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On a class of singularly perturbed partly dissipative reaction-diffusion systems

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Abstract

We consider the singularly perturbed partly dissipative reaction-diffusion system $\varepsilon^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=g(u,v,x,t,\varepsilon)$, $\frac{\partial v}{\partial t}=f(u,v,x,t,\varepsilon)$ under the condition that the degenerate equation $g(u,v,t,0)=0$ has two solutions $u=\varphi_i(v,x,t), i=1,2$, that intersect (exchange of stabilities) and that v is a vector. Our main result concerns existence and asymptotic behavior in ε of the solution of the initial boundary value problem under consideration. The proof is based on the method of asymptotic lower and upper solutions.

1 Introduction.

We consider the singularly perturbed partly dissipative reaction-diffusion system

$$\begin{aligned}\varepsilon^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right) &= g(u,v,x,t,\varepsilon),\\ \frac{\partial v}{\partial t} &= f(u,v,x,t,\varepsilon),\end{aligned}\tag{1.1}$$

where u is a scalar and v is a vector, ε is a small positive parameter. Partly dissipative systems are often used to model reaction-diffusion processes in different fields (chemical kinetics, biology, astrophysics) when the effect of diffusion of some species is negligible (see, for example, [11, 12, 7, 8, 6, 13, 5]).
If the degenerate equation

$$g(u,v,x,t,0)=0$$

has an isolated solution with respect to u then the standard theory (see [15]) can be applied to derive asymptotic properties for the solution of initial boundary value problems to system (1.1).
In what follows we consider system (1.1) under the assumption that the degenerate equation has two intersecting solutions. This assumption implies an exchange of stabilities for families of equilibria of the associated differential equation to (1.1). The present paper continues the investigations of the authors in [3] where the case has been treated that v is a scalar. The main results of the present paper concern existence and asymptotic behavior in ε of the solution of some initial boundary value problem related to system (1.1). The proof of our results is based on the method of asymptotic lower and upper solutions.
The paper is organized as follows: In section 2 we formulate the problem and the

corresponding assumptions. We also describe the method of asymptotic lower and upper solutions. Section 3 contains the mains results: existence of a unique solution and it asymptotic behavior. In section 4 we illustrate our approach by considering an example.

2 Formulation of the problem. Assumptions.

We study the singularly perturbed nonlinear initial boundary value problem

$$\begin{aligned}
\varepsilon^2 \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) &= g(u, v, x, t, \varepsilon), \\
\frac{\partial v}{\partial t} &= f(u, v, x, t, \varepsilon), \\
(x,t) \in Q &:= \{(x,t) \in R^2 : 0 < x < 1,\ 0 < t \le T\}, \\
\varepsilon \in I_{\varepsilon_0} &:= \{\varepsilon \in R : 0 < \varepsilon \le \varepsilon_0 \ll 1\}, \\
\frac{\partial u}{\partial x}(0, t, \varepsilon) &= \frac{\partial u}{\partial x}(1, t, \varepsilon) = 0 \quad \text{for} \quad 0 < t \le T, \\
u(x, 0, \varepsilon) &= u^0(x),\ v(x, 0, \varepsilon) = v^0(x) \quad \text{for} \quad 0 \le x \le 1,
\end{aligned} \tag{2.1}$$

where u is a scalar, and, in order to simplify the representation, v is a two-dimensional vector. Let $v = (y, z),\ f = (f_1, f_2)$.
We consider (2.1) under the following assumptions:

(A_0). $g \in C^2(D, R)$, $f \in C^2(D, R^2)$ *where* $D := R \times R^2 \times [0,1] \times [0,T] \times \overline{I}_{\varepsilon_0}$, $u^0 \in C^2([0,1], R)$, $v^0 \in C^2([0,1], R^2)$.

If we set $\varepsilon = 0$ in (2.1) then we get the degenerate system

$$\begin{aligned}
0 &= g(u, v, x, t, 0), \\
\frac{dv}{dt} &= f(u, v, x, t, 0).
\end{aligned} \tag{2.2}$$

Concerning the solution set of the equation

$$g(u, v, x, t, 0) = 0 \tag{2.3}$$

we assume

(A_1). *Equation* (2.3) *has exactly two solutions* $u = \varphi_1(v, x, t)$ *and* $u = \varphi_2(v, x, t)$ *defined for* $(v, x, t) \in \overline{G}_v \times \overline{Q}$, *where* G_v *is some bounded open region in* R^2, *and where* φ_1 *and* φ_2 *have the same smoothness properties as* g.

Throughout the following w will be a placeholder for the components y and z of v. From assumption (A_1) we obtain for $(v, x, t) \in \overline{G}_v \times \overline{Q}$, and for $i = 1, 2$:

$$\begin{aligned} g(\varphi_i(v,x,t), v, x, t, 0) &\equiv 0, \\ g_u(\varphi_i(v,x,t), v, x, t, 0)\, \frac{\partial \varphi_i}{\partial w}(v,x,t) + g_w(\varphi_i(v,x,t), v, x, t, 0) &\equiv 0. \end{aligned} \tag{2.4}$$

Different from the classical theory (see [14],[15]), assumption (A_1) does not require that the solutions φ_1 and φ_2 are isolated. In what follows we assume that the surfaces $u = \varphi_1(y, z, x, t)$ and $u = \varphi_2(y, z, x, t)$ intersect in a smooth surface whose projection into the (y, z, x, t)-space can be described by $y = s(z, x, t)$.

(A_2). *There exist an interval I_z and a smooth function $s : \overline{I}_z \times \overline{Q} \to R$ such that for all $(z, x, t) \in \overline{I}_z \times \overline{Q}$*

$$\begin{aligned} \varphi_1(y,z,x,t) &= \varphi_2(y,z,x,t) \quad \text{for} \quad y = s(z,x,t), \\ \varphi_1(y,z,x,t) &> \varphi_2(y,z,x,t) \quad \text{for} \quad y < s(z,x,t), \\ \varphi_1(y,z,x,t) &< \varphi_2(y,z,x,t) \quad \text{for} \quad y > s(z,x,t). \end{aligned}$$

The differential equation

$$\frac{du}{d\tau} = g(u, v, x, t, 0), \tag{2.5}$$

where v, x, t are considered as parameters, is said to be the associated equation to (2.1). It follows from hypothesis (A_1) that $u = \varphi_i(v, x, t), i = 1, 2$, are families of equilibria of (2.5). The families φ_i are stable (unstable) if $g_u(\varphi_i, v, x, t, 0)$ is negative (positive). Concerning the stability of these families we assume

(A_3). *For $(z, x, t) \in \overline{I}_z \times \overline{Q}$ it holds*

$$\begin{aligned} &g_u(\varphi_1(y,z,x,t), y, z, x, t, 0) < 0, \\ &g_u(\varphi_2(y,z,x,t), y, z, x, t, 0) > 0 \end{aligned} \qquad \textit{for} \;\; y < s(z,x,t),$$

$$\begin{aligned} &g_u(\varphi_1(y,z,x,t), y, z, x, t, 0) > 0, \\ &g_u(\varphi_2(y,z,x,t), y, z, x, t, 0) < 0 \end{aligned} \qquad \textit{for} \;\; y > s(z,x,t).$$

From assumption (A_3) we get that $g_u(u, y, z, t, 0)$ changes its sign on the surface $y = s(z, x, t)$ where $u = \varphi_1(y, z, x, t)$ and $u = \varphi_2(y, z, x, t)$ intersect. This sign change of g_u expresses an exchange of stabilities of the families of equilibria to the associated equation (2.5).
From (A_3) we get for $(x, t) \in \overline{Q}$ and for $i = 1, 2$

$$g_u(\varphi_i(s(z,x,t), z, x, t), s(z,x,t), z, x, t, 0) \equiv 0.$$

In what follows we will construct the so-called composed stable solution to the degenerate system (2.2). To this purpose we assume $y^0(x) \neq s(z^0(x), x, 0)$ for all $x \in [0, 1]$. First we consider the case

$$y^0(x) < s(z^0(x), x, 0) \quad \text{for} \quad 0 \le x \le 1. \tag{2.6}$$

(A$_4$). *For* $x \in [0,1]$, *the initial value problem*

$$\begin{aligned} \frac{dv}{dt} &= f(\varphi_1(v,x,t),v,x,t,0), \\ & v(x,0) = v^0(x) \end{aligned} \tag{2.7}$$

where $v^0(x)$ *satisfies* (2.6) *has a unique solution* $v = v_1 = (y_1(x,t), z_1(x,t))$ *defined on* $\overline{Q}$ *such that* $z_1(x,t) \in I_z$ *for* $(x,t) \in \overline{Q}$ *where* I_z *is the interval introduced in hypothesis* (A$_2$). *There exists a smooth curve* $\mathcal{C}$ *defined by* $\mathcal{C} := \{(x,t) \in \overline{Q} : t = t_c(x),\ 0 \le x \le 1\}$ *where* $t_c \in C^2[0,1]$ *and such that*

$$\begin{array}{rll} 0 < t_c(x) < T & \textit{for} & 0 \le x \le 1, \\ y_1(x,t) < s(z_1(x,t),x,t) & \textit{for} & 0 \le t < t_c(x),\ 0 \le x \le 1, \\ y_1(x,t) = s(z_1(x,t),x,t) & \textit{for} & t = t_c(x),\ 0 \le x \le 1, \\ y_1(x,t) > s(z_1(x,t),x,t) & \textit{for} & t_c(x) < t \le T,\ 0 \le x \le 1. \end{array} \tag{2.8}$$

Assumption (A$_4$) says that the surfaces $y = y_1(x,t)$ and $y = s(z_1(x,t),x,t)$ intersect in a curve whose projection into $\overline{Q}$ can be described by $t = t_c(x)$. We denote by Q_1 and Q_2 all points (x,t) of $\overline{Q}$ satisfying $t < t_c(x)$ and $t > t_c(x)$ respectively (see Fig 1.).

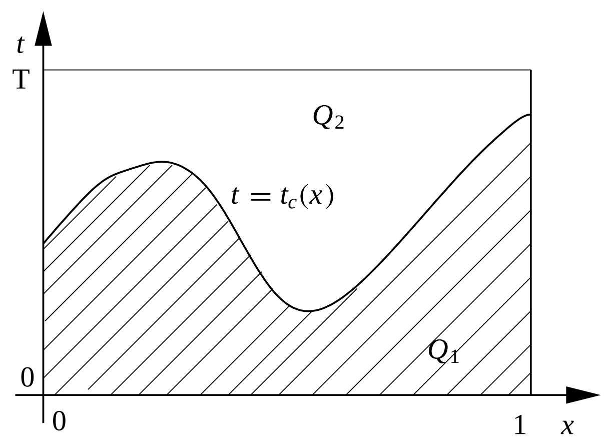

Fig. 1: Decomposition of $\overline{Q}$ into Q_1 and Q_2 by the curve $\mathcal{C}$.

(A$_5$). *For* $x \in [0,1]$, *the initial value problem*

$$\begin{aligned} \frac{dv}{dt} &= f(\varphi_2(v,x,t),v,x,t,0), \\ y(x,t_c(x)) &= s(z_1(x,t_c(x)),x,t_c(x)), \\ z(x,t_c(x)) &= z_1(x,t_c(x)) \end{aligned} \tag{2.9}$$

has a unique solution $v = v_2(x,t) = (y_2(x,t), z_2(x,t))$ *defined on* $\overline{Q}$ *such that*

$$\begin{array}{rcl} z_2(x,t) \in I_z & \textit{for} & (x,t) \in \overline{Q}, \\ y_2(x,t) > s(z_2(x,t),x,t) & \textit{for} & (x,t) \in Q_2, \\ y_2(x,t) < s(z_2(x,t),x,t) & \textit{for} & (x,t) \in Q_1. \end{array} \tag{2.10}$$

Let

$$\begin{array}{rcll} \psi_1(x,t) & := & \varphi_1(v_1(x,t),x,t) & \text{for} \quad (x,t) \in \overline{Q}, \\ \psi_2(x,t) & := & \varphi_2(v_2(x,t),x,t) & \text{for} \quad (x,t) \in \overline{Q}. \end{array} \tag{2.11}$$

From assumption (A_2) and from (2.11) we obtain

$$\psi_1(x,t) \equiv \psi_2(x,t) \quad \text{on} \quad \mathcal{C}. \tag{2.12}$$

Now, we introduce the functions $\hat{u}(x,t)$ and $\hat{v}(x,t)$ by

$$\begin{array}{rcl} \hat{u}(x,t) & := & \begin{cases} \psi_1(x,t) & \text{for} \quad (x,t) \in \overline{Q}_1, \\ \psi_2(x,t) & \text{for} \quad (x,t) \in \overline{Q}_2, \end{cases} \\ \hat{v}(x,t) & := & \begin{cases} v_1(x,t) & \text{for} \quad (x,t) \in \overline{Q}_1, \\ v_2(x,t) & \text{for} \quad (x,t) \in \overline{Q}_2. \end{cases} \end{array} \tag{2.13}$$

We suppose that $\hat{v}(x,t) \in G_v$ for $(x,t) \in \overline{Q}$ where G_v is the region introduced in hypothesis (A_1).

The vector-function $(\hat{u}(x,t),\ \hat{v}(x,t))$ is referred to as the *composed stable solution* of the degenerate system (2.2).

The function $\hat{v}(x,t)$ is obviously continuously differentiable for $(x,t) \in \overline{Q}$, but $\hat{u}(x,t)$ is in general not smooth on the curve $\mathcal{C}$.

For the sequel it is convenient to introduce the following notation: the symbol "ˆ" over g and f or some derivative of g and f denotes that we have to consider the arguments (u,v,ε) at $(\hat{u}(x,t),\hat{v}(x,t),0)$.
It follows from assumption (A_1) that

$$\hat{g}(x,t) := g(\hat{u}(x,t),\hat{v}(x,t),x,t,0) \equiv 0 \quad \text{in} \quad \overline{Q}, \tag{2.14}$$

by assumption (A_3) we have

$$\hat{g}_u(x,t) < 0 \quad \text{in} \quad \overline{Q}\backslash\mathcal{C}, \tag{2.15}$$

$$\hat{g}_u(x,t) \equiv 0 \quad \text{on } \mathcal{C}. \tag{2.16}$$

Remark 2.1 *The case* $y^0(x) > s(z^0(x), x, 0)$ *can be treated analogously. In that case we have to use the function* $\varphi_2(v, x, t)$ *to construct* $v_1(x, t)$ *(see assumption* (A_4)*) and the function* $\varphi_1(v, x, t)$ *to construct* $v_2(x, t)$ *(see assumption* (A_5)*). The case when* $y^0(x) = s(z^0(x), x, 0)$ *for some* x *requires a special treatment.*

In what follows we prove that under the hypotheses $(A_1) - (A_5)$ and under some additional assumptions (see $(A_6) - (A_8)$ below) problem (2.1) has a solution $(u(x, t, \varepsilon), v(x, t, \varepsilon))$ satisfying

$$\begin{aligned} \lim_{\varepsilon \to 0} u(x, t, \varepsilon) &= \hat{u}(x, t) \quad \text{in} \quad \overline{Q} \backslash \{t = 0, 0 \le x \le 1\}, \\ \lim_{\varepsilon \to 0} v(x, t, \varepsilon) &= \hat{v}(x, t) \quad \text{in} \quad \overline{Q}. \end{aligned} \tag{2.17}$$

Concerning the initial condition $u^0(x)$ for $u(x, t, \varepsilon)$ we assume

(A_6). *For* $x \in [0, 1]$, $u^0(x)$ *lies in the basin of attraction of the equilibrium point* $\varphi_1(v^0(x), x, 0)$ *of the associated equation* (2.5) *for* $v = v^0(x)$, $t = 0$.

Assumption (A_6) implies that for $v = v^0(x)$, $t = 0$ equation (2.5) with the initial condition

$$u(x, 0) = u^0(x)$$

has a unique solution $u = \overline{u}(x, \tau)$ defined for $\tau \ge 0$, and such that $\lim_{\tau \to \infty} \overline{u}(x, \tau) = \varphi_1(v^0(x), x, 0)$. Finally, we assume

(A_7). $\hat{g}_{uu}(x, t) := g_{uu}(\hat{u}(x, t), \hat{v}(x, t), x, t, 0) < 0$ *on* $\mathcal{C}$.

(A_8). $\hat{g}_\varepsilon(x, t) > 0$ *on* $\mathcal{C}$.

Concerning assumption (A_8) we would like to mention that the sign of $\hat{g}_\varepsilon(x, t)$ on $\mathcal{C}$ plays an important role (see [1] - [4]).

Our approach to prove the asymptotic behavior of the solution of problem (2.1) is based on the concept of ordered lower and upper solutions. Let us recall its definition [11].

Definition 2.1 *Let the vector-functions* $\alpha(.) := \alpha(x, t, \varepsilon) = (\alpha^u(.), \alpha^v(.)) := (\alpha^u(.), \alpha^y(.), \alpha^z(.))$ *and* $\beta(.) := (\beta^u(.), \beta^v(.))$ *be defined for* $(x, t, \varepsilon) \in \overline{Q} \times I_{\varepsilon_1}, \varepsilon_1 \le \varepsilon_0$, *and satisfy the smoothness conditions* $\alpha^u, \beta^u \in C^{2,1,0}_{x,t,\varepsilon}(Q \times I_{\varepsilon_1}) \cap C^{1,0,0}_{x,t,\varepsilon}(\overline{Q} \times I_{\varepsilon_1})$, $\alpha^y, \alpha^z, \beta^y, \beta^z \in C^{0,1,0}_{x,t,\varepsilon}(Q \times I_{\varepsilon_1}) \cap C^{0,0,0}_{x,t,\varepsilon}(\overline{Q} \times I_{\varepsilon_1})$. *Furthermore, we assume* $\alpha^u(.) \le \beta^u(.), \alpha^v(.) \le \beta^v(.)$ *for* $(x, t, \varepsilon) \in \overline{Q} \times I_{\varepsilon_1}$ *(where the last inequality has to be understood componentwise).*
Let the operators L, M, N *be defined by*

$$(Lw)(x, t, \varepsilon) \;:=\; \varepsilon^2 \left(\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} \right) - g(w, y, z, x, t, \varepsilon), \tag{2.18}$$

$$(Mw)(x,t,\varepsilon) \quad := \quad \frac{\partial w}{\partial t} - f_1(u,w,z,x,t,\varepsilon), \tag{2.19}$$

$$(Nw)(x,t,\varepsilon) \quad := \quad \frac{\partial w}{\partial t} - f_2(u,y,w,x,t,\varepsilon). \tag{2.20}$$

Then, $\alpha(x,t,\varepsilon)$ *and* $\beta(x,t,\varepsilon)$ *are called ordered lower and upper solutions of problem* (2.1) *respectively, if they satisfy the following inequalities*

$$(L\alpha^u)(.) \le 0 \le (L\beta^u)(.) \quad \text{for } (x,t,\varepsilon) \in Q \times I_{\varepsilon_1}, \alpha^v \le v \le \beta^v, \tag{2.21}$$

$$(M\alpha^y)(.) \le 0 \le (M\beta^y)(.) \quad \text{for} \quad \begin{array}{l} (x,t,\varepsilon) \in Q \times I_{\varepsilon_1}, \\ \alpha^u \le u \le \beta^u, \alpha^z \le z \le \beta^z, \end{array} \tag{2.22}$$

$$(N\alpha^z)(.) \le 0 \le (N\beta^z)(.) \quad \text{for} \quad \begin{array}{l} (x,t,\varepsilon) \in Q \times I_{\varepsilon_1}, \\ \alpha^u \le u \le \beta^u, \alpha^y \le y \le \beta^y, \end{array} \tag{2.23}$$

$$\frac{\partial \alpha^u}{\partial x}(0,t,\varepsilon) \ge 0 \ge \frac{\partial \beta^u}{\partial x}(0,t,\varepsilon), \qquad \begin{array}{l} \frac{\partial \alpha^u}{\partial x}(1,t,\varepsilon) \le 0 \le \frac{\partial \beta^u}{\partial x}(1,t,\varepsilon) \\ \text{for } (t,\varepsilon) \in [0,T] \times I_{\varepsilon_1}, \end{array} \tag{2.24}$$

$$\begin{array}{l} \alpha^u(x,0,\varepsilon) \le u^0(x) \le \beta^u(x,0,\varepsilon), \\ \alpha^v(x,0,\varepsilon) \le v^0(x) \le \beta^v(x,0,\varepsilon), \end{array} \qquad \text{for } (x,\varepsilon) \in [0,1] \times I_{\varepsilon_1}. \tag{2.25}$$

This definition can be obviously adapted to any subdomain of $\overline{Q}$.

Remark 2.2 *It is known* (see, for example, [11]) *that the existence of ordered lower and upper solutions of* (2.1) *implies the existence of a unique solution* $(u(x,t,\varepsilon)$, $v(x,t,\varepsilon))$ *of* (2.1) *satisfying for* $(x,t,\varepsilon) \in \overline{Q} \times I_{\varepsilon_1}$

$$\begin{aligned} \alpha^u(x,t,\varepsilon) &\le u(x,t,\varepsilon) \le \beta^u(x,t,\varepsilon), \\ \alpha^y(x,t,\varepsilon) &\le y(x,t,\varepsilon) \le \beta^y(x,t,\varepsilon), \\ \alpha^z(x,t,\varepsilon) &\le z(x,t,\varepsilon) \le \beta^z(x,t,\varepsilon). \end{aligned}$$

The goal of the following investigations is to characterize the asymptotic behavior of the solution of the initial boundary value problem (2.1), in particular, we prove the limit behavior (2.17) by constructing appropriate lower and upper solutions.

3 Existence and asymptotic behavior of the solution.

In this section we will prove that problem (2.1) has a unique solution. Taking into account an initial layer correction we can show that for small ε the solution of (2.1) is close to the composed stable solution $(\hat{u}(x,t), \hat{v}(x,t))$.

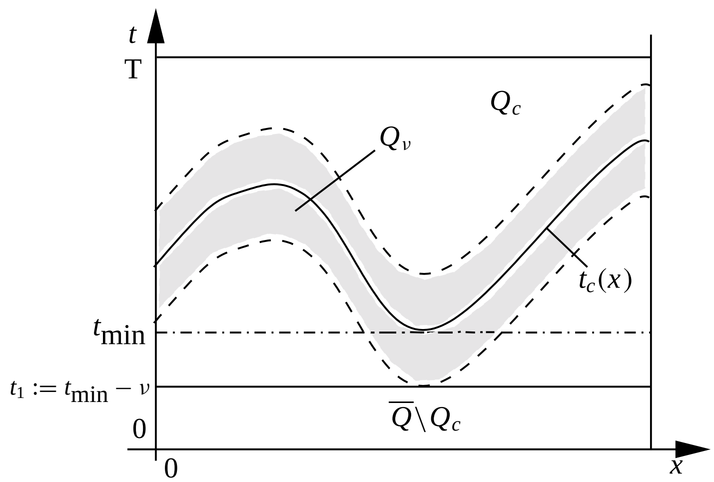

Fig. 2: Decomposition of $\overline{Q}$.

In order to be able to formulate our main result we decompose the domain $\overline{Q}$ into two disjoint sub-domains Q_c and $Q \backslash Q_c$ and introduce a function which represents an approximation of the initial layer correction.

First we decompose $\overline{Q}$. Let t_{min} be the minimum of the function $t_c(x)$ in [0,1], let ν be any small positive number such that $t_1 := t_{min} - \nu$ is positive. Let Q_c be the domain defined by $Q_c := \{(x,t) \in R^2 : 0 < x < 1,\ t_1 < t \leq T\}$, (see Fig. 2).

Next we introduce an initial layer correction. According to [15] we define the zeroth order initial layer function $\Pi_0(x,\tau)$ $(\tau = t/\varepsilon^2)$ as the solution of the initial value problem ($x \in [0,1]$ has to be considered as a parameter)

$$\begin{aligned} \frac{d\Pi_0}{d\tau} &= g(\psi_1(x,0) + \Pi_0, v^0(x), x, 0, 0), \quad \tau > 0, \\ \Pi_0(x,0) &= u^0(x) - \psi_1(x,0). \end{aligned} \tag{3.1}$$

By (2.11) we have $\psi_1(x,0) = \varphi_1(v^0(x), x, 0)$. Thus, from assumptions (A_3) and (A_6) it follows that the initial value problem (3.1) has a solution which satisfies the estimate $|\Pi_0(x,\tau)| < c\exp(-\kappa\tau)$ for $\tau \geq 0$ where c and κ are some positive constants.

Theorem 3.1 *Assume hypotheses* (A_0)–(A_8) *to be valid. Then, for sufficiently small* ε*, the initial boundary value problem* (2.1) *has a unique solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ *satisfying*

$$u(x,t,\varepsilon) = \begin{cases} \hat{u}(x,t) + \Pi_0(x,\tau) + O(\varepsilon) & \text{for} \quad (x,t) \in \overline{Q} \backslash Q_c, \\ \hat{u}(x,t) + O(\sqrt{\varepsilon}) & \text{for} \quad (x,t) \in \overline{Q}_c, \end{cases} \tag{3.2}$$

$$v(x,t,\varepsilon) = \begin{cases} \hat{v}(x,t) + O(\varepsilon) & \text{for} \quad (x,t) \in \overline{Q} \backslash Q_c, \\ \hat{v}(x,t) + O(\sqrt{\varepsilon}) & \text{for} \quad (x,t) \in \overline{Q}_c. \end{cases} \tag{3.3}$$

Corollary 3.1 *From* (3.2), (3.3) *it is obvious that the relations* (2.17) *hold.*

Proof. The proof Theorem 3.1 consists of two steps. In the first step we consider the initial boundary value problem (2.1) in the sub-domain $\overline{Q} \setminus Q_c$. From our assumptions it follows that the exchange of stabilities takes place in Q_c. Therefore, we can apply the standard theory [15] to solve the initial boundary value problem in $\overline{Q} \setminus Q_c$. We get the following result.

Lemma 3.1 *Assume hypotheses* (A_0) – (A_4), (A_6) *to be valid. Then, for sufficiently small* ε*, the initial boundary value problem* (2.1) *has a unique solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ *in* $\overline{Q} \setminus Q_c$ *satisfying*

$$\begin{aligned} u(x,t,\varepsilon) &= \hat{u}(x,t) + \Pi_0(x,\tau) + O(\varepsilon), \\ v(x,t,\varepsilon) &= \hat{v}(x,t) + O(\varepsilon). \end{aligned} \tag{3.4}$$

Let $u^1(x,\varepsilon) := u(x,t_1,\varepsilon), v^1(x,\varepsilon) := v(x,t_1,\varepsilon)$. Now we consider the initial boundary value problem (2.1) in $\overline{Q}_c$ with the initial conditions $u(x,t_1,\varepsilon) = u^1(x,\varepsilon)$, $v(x,t_1,\varepsilon) = v^1(x,\varepsilon)$ for $0 \leq x \leq 1$.

Our approach to study this problem is based on the method of ordered lower and upper solutions. We construct these solutions for (2.1) by means of the composed stable solution $(\hat{u}(x,t), \hat{v}(x,t))$.
As we noticed above, $\hat{u}(x,t)$ in general is not smooth on the curve $\mathcal{C}$. In order to be able to use $\hat{u}(x,t)$ for the construction of lower and upper solutions we have to smooth $\hat{u}(x,t)$ in some neighbourhood Q_ν of $\mathcal{C}$. Let Q_ν be defined by $Q_\nu := \{(x,t) \in \overline{Q} : |t - t_c(x)| < \nu, 0 \leq x \leq 1\}$, where ν is any sufficiently small positive number such that Q_ν has no common point with $t = T$ (see Fig. 2).

Using the function

$$\omega(\xi) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi} \exp(-s^2) ds,$$

where

$$\xi := (t - t_c(x))/\varepsilon^a, \quad a \in (1/2, 1)$$

we introduce the function $\tilde{u}$ by

$$\tilde{u}(x,t,\varepsilon) := \psi_1(x,t)\omega(-\xi) + \psi_2(x,t)\omega(\xi). \tag{3.5}$$

It is easy to show that $\tilde{u}$ is smooth and satisfies

$$\tilde{u}(x,t,\varepsilon) = \hat{u}(x,t) + \eta(x,t,\varepsilon), \tag{3.6}$$

where

$$\eta(x,t,\varepsilon) \quad = \quad \begin{cases} O(\varepsilon^a) & \text{for} \quad (x,t) \in Q_\nu, \\ O(\exp -(\nu/\varepsilon)) & \text{for} \quad (x,t) \in \overline{Q} \setminus Q_\nu \end{cases} \tag{3.7}$$

(see [1]).

Now we construct lower and upper solutions for (2.1) in $\overline{Q}_c$ by using the smooth function $\tilde{u}$ as follows

$$\begin{aligned} \beta^u(x,t,\varepsilon) &:= \tilde{u}(x,t,\varepsilon) + \sqrt{\varepsilon}\gamma h(x,t) + \varepsilon^a z(x,\varepsilon), \\ \alpha^u(x,t,\varepsilon) &:= \tilde{u}(x,t,\varepsilon) - \sqrt{\varepsilon}\sigma h(x,t) - \varepsilon^a z(x,\varepsilon), \\ \beta^w(x,t,\varepsilon) &:= \hat{w}(x,t) + \sqrt{\varepsilon}\sigma^2 h(x,t), \\ \alpha^w(x,t,\varepsilon) &:= \hat{w}(x,t) - \sqrt{\varepsilon}\sigma^2 h(x,t), \end{aligned} \tag{3.8}$$

where w is a placeholder for y and z,

$$\begin{aligned} h(x,t) &:= \exp(\lambda(t - t_c(x))), \\ z(x,\varepsilon) &:= \exp(-kx/\varepsilon^a) + \exp(-k(1-x)/\varepsilon^a) \end{aligned} \tag{3.9}$$

are positive functions in $\overline{Q} \times I_{\varepsilon_1}$, $\gamma, \sigma, \lambda, k$ are positive numbers. We will determine these numbers in such a way that α and β will be ordered lower and upper solutions, i.e. they will satisfy all conditions of Definition 2.1 in $\overline{Q}_c$.

It is obvious that for any positive γ, σ, λ and k we have

$$\alpha^u(x,t,\varepsilon) \le \beta^u(x,t,\varepsilon), \quad \alpha^w(x,t,\varepsilon) \le \beta^w(x,t,\varepsilon) \quad \text{in} \quad \overline{Q}_c,$$

hence, if $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ are lower and upper solutions for (2.1) then they are ordered.

Taking into account the exponential decay of $\Pi_0(x,\tau)$ we get from (3.8), (3.4) for sufficiently small ε

$$\begin{aligned} \alpha^u(x,t_1,\varepsilon) &\le u(x,t_1,\varepsilon) = u^1(x,\varepsilon) \le \beta^u(x,t_1,\varepsilon), \\ \alpha^v(x,t_1,\varepsilon) &\le v(x,t_1,\varepsilon) = v^1(x,\varepsilon) \le \beta^v(x,t_1,\varepsilon). \end{aligned}$$

Consequently, the inequalities (2.25) for the initial data hold.

From (2.11) and (2.12) we obtain

$$\psi_2(x,t) - \psi_1(x,t) = O(|t - t_c(x)|).$$

Using this relation it can be shown (see [1, 4]) that

$$\varepsilon^2\left(\frac{\partial\tilde u}{\partial t}-\frac{\partial^2\tilde u}{\partial x^2}\right)=\begin{cases} O(\varepsilon^{2-a})) & \text{in} \quad Q_\nu, \\ O(\varepsilon^2) & \text{in} \quad Q_c\backslash Q_\nu. \end{cases} \tag{3.10}$$

From (3.9) we get

$$\begin{aligned} \varepsilon^2\varepsilon^{1/2}\left(\frac{\partial h}{\partial t}-\frac{\partial^2 h}{\partial x^2}\right) &= O(\varepsilon^{5/2}) \quad \text{in} \quad Q_c, \\ \varepsilon^2\varepsilon^{a}\left(\frac{\partial z}{\partial t}-\frac{\partial^2 z}{\partial x^2}\right) &= O(\varepsilon^{2-a}) \quad \text{in} \quad Q_c. \end{aligned} \tag{3.11}$$

Thus, because of $1/2 < a < 1$, we obtain from (3.8) - (3.11)

$$\varepsilon^2\left(\frac{\partial\beta^u}{\partial t}-\frac{\partial^2\beta^u}{\partial x^2}\right)=O(\varepsilon^{2-a})=o(\varepsilon) \quad \text{in} \quad Q_c, \tag{3.12}$$

$$\varepsilon^2\left(\frac{\partial\alpha^u}{\partial t}-\frac{\partial^2\alpha^u}{\partial x^2}\right)=O(\varepsilon^{2-a})=o(\varepsilon) \quad \text{in} \quad Q_c. \tag{3.13}$$

Now we check that $\alpha^u(x,t,\varepsilon)$ and $\beta^u(x,t,\varepsilon)$ satisfy the inequalities (2.21) in Q_ν for sufficiently small ε.
To treat the expression $g(\beta^u(x,t,\varepsilon),v,x,t,\varepsilon)$ in $L\beta^u$ we use the relations $\tilde u(x,t,\varepsilon)=\hat u(x,t)+O(\varepsilon^a)$ which follows from (3.6) and (3.7) and $\varepsilon^a z(x,\varepsilon)=O(\varepsilon^a)$ in Q_ν due to (3.9). Moreover, we note that the set of all w satisfying $\alpha^w(x,t,\varepsilon)\le w\le\beta^w(x,t,\varepsilon)$ can be represented in the form (see (3.8))

$$\begin{aligned} y &= \hat y(x,t)+\sqrt{\varepsilon}\sigma^2 h(x,t)\theta_1 \quad \text{for} \quad -1\le\theta_1\le 1, \\ z &= \hat z(x,t)+\sqrt{\varepsilon}\sigma^2 h(x,t)\theta_2 \quad \text{for} \quad -1\le\theta_2\le 1. \end{aligned}$$

Thus, we have

$$\begin{aligned} & g(\beta^u(x,t,\varepsilon),y,z,x,t,\varepsilon)= \\ & g(\hat u+\sqrt{\varepsilon}\gamma h(x,t)+O(\varepsilon^a),\hat y+\sqrt{\varepsilon}\sigma^2 h(x,t)\theta_1,\hat z+\sqrt{\varepsilon}\sigma^2 h(x,t)\theta_2,x,t,\varepsilon)= \\ & \hat g(x,t)+\sqrt{\varepsilon}\Big[\hat g_u(x,t)\Big(\gamma+O(\varepsilon^{a-1/2})\Big)+(\hat g_y(x,t)\theta_1+\hat g_z(x,t)\theta_2)\sigma^2\Big]h(x,t)+ \\ & +\frac{1}{2}\varepsilon\Big[\hat g_{uu}(x,t)\gamma^2+2(\hat g_{uy}(x,t)\theta_1+\hat g_{uz}(x,t)\theta_2)\gamma\sigma^2+ \\ & (2\hat g_{yz}(x,t)\theta_1^2+\hat g_{yy}(x,t)\theta_1\theta_2+\hat g_{zz}(x,t)\theta_2^2)\sigma^4\Big]h^2(x,t)+\varepsilon\hat g_\varepsilon(x,t)+o(\varepsilon). \end{aligned} \tag{3.14}$$

Our goal is to prove $g(\beta^u(x,t,\varepsilon),v,x,t,\varepsilon)=-c\varepsilon+o(\varepsilon)$ for $(x,t)\in Q_\nu$ and some positive constant c.
From (2.4) we get

$$\hat g_w(x,t)=-\hat g_u(x,t)\hat\varphi_w(x,t), \tag{3.15}$$

where

$$\hat{\varphi}_w(x,t) = \begin{cases} \varphi_{1w}(v_1(x,t),x,t) & \text{in} \quad Q_1, \\ \varphi_{2w}(v_2(x,t),x,t) & \text{in} \quad Q_2. \end{cases}$$

Since $\hat{\varphi}_w(x,t)$ is uniformly bounded in $\overline{Q}$, $|\theta_i| \le 1$, $i = 1,2$, we have by (3.15) and (2.15), (2.16) for any fixed σ and for sufficiently large γ

$$\begin{aligned} &\hat{g}_u(x,t)(\gamma + O(\varepsilon^{a-1/2})) + (\hat{g}_y(x,t)\theta_1 + \hat{g}_z(x,t)\theta_2)\sigma^2 = \\ &= \hat{g}_u(x,t)[\gamma + O(\varepsilon^{a-1/2}) - (\hat{\varphi}_y(x,t)\theta_1 + \hat{\varphi}_z(x,t)\theta_2)\sigma^2] \le 0. \end{aligned} \tag{3.16}$$

According to assumption (A_7) there is a positive constant c_ν such that for sufficiently small ν

$$\hat{g}_{uu}(x,t) \le -c_\nu < 0 \quad \text{in} \quad Q_\nu. \tag{3.17}$$

Hence, for sufficiently large γ, we have for $(x,t) \in Q_\nu$

$$\begin{aligned} &\gamma[\hat{g}_{uu}(x,t)\gamma + (\hat{g}_{uy}(x,t)\theta_1 + \hat{g}_{uz}(x,t)\theta_2)2\sigma^2 + \\ &\gamma^{-1}(\hat{g}_{yy}(x,t)\theta_1^2 + 2\hat{g}_{yz}(x,t)\theta_1\theta_2 + \hat{g}_{zz}(x,t)\theta_2^2)\sigma^4] < -2\gamma\overline{c}, \end{aligned} \tag{3.18}$$

where $\overline{c}$ is some positive constant.
Now we set $\lambda = 1/\nu$. Then, by (3.9), it holds

$$e^{-1} \le h(x,t) \le e \quad \text{for} \quad (x,t) \in \overline{Q}_\nu. \tag{3.19}$$

Under our smoothness assumption there is a positive constant c_g such that

$$|\hat{g}_\varepsilon(x,t)| \le c_g \quad \text{for} \quad (x,t) \in \overline{Q}_\nu. \tag{3.20}$$

By (2.14), (3.15) - (3.20) we get from (3.14)

$$g(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon) < -(\gamma\overline{c}e^{-2} - c_g)\varepsilon + o(\varepsilon). \tag{3.21}$$

Taking into account (3.12) and (3.21) we have for sufficiently small ν and ε and for sufficiently large γ

$$\begin{aligned} (L\beta^u)(x,t,\varepsilon) &\equiv \varepsilon^2\left(\frac{\partial\beta^u}{\partial t} - \frac{\partial^2\beta^u}{\partial x^2}\right) - g(\beta^u(x,t,\varepsilon), v, x, t, \varepsilon) \\ &> (\gamma\overline{c}e^{-2} - c_g)\varepsilon + o(\varepsilon) \ge 0, \end{aligned}$$

i.e. the inequality (2.21) holds for β^u in Q_ν.

Now we verify the inequality (2.21) for α^u in Q_ν. Using (3.8), (3.13), and a representation for $g(\alpha^u(x,t,\varepsilon), v, x, t, \varepsilon)$ similar to (3.14) we get

$$
\begin{aligned}
L\alpha^u(x,t,\varepsilon) \equiv \varepsilon^2\left(\frac{\partial \alpha^u}{\partial t} - \frac{\partial^2 \alpha^u}{\partial x^2}\right) - g(\alpha^u(x,t,\varepsilon), v, x, t, \varepsilon) = \\
\sqrt{\varepsilon}\hat{g}_u(x,t)\Big[\sigma + O(\varepsilon^{a-1/2}) + (\hat{\varphi}_y(x,t)\theta_1 + \hat{\varphi}_z(x,t)\theta_2)\sigma^2\Big]h(x,t) \\
- \quad \frac{1}{2}\sigma^2\varepsilon\Big[\hat{g}_{uu}(x,t) - 2(\hat{g}_{uy}(x,t)\theta_1 + \hat{g}_{uz}(x,t)\theta_2)\sigma + \\
(\hat{g}_{yy}(x,t)\theta_1^2 + 2\hat{g}_{yz}(x,t)\theta_1\theta_2 + \hat{g}_{zz}(x,t)\theta_2^2)\sigma^2\Big]h^2(x,t) - \varepsilon\hat{g}_\varepsilon(x,t) + o(\varepsilon).
\end{aligned} \tag{3.22}
$$

There is a sufficiently small σ_0 such that for $0 < \sigma \leq \sigma_0$

$$
1 + \sigma(\hat{\varphi}_y(x,t)\theta_1 + \hat{\varphi}_z(x,t)\theta_2) \geq 1/2 \quad \text{for} \quad (x,t) \in Q_\nu,\ |\theta_i| \leq 1,\ i = 1,2.
$$

Thus, because of $a - 1/2 > 0$ and taking into account (2.15), (2.16) and (3.9), we have for sufficiently small ε

$$
\hat{g}_u(x,t)\Big[\sigma + O(\varepsilon^{a-1/2}) + (\hat{\varphi}_y(x,t) + \hat{\varphi}_z(x,t))\sigma^2\theta\Big]h(x,t) \leq 0. \tag{3.23}
$$

By assumption (A_8) there is a positive constant k_g such that for sufficiently small ν

$$
-\hat{g}_\varepsilon(x,t) \leq -k_g < 0 \quad \text{for} \quad (x,t) \in Q_\nu.
$$

Now we choose σ_0 so small that for $0 < \sigma \leq \sigma_0$ and for $(x,t) \in Q_\nu$

$$
\begin{aligned}
&\frac{1}{2}\sigma^2\Big|\hat{g}_{uu}(x,t) - 2(\hat{g}_{uy}(x,t)\theta_1 + \hat{g}_{uz}(x,t)\theta_2)\sigma) + \\
&(\hat{g}_{yy}(x,t)\theta_1^2 + 2\hat{g}_{yz}(x,t)\theta_1\theta_2 + \hat{g}_{zz}(x,t)\theta_2^2)\sigma^2\Big|h^2(x,t) \leq k_g/2 \quad \text{for} \quad (x,t) \in Q_\nu.
\end{aligned} \tag{3.24}
$$

Therefore, for $0 < \sigma \leq \sigma_0$, and for sufficiently small ε we get from (3.22), (3.23), and (3.24)

$$
(L\alpha^u)(x,t,\varepsilon) \leq 0,
$$

i.e. inequality (2.21) is satisfied for α^u in Q_ν.

Now we will prove that α^u and β^u will satisfy the inequalities (2.21), (2.22) in $Q_c \backslash Q_\nu$. From (3.14) we get

$$
g(\beta^u(.), v, x, t, \varepsilon) = \sqrt{\varepsilon}\Big[\hat{g}_u(x,t)\gamma + (\hat{g}_y(x,t)\theta_1 + \hat{g}_z(x,t)\theta_2)\sigma^2\Big]h(x,t) + o(\sqrt{\varepsilon}). \tag{3.25}
$$

It follows from (2.15) that there is a positive constant c_1 such that for sufficiently large γ

$$
\hat{g}_u(x,t)\gamma + (\hat{g}_y(x,t)\theta_1 + \hat{g}_z(x,t)\theta_2)\sigma^2 \leq -c_1 \quad \text{in} \quad Q_c \backslash Q_\nu. \tag{3.26}
$$

Therefore, by (2.18), (3.12), (3.25), and (3.26) we have for γ sufficiently large and ε sufficiently small

$$
(L\beta^u)(x,t,\varepsilon) \geq 0 \quad \text{in} \quad Q_c \backslash Q_\nu.
$$

Analogously, we get from (3.22) for $(x,t) \in Q_c \backslash Q_\nu$ and for σ and ε sufficiently small

$$(L\alpha^u)(x,t,\varepsilon) = \sqrt{\varepsilon}\hat{g}_u(x,t)(\sigma + (\hat{\varphi}_y(x,t)\theta_1 + \hat{\varphi}_z(x,t)\theta_2)\sigma^2)h(x,t) + o(\sqrt{\varepsilon}) \le 0.$$

Thus, the inequalities (2.21) for α^u, β^u hold in $Q_c \backslash Q_\nu$.

Now we verify the inequality (2.22) in Q_c. For u and z we use the representations

$$\begin{aligned} u &= \hat{u}(x,t) + \sqrt{\varepsilon}h(x,t)\kappa + O(\varepsilon^a) \quad \text{for} \quad -\sigma \le \kappa \le \gamma, \\ u &= \hat{z}(x,t) + \sqrt{\varepsilon}\sigma^2 h(x,t)\theta_2 \quad \text{for} \quad -1 \le \theta_2 \le 1. \end{aligned}$$

By (2.19) and (3.8) we have

$$\begin{aligned} (M\beta^y)(.) \equiv \frac{\partial \beta^y}{\partial t} - f_1(u, \beta^y(x,t,\varepsilon), z, x, t, \varepsilon) = \frac{\partial \hat{y}}{\partial t} + \sqrt{\varepsilon}\frac{\sigma^2}{\nu}h(x,t) - \\ f_1(\hat{u}(x,t) + \sqrt{\varepsilon}h(x,t)\kappa + O(\varepsilon^a), \hat{y} + \sqrt{\varepsilon}\sigma^2 h(x,t), \hat{z} + \sqrt{\varepsilon}\sigma^2 h(x,t)\theta_2, x, t, \varepsilon). \end{aligned} \tag{3.27}$$

Using the representation

$$\begin{aligned} &f_1(\hat{u}(x,t) + \sqrt{\varepsilon}h(x,t)\kappa + O(\varepsilon^a), \hat{y} + \sqrt{\varepsilon}\sigma^2 h(x,t), \hat{z} + \sqrt{\varepsilon}\sigma^2 h(x,t)\theta_2, x, t, \varepsilon) = \\ &\hat{f}_1(x,t) + \sqrt{\varepsilon}\Big[\hat{f}_{1u}(x,t)\kappa + (\hat{f}_{1y}(x,t) + \hat{f}_{1z}(x,t)\theta_2)\sigma^2\Big]h(x,t) + o(\sqrt{\varepsilon}) \end{aligned}$$

and taking into account

$$\frac{\partial \hat{y}}{\partial t} - \hat{f}_1(x,t) \equiv 0$$

we get from (3.27)

$$(M\beta^y)(.) = \sqrt{\varepsilon}\Big[\frac{\sigma^2}{\nu} - \hat{f}_{1u}(x,t)\kappa - (\hat{f}_{1y}(x,t) + \hat{f}_{1z}(x,t)\theta_2)\sigma^2\Big]h(x,t) + o(\sqrt{\varepsilon}). \tag{3.28}$$

To given $\sigma > 0$ we choose ν so small such that

$$\Big[\frac{\sigma^2}{\nu} - \hat{f}_{1u}(x,t)\kappa - (\hat{f}_{1y}(x,t) + \hat{f}_{1z}(x,t)\theta_2)\sigma^2\Big]h(x,t) \ge c_2 \quad \text{for} \quad (x,t) \in Q_c,$$

where c_2 is some positive number. Thus, for sufficiently small ε, we have

$$(M\beta^y)(x,t,\varepsilon) \ge 0 \quad \text{for} \quad (x,t) \in Q_c.$$

Similarly we can verify the inequality (2.22) for α^y.
The verification of the inequalities (2.23) for the operator N follows exactly the same line as before.

Finally, we verify the inequalities (2.24). If we differentiate β^u with respect to x at $x = 0$ and $x = 1$ respectively, we get from (3.8)

$$\begin{aligned} \frac{\partial \beta^u}{\partial x}(0,t,\varepsilon) &= \frac{\partial \tilde{u}}{\partial x}(0,t,\varepsilon) - k + O(\sqrt{\varepsilon}), \\ \frac{\partial \beta^u}{\partial x}(1,t,\varepsilon) &= \frac{\partial \tilde{u}}{\partial x}(1,t,\varepsilon) + k + O(\sqrt{\varepsilon}). \end{aligned}$$

Using (3.5) and (3.6) it can be shown that there exists a positive constant c_3 such that

$$\left|\frac{\partial \tilde{u}}{\partial x}(x,t,\varepsilon)\right| \le c_3 \quad \text{for} \quad (x,t) \in \overline{Q}.$$

Consequently, the inequalities (2.24) for β^u from Definition 2.1 are satisfied if k is chosen sufficiently large and ε is sufficiently small. The inequalities (2.24) for α^u can be verified in a similar way.

From our considerations above it follows that the functions $\alpha(x,t,\varepsilon), \beta(x,t,\varepsilon)$ fulfil all conditions of Definition 2.1 , and we can conclude that for sufficiently small ε there exists a unique solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of problem (2.1) satisfying for $(x,t) \in \overline{Q_c}$

$$\begin{aligned} \alpha^u(x,t,\varepsilon) &\le u(x,t,\varepsilon) \le \beta^u(x,t,\varepsilon), \\ \alpha^v(x,t,\varepsilon) &\le v(x,t,\varepsilon) \le \beta^v(x,t,\varepsilon). \end{aligned}$$

From these inequalities and from (3.8) it follows that the representations (3.2) and (3.3) for $u(x,t,\varepsilon)$ and $v(x,t,\varepsilon)$ in $\overline{Q_c}$ are valid. This completes the proof of Theorem 3.1.

4 Example.

Consider the initial boundary value problem

$$\begin{aligned} \varepsilon^2\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) &= g(u,v,x,t,\varepsilon) \equiv -u(u-y+x+t+2) + \varepsilon I(x,t), \\ \frac{\partial y}{\partial t} &= f_1(u,v,x,t,\varepsilon) \equiv u+2, \\ \frac{\partial z}{\partial t} &= f_2(u,v,x,t,\varepsilon) \equiv y, \qquad (4.1) \\ (x,t) \in Q &:= \{(x,t) \in R^2 : 0 < x < 1, 0 < t \le T\},\ T > 2, \\ \frac{\partial u}{\partial x}(0,t,\varepsilon) &= \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0 \quad \text{for} \quad 0 < t \le T, \\ u(x,0,\varepsilon) = u^0(x) &> 0,\ y(x,0,\varepsilon) = y^0(x) \equiv 1, z(x,0,\varepsilon) = z^0(x) \equiv 0 \ \text{ for } 0 \le x \le 1, \end{aligned}$$

where $I : \overline{Q} \to R$ is smooth and positive, u^0 is a smooth function on $0 \le x \le 1$. The degenerate equation

$$-u(u-y+x+t+2) = 0$$

has two solutions

$$u = \varphi_1(y,z,x,t) \equiv 0 \qquad \text{and} \qquad u = \varphi_2(y,z,x,t) \equiv y-x-t-2 \qquad (4.2)$$

intersecting in a smooth surface with the representation

$$y = s(z, x, t) \equiv x + t + 2.$$

The inequality $\varphi_1(y, z, x, t) > (<) \varphi_2(y, z, x, t)$ holds for $y < (>) s(z, x, t), (x, t) \in \overline{Q}$, i.e. the assumptions (A_1) and (A_2) are fulfilled.

From (4.1) and (4.2) we get

$$g_u(\varphi_1(y, z, x, t), x, t, 0) \equiv y - x - t - 2 \equiv -g_u(\varphi_2(y, z, x, t), x, t, 0).$$

Obviously we have for $(x, t) \in \overline{Q}$

$$\begin{aligned} g_u(\varphi_1(v, x, t), x, t, 0) &< 0, \; g_u(\varphi_2(v, x, t), x, t, 0) > 0 \;\text{ for }\; y < s(z, x, t), \\ g_u(\varphi_1(v, x, t), x, t, 0) &> 0, \; g_u(\varphi_2(v, x, t), x, t, 0) < 0 \;\text{ for }\; y > s(z, x, t), \end{aligned}$$

i.e. assumption (A_3) holds.

Note that $1 \equiv y^0(x) < s(z, x, 0) = x + 2$ for $x \in [0, 1]$, $f_1(\varphi_1(v, x, t), v, x, t, 0) \equiv 2$, $f_2(\varphi_1(v, x, t), v, x, t, 0) \equiv y$. Therefore, the initial value problem for $y_1(x, t), z_1(x, t)$ reads

$$\frac{dy_1}{dt} = 2, \frac{dz_1}{dt} = y_1, \quad 0 < t \leq T; \;\; y_1(x, 0) = 1, \; z_1(x, 0) = 0.$$

It has the solution

$$y_1(x, t) = 2t + 1, \; z_1(x, t) = t^2 + t.$$

The equation

$$y_1(x, t) = s(z_1(x, t), x, t), \text{ i.e. } 2t + 1 = x + t + 2$$

defines the curve $\mathcal{C}$:

$$t = t_c(x) = x + 1.$$

It is obvious that

$$y_1(x, t) < s(z_1(x, t), x, t) \;\text{ for }\; 0 \leq t < t_c(x)$$

and

$$y_1(x, t) > s(z_1(x, t), x, t) \;\text{ for }\; t_c(x) < t \leq T,$$

i.e. assumption (A_4) is fulfilled.

From $f_1(\varphi_2(v, x, t), v, x, t, 0) \equiv y - x - t$, $f_2(\varphi_2(v, x, t), v, x, t, 0) \equiv y$ and $y_1(x, t_c(x)) \equiv 2x + 3, z_1(x, t_c(x)) \equiv (x + 1)(x + 2)$ it follows that the initial value problem for $y_2(x, t), z_2(x, t)$ reads

$$\frac{dy_2}{dt} = y_2 - x - t, \frac{dz_2}{dt} = y_2, \; y_2(x, t_c(x)) = 2x + 3, z_2(x, t_c(x)) = (x + 1)(x + 2).$$

Its solution is

$$y_2(x, t) = \exp(t - x - 1) + x + t + 1, \; z_2(x, t) = \exp(t - x - 1) + \tilde{z}_2(x, t),$$

where $\tilde{z}_2(x,t) = \frac{1}{2}t^2 + t(x+1) - \frac{1}{2}(x^2+1)$.

It is easy to check that

$$y_2(x,t) > s(z_2(x,t),x,t) \ \text{ for } \ t_c(x) < t \leq T \quad (\text{ i.e. in } Q_2),$$

and

$$s_2(x,t) < s(z_2(x,t),x,t) \quad \text{for} \ \ 0 \leq t < t_c(x) \quad (\text{ i.e. in } Q_1).$$

Therefore, assumption (A_5) holds.

In our example the composed stable solution has the form

$$\hat{u}(x,t) = \begin{cases} \psi_1(x,t) \equiv 0 \quad \text{in } \overline{Q}_1, \\ \psi_2(x,t) \equiv \exp(t-x-1)-1 \ \text{ in } \ \overline{Q}_2, \end{cases} \tag{4.3}$$

$$\hat{y}(x,t) = \begin{cases} y_1(x,t) \equiv 2t+1 \quad \text{in } \overline{Q}_1, \\ y_2(x,t) \equiv \exp(t-x-1)+x+t+1 \ \text{ in } \ \overline{Q}_2, \end{cases} \tag{4.4}$$

$$\hat{z}(x,t) = \begin{cases} z_1(x,t) \equiv t^2+t \quad \text{in } \overline{Q}_1, \\ z_2(x,t) \equiv \exp(t-x-1)+\tilde{z}_2(x,t) \text{ in } \overline{Q}_2. \end{cases} \tag{4.5}$$

Now we verify the hypotheses (A_6) - (A_8). The associated equation (2.5) for $y = y^0(x) \equiv 1, z = z^0(x) \equiv 0, \ t = 0$ reads

$$\frac{du}{d\tau} = -u(u+x+1), \ \tau > 0.$$

It is easy to see that the solution $\overline{u}(x,\tau)$ of this equation with the initial condition

$$\overline{u}(x,0) = u^0(x) > 0$$

exists for $\tau > 0$ and tends to $\varphi_1(v^0(x),x,0) = 0$ as $\tau \to \infty$. Hence, assumption (A_6) is fulfilled.

Moreover, the solution $\Pi_0(x,\tau)$ of problem (3.1) which reads in our case

$$\frac{d\Pi_0}{d\tau} = -\Pi_0(\Pi_0+x+1), \ \tau > 0; \ \Pi_0(x,0) = u^0(x)$$

can be found in the explicit form

$$\Pi_0(x,\tau) = u^0(x)(x+1)\Big[u^0(x)(1-\exp(-(x+1))\tau)+x+1\Big]^{-1}\exp(-(x+1)\tau). \tag{4.6}$$

Assumptions (A_7) and (A_8) are obviously satisfied since

$$g_{uu} \equiv -2 < 0 \ \text{ in } \ \overline{Q} \ \text{ and } \ g_\varepsilon \equiv I(x,t) > 0 \ \text{ in } \ \overline{Q}.$$

Thus, all assumptions (A_0)-(A_8) of Theorem 3.1 are fulfilled. Therefore, the initial boundary value problem (2.1) for our example has a unique solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ satisfying (3.2), (3.3) where $\hat{u}(x,t), \hat{v}(x,t)$, and $\Pi_0(x,\tau)$ are defined in (4.3) - (4.6).

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