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Boundary-Oriented Subdifferential Characterization of Calmness for Convex Systems

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Abstract

We study subdifferential characterizations of the calmness property for multifunctions representing convex constraint systems in a Banach space. Extending earlier work in finite dimensions, we show that - in contrast to the stronger Aubin property of a multifunction (or metric regularity of its inverse), calmness can be ensured by corresponding weaker constraint qualifications which are based on boundaries of subdifferentials and normal cones only rather than on the full objects.

1 Introduction

Following [15] (p.399), a multifunction $M : Y \rightrightarrows X$ between metric spaces X, Y is *calm* at some point $(\bar{y}, \bar{x})$ of its graph if there exist neighborhoods $\mathcal{V}, \mathcal{U}$ of $\bar{y}, \bar{x}$, respectively, and some $L > 0$ such that the corresponding distance functions satisfy

$$d(x, M(\bar{y})) \leq L d(y, \bar{y}) \qquad \forall x \in M(y) \cap \mathcal{U} \quad \forall y \in \mathcal{V}. \tag{1}$$

With $\mathcal{U} := X$, calmness reduces to the upper Lipschitz property of multifunctions introduced by Robinson [14]. Obviously, calmness is also weaker than the well-known Aubin property of multifunctions

$$d(x, M(y')) \leq L d(y, y') \qquad \forall x \in M(y) \cap \mathcal{U} \quad \forall y, y' \in \mathcal{V}. \tag{2}$$

(in particular, $M(y) = \emptyset$ for y close to but different from $\bar{y}$ is possible under calmness but violates the Aubin property). As a stability concept, calmness of multifunctions is important for issues related with optimality conditions, stability of solutions to parametric optimization problems or conditioning. For instance, in the context of finite dimensional optimization problems with Lipschitzian data (inequalities, equations and objective function), calmness of the constraint mapping defined by right-hand-side perturbations of inequalities and equations implies calmness of the optimization problem in the sense of Clarke and, hence, ensures the existence of (nondegenerate) Lagrange multipliers at local solutions (see [4], Prop. 6.4.4).

If M is a polyhedral multifunction, then it is automatically calm (see [14]). Apart from this special class, certain conditions have to hold true in order to ensure calmness, and it seems natural to characterize these conditions in terms of well-known objects from nonsmooth analysis such as (co-) derivatives, (sub-) differentials or tangent or normal cones. Similar characterizations have been successfully established for the stronger Aubin property. In finite dimensions, for instance, (2) is equivalent to each of the following two conditions due to Mordukhovich [10] and Aubin (see, e.g. [1] and [5], Corollary 1.19 for sufficiency), respectively:

$$
\begin{aligned}
D^*M(\bar{y},\bar{x})(0) &= \{0\} && (3)\\
\exists \alpha,\beta>0:\ B(0,1) &\subseteq D_-M^{-1}(\bar{x},\bar{y})(B(0,\alpha))\ \forall x,y\in \operatorname{Gph} M\cap B((\bar{x},\bar{y}),\beta) && (4)
\end{aligned}
$$

Here, D^* and D_- refer to Mordukhovich's coderivative and to the contingent derivative, respectively, while B refers to appropriate closed balls. As coderivatives relate to normal cones whereas derivatives are associated with tangent cones, the first criterion above is of dual nature and the second one is of primal nature. The question arises if the criteria above can be modified appropriately in order to characterize the weaker calmness property (1) rather than (2). An answer was given in [6] for dual characterizations of (1) in the special case of finite dimensional multifunctions of the type

$$M(y) := \{x \in C | g(x) + y \in D\},$$

where $C \subseteq \mathbb{R}^p, D \subseteq \mathbb{R}^m$ are closed subsets and $g : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. This is the typical structure of constraint systems in nonlinear optimization or complementarity problems. In this special case, Mordukhovich's criterion (3) for the Aubin property takes the form

$$\bigcup_{y^* \in N_D(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(y^*) \cap (-N_C(\bar{x})) = \emptyset,$$

where N refers to Mordukhovich's normal cone. It was shown in [6] that under mild assumptions, calmness is implied by the weaker condition

$$\bigcup_{y^* \in N_D(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(y^*) \cap (-\operatorname{bd} N_C(\bar{x})) = \emptyset,$$

where 'bd' refers to the topological boundary. Hence, passing from Lipschitzian stability to upper Lipschitzian stability, is reflected in a transition from certain geometric objects to their boundaries. This fact becomes most evident for the simple case of one single inequality $g(x)+y \leq 0$ (i.e., $D = \mathbb{R}_-$): if g (as a function) and C (as a set) are regular in the sense of Clarke, then calmness of M holds true at some point $(0, \bar{x})$ with $g(\bar{x}) = 0$ provided that $\operatorname{bd} \partial g(\bar{x}) \cap (-\operatorname{bd} N_C(\bar{x})) = \emptyset$. Here, '$\partial$' refers to either Mordukhovich's or Clarke's subdifferential (which coincide due to regularity). This last constraint qualification can be opposed again to the corresponding criterion of the Aubin property which now takes the form $\partial g(\bar{x}) \cap (-N_C(\bar{x})) = \emptyset$. For absent abstract constraints ($C = \mathbb{R}^p$) the calmness condition reduces to $0 \notin \operatorname{bd} \partial g(\bar{x})$. In particular, for convex g, a (nondegenerate) multiplier rule can be obtained under this 'weak Slater condition' (as opposed to the classical Slater condition $0 \notin \partial g(\bar{x})$ which ensures the stronger Aubin property).

The aim of this paper is to study possible infinite-dimensional extensions of the previous results. For the single inequality (plus abstract constraints) in a Banach space setting it turns out that, even for Clarke-regular data, the mentioned constraint qualification $\mathrm{bd}\,\partial g(\bar{x}) \cap (-\mathrm{bd}\,N_C(\bar{x})) = \emptyset$ no longer implies calmness. It does so, however, for convex data, and in this case it can be even weakened again. This gives an improvement even for the finite-dimensional case. Therefore, the focus of the paper is on convex constraint systems.

2 Notation

Throughout this paper, X will denote some Banach space and X^* its dual being endowed with the strong topology. In these spaces, $B(\alpha, \beta)$ and $B^*(\alpha, \beta)$ are the closed balls around α with radius β,whereas $B^0(\alpha, \beta)$ refers to the corresponding open ball in X. By i_S we denote the indicator function of a closed set $S \subseteq X$ and by epi$f$ the epigraph of some function $f : X \to \mathbb{R} \cup \{\infty\}$. $N(S; x), \partial f$ and $\partial^\infty f$ refer to the normal cone to S at some $x \in S$ and to the usual and singular subdifferentials of f, respectively, all of them in the sense of convex analysis. In contrast, ∂^c represents Clarke's subdifferential. 'bd' and 'int' are the topological boundary and interior. For a multifunction $M : X \rightrightarrows Y$ between Banach spaces,

$$\begin{aligned} \mathrm{Gph}\,M &= \{(x,y) \in X \times Y | y \in M(x)\} \\ \mathrm{range}\,M &= \{y \in Y | \exists x \in X, y \in M(x)\} \\ M^{-1} &: Y \rightrightarrows X; \quad M^{-1}(y) = \{x \in X | y \in M(x)\} \end{aligned}$$

denote its graph, its range and its inverse, respectively.

3 Convex constraint systems with a perturbed inequality

In this section, we consider constraint systems involving a fixed abstract constraint set and an inequality which is subject to perturbations. More precisely, we are interested in the calmness property (1) of the multifunction

$$M(y) := \{x \in C \mid f(x) \le y\} \quad (y \in \mathbb{R}), \tag{5}$$

where C is a closed, convex subset of some Banach space X and f is a convex, lower semicontinuous function. First, we state an auxiliary result. Recall from [2], that a set $S \subseteq X$ is *compactly epi-Lipschitzian* at some $x^0 \in S$ if there exist a norm-compact set K and a constant $r > 0$ such that

$$S \cap B(x^0, r) + B(0, tr) \subseteq S - tK \quad \forall t \in (0, r).$$

Lemma 3.1 *For C and f as introduced above, the sum rule*

$$\partial(f + i_C)(\bar{x}) \subseteq \partial f(\bar{x}) + N(C; \bar{x})$$

applies if the following constraint qualification is satisfied:

$$\left.\begin{array}{l}\partial^\infty f(\bar{x}) \cap -N(C;\bar{x}) = \{0\} \quad \text{and} \\ C \text{ or } \operatorname{epi} f \textit{ is compactly epi-Lipschitzian at } \bar{x}\end{array}\right\} \quad (CQ^*)$$

Proof. Define two closed and convex subsets of $X \times \mathbb{R}$ by $D_1 = \operatorname{epi} f$ and $D_2 = C \times \mathbb{R}$. The first part of (CQ*) implies that

$$N(D_1;(\bar{x};f(\bar{x}))) \cap -N(D_2;(\bar{x};f(\bar{x}))) = \{0\}.$$

Along with the second part of (CQ*), this last relation is sufficient for the intersection rule

$$N(D_1 \cap D_2;(\bar{x};f(\bar{x}))) \subseteq N(D_1;(\bar{x};f(\bar{x}))) + N(D_2;(\bar{x};f(\bar{x})))$$

(see [7], Cor. 4.5). Now, let $x^* \in \partial(f+i_C)(\bar{x})$ be arbitrarily given, i.e., $\langle x^*, x-\bar{x}\rangle \leq f(x) - f(\bar{x})$ for all $x \in C$. Consequently,

$$\langle (x^*,-1),(x,t)-(\bar{x},f(\bar{x})\rangle \leq 0 \quad \forall x \in C \ \forall t \geq f(x).$$

In other words, $(x^*,-1) \in N(D_1 \cap D_2;(\bar{x};f(\bar{x})))$, and the above intersection rule ensures that $(x^*,-1) = (y^*,r) + (z^*,t)$ for certain $(y^*,r) \in N(D_1;(\bar{x};f(\bar{x})))$ and $(z^*,t) \in N(D_2;(\bar{x};f(\bar{x})))$. By definition of D_2 one gets $t=0$ and $z^* \in N(C;\bar{x})$. It results that $r=-1$, hence $y^* \in \partial f(\bar{x})$ by definition of D_1. Summarizing, $x^* \in \partial f(\bar{x}) + N(C;\bar{x})$, as was to be shown. ■

Remark 3.2 *The constraint qualification (CQ*) in Lemma 3.1 is always satisfied if the convex function f is continuous at $\bar{x}$ or $\bar{x}$ is an interior point of C. The second part of (CQ*) holds true whenever X is finite dimensional or the convex set C has nonempty interior.*

Theorem 3.3 *With the setting introduced above, the multifunction M in (5) is calm at a point $(0,\bar{x}) \in \operatorname{Gph} M$ of its graph if one of the following conditions is satisfied:*

$$f(\bar{x}) < 0 \tag{6}$$
$$\operatorname{bd}\partial f(\bar{x}) \cap -\operatorname{bd} N(C;\bar{x}) \neq \partial f(\bar{x}) \cap -N(C;\bar{x}) \tag{7}$$
$$\operatorname{bd}\partial f(\bar{x}) \cap -\operatorname{bd} N(C;\bar{x}) = \emptyset \quad \text{and} \quad (CQ^*) \tag{8}$$

Proof. From $(0,\bar{x}) \in \operatorname{Gph} M$ it follows that $\bar{x} \in C$ and $f(\bar{x}) \leq 0$. In case of (6), it follows that

$$0 \in \operatorname{int}[f(\bar{x}),\infty) \subseteq \operatorname{int}\operatorname{range} M^{-1}. \tag{9}$$

Since M has closed and convex graph, this last relation implies metric regularity of M^{-1} at $(\bar{x},0)$ by the Robinson-Ursescu Theorem ([12],[16]). However, metric regularity of M^{-1} at $(\bar{x},0)$ is equivalent to M having the Aubin property at $(0,\bar{x})$ (cf. [3],[11],[15]), which in turn implies calmness of M at $(0,\bar{x})$. Hence, in the sequel we assume that $f(\bar{x}) = 0$. Suppose next that (7) is satisfied. Then, since both $\partial f(\bar{x})$ and $-N(C;\bar{x})$ are (strongly) closed in X^*, it holds that

$$\operatorname{int}\partial f(\bar{x}) \cap -N(C;\bar{x}) \neq \emptyset \quad \text{or} \quad \partial f(\bar{x}) \cap -\operatorname{int} N(C;\bar{x}) \neq \emptyset. \tag{10}$$

If the first condition of (10) holds true, then choose $x^* \in \operatorname{int}\partial f(\bar{x}) \cap -N(C;\bar{x})$. Accordingly, there exists some $\alpha > 0$ such that $B^*(x^*;\alpha) \subseteq \partial f(\bar{x})$. In other words:

$$\langle x^* + \alpha p^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) = f(x) \quad \forall p^* \in B^*(0;1)\ \forall x \in X.$$

It follows that

$$\langle p^*, x - \bar{x}\rangle \leq \alpha^{-1}(f(x) - \langle x^*, x - \bar{x}\rangle) \leq \alpha^{-1} f(x) \quad \forall p^* \in B^*(0;1)\ \forall x \in C,$$

since $x^* \in -N(C;\bar{x})$. Consequently,

$$\|x - \bar{x}\| \leq \alpha^{-1} f(x) \quad \forall x \in C \quad \text{and} \quad f(x) \geq 0 \quad \forall x \in C, \tag{11}$$

so it follows the desired calmness property of M (with $\mathcal{U} := X$ and $\mathcal{V} := \mathbb{R}$ in (1)):

$$d(x, M(0)) \leq \|x - \bar{x}\| \leq \alpha^{-1} y = \alpha^{-1} d(y, 0) \quad \forall y \in \mathbb{R}\ \forall x \in M(y).$$

If the second condition of (10) holds true, then choose $x^* \in \partial f(\bar{x}) \cap -\operatorname{int} N(C;\bar{x})$. Now, there is some $\alpha > 0$ such that $B^*(x^*;\alpha) \subseteq -N(C;\bar{x})$, hence

$$\langle x^* - \alpha p^*, x - \bar{x}\rangle \geq 0 \quad \text{or} \quad \langle p^*, x - \bar{x}\rangle \leq \alpha^{-1}\langle x^*, x - \bar{x}\rangle \quad \forall p^* \in B^*(0;1)\ \forall x \in C.$$

Due to $x^* \in \partial f(\bar{x})$, this yields $\|x - \bar{x}\| \leq \alpha^{-1}\langle x^*, x - \bar{x}\rangle \leq \alpha^{-1} f(x)$ for all $x \in C$. In this way, we end up once more at relation (11) and, hence, at calmness of M at $(0, \bar{x})$ as above.

Finally, assume that (8) holds true. If $0 \in \operatorname{int}\partial f(\bar{x})$, then - because of $0 \in \partial f(\bar{x}) \cap -N(C;\bar{x})$ - (7) is satisfied and calmness of M follows as shown before. Suppose that $0 \in \operatorname{bd}\partial f(\bar{x})$. In case that $N(C;\bar{x}) = X^*$, calmness of M follows again from (7). In the opposite case $N(C;\bar{x}) \neq X^*$ it always holds that $0 \in -\operatorname{bd} N(C;\bar{x})$ which gives a contradiction to (8). It remains to check the case

$$0 \notin \partial f(\bar{x}). \tag{12}$$

Then, two possibilities are left:

$$\partial f(\bar{x}) \cap -N(C;\bar{x}) = \emptyset \quad \text{or} \quad \partial f(\bar{x}) \subseteq -\operatorname{int} N(C;\bar{x}). \tag{13}$$

To verify this alternative, assume that none of the two conditions is satisfied. Then, there exist $x_1^*, x_2^* \in \partial f(\bar{x})$ such that $x_1^* \in -N(C;\bar{x})$ and $x_2^* \notin -\operatorname{int} N(C;\bar{x})$. The convexity of $\partial f(\bar{x})$ and $-N(C;\bar{x})$ guarantees the existence of some x^* (on the line segment $[x_1^*, x_2^*]$), such that $x^* \in \partial f(\bar{x}) \cap -\operatorname{bd} N(C;\bar{x})$. By the cone property of $N(C;\bar{x})$, one has that $tx^* \in -\operatorname{bd} N(C;\bar{x})$ for all $t > 0$. Due to the closedness of $\partial f(\bar{x})$, there must be some $t^* > 0$ such that $t^* x^* \notin \partial f(\bar{x})$ (otherwise a contradiction with (12)). But then, since $x^* \in \partial f(\bar{x})$, there must exist some $\hat{t} > 0$ such that $\hat{t}x^* \in \operatorname{bd} \partial f(\bar{x})$. At the same time, $\hat{t}x^* \in -\operatorname{bd} N(C;\bar{x})$, whence a contradiction to (8), and (13) must hold true.

Now, the first case of (13) implies the existence of some $x' \in C$ such that $f(x') < 0$ (Slater's condition). Indeed, negating Slater's condition means that $\bar{x}$ is a minimum of f over C or, equivalently, a free minimum of the lower semicontinuous function $f + i_C$. Consequently,

$$0 \in \partial(f + i_C)(\bar{x}) \subseteq \partial f(\bar{x}) + N(C;\bar{x}),$$

where we applied Lemma 3.1. However, the obtained relation contradicts the first case of (13). Hence, Slater's condition is satisfied and one has (9) with $\bar{x}$ replaced by x'. Consequently, calmness of M at $\bar{x}$ follows as in the lines below (9).

Concerning the second case of (13), assume first that $\partial f(\bar{x}) = \emptyset$. Then, we are back to the already considered first case of (13). Finally, if $\partial f(\bar{x}) \neq \emptyset$, then the second case of (13) along with (8) yields (7) and calmness of M at $(0,\bar{x})$ follows again. ■

For missing abstract constraints, a much simpler characterization of calmness can be derived from Theorem 3.3:

Corollary 3.4 *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ a convex, lower semicontinuous function. Then, the multifunction $M(y) := f^{-1}(-\infty, y]$ is calm at a point $(0,\bar{x})$ with $f(\bar{x}) \leq 0$ if*

$$f(\bar{x}) < 0 \quad \text{or} \quad 0 \notin \operatorname{bd} \partial f(\bar{x}). \tag{14}$$

Proof. The first condition of (14) coincides with (6), thus it suffices to consider the second condition of (14). Evidently, in the setting of (5), we have $C = X$, hence $N(C;\bar{x}) = \operatorname{bd} N(C;\bar{x}) = \{0\}$. Along with $0 \notin \operatorname{bd} \partial f(\bar{x})$, this provides

$$\operatorname{bd} \partial f(\bar{x}) \cap -\operatorname{bd} N(C;\bar{x}) = \emptyset,$$

hence (8) is satisfied (note that (CQ*) is trivially satisfied in the context of this corollary, see Remark 3.2). ■

Note that in the setting of Corollary 3.4, we have the following implications

$$(7) \Longrightarrow 0 \in \operatorname{int} \partial f(\bar{x}) \Longrightarrow (8) \Longrightarrow (14).$$

Hence, in contrast to the alternative of conditions (7) and (8) in Theorem 3.3, there is no use of considering (7) in addition to (14) here . In the general setting of Theorem 3.3, however, it is no longer true that (7) implies (8) as can be seen from Example 3.6 below.

Remark 3.5 *For finite dimensional X, condition (8) - with the convex subdifferential replaced by Clarke one's - was shown in [6] to be sufficient for calmness of* *the multifunction M if f is locally Lipschitzian and both f and C are regular in the sense of Clarke. Theorem 3.3 demonstrates that this condition can be weakened to '(7) or (8)' in the convex case even if X is infinite dimensional. More precisely, one has the following structure of constraint qualifications here (assuming that f is continuous at $\bar{x} \in C$ and $f(\bar{x}) = 0$):*

$$\begin{array}{ccccc} \partial f(\bar{x}) \cap -N(C;\bar{x}) = \emptyset & \Longrightarrow & (8) & \Longrightarrow & (7) \text{ or } (8) \\ \Updownarrow & & & & \\ \textit{Slater's condition} & & & & \Downarrow \\ \Updownarrow & & & & \\ \textit{Aubin property of } M \text{ at } (0,\bar{x}) & & & & \textit{calmness of } M \text{ at } (0,\bar{x}) \end{array} \qquad (15)$$

We continue by some examples.

Example 3.6 *All constraint qualifications considered in Remark 3.5 are strictly different. Setting, for instance, $f(x) = |x|$, $C = \mathbb{R}$, $\bar{x} = 0$, Slater's condition is obviously violated (and also $0 \in \partial f(\bar{x}) \cap -N(C;\bar{x}) \neq \emptyset$), whereas (8) holds true:*

$$\operatorname{bd} \partial f(\bar{x}) \cap -\operatorname{bd} N(C;\bar{x}) = \{-1,1\} \cap \{0\} = \emptyset.$$

Indeed, M is calm at $(0,\bar{x})$ but fails to have the Aubin property there. Another example is $f(x) = f(x_1,x_2) = \|x\|$, $C = \{(x_1,x_2) \mid x_1 \geq 0\}$. Then,

$$\begin{aligned} \operatorname{bd} \partial f(\bar{x}) \cap -\operatorname{bd} N(C;\bar{x}) &= \{(x_1,x_2) \mid x_1^2 + x_2^2 = 1,\ x_1 \geq 0, x_2 = 0\} = \{(1,0)\}, \\ \partial f(\bar{x}) \cap -N(C;\bar{x}) &= \{(x_1,x_2) \mid x_1^2 + x_2^2 \leq 1,\ x_1 \geq 0, x_2 = 0\} \\ &= \operatorname{conv}\{(0,0),(1,0)\}. \end{aligned}$$

Hence, (8) is violated here, whereas (7) is satisfied and thus, Theorem 3.3 ensures calmness of M at $(0,\bar{x})$. Again, M fails to have the Aubin property.

The following example demonstrates that Theorem 3.3 provides just a sufficient but not a necessary condition for the calmness of the multifunction M considered there.

Example 3.7 *Let $X = C = \mathbb{R}$, $\bar{x} = 0$ and $f(x) = \max\{x,0\}$. Then, $(0,\bar{x}) \in \operatorname{Gph} M$, $f(\bar{x}) = 0$ and $M(0) = \mathbb{R}_-$. One has $M(y) = \emptyset$ for $y < 0$ and $M(y) = (-\infty, y]$ for $y \geq 0$, hence, $d(x, M(0)) \leq d(y,0)$ for all $y \in \mathbb{R}$ and all $x \in M(y)$. This means calmness of M at $(0,\bar{x})$. On the other hand, since $\partial f(\bar{x}) = [0,1]$, (14) is violated, which implies violation of both (8) and (7).*

Note that, in the last example, M was a polyhedral multifunction, hence it seems that one cannot recover by Theorem 3.3 Robinson's result mentioned in the introduction. However, this will be possible after some modification following the ideas of [8].

The next example requires some technical work. It illustrates the limitation of Theorem 3.3 to convex data. In finite dimensions, the condition '$f(\bar{x}) < 0$ or $0 \notin$

bd $\partial^c f(\bar{x})$' (i.e., (14) with the convex replaced by Clarke's subdifferential) was found in [6] to ensure calmness of multifunctions (5) without abstract constraints (i.e., $C = X$) as long as f is regular at $\bar{x}$ in the sense of Clarke. This is no longer true in infinite dimensions unless the data are restricted to be convex as in Corollary 3.4.

Example 3.8 *For $k \in \mathbb{N}$, let $\tau_k \in (0, k^{-2})$ be the unique solution of $\tau + k\sqrt{\tau} = 1$. Define the sequence of real functions*

$$\varphi_k(\tau) := \begin{cases} |\tau|(1 - k\sqrt{|\tau|}) & \text{if } \tau \in [-\tau_k, \tau_k] \\ \tau_k^2 & \text{if } |\tau| \geq \tau_k \end{cases} .$$

Elementary analysis shows that each φ_k is (globally) Lipschitz continuous with modulus 1 and regular at zero in the sense of Clarke (close to the origin, each φ_k can be represented as the maximum of two C^1- functions). Furthermore,

$$\varphi_k \geq 0\,, \quad \varphi_k(\tau) = 0 \Longleftrightarrow \tau = 0 \quad \text{and} \quad \varphi_k(\tau_k) = \tau_k^2 \quad \forall k \in \mathbb{N}\, \forall \tau \in \mathbb{R}. \tag{16}$$

Now, let $X = l^1$ and define $f : X \to \mathbb{R}$ by $f(x) := \sum_{k=1}^{\infty} \varphi_k(x_k)$. Evidently, $f(0) = 0$ by (16). Since $\varphi_k(\tau) \leq \tau_k^2 \leq k^{-4}$ for all $\tau \in \mathbb{R}$ and all $k \in \mathbb{N}$, f is well defined. For arbitrary $x, y \in X$, one has

$$\begin{aligned} |f(x) - f(y)| &= |\sum_{k=1}^{\infty}(\varphi_k(x_k) - \varphi_k(y_k))| \leq \sum_{k=1}^{\infty} |\varphi_k(x_k) - \varphi_k(y_k)| \\ &\leq \sum_{k=1}^{\infty} |x_k - y_k| = \|x - y\|_1, \end{aligned}$$

hence f is (globally) Lipschitz continuous with modulus 1.

Next, we calculate Clarke's directional derivative $f^0(0; h)$ of f at zero in arbitrary direction $h \in X$. By definition (see [4]), one has

$$\begin{aligned} f^0(0; h) &= \limsup_{t \downarrow 0, x \to 0} \frac{f(x + th) - f(x)}{t} = \lim_{n \to \infty} \frac{f(x^{(n)} + t^{(n)}h) - f(x^{(n)})}{t^{(n)}} \\ &= \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{\varphi_k(x_k^{(n)} + t^{(n)}h_k) - \varphi_k(x_k^{(n)})}{t^{(n)}}, \end{aligned} \tag{17}$$

where $x^{(n)} \to 0$ and $t^{(n)} \downarrow 0$ are suitable sequences realizing the above limsup as a limit. Now, we fix an arbitrary $k' \in \mathbb{N}$. Assume that there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\varphi_{k'}(x_{k'}^{(n)} + t^{(n)}h_{k'}) - \varphi_{k'}(x_{k'}^{(n)})}{t^{(n)}} \leq \frac{\varphi_{k'}(t^{(n)}h_{k'})}{t^{(n)}} - \varepsilon \quad \forall n \geq n_0. \tag{18}$$

In order to lead (18) to a contradiction, define a sequence $\tilde{x}^{(n)} \in X$ by

$$\tilde{x}_k^{(n)} := \begin{cases} x_k^{(n)} & k \neq k' \\ 0 & k = k' \end{cases} \qquad \forall k, n \in \mathbb{N}.$$

It follows, that $\tilde{x}^{(n)} \to 0$ and, in view of (16),

$$\begin{aligned} \frac{f(\tilde{x}^{(n)}+t^{(n)}h)-f(\tilde{x}^{(n)})}{t^{(n)}} &= \sum_{k=1,k\neq k'}^{\infty} \frac{\varphi_k(x_k^{(n)}+t^{(n)}h_k)-\varphi_k(x_k^{(n)})}{t^{(n)}} + \frac{\varphi_{k'}(t^{(n)}h_{k'})}{t^{(n)}} \\ &\geq \sum_{k=1}^{\infty} \frac{\varphi_k(x_k^{(n)}+t^{(n)}h_k)-\varphi_k(x_k^{(n)})}{t^{(n)}} + \varepsilon \\ &= \frac{f(x^{(n)}+t^{(n)}h)-f(x^{(n)})}{t^{(n)}} + \varepsilon, \end{aligned}$$

for $n \geq n_0$ whence the contradiction with (17)

$$\limsup_{t\downarrow 0, x\to 0} \frac{f(x+th)-f(x)}{t} \geq f^0(0;h)+\varepsilon.$$

Therefore, we may negate (18) in order to obtain a subsequence symbolized by the index $m(n)$ such that

$$\begin{aligned} &\liminf_{n\to\infty} \frac{\varphi_{k'}(x_{k'}^{(m(n))}+t^{(m(n))}h_{k'})-\varphi_{k'}(x_{k'}^{(m(n))})}{t^{(m(n))}} \\ \geq\ &\lim_{n\to\infty} \frac{\varphi_{k'}(t^{(m(n))}h_{k'})}{t^{(m(n))}} = d\varphi_{k'}(0;h_{k'}) = \varphi_{k'}^0(0;h_{k'}) \\ \geq\ &\limsup_{n\to\infty} \frac{\varphi_{k'}(x_{k'}^{(m(n))}+t^{(m(n))}h_{k'})-\varphi_{k'}(x_{k'}^{(m(n))})}{t^{(m(n))}}, \end{aligned} \tag{19}$$

where '$d\varphi_{k'}$' refers to the usual directional derivative which, by the already stated regularity of $\varphi_{k'}$ in the sense of Clarke, exists and coincides with $\varphi_{k'}^0$. From the definition of $\varphi_{k'}$ one calculates $d\varphi_{k'}(0;h_{k'}) = |h_{k'}|$. Since k' was arbitrarily fixed, (19) provides

$$\lim_{n\to\infty} \frac{\varphi_k(x_k^{(m(n))}+t^{(m(n))}h_k)-\varphi_k(x_k^{(m(n))})}{t^{(m(n))}} = |h_k| \quad \forall k \in \mathbb{N}.$$

This finally allows to interchange limit and summation in the last term of (17) (upon passing to the subsequence $m(n)$ there too):

$$f^0(0;h) = \sum_{k=1}^{\infty} |h_k| = \|h\|_1 \quad \forall h \in X.$$

Consequently, $\partial^c f(0) = B_1$, where ∂^cdenotes Clarke's subdifferential and B_1 is the unit ball in X.

Next, we verify that f is regular at 0 in the sense of Clarke. To this aim, we calculate its usual directional derivative at 0 in arbitrary direction h. Since for each sequence $t^{(n)} \downarrow 0$ it holds that

$$\lim_{n\to\infty} \frac{\varphi_k(t^{(n)}\tilde{h})}{t^{(n)}} = d\varphi_k(0;\tilde{h}) = |\tilde{h}| \quad \forall \tilde{h} \in \mathbb{R}\,\forall k \in \mathbb{N},$$

one may interchange limit and summation once more:

$$\|h\|_1 = \sum_{k=1}^{\infty} \lim_{n\to\infty} \frac{\varphi_k(t^{(n)}h_k)}{t^{(n)}} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\varphi_k(t^{(n)}h_k)}{t^{(n)}} = \lim_{n\to\infty} \frac{f(t^{(n)}h) - f(0)}{t^{(n)}}.$$

As $t^{(n)} \downarrow 0$ was arbitrary, it follows that $df(0;h) = \|h\|_1 = f^0(0;h)$, hence f is regular in the sense of Clarke.

Finally, we consider the multivalued mapping $M : \mathbb{R} \rightrightarrows X$ defined by $M(t) := \{x \in X \mid f(x) \le t\}$. This is exactly the setting of (5) with abstract constraints missing ($X = C$). By definition of f and (16), one has

$$f(x) \ge 0 \quad \forall x \in X \quad \text{and} \quad f(x) = 0 \Longleftrightarrow x = 0.$$

Hence, $M(0) = \{0\}$. Define a sequence $z^{(n)} = (0, \ldots, 0, \tau_n, 0, 0, \ldots) \in X$, with τ_n at position n. Then, again by (16),

$$d(z^{(n)}, M(0)) = \|z^{(n)}\|_1 = \tau_n \quad \text{and} \quad f(z^{(n)}) = \varphi_n(\tau_n) = \tau_n^2 \quad \forall n \in \mathbb{N}.$$

Putting $y^{(n)} := f(z^{(n)})$, we have constructed sequences $z^{(n)}, y^{(n)}$ such that $z^{(n)} \in M(y^{(n)})$, $z^{(n)} \to 0$, $y^{(n)} \to 0$ (because of $\tau_n \in (0, n^{-2})$). From here, we derive that M fails to be calm at $(0,0)$:

$$d(z^{(n)}, M(0)) = \tau_n^{-1} f(z^{(n)}) = \tau_n^{-1} d(f(z^{(n)}), 0) \ge n^2 d(f(z^{(n)}), 0)$$

(again by $\tau_n \in (0, n^{-2})$), which contradicts (1). On the other hand, we have seen that $\partial^c f(0) = B_1$, hence $0 \in \operatorname{int} \partial^c f(0)$ and the constraint qualification '$f(\bar{x}) < 0$ or $0 \notin \operatorname{bd} \partial^c f(\bar{x})$' - which was sufficient for calmness in the regular, finite dimensional and in the convex, infinite dimensional case - is evidently satisfied. However, the same constraint qualification (to which the conditions (8) and (7) reduce when $C = X$) does not imply calmness in the regular, infinite dimensional case, as was shown in this example.

The next result is an immediate application of Theorem 3.3 to the characterization of calmness for non-structured multifunctions.

Corollary 3.9 *Let X be a Banach space, Y a metric space, $M : X \rightrightarrows Y$ a multifunction with closed values and $(\bar{x}, 0) \in \operatorname{Gph} M$. Assume further that*

1. *The distance function $d(0, M(\cdot))$ is convex and lower semicontinuous in a neighborhood of $\bar{x}$.*
2. *$0 \notin \operatorname{bd} \partial d(0, M(\cdot))(\bar{x})$.*

Then, M^{-1} is calm at $(0, \bar{x})$ (or, equivalently, M is metrically regular at $(\bar{x}, 0)$).

Proof. Corollary 3.4 immediately provides calmness at $(0, \bar{x})$ of the multifunction $P : \mathbb{R} \rightrightarrows X$ defined by

$$P(t) := \{x \in X \mid d(0, M(x)) \leq t\}$$

This means existence of some $L > 0$, $\varepsilon > 0$ such that

$$d(x, P(0)) \leq L|t| \quad \forall t \in (-\varepsilon, \varepsilon)\ \forall x \in B^0(\bar{x}; \varepsilon) \cap P(t).$$

Since $P(0) = M^{-1}(0)$ and $M^{-1}(y) \subseteq P(d(0, y))$ for all $y \in Y$, it follows the calmness of M^{-1} at $(0, \bar{x})$:

$$d(x, M^{-1}(0)) = d(x, P(0)) \leq Ld(0, y) \quad \forall y \in B^0(0; \varepsilon)\ \forall x \in B^0(\bar{x}; \varepsilon) \cap M^{-1}(y).$$

■

Note that condition 2. in the above corollary is far removed from being necessary for calmness or even the stronger Aubin property.

Example 3.10 *Consider $M(x) := [x, \infty)$ at $(0, 0) \in \operatorname{Gph} M$. Since $d(0, M(x)) = \max\{0, x\}$, condition 1. of the last corollary is satisfied whereas condition 2 is violated. On the other hand, the inverse multifunction $M^{-1}(y) = \{x | x \leq y\}$ is easily seen to satisfy the Aubin property (2) and, hence, calmness at $(0, 0)$.*

4 Calmness of the intersection of two sets

In this section, we turn to the calmness property with respect to two sets. To this aim, let $C, D \subseteq X$ closed, convex subsets such that $\bar{x} \in C \cap D$. We want to characterize calmness of the multivalued mapping $Q : \mathbb{R} \rightrightarrows X$ defined by

$$Q(t) := \{x \in X \mid d(x, C) + d(x, D) \leq t\}$$

at the point $(0, \bar{x}) \in \operatorname{Gph} Q$.

Lemma 4.1 *Q is calm at $(0, \bar{x}) \in \operatorname{Gph} Q$ provided that*

$$\operatorname{int} N(D; \bar{x}) \cap -N(C; \bar{x}) \neq \emptyset. \tag{20}$$

Proof. Choose $x^* \in \operatorname{int} N(D; \bar{x}) \cap -N(C; \bar{x})$. From $x^* \in \operatorname{int} N(D; \bar{x})$, it follows similar to the proof of Theorem 3.3 the existence of some $\alpha > 0$ such that

$$\alpha\|x - \bar{x}\| + \langle x^*, x - \bar{x}\rangle \leq 0 \quad \forall x \in D.$$

Hence, $\bar{x}$ is a minimizer of the function $\langle x^*, \bar{x} - \cdot\rangle - \alpha\| \cdot - \bar{x}\|$ on the set D. Now, using a well-known penalization argument, which appeals to the Lipschitz constant of the function involved, it follows the existence of some $\varepsilon > 0$ such that

$$\langle x^*, \bar{x} - x\rangle - \alpha\|x - \bar{x}\| + (\|x^*\| + \alpha)d(x, D) \geq 0 \quad \forall x \in B(\bar{x}; \varepsilon),$$

whence, by $x^* \in -N(C;\bar{x})$,

$$-\alpha\|x-\bar{x}\| + (\|x^*\|+\alpha)d(x,D) \geq 0 \quad \forall x \in B(\bar{x};\varepsilon)\cap C.$$

In other words, $\bar{x}$ is a local minimizer of the function $-\alpha\|\cdot-\bar{x}\|+(\|x^*\|+\alpha)d(\cdot,D)$ on the set C. Now, upon repeating the same penalization argument, one arrives at

$$-\alpha\|x-\bar{x}\| + (\|x^*\|+\alpha)d(x,D) + (\|x^*\|+2\alpha)d(x,C) \geq 0 \quad \forall x \in B(\bar{x};\varepsilon')$$

for some $\varepsilon' > 0$. This, however, is the desired calmness property

$$d(x,Q(0)) \leq \|x-\bar{x}\| \leq \alpha^{-1}(\|x^*\|+2\alpha)(d(x,D)+d(x,C)) \leq \alpha^{-1}(\|x^*\|+2\alpha)|t|$$

which holds true for all $t \in \mathbb{R}$ and all $x \in B(\bar{x};\varepsilon')\cap Q(t)$. ■

Next, we need an auxiliary result which is of independent interest.

Lemma 4.2 *If one of the sets C or D is compactly epi-Lipschitzian in a neighborhood of $\bar{x}$, then*

$$N(D;\bar{x}) \cap -N(C;\bar{x}) = \{0\} \Longleftrightarrow 0 \in \operatorname{int}(D - C\cap B(\bar{x},1))$$

Proof. ($\Longrightarrow$) For symmetry reasons, one may take, e.g., D to be compactly epi-Lipschitzian in a neighborhood of $\bar{x}$. Assume that

$$0 \notin \operatorname{int}(D - C\cap B(\bar{x},1)) = \operatorname{int}\overline{(D - C\cap B(\bar{x},1))}$$

(the equality follows from [13], Lemma 1). Accordingly, there exists a sequence $b_n \to 0$ with

$$b_n \notin \overline{D - C\cap B(\bar{x},1))}.$$

The separation theorem, provides a corresponding sequence $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and

$$\langle x_n^*, b_n\rangle \leq \langle x_n^*, d-\bar{x}\rangle \quad \forall d \in D, \quad \langle x_n^*, b_n\rangle \leq \langle x_n^*, \bar{x}-c\rangle \quad \forall c \in C\cap B(\bar{x},1). \quad (21)$$

The first relation of (21) yields that $\langle x_n^*, \bar{x}\rangle \leq \inf_{d\in D}\langle x_n^*, d\rangle + \|b_n\|$. Now, Ekeland's variational principle provides a sequence $d_n \in D$ such that

$$\|d_n - \bar{x}\| \leq \sqrt{\|b_n\|} \quad \text{and} \quad \langle x_n^*, d_n\rangle \leq \langle x_n^*, d\rangle + \sqrt{\|b_n\|}\,\|d_n - d\| \quad \forall d \in D. \quad (22)$$

The second relation of (22) entails that $-x_n^* \in N(D;d_n) + B^*(0,\sqrt{\|b_n\|})$, hence there are sequences $z_n^* \in N(D;d_n)$ and b_n^* with $\|b_n^*\| \leq \sqrt{\|b_n\|}$ such that $z_n^* + x_n^* + b_n^* = 0$. In particular; $\|z_n^*\| \to 1$. Thus, the sequence z_n^* is bounded and, hence, there exists a weak* convergent subnet $z_\lambda^* \rightharpoonup_{w^*} z^*$. Now, since $z_\lambda^* \in N(D;d_\lambda)$, this last convergence along with $d_\lambda \to \bar{x}$ (see first relation of (22)) and the very definition of the normal cone to convex sets yield that $z^* \in N(D;\bar{x})$. Now, the assumed property of D being compactly epi-Lipschitzian in a neighborhood $\mathcal{V}_{\bar{x}}$ of $\bar{x}$ results in the inclusion

$$N(D;x) \subseteq \{x^* \mid \|x^*\| \leq \max_{i=1,\ldots,k}\langle x^*, h_i\rangle\} \quad \forall x \in \mathcal{V}_{\bar{x}} \cap D$$

for certain $h_i \in X$ $(i = 1, \ldots, k)$. From $d_\lambda \to \bar{x}$, one derives that

$$\max_{i=1,\ldots,k} \langle z^*_\lambda, h_i \rangle \geq \|z^*_\lambda\|.$$

Consequently, $z^* \neq 0$. On the other hand, we also have that $x^*_\lambda = -z^*_\lambda - b^*_\lambda \rightharpoonup_{w^*} -z^*$ which together with the second part of (21) provides

$$\langle -z^*, \bar{x} - c \rangle \leftharpoondown_{w^*} \langle x^*_\lambda, \bar{x} - c \rangle \geq \langle x^*_\lambda, b_\lambda \rangle \to 0 \quad \forall c \in C \cap B(\bar{x}, 1),$$

whence $z^* \in -N(C; \bar{x})$. Summarizing, there is some $z^* \neq 0$ with $z^* \in N(D; \bar{x}) \cap -N(C; \bar{x})$. This contradicts our assumption.
$(\Longleftarrow)$
Choose an arbitrary $x^* \in N(D; \bar{x}) \cap -N(C; \bar{x})$. Then,

$$\langle x^*, d - \bar{x} \rangle \leq 0 \quad \forall d \in D \quad \text{and} \quad \langle x^*, \bar{x} - c \rangle \leq 0 \quad \forall c \in C.$$

In other words, $\langle x^*, d - c \rangle \leq 0$ for all $d \in D$ and all $c \in C$. However, since by assumption $0 \in \operatorname{int}(D - C)$, it results that $x^* = 0$, as was to be shown. ■

Theorem 4.3 *Let one of the sets C or D be compactly epi-Lipschitzian at $\bar{x}$. Then, Q is calm at $(0, \bar{x})$ under the following condition:*

$$\operatorname{bd} N(D; \bar{x}) \cap -\operatorname{bd} N(C; \bar{x}) = \{0\}. \tag{23}$$

Proof. In case that $N(D; \bar{x}) \cap -N(C; \bar{x}) = \{0\}$, Lemma 4.2 ensures that $0 \in \operatorname{int}(D - C)$. Since $D - C$ equals the range of the multifunction $M : X \rightrightarrows X$ defined by

$$M(x) = \begin{cases} -x + D & x \in C \\ \emptyset & x \notin C \end{cases},$$

we have $0 \in \operatorname{int} \operatorname{range} M$ and the Robinson-Ursescu Theorem yields metric regularity of M at the point $(\bar{x}, 0)$ of its graph. This property means the existence of $L, \varepsilon > 0$ such that

$$d(x, M^{-1}(y)) \leq L\, d(y, M(x)) \quad \forall x \in B(\bar{x}, \varepsilon) \ \forall y \in B(0, \varepsilon).$$

With $M^{-1}(y) = C \cap (D - y)$, and fixing $y := 0$, one arrives at

$$d(x, C \cap D) \leq L\, d(x, D) \quad \forall x \in B(\bar{x}, \varepsilon) \cap C, \text{ hence}$$

$$d(x, C \cap D) \leq (L + 1)\,(d(x, D) + d(x, C)) \quad \forall x \in B(\bar{x}, \varepsilon).$$

This, of course, is calmness of the multifunction Q at $(0, \bar{x})$.
Otherwise $(N(D; \bar{x}) \cap -N(C; \bar{x}) \neq \{0\})$, (23) implies that

$$\operatorname{int} N(D; \bar{x}) \cap -N(C; \bar{x}) \neq \emptyset \quad \text{or} \quad N(D; \bar{x}) \cap -\operatorname{int} N(C; \bar{x}) \neq \emptyset.$$

In both cases, Lemma 4.1 yields the desired result. ■

5 The differentiable nonconvex case

In this section we briefly return to the constraint system (5) with a convex closed subset $C \subseteq X$ as before but with a (strictly) differentiable function f. Theorem 3.3 has shown that, in the completely convex case (C and f), each of the constraint qualifications (8), (7) is sufficient for calmness of (5). On the other hand, we know by Example 3.8 that none of the two conditions ensures calmness if f is just regular in the sense of Clarke. Since, in that example, f was non-differentiable, the question arises if a positive result can be expected in the smooth case. The answer is affirmative even for a finite number of inequalities.

Theorem 5.1 *Consider a multifunction $M : \mathbb{R}^m \rightrightarrows X$ defined by*

$$M(y) := \{x \in C \mid f(x) \le y\} \quad (y \in \mathbb{R}^m),$$

where $C \subseteq X$ is convex and closed and $f : X \to \mathbb{R}^m$ is strictly differentiable. Then, the constraint qualification

$$\operatorname{conv} \{\nabla f_i(\bar{x})\}_{i \in I(\bar{x})} \cap -\operatorname{bd} N(C;\bar{x}) = \emptyset \tag{24}$$

implies calmness of M at $(0,\bar{x}) \in \operatorname{Gph} M$. Here, f_i denote the components of f and $I(x) = \{i \in \{1,\dots,m\} | f_i(x) = 0\}$ refers to the set of active indices.

Proof. Assume first that $\operatorname{conv}\{\nabla f_i(\bar{x})\}_{i\in I(\bar{x})} \cap -N(C;\bar{x}) = \emptyset$. Then, the strict differentiability assumption on f allows to apply Theorem 2.4. in [9] in order to derive metric regularity of M^{-1} at $(\bar{x},0)$ which is equivalent to the Aubin property of M at $(0,\bar{x})$ and, hence, implies calmness of M at $(0,\bar{x})$. In the opposite case, (24) guarantees the existence of some $x^* \in \operatorname{conv}\{\nabla f_i(\bar{x})\}_{i\in I(\bar{x})} \cap -\operatorname{int} N(C;\bar{x})$. Accordingly, there exist $\lambda_i \ge 0$ ($i \in I(\bar{x})$) with $\sum_{i\in I(\bar{x})} \lambda_i = 1$ as well as $\varepsilon > 0$ such that

$$x^* = \sum_{i\in I(\bar{x})} \lambda_i \nabla f_i(\bar{x}) \quad \text{and} \quad \varepsilon\|x-\bar{x}\| \le \langle x^*, x-\bar{x}\rangle \quad \forall x \in C.$$

Due to the differentiability assumption on f and to the finiteness of $I(\bar{x})$ there is some $\eta > 0$ such that

$$f_i(x) - f_i(\bar{x}) \ge \langle \nabla f_i(\bar{x}), x-\bar{x}\rangle - \frac{\varepsilon}{2}\|x-\bar{x}\| \quad \forall x \in B(\bar{x},\eta)\ \forall i \in I(\bar{x}).$$

Using that $f_i(\bar{x}) = 0$ for $i \in I(\bar{x})$ it holds for all $x \in C \cap B(\bar{x},\eta)$ that

$$\max_{i\in I(\bar{x})} f_i(x) \ge \sum_{i\in I(\bar{x})} \lambda_i f_i(x) \ge \sum_{i\in I(\bar{x})} \lambda_i \langle \nabla f_i(\bar{x}), x-\bar{x}\rangle - \frac{\varepsilon}{2}\|x-\bar{x}\| \ge \frac{\varepsilon}{2}\|x-\bar{x}\|.$$

Measuring, without loss of generality, the distance in $\mathbb{R}^m$ with respect to the maximum norm, one has for all $x \in M(y) \cap B(\bar{x},\eta)$ and all $y \in \mathbb{R}^m$:

$$d(x, M(0)) \le \|x-\bar{x}\| \le \frac{2}{\varepsilon} \max_{i\in I(\bar{x})} f_i(x) \le \frac{2}{\varepsilon} \max_{i=1,\dots,m} |y_i| = \frac{2}{\varepsilon} d(y,0).$$

This, however, is calmness of M at $(0, \bar{x})$. ■

The last result shows that the ideas of the completely convex case can be extended to differentiable inequalities. With a single inequality which is differentiable and convex, (24) reduces to (8) (without the need of the additional constraint qualification (CQ*)). One might ask about an alternative condition in the sense of (7) for the differentiable case as well. However, the closedness of the normal cone immediately provides that the differentiable formulation of (7) implies (8), hence the two conditions are not independent as in the convex (nonsmooth) setting. Finally, we note that for finite dimensional X, (24) can be weakened to the condition

$$\mathrm{bd}\,\mathrm{conv}\,\{\nabla f_i(\bar{x})\}_{i \in I(\bar{x})} \cap -\mathrm{bd}\, N(C; \bar{x}) = \emptyset.$$

(see [6], Th. 9). In inifinite dimensions, the interior of the convex hull involved is empty, hence this last relation is equivalent with (24).

6 Conclusion

The conditions for calmness formulated in this paper (in particular (7),(8), (14) and (24)) can be interpreted as constraint qualifications ensuring (nondegenerate) multiplier rules in optimization problems where the corresponding multifunctions figure as constraint systems subject to perturbations. If, for instance, $\bar{x}$ is a solution of the convex optimization problem

$$\min\{g(x) | x \in C, \ f(x) \leq 0\}$$

with convex and continuous objective g, then the constraint qualification '(7) or (8)' - which is much weaker than Slater's condition, see (15) - entails the existence of some $\lambda \in \mathbb{R}_+$ such that $0 \in \partial g(\bar{x}) + \lambda \partial f(\bar{x}) + N(C; \bar{x})$. A similar statement holds true for differentiable optimization problems with constraint sets as in Theorem 5.1 under condition (24). Finally, it is noted that also conditioning results in the sense of linear growth of the objective locally around the solution set of an optimization problem can be obtained from the same conditions in the respective settings. This has an immediate impact on quantitative stability of solution sets.

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