{\alpha}{2} + \frac{\beta^2}{4\omega^2}) - \frac{p^2\gamma^2}{4}, \tag{8.5}$$

$$b_1^1 = -\frac{1}{2}p\omega a(\theta),\ a_1^2 = p(\frac{\beta^2}{8\omega^2} - \frac{\gamma^2}{8}) - \frac{p^2}{16}(\frac{\beta^2}{\omega^2} - \gamma^2),\ b_1^2 = 0.$$

So,

$$c = \frac{1}{2}a_1^0 + a_1^1\cos 2\varphi - \frac{1}{2}p\omega a\sin 2\varphi + a_1^2\cos 4\varphi, \tag{8.6}$$

where $a_1^0,\ a_1^1,\ a_1^2$ are given by (8.5).

Now systems (4.8)-(4.9) acquire the form

$$\dot{A}_1^1 - 2\omega B_1^1 = -a_1^1,\ \dot{B}_1^1 + 2\omega A_1^1 = \frac{1}{2}p\omega a, \tag{8.7}$$

(8.8) $$\dot{A}_1^2 - 4\omega B_1^2 = -a_1^2,\ \dot{B}_1^2 + 4\omega A_1^2 = 0.$$

Clearly, if $4\omega \neq 1,\ 2, ...$, the systems (8.7), (8.8) possess 2π-periodic solutions and consequently (see Theorem 7.1 and condition (7.5) with $l = 2\pi$) formula (7.6) holds. Consider the case of constant β and γ, i.e. the case of constant intensities for random excitations in the Hill equation. Then a_1^2 is a constant, system (8.8) has a 2π-periodic solution, and the condition $2\omega \neq 1,\ 2, ...$ ensures (7.6). So, we get

Proposition 8.1. *For the Hill equation with general random excitations, if*

$$4\omega \neq 1,\ 2, ...,$$

then

(8.9) $$g^\varepsilon(p) = \varepsilon g_1(p) + O(\varepsilon^2).$$

For the Hill equation with random excitations of constant intensities, formula (8.9) *holds if*

$$2\omega \neq 1,\ 2, ...$$

Consider now the Mathieu equation with random excitations of constant intensities and with constant friction, i.e. with constant α, β, and γ. Then in (8.7)-(8.8) a_1^1 and a_1^2 are constants and we recall that $a(\theta) = \cos\theta$. Clearly the following statement is true.

Proposition 8.2. *For the Mathieu equation with random excitations of constant intensities and with constant damping, the condition*

$$\omega \neq \frac{1}{2}$$

implies (8.9) *with*

$$g_1(p) = p(-\frac{\alpha}{2} + \frac{\beta^2}{8\omega^2} + \frac{\gamma^2}{8}) + \frac{p^2}{16}(\frac{\beta^2}{\omega^2} + 3\gamma^2).$$

Let us note that the value $\omega = 1/2$ corresponds to the strongest parametric resonance (see, e.g., [7]).

For Mathieu's equation with constant α, β, γ, and $\omega \neq 1/2$, it is not difficult to find a function $f_1(\theta, \varphi; p)$ periodic in θ and φ with $f_1(0, 0; p) = 0$ from (4.6)-(4.9). One should take into account (8.7)-(8.8) with $a(\theta) = \cos\theta$ and the equation $\dot{A}_1^0 = 0$ as $g_1 = \frac{1}{2}a_1^0$ in this case. If $\omega \neq 1/2$, the system (8.7)-(8.8) always has a 2π-periodic solution which is not unique for some ω. Let us take the following solution

(8.10)
$$f_1(\theta, \varphi; p) = \frac{p\omega^2}{4\omega^2 - 1}(-1 + \cos\theta\cos 2\varphi) + (\frac{-p\omega}{2(4\omega^2 - 1)}\sin\theta + \frac{a_1^1}{2\omega})\sin 2\varphi + \frac{a_1^2}{4\omega}\sin 4\varphi.$$

We continue the consideration of the Mathieu equation under

(8.11) $$\omega = 1,\ \beta^2 = \gamma^2 = \alpha.$$

We have

$$g_1(p) = \frac{\alpha p}{4}(p-1). \tag{8.12}$$

Due to (7.7), the stability index satisfies

$$\gamma^\varepsilon = 1 + O(\varepsilon),$$

and if $0 < p < 1$, then for all sufficiently small ε the considered system is p-stable. Let us apply Remark 7.4 in case (8.11) with $\alpha = 1$. In this case

$$a_1^1 = \frac{p}{4}(3-p),\ a_1^2 = 0,$$

$$f_1(\theta, \varphi; p) = \frac{p}{3}(-1 + \cos\theta\cos 2\varphi) + (-\frac{p}{6}\sin\theta + \frac{p}{8}(3-p))\sin 2\varphi,$$

$$L_2 = \frac{1}{2}\cos^2\varphi\frac{\partial^2}{\partial\varphi^2} + (-\cos\theta\cos^2\varphi - \frac{3}{4}\sin 2\varphi + \frac{p}{2}\sin 2\varphi)\frac{\partial}{\partial\varphi}$$

$$+(-\frac{p}{4} + \frac{3p}{4}\cos 2\varphi + \frac{p^2}{2}\sin^2\varphi - \frac{p}{2}\cos\theta\sin 2\varphi).$$

The following rough bounds for f_1 and $L_2 f_1$ under, for example $p = 1/2$, are evident:

$$\max_{\theta,\varphi}|f_1| < 0.6,\ \max_{\theta,\varphi}|\frac{\partial f_1}{\partial\varphi}| < 0.9,\ \max_{\theta,\varphi}|\frac{\partial^2 f_1}{\partial\varphi^2}| < 1.7,\ \max_{\theta,\varphi}|L_2 f_1| < 2.8.$$

Using these calculations and Remark 7.4, we obtain that the considered equation is 1/2-stable if $0 < \varepsilon < 0.02$.

For the convenience of the reader consider also the Ito equation

$$dX^\varepsilon = \omega J X^\varepsilon dt + \varepsilon A(t) X^\varepsilon dt + \sqrt{\varepsilon}\sum_{r=1}^{q} A_r(t) X^\varepsilon dw_r(t). \tag{8.13}$$

The formula for $g_1^{Ito}(p)$ has the form

$$g_1^{Ito}(p) = \frac{1}{8l}p^2\sum_{r=1}^{q}\int_0^l (\frac{3}{2}(a_r^{11})^2 + \frac{3}{2}(a_r^{22})^2 + a_r^{11}a_r^{22} + \frac{1}{2}(a_r^{12} + a_r^{21})^2)d\theta \tag{8.14}$$

$$+\frac{1}{2l}p\int_0^l (a_{11} + a_{22})d\theta + \frac{1}{8l}p\sum_{r=1}^{q}\int_0^l ((a_r^{12} - a_r^{21})^2 - (a_r^{11} + a_r^{22})^2)d\theta.$$

The conditions on ω in Propositions 8.1 and 8.2 ensuring the expansions of the moment Lyapunov exponent with exactness $O(\varepsilon^2)$ remain true for the Hill and Mathieu equations in the Ito form as well. For the Mathieu equation in the Ito form

$$\ddot{X} = -\omega^2(1 + \varepsilon a\cos t)X - \varepsilon\alpha\dot{X} + \sqrt{\varepsilon}\beta X\dot{w}_1 + \sqrt{\varepsilon}\gamma\dot{X}\dot{w}_2,$$

with constant coefficients α, β, and γ we get

$$g_1^{Ito}(p) = p(-\frac{\alpha}{2} + \frac{\beta^2}{8\omega^2} - \frac{\gamma^2}{8}) + \frac{p^2}{16}(\frac{\beta^2}{\omega^2} + 3\gamma^2).$$

For example, under (8.11) we obtain $g_1^{Ito}(p) = \frac{p\alpha}{4}(p-2)$ instead of (8.12).

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