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An analytic approach to a generalized Naghdi shell model

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Abstract

In this paper a shell model of generalized Naghdi type is studied which requires only low regularity conditions. It is shown that the corresponding system of linear variational equations (representing a boundary value problem for a linear system of six partial differential equations on the shell) admits a unique solution. The main step in the proof is to show the coercivity of the corresponding bilinear form which is equivalent to a Korn inequality in curvilinear coordinates. In this paper, a direct approximation argument is used for the proof of coercivity.

1 Introduction

In this work, we propose a model of a thin shell which may be viewed as a direct generalization of the classical Naghdi model or of the Reissner–Mindlin linear plate model. We underline that our work enters the class of hierarchical models, and we quote the treatise of Ciarlet [3], [4] for a detailed presentation of the subject.

In our study, we are motivated by several aims. First, we relax the regularity assumptions which are generally used in the literature on shells and curved rods, namely the three times differentiability of the middle surface or of the line of centroids. Our assumption requires just piecewise C^2 - surfaces and it may be compared with the recent works of Geymonat and Sanchez-Palencia [6], Blouza [1], Le Dret and Blouza [5], Ignat, Sprekels and Tiba [7], although the methods are very different. In **Remark 2.1**, we point out that even slightly less regularity is enough for our present approach to work.

Another scope of this paper is to provide a simplified proof for the existence and uniqueness theorem for the shell equation. Consequently, we have minimized, as much as possible, the use of elements from differential geometry and even from mechanics, and this explains the title of our work. In this sense, we have made the simplifying assumption that the middle surface of the shell is given by the graph of a function, which still allows for a large class of applications. We also point out that the main reason behind this hypothesis is our intention to study, in a subsequent work, shape optimization questions associated to our model. Taking into account the complexity and the novelty of the problem and of the approach, such a simplification seems reasonable.

In Section 2, we perform a detailed description of our partially clamped shell model, and we state the main result. Section 3 provides the rather lengthy existence proof, organized as a sequence of lemmas.

Finally, we mention that in the case of arches it was shown in Sprekels and Tiba [10], and Ignat, Sprekels and Tiba [8], that Lipschitz regularity suffices for the existence results; besides, a complete optimization theory was developed.

2 The model

Let $\omega \subset \mathbb{R}^2$ be an open bounded connected set, not necessarily simply connected, with Lipschitz boundary $\partial\omega$. For $\varepsilon > 0$, we define $\Omega := \omega \times]-\varepsilon, \varepsilon[\subset \mathbb{R}^3$, satisfying the same assumptions as ω. We denote by $(x_1, x_2) \in \omega$, $x_3 \in]-\varepsilon, \varepsilon[$, $\bar{x} = (x_1, x_2, x_3) \in \Omega$, the independent variables.

Let $p : \omega \to \mathbb{R}$ denote a piecewise $C^2(\bar{\omega})$-mapping whose graph represents the middle surface $\mathcal{S}$ of the shell. We consider the geometric transformation $F : \Omega \to \mathbb{R}^3$,

$$F(\bar{x}) := \bar{\pi}(x_1, x_2) + x_3\, \bar{n}(x_1, x_2), \tag{2.1}$$

with $\bar{\pi} = (\pi_1, \pi_2, \pi_3) = (x_1, x_2, p(x_1, x_2))$, and with $\bar{n} = (n_1, n_2, n_3)$ denoting the normal vector to $\mathcal{S}$ in the point $\bar{\pi}(x_1, x_2)$. Notice that the vectors $\dfrac{\partial\bar{\pi}}{\partial x_1} = (1, 0, p_1)$, $\dfrac{\partial\bar{\pi}}{\partial x_2} = (0, 1, p_2)$, where $p_1 := \dfrac{\partial p}{\partial x_1}$ and $p_2 := \dfrac{\partial p}{\partial x_2}$, are always linearly independent; consequently,

$$\begin{aligned} \bar{n} &= \frac{\dfrac{\partial\bar{\pi}}{\partial x_1} \wedge \dfrac{\partial\bar{\pi}}{\partial x_2}}{\left| \dfrac{\partial\bar{\pi}}{\partial x_1} \wedge \dfrac{\partial\bar{\pi}}{\partial x_2} \right|_{\mathbb{R}^3}} \\ &= \left(-\frac{p_1}{\sqrt{1+p_1^2+p_2^2}}, -\frac{p_2}{\sqrt{1+p_1^2+p_2^2}}, \frac{1}{\sqrt{1+p_1^2+p_2^2}} \right). \end{aligned} \tag{2.2}$$

Here "$\wedge$" is the exterior product in $\mathbb{R}^3$, while $|\cdot|_{\mathbb{R}^K}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^K}$ denote norm and scalar product,respectively, in the Euclidian space $\mathbb{R}^K$. In addition, standard notations for vectors, matrices, and so on, will be used throughout the text.

Assume that $\partial\omega$ is divided in two nonoverlapping open parts γ_0, γ_1. We introduce the notations $\Gamma_0 := \gamma_0 \times]-\varepsilon, \varepsilon[$, $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, as well as

$$\hat{\Omega} := F(\Omega), \quad \hat{\Gamma}_0 := F(\Gamma_0), \qquad \hat{\Gamma}_1 := F(\Gamma_1). \tag{2.3}$$

Under our subsequent assumptions (see (2.15)), F is a homeomorphism, and $\hat{\Omega}$ is an open connected bounded set in $\mathbb{R}^3$ representing the shell and having the Lipschitz boundary $\partial\hat{\Omega} := \overline{\hat{\Gamma}_0} \cup \overline{\hat{\Gamma}_1}$. For $\hat{\Omega}$, we introduce the Hilbert space

$$V(\hat{\Omega}) = \left\{ \hat{v} \in H^1(\hat{\Omega})^3 ;\, \hat{v}|_{\hat{\Gamma}_0} = 0 \right\}, \tag{2.4}$$

and the linear elasticity system in the weak formulation,

$$\int_{\hat\Omega} \Big[\lambda\, \hat e_{pp}(\hat u)\, \hat e_{qq}(\hat v) \;+\; 2\,\mu\, \hat e_{ij}(\hat u)\, \hat e_{ij}(\hat v)\Big]\, d\hat x$$

$$= \int_{\hat\Omega} \hat f_i\, \hat v_i\, d\hat x \;+\; \int_{\hat\Gamma_1} \hat h_i\, \hat v_i\, d\hat\Gamma\,, \quad \forall\; \hat v \,\in\, V(\hat\Omega)\,. \tag{2.5}$$

Here, $\lambda \geq 0\,, \mu > 0$ are the Lamé constants of the material, $\hat f_i \in L^2(\hat\Omega)$ are the body forces, $\hat h_i \in L^2(\hat\Gamma_1)$ are the surface tractions, and the summation convention is used. The components of the linearized strain or change of metric tensor are given by

$$\hat e_{ij}(\hat v) \;=\; \frac{1}{2}\Big(\frac{\partial \hat v_i}{\partial \hat x_j} \;+\; \frac{\partial \hat v_j}{\partial \hat x_i}\Big)\,, \quad i,j \;=\; \overline{1,3}\,. \tag{2.6}$$

Our main *geometric assumption* is that the displacement $\hat u \in V(\hat\Omega)$ has the form

$$\hat u(\hat x) \;=\; \bar u(x_1,x_2) \;+\; x_3\, \bar r(x_1,x_2)\,, \quad \hat x \,\in\, \hat\Omega\,, \tag{2.7}$$

with $\bar x \;=\; (x_1,x_2,x_3) \;=\; F^{-1}(\hat x)$, and where $\bar u \;=\; (u_1,u_2,u_3)$ and $\bar r \;=\; (r_1,r_2,r_3)$ belong to the space

$$V(\omega) := \{\bar v = (v_1,v_2,v_3) \in H^1(\omega)^3\,;\; \bar v|_{\gamma_0} = 0\}\,. \tag{2.8}$$

This means that we are looking for solutions in the infinite dimensional subspace

$$\tilde V(\hat\Omega) := \{\hat u \in V(\hat\Omega)\,;\; \hat u \quad \text{is of the form (2.7) }\}\,. \tag{2.9}$$

Note that $\tilde V(\hat\Omega)$ can through the relation (2.7) be identified with the product space $V(\omega)^2 := V(\omega) \times V(\omega)$. Therefore, instead of working in the space $\tilde V(\hat\Omega)$, we can always work in $V(\omega)^2$. We will do this repeatedly later in this paper.

From the geometrical point of view, it should be clear that $\bar u$ represents the displacement of the middle surface $\mathcal{S}$ of the shell, while $\bar r$ is the modification of the points along the normal $\bar n(x_1,x_2)$ which are assumed to remain on a line. Note also that the form (2.7) allows for both dilation and contraction of the elastic material, and that it constitutes a generalization of the standard assumptions associated with the so-called *Naghdi model* (cf. Ciarlet [4], Blouza [1]).

Let us now collect some properties of the transformation F. The Jacobian $J := DF$ of F is given by

$$J(\bar x) \;=\; \begin{bmatrix} 1 + x_3\,\dfrac{\partial n_1}{\partial x_1} & x_3\,\dfrac{\partial n_1}{\partial x_2} & n_1 \\[2ex] x_3\,\dfrac{\partial n_2}{\partial x_1} & 1{+}x_3\,\dfrac{\partial n_2}{\partial x_2} & n_2 \\[2ex] p_1 + x_3\,\dfrac{\partial n_3}{\partial x_1} & p_2 + x_3\,\dfrac{\partial n_3}{\partial x_2} & n_3 \end{bmatrix}. \tag{2.10}$$

We recall the relations

$$n_1 = -n_3 p_1, \quad n_2 = -n_3 p_2, \tag{2.11}$$

$$\frac{\partial \bar{n}}{\partial x_1}(x_1, x_2) = \frac{\partial n_1}{\partial x_1} \frac{\partial \bar{\pi}}{\partial x_1} + \frac{\partial n_2}{\partial x_1} \frac{\partial \bar{\pi}}{\partial x_2}, \tag{2.12}$$

$$\frac{\partial \bar{n}}{\partial x_2}(x_1, x_2) = \frac{\partial n_1}{\partial x_2} \frac{\partial \bar{\pi}}{\partial x_1} + \frac{\partial n_2}{\partial x_2} \frac{\partial \bar{\pi}}{\partial x_2}, \tag{2.13}$$

which are easy consequences of (2.2) and of $|\bar{n}|_{I\!R^3} = 1$ which implies that $\left\langle \bar{n}, \frac{\partial \bar{n}}{\partial x_i} \right\rangle_{I\!R^3} = 0$. Hence, $\frac{\partial \bar{n}}{\partial x_i}$ is orthogonal to $\bar{n}$ and generated by $\frac{\partial \bar{\pi}}{\partial x_1}$ and $\frac{\partial \bar{\pi}}{\partial x_2}$, $i = 1, 2$. Notice that (2.12), (2.13) are special cases of the equations of movement of the local frame on the surface $\mathcal{S}$, see Cartan [2]. The coefficients $\frac{\partial n_i}{\partial x_\alpha}$, $i = \overline{1,3}$, $\alpha = \overline{1,2}$, may be interpreted as various curvatures of $\mathcal{S}$.

Using (2.10)–(2.13), one can easily check that

$$\begin{aligned} \det J(\bar{x}) = & \left[1 + x_3 \left(\frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2}\right) + x_3^2 \left(\frac{\partial n_1}{\partial x_1} \frac{\partial n_2}{\partial x_2} - \frac{\partial n_1}{\partial x_2} \frac{\partial n_2}{\partial x_1}\right)\right] \\ & \cdot \sqrt{1 + p_1^2 + p_2^2} \,. \end{aligned} \tag{2.14}$$

Since $p \in W^{2,\infty}(\omega)$, it follows from (2.14) that if $\varepsilon > 0$ is assumed to be "small", then

$$\det J(\bar{x}) \geq c > 0 \quad \forall \bar{x} \in \Omega. \tag{2.15}$$

This justifies the definition (2.3) of the shell $\hat{\Omega}$ via the geometric transformation F from (2.1). From now on, we will always assume that $0 < \varepsilon < 1$ is small enough to guarantee the validity of (2.15).

In the next section, the inverse of J and the Jacobian of F^{-1} will be needed. We denote them by

$$J(\bar{x})^{-1} = (h_{ij}(\bar{x}))_{i,j=\overline{1,3}}, \quad D F^{-1}(\hat{x}) = (d_{ij}(\hat{x}))_{i,j=\overline{1,3}}. \tag{2.16}$$

Their calculation is tedious (but straightforward), and we just list some elements of $J(\bar{x})^{-1} \cdot \det J(\bar{x})$ (with obvious notations):

$$\tilde{h}_{11} = n_3 \left(1 + x_3 \frac{\partial n_2}{\partial x_2}\right) - n_2 \left(p_2 + x_3 \frac{\partial n_3}{\partial x_2}\right), \tag{2.17}$$

$$\tilde{h}_{21} = n_2 \left(p_1 + x_3 \frac{\partial n_3}{\partial x_1}\right) - n_3 x_3 \frac{\partial n_2}{\partial x_1}, \tag{2.18}$$

$$\tilde{h}_{31} \;=\; x_3\,\frac{\partial n_2}{\partial x_1}\left(p_2\;+\;x_3\,\frac{\partial n_3}{\partial x_2}\right)\;-\;\left(1\;+\;x_3\,\frac{\partial n_2}{\partial x_2}\right)\left(p_1\;+\;x_3\,\frac{\partial n_3}{\partial x_1}\right), \tag{2.19}$$

$$\tilde{h}_{32} \;=\; x_3\,\frac{\partial n_1}{\partial x_2}\left(p_1\;+\;x_3\,\frac{\partial n_3}{\partial x_1}\right)\;-\;\left(1\;+\;x_3\,\frac{\partial n_1}{\partial x_1}\right)\left(p_2\;+\;x_3\,\frac{\partial n_3}{\partial x_2}\right). \tag{2.20}$$

We introduce the vectorial mapping $\bar{w} : \Omega \to I\!R^3$ by

$$\bar{w}(\bar{x}) \;=\; \bar{u}(x_1,x_2)\;+\;x_3\,\bar{r}(x_1,x_2)\,,\quad \bar{x}\,\in\,\Omega\,, \tag{2.21}$$

so that

$$\hat{u}(\hat{x}) \;=\; \bar{w}(F^{-1}(\hat{x}))\,,\quad \hat{x}\,\in\,\hat{\Omega}\,. \tag{2.22}$$

The Jacobian of $\bar{w}$ is

$$D\bar{w}(\bar{x}) \;=\; \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1}+x_3\dfrac{\partial r_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2}+x_3\dfrac{\partial r_1}{\partial x_2} & r_1 \\ \dfrac{\partial u_2}{\partial x_1}+x_3\dfrac{\partial r_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2}+x_3\dfrac{\partial r_2}{\partial x_2} & r_2 \\ \dfrac{\partial u_3}{\partial x_1}+x_3\dfrac{\partial r_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2}+x_3\dfrac{\partial r_3}{\partial x_2} & r_3 \end{bmatrix}. \tag{2.23}$$

We infer that, for $\bar{x} = F^{-1}(\hat{x})$,

$$\begin{aligned} & D\,\hat{u}(\hat{x}) \;=\; D\,\bar{w}\left(F^{-1}(\hat{x})\right)\,D\,F^{-1}(\hat{x}) \;=\; D\,\bar{w}(\bar{x})\,\cdot\,(d_{ij}(\hat{x}))_{i,j=\overline{1,3}} \\ =\;& D\,\bar{w}\left(F^{-1}(\hat{x})\right)\,J\left(F^{-1}(\hat{x})\right)^{-1} \;=\; D\,\bar{w}(\bar{x})\,J(\bar{x})^{-1} \\ =\;& D\,\bar{w}(\bar{x})\;\cdot\,(h_{ij}(\bar{x}))_{i,j=\overline{1,3}}\,. \end{aligned} \tag{2.24}$$

Consequently, we have (again $\bar{x} = F^{-1}(\hat{x})$)

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_j}(\hat{x}) = \left\langle \left(\frac{\partial u_i}{\partial x_1}+x_3\frac{\partial r_i}{\partial x_1}\,,\;\frac{\partial u_i}{\partial x_2}+x_3\frac{\partial r_i}{\partial x_2}\,,\;r_i\right),(d_{1j}(\hat{x})),d_{2j}(\hat{x}),d_{3j}(\hat{x}))\right\rangle_{I\!R^3}. \tag{2.25}$$

To arrive at our final model, we now restrict the set of admissible test functions $\hat{v} \in V(\hat{\Omega})$. In accordance with the expected special form (2.7) of the displacement, we consider test functions $\hat{v} \in \tilde{V}(\hat{\Omega})$,

$$\hat{v}(\hat{x}) = \bar{\mu}(x_1,x_2)\;+\;x_3\,\bar{\rho}(x_1,x_2)\,,\quad \hat{x} \in \hat{\Omega}\,, \tag{2.26}$$

where $\bar{x} \;=\; F^{-1}(\hat{x})$ and $\bar{\mu} \;=\; (\mu_1,\mu_2,\mu_3)\,,\;\bar{\rho} \;=\; (\rho_1,\rho_2,\rho_3)\;\in\;V(\omega)$. As $\hat{u},\hat{v}\;\in\;V(\hat{\Omega})$, we can insert $\hat{u},\hat{v}$ in (2.5) in order to obtain the bilinear form governing our generalized Naghdi model,

$$
\begin{aligned}
B(\hat{u},\hat{v}) \;=\; & \lambda \int\limits_{\hat{\Omega}} \Big\{ \sum_{i=1}^{3} \Big[\Big(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \Big) d_{1i} + \Big(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \Big) d_{2i} + r_i d_{3i} \Big] \Big\} \\
& \cdot \Big\{ \sum_{j=1}^{3} \Big[\Big(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \rho_j}{\partial x_1} \Big) d_{1j} + \Big(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \rho_j}{\partial x_2} \Big) d_{2j} + \rho_j d_{3j} \Big] \Big\} \, d\hat{x} \\
& + 2\,\mu \int\limits_{\hat{\Omega}} \sum_{i=1}^{3} \Big[\Big(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \Big) d_{1i} + \Big(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \Big) d_{2i} + r_i d_{3i} \Big] \\
& \qquad \cdot \Big[\Big(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \rho_i}{\partial x_1} \Big) d_{1i} + \Big(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \rho_i}{\partial x_2} \Big) d_{2i} + \rho_i d_{3i} \Big] \, d\hat{x} \\
& + \mu \int\limits_{\hat{\Omega}} \Big\{ \Big[\Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) d_{12} + \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) d_{22} + r_1 d_{32} \\
& \qquad + \Big(\frac{\partial u_2}{\partial x_1} + x_3 \frac{\partial r_2}{\partial x_1} \Big) d_{11} + \Big(\frac{\partial u_2}{\partial x_2} + x_3 \frac{\partial r_2}{\partial x_2} \Big) d_{21} + r_2 d_{31} \Big] \\
& \qquad \cdot \Big[\Big(\frac{\partial \mu_1}{\partial x_1} + x_3 \frac{\partial \rho_1}{\partial x_1} \Big) d_{12} + \Big(\frac{\partial \mu_1}{\partial x_2} + x_3 \frac{\partial \rho_1}{\partial x_2} \Big) d_{22} + \rho_1 d_{32} \\
& \qquad + \Big(\frac{\partial \mu_2}{\partial x_1} + x_3 \frac{\partial \rho_2}{\partial x_1} \Big) d_{11} + \Big(\frac{\partial \mu_2}{\partial x_2} + x_3 \frac{\partial \rho_2}{\partial x_2} \Big) d_{21} + \rho_2 d_{31} \Big] \\
& \qquad + \Big[\Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) d_{13} + \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) d_{23} + r_1 d_{33} \\
& \qquad + \Big(\frac{\partial u_3}{\partial x_1} + x_3 \frac{\partial r_3}{\partial x_1} \Big) d_{11} + \Big(\frac{\partial u_3}{\partial x_2} + x_3 \frac{\partial r_3}{\partial x_2} \Big) d_{21} + r_3 d_{31} \Big] \\
& \qquad \cdot \Big[\Big(\frac{\partial \mu_1}{\partial x_1} + x_3 \frac{\partial \rho_1}{\partial x_1} \Big) d_{13} + \Big(\frac{\partial \mu_1}{\partial x_2} + x_3 \frac{\partial \rho_1}{\partial x_2} \Big) d_{23} + \rho_1 d_{33} \\
& \qquad + \Big(\frac{\partial \mu_3}{\partial x_1} + x_3 \frac{\partial \rho_3}{\partial x_1} \Big) d_{11} + \Big(\frac{\partial \mu_3}{\partial x_2} + x_3 \frac{\partial \rho_3}{\partial x_2} \Big) d_{21} + \rho_3 d_{31} \Big] \\
& \qquad + \Big[\Big(\frac{\partial u_2}{\partial x_1} + x_3 \frac{\partial r_2}{\partial x_2} \Big) d_{13} + \Big(\frac{\partial u_2}{\partial x_2} + x_3 \frac{\partial r_2}{\partial x_2} \Big) d_{23} + r_2 d_{33} \\
& \qquad + \Big(\frac{\partial u_3}{\partial x_1} + x_3 \frac{\partial r_3}{\partial x_2} \Big) d_{12} + \Big(\frac{\partial u_3}{\partial x_2} + x_3 \frac{\partial r_3}{\partial x_2} \Big) d_{22} + r_3 d_{32} \Big] \\
& \qquad \cdot \Big[\Big(\frac{\partial \mu_2}{\partial x_1} + x_3 \frac{\partial \rho_2}{\partial x_1} \Big) d_{13} + \Big(\frac{\partial \mu_2}{\partial x_2} + x_3 \frac{\partial \rho_2}{\partial x_2} \Big) d_{23} + \rho_2 d_{33} \\
& \quad + \Big(\frac{\partial \mu_3}{\partial x_1} + x_3 \frac{\partial \rho_3}{\partial x_1} \Big) d_{12} + \Big(\frac{\partial \mu_3}{\partial x_2} + x_3 \frac{\partial \rho_3}{\partial x_2} \Big) d_{22} + \rho_3 d_{32} \Big] \Big\} \; d\hat{x} \; . \qquad (2.27)
\end{aligned}
$$

The generalized Naghdi model of a partially clamped shell is now finally obtained by (2.7), (2.26), (2.27), and by the variational equation

$$B(\hat{u},\hat{v}) = \int\limits_{\hat{\Omega}} \hat{f}_i\, \hat{v}_i\, d\hat{x} \; + \int\limits_{\hat{\Gamma}_1} \hat{h}_i\, \hat{v}_i\, d\hat{\Gamma} \quad \forall\, \hat{v} \in \tilde{V}(\hat{\Omega})\,. \tag{2.28}$$

We underline that (2.28) is a projection of the general elasticity system (2.5) from $V(\hat{\Omega})$ onto the infinite dimensional subspace $\tilde{V}(\hat{\Omega})$. This process is reminiscent to the finite element approximation method where the projection subspaces are however only finite dimensional. We also note that with the bilinear form B acting on $\tilde{V}(\hat{\Omega}) \times \tilde{V}(\hat{\Omega})$ we can associate a bilinear form $\mathcal{B}$ acting on $V(\omega)^2 \times V(\omega)^2$ through the identity

$$\mathcal{B}((\bar{u},\bar{r}),(\bar{\mu},\bar{\rho})) = B(\hat{u},\hat{v})\,. \tag{2.29}$$

In what follows, we will mainly work with the bilinear form B even if $\mathcal{B}$ is actually meant. From this no confusion will arise.

After a standard change of variables, using also (2.22), we can rewrite the bilinear forms B and $\mathcal{B}$, respectively, as

$$\begin{aligned}
& B(\hat{u},\hat{v}) \;=\; \mathcal{B}((\bar{u},\bar{r}),(\bar{\mu},\bar{\rho})) \\
= \quad & \lambda \int\limits_{\Omega} \Big\{ \sum_{i=1}^{3} \Big[\Big(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \Big) h_{1i} + \Big(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \Big) h_{2i} + r_i h_{3i} \Big] \Big\} \\
& \quad \cdot \Big\{ \sum_{j=1}^{3} \Big[\Big(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \rho_j}{\partial x_1} \Big) h_{1j} + \Big(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \rho_j}{\partial x_2} \Big) h_{2j} + \rho_j h_{3j} \Big] \Big\} \, |\det J(\bar{x})| \, d\bar{x} \\
& + 2\,\mu \int\limits_{\Omega} \sum_{i=1}^{3} \Big[\Big(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \Big) h_{1i} + \Big(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \Big) h_{2i} + r_i h_{3i} \Big] \\
& \qquad \cdot \Big[\Big(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \rho_i}{\partial x_1} \Big) h_{1i} + \Big(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \rho_i}{\partial x_2} \Big) h_{2i} + \rho_i h_{3i} \Big] \, |\det J(\bar{x})| \, d\bar{x} \\
& + \mu \int\limits_{\Omega} \Big\{ \Big[\Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) h_{12} + \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) h_{22} + r_1 h_{32} \\
& \qquad + \Big(\frac{\partial u_2}{\partial x_1} + x_3 \frac{\partial r_2}{\partial x_1} \Big) h_{11} + \Big(\frac{\partial u_2}{\partial x_2} + x_3 \frac{\partial r_2}{\partial x_2} \Big) h_{21} + r_2 h_{31} \Big] \\
& \qquad \cdot \Big[\Big(\frac{\partial \mu_1}{\partial x_1} + x_3 \frac{\partial \rho_1}{\partial x_1} \Big) h_{12} + \Big(\frac{\partial \mu_1}{\partial x_2} + x_3 \frac{\partial \rho_1}{\partial x_2} \Big) h_{22} + \rho_1 h_{32} \\
& \qquad + \Big(\frac{\partial \mu_2}{\partial x_1} + x_3 \frac{\partial \rho_2}{\partial x_1} \Big) h_{11} + \Big(\frac{\partial \mu_2}{\partial x_2} + x_3 \frac{\partial \rho_2}{\partial x_2} \Big) h_{21} + \rho_2 h_{31} \Big] \\
& \qquad + \Big[\Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) h_{13} + \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) h_{23} + r_1 h_{33}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{\partial u_3}{\partial x_1}+x_3\frac{\partial r_3}{\partial x_1}\Big)h_{11}+\Big(\frac{\partial u_3}{\partial x_2}+x_3\frac{\partial r_3}{\partial x_2}\Big)h_{21}+r_3h_{31}\Big]\\
&\cdot\Big[\Big(\frac{\partial \mu_1}{\partial x_1}+x_3\frac{\partial \rho_1}{\partial x_1}\Big)h_{13}+\Big(\frac{\partial \mu_1}{\partial x_2}+x_3\frac{\partial \rho_1}{\partial x_2}\Big)h_{23}+\rho_1h_{33}\\
&+\Big(\frac{\partial \mu_3}{\partial x_1}+x_3\frac{\partial \rho_3}{\partial x_1}\Big)h_{11}+\Big(\frac{\partial \mu_3}{\partial x_2}+x_3\frac{\partial \rho_3}{\partial x_2}\Big)h_{21}+\rho_3h_{31}\Big]\\
&+\Big[\Big(\frac{\partial u_2}{\partial x_1}+x_3\frac{\partial r_2}{\partial x_1}\Big)h_{13}+\Big(\frac{\partial u_2}{\partial x_2}+x_3\frac{\partial r_2}{\partial x_2}\Big)h_{23}+r_2h_{33}\\
&+\Big(\Big(\frac{\partial u_3}{\partial x_1}+x_3\frac{\partial r_3}{\partial x_1}\Big)h_{12}+\Big(\frac{\partial u_3}{\partial x_2}+x_3\frac{\partial r_3}{\partial x_2}\Big)h_{22}+r_3h_{32}\Big]\\
&\cdot\Big[\Big(\frac{\partial \mu_3}{\partial x_1}+x_3\frac{\partial \rho_3}{\partial x_1}\Big)h_{12}+\Big(\frac{\partial \mu_3}{\partial x_2}+x_3\frac{\partial \rho_3}{\partial x_2}\Big)h_{22}+\rho_3h_{32}\\
&+\Big(\frac{\partial \mu_2}{\partial x_1}+x_3\frac{\partial \rho_2}{\partial x_1}\Big)h_{13}+\Big(\frac{\partial \mu_2}{\partial x_2}+x_3\frac{\partial \rho_2}{\partial x_2}\Big)h_{23}+\rho_2h_{33}\Big]\Big\}\,|\det J(\bar{x})|\,d\bar{x}\,.
\end{aligned}
\tag{2.30}
$$

Remark 2.1 It is here that the piecewise $C^2(\bar{\omega})$-regularity of $p(\cdot,\cdot)$ is in fact used. However, this assumption may be slightly relaxed using more refined change of variables theorems (see, for instance, Rudin [9], p. 153).

By performing a similar change of variables in the right-hand side of (2.28), the generalized Naghdi model can be expressed directly on the domain Ω. The computations are rather tedious and, for the sake of brevity, we do not give them in detail, here. The reader may get a hint in this direction from the arguments developed in the next section.

We close the second part by stating the main result of this paper:

Theorem 2.1 *If* $\varepsilon > 0$ *is sufficiently small, then the generalized Naghdi model* (2.28) *has a unique solution of the form* $\hat{u}(\hat{x}) = \bar{u}(x_1,x_2) + x_3\,\bar{r}(x_1,x_2)$ *with* $(\bar{u},\bar{r}) \in V(\omega)^2$ *and* $\bar{x} = F^{-1}(\hat{x})$.

This result will be a consequence of the Lax-Milgram lemma applied to the bilinear form (2.30). To this end, we have to show its coercivity.

3 Proof of coercivity

In what follows, we shall fix $\lambda = 0$, $\mu = \frac{1}{2}$, without loss of generality. The classical Korn's inequality with boundary conditions (cf. Ciarlet [3]) yields that

$$
B(\hat{u},\hat{u}) \ \ge\ \int\limits_{\hat{\Omega}} \sum_{i,j=1}^{3} |\hat{e}_{ij}(\hat{u})|^2\,d\hat{x} \ \ge\ c(\hat{\Omega},\hat{\Gamma}_0)\ \|\hat{u}\|^2_{H^1(\hat{\Omega})} \quad \forall\,\hat{u}\in V(\hat{\Omega})\,. \tag{3.1}
$$

Since $\hat{u}|_{\hat{\Gamma}_0} = 0$, we may replace $\|\hat{u}\|_{H^1(\hat{\Omega})}$ by the equivalent norm

$$|\hat{u}|^2_{H^1(\hat{\Omega})} := \sum_{i,j=1}^{3} \int_{\hat{\Omega}} \left| \frac{\partial \hat{u}_i}{\partial \hat{x}_j} \right|^2 d\hat{x} \, . \tag{3.2}$$

Lemma 3.1 *If $\hat{u}$ has the form* (2.7)*, then*

$$\begin{aligned} |\hat{u}|^2_{H^1(\hat{\Omega})} \; &= \; \int\limits_{\Omega} \sum_{i,j=1}^{3} \Big[\Big(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \Big) h_{1j}(\bar{x}) + \Big(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \Big) h_{2j}(\bar{x}) \\ &\qquad + r_i(\bar{x}) h_{3j}(\bar{x}) \Big]^2 \, |\det J(\bar{x})| \, d\bar{x} \, . \end{aligned} \tag{3.3}$$

Proof. This is the consequence of (3.2) and of the change of variables in the integral, similar to that performed in (2.27), (2.28). □

Our aim is to obtain an estimate directly involving the norms of $\bar{u}, \bar{r} \in V(\omega)$. While Korn's inequality estimates the symmetrized gradients $\bar{e}_{ij}$ in terms of the $H^1(\hat{\Omega})$ - norm, our task is more complicated owing to the presence of the nonconstant coefficients h_{ij} appearing in (3.3). In the literature, such inequalities are called *Korn's inequalities in curvilinear coordinates*, see Ciarlet [4]. Here we indicate a direct approach based on a special approximation of the coefficients h_{ij}.

To this end, recall (2.2) and the fact that $\left\langle \bar{n}, \frac{\partial \bar{n}}{\partial x_i} \right\rangle_{I\!R^3} = 0$ for $i = 1, 2$. Hence, we can conclude by a direct calculation that

$$J(\bar{x}) \; = \; \begin{bmatrix} 1 & 0 & n_1 \\ 0 & 1 & n_2 \\ p_1 & p_2 & n_3 \end{bmatrix} \begin{bmatrix} 1 + x_3 \dfrac{\partial n_1}{\partial x_1} & x_3 \dfrac{\partial n_1}{\partial x_2} & 0 \\ x_3 \dfrac{\partial n_2}{\partial x_1} & 1 + x_3 \dfrac{\partial n_2}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \, . \tag{3.4}$$

Apparently, the first matrix does not depend on x_3, while the second matrix is a perturbation of the identity matrix for small values of $|x_3|$. By virtue of the relations (2.9), we also have

$$\begin{bmatrix} 1 & 0 & n_1 \\ 0 & 1 & n_2 \\ p_1 & p_2 & n_3 \end{bmatrix}^{-1} = \frac{1}{\sqrt{1 + p_1^2 + p_2^2}} \begin{bmatrix} n_3 - n_2 p_2 & n_1 p_2 & -n_1 \\ n_2 p_1 & n_3 - n_1 p_1 & -n_2 \\ -p_1 & -p_2 & 1 \end{bmatrix} \tag{3.5}$$

We now approximate the coefficients h_{ij}, $i,j = \overline{1,3}$ by the elements of the matrix $H = (h^0_{ij})_{i,j=\overline{1,3}}$ which is defined by the right-hand side of equation (3.5). From

(2.2) and (3.5), we obtain

$$H = \frac{1}{\sqrt{1+p_1^2+p_2^2}} \begin{bmatrix} \dfrac{1+p_2^2}{\sqrt{1+p_1^2+p_2^2}} & \dfrac{-p_1p_2}{\sqrt{1+p_1^2+p_2^2}} & \dfrac{p_1}{\sqrt{1+p_1^2+p_2^2}} \\ \dfrac{-p_1p_2}{\sqrt{1+p_1^2+p_2^2}} & \dfrac{1+p_1^2}{\sqrt{1+p_1^2+p_2^2}} & \dfrac{p_2}{\sqrt{1+p_1^2+p_2^2}} \\ -p_1 & -p_2 & 1 \end{bmatrix} . \quad (3.6)$$

Obviously, $\det H = \sqrt{1+p_1^2+p_2^2}$, and therefore the quadratic form

$$\mathcal{K}(\bar{u},\bar{r}) := \int_\Omega \sum_{i,j=1}^{3} \Big[\Big(\frac{\partial u_i}{\partial x_i} + x_3\frac{\partial r_i}{\partial x_1}\Big)h_{ij}^0 + \Big(\frac{\partial u_i}{\partial x_2} + x_3\frac{\partial r_i}{\partial x_2}\Big)h_{2j}^0 + r_i h_{3j}^0\Big]^2 \cdot \sqrt{1+p_1^2+p_2^2}\, d\bar{x}\,, \quad (3.7)$$

where $(\bar{u},\bar{r}) \in V(\omega)^2$, constitutes an approximation to the one given in (3.3). It thus makes sense to study this form instead of (3.3) first.

Taking into account that all the functions appearing in (3.7) are independent of x_3, we can perform the integration with respect to x_3 to obtain

$$\begin{aligned}\mathcal{K}(\bar{u},\bar{r}) = {} & 2\,\varepsilon \int_\omega \sum_{i,j=1}^{3} \Big(\frac{\partial u_i}{\partial x_1}h_{1j}^0 + \frac{\partial u_i}{\partial x_2}h_{2j}^0 + r_i h_{3j}^0\Big)^2 \sqrt{1+p_1^2+p_2^2}\; dx_1\,dx_2 \\ & + \frac{2\varepsilon^3}{3} \int_\omega \sum_{i,j=1}^{3} \Big(\frac{\partial r_i}{\partial x_1}h_{1j}^0 + \frac{\partial r_i}{\partial x_2}h_{2j}^0\Big)^2 \sqrt{1+p_1^2+p_2^2}\; dx_1\,dx_2\,. \end{aligned} \quad (3.8)$$

Lemma 3.2 *The quadratic form $\mathcal{K}$ defines a norm on $V(\omega)^2$ through the identity $\|(\bar{u},\bar{r})\| := \sqrt{\mathcal{K}(\bar{u},\bar{r})}$, for $(\bar{u},\bar{r}) \in V(\omega)^2$.*

Proof. Due to the quadratic structure of $\mathcal{K}$, we only need to show that $\mathcal{K}(\bar{u},\bar{r}) = 0$ implies that $(\bar{u},\bar{r}) = (0,0)$ almost everywhere in ω.

We just prove that $\bar{r} = 0$; the argument for $\bar{u}$ is similar. We have

$$\frac{\partial r_i}{\partial x_1}h_{1j}^0 + \frac{\partial r_i}{\partial x_2}h_{2j}^0 = 0\,, \quad i,j = \overline{1,3}\,, \quad \text{a.e. in} \quad \omega\,. \quad (3.9)$$

Let i be fixed. Multiplying (3.9) by $-p_1$ for $j = 3$, and adding the result to relation (3.9) for $j = 1$, we obtain from (3.6) that $\dfrac{\partial r_i}{\partial x_1} = 0$ a.e. in ω. Likewise, multiplication of (3.9) by $-p_2$ for $j = 3$, and addition to relation (3.9) for $j = 2$, yield that $\dfrac{\partial r_i}{\partial x_2} = 0$ a.e. in ω. Since $r_i|_{\gamma_o} = 0$, we conclude that $r_i = 0$ a.e. in ω. □

Lemma 3.3 *There is some* $\hat{c} > 0$ *such that*

$$\int_\omega \sum_{i,j=1}^{3} \Big(\frac{\partial v_i}{\partial x_1} h^0_{1j} + \frac{\partial v_i}{\partial x_2} h^0_{2j} \Big)^2 \, dx_1 \, dx_2 \;\geq\; \hat{c} \, |\bar{v}|^2_{H^1(\omega)^3} \,, \quad \forall \, \bar{v} \in V(\omega) \,, \tag{3.10}$$

where

$$|\bar{v}|^2_{H^1(\omega)^3} \;:=\; \sum_{i=1}^{3} \int_\omega |\nabla v_i|^2 \, dx_1 \, dx_2 \,. \tag{3.11}$$

Proof. Notice at first that, owing to the zero boundary conditions on γ_0, the norm $|\cdot|_{H^1(\omega)^3}$ is equivalent on $V(\omega)$ to the usual norm of $H^1(\omega)^3$.

We consider the linear space

$$W := \Big\{ \bar{v} \in L^2(\omega)^3 \, ; \, \frac{\partial v_i}{\partial x_1} h^0_{1j} + \frac{\partial v_i}{\partial x_2} h^0_{2j} \in L^2(\omega) \,, \;\; i, j = \overline{1,3} \,, \;\; \bar{v}|\gamma_0 \;=\; 0 \Big\} \,. \tag{3.12}$$

Arguing as in the proof of **Lemma 3.2**, we can infer that

$$\|\bar{v}\|_W := \left(\int_\omega \sum_{i,j=1}^{3} \Big(\frac{\partial v_i}{\partial x_1} h^0_{1j} + \frac{\partial v_i}{\partial x_2} h^0_{2j} \Big)^2 \, dx_1 \, dx_2 \right)^{1/2} \tag{3.13}$$

defines a norm on W. Clearly, we have $V(\omega) \subset W$, and for any $\bar{v} \in V(\omega)$ it holds

$$\|\bar{v}\|_W \;\leq\; M \, |\bar{v}|_{H^1(\omega)^3} \,, \tag{3.14}$$

with some fixed $M > 0$. We now show that also $W \subset V(\omega)$, i.e. that $W = V(\omega)$. To this end, suppose that $\bar{v} \in W$, and let

$$f_{ij} := \frac{\partial v_i}{\partial x_1} h^0_{1j} + \frac{\partial v_i}{\partial x_2} h^0_{2j} \,, \quad i, j = \overline{1,3} \,. \tag{3.15}$$

Then $f_{ij} \in L^2(\omega)$, $i, j = \overline{1,3}$. Now let i be fixed. As in the proof of **Lemma 3.2**, we multiply (3.15) by $-p_1$ for $j = 3$ and add the result to (3.15) for $j = 1$, to find that

$$\frac{\partial v_i}{\partial x_1} \;=\; \frac{-p_1 \, f_{i3} \;+\; f_{i1}}{\sqrt{1 + p_1^2 + p_2^2}} \in L^2(\omega) \,. \tag{3.16}$$

Similarly, we prove that also $\frac{\partial v_i}{\partial x_2} \in L^2(\omega)$. In conclusion, $v_i \in H^1(\omega)$ (which also makes the boundary condition $\bar{v}_{|\gamma_0} = 0$ meaningful), and thus $\bar{v} \in V(\omega)$.

We now consider the identity mapping I acting between the Banach space $(V(\omega)$, $|\cdot|_{H^1(\omega)^3})$ and the normed space $(W, \|\cdot\|_W)$. Clearly, I is linear and injective, and we have just shown its surjectivity. Besides, (3.14) implies that I is continuous. Therefore, if $(W, \|\cdot\|_W)$ is also complete, i.e. a Banach space, then it follows from the open mapping theorem that also the inverse I^{-1} is continuous which then proves (3.10).

To prove the completeness, take any $\|\cdot\|_W$-Cauchy sequence $\{\bar{v}^n\} \subset W$. Then, for $i, j = \overline{1,3}$,

$$q_{ij}^{n,m} := \left(\frac{\partial v_i^n}{\partial x_1} - \frac{\partial v_i^m}{\partial x_1}\right) h_{1j}^0 + \left(\frac{\partial v_i^n}{\partial x_2} - \frac{\partial v_i^m}{\partial x_2}\right) h_{2j}^0 \to 0, \quad n, m \to \infty, \tag{3.17}$$

in $L^2(\omega)$. Using the same argument as in the derivation of (3.15), we have, for $i = \overline{1,3}$,

$$\frac{\partial(v_i^n - v_i^m)}{\partial x_1} = \frac{-p_1\, q_{i3}^{n,m} + q_{i1}^{n,m}}{\sqrt{1 + p_1^2 + p_2^2}}, \tag{3.18}$$

which converges to 0 in $L^2(\omega)$ as $n, m \to \infty$. Arguing similarly for $\frac{\partial(v_i^n - v_i^m)}{\partial x_2}$, we conclude that $\{\bar{v}^n\}$ is a Cauchy sequence in $(V(\omega), |\cdot|_{H^1(\omega)^3})$, hence convergent to some $\bar{v} \in V(\omega)$. By (3.14), $\|\bar{v}_n - \bar{v}\|_W \to 0$, which concludes the proof of the assertion. □

Lemma 3.4 *$\mathcal{K}$ is coercive on $V(\omega)^2$ equipped with the usual $H_0^1(\omega)^6$-norm.*

Proof Let $(\bar{u}, \bar{r}) \in V(\omega)$. Using Young's inequality and **Lemma 3.3**, we have with some $\hat{C} > 0$,

$$\begin{aligned}
\mathcal{K}(\bar{u}, \bar{r}) \;\geq\;& \varepsilon \int_\omega \sum_{i,j=1}^{3} \Big(\frac{\partial u_i}{\partial x_i} h_{1j}^0 + \frac{\partial u_i}{\partial x_2} h_{2j}^0\Big)^2 \sqrt{1 + p_1^2 + p_2^2}\; dx_1\, dx_2 \\
& - \hat{C}\,\varepsilon \int_\omega \sum_{i,j=1}^{3} r_i^2 (h_{3j}^0)^2 \sqrt{1 + p_1^2 + p_2^2}\; dx_1\, dx_2 \\
& + \frac{2\,\varepsilon^3}{3} \int_\omega \sum_{i,j=1}^{3} \Big(\frac{\partial r_i}{\partial x_i} h_{1j}^0 + \frac{\partial r_i}{\partial x_2} h_{2j}^0\Big)^2 \sqrt{1 + p_1^2 + p_2^2}\; dx_1\, dx_2 \\
\geq\;& \hat{c}\,\varepsilon \int_\omega \sum_{i=1}^{3} \Big\{\Big|\frac{\partial u_i}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_i}{\partial x_2}\Big|^2\Big\}\, dx_1\, dx_2 \\
& + \frac{2\,\hat{c}\,\varepsilon^3}{3} \int_\omega \sum_{i=1}^{3} \Big\{\Big|\frac{\partial r_i}{\partial x_1}\Big|^2 + \Big|\frac{\partial r_i}{\partial x_2}\Big|^2\Big\}\, dx_1\, dx_2 \\
& - \hat{C}\,\varepsilon \int_\omega \sum_{i,j=1}^{3} r_i^2 (h_{3j}^0)^2 \sqrt{1 + p_1^2 + p_2^2}\; dx_1\, dx_2\,.
\end{aligned} \tag{3.19}$$

Now assume that $\mathcal{K}$ is not coercive in the $H_0^1(\omega)^6$ - norm. Then there exists a sequence $\{(\bar{u}^n, \bar{r}^n)\} \subset V(\omega)^2$ satisfying

$$\sum_{i=1}^{3} \int_{\omega} \left(\left| \frac{\partial u_1^n}{\partial x_1} \right|^2 + \left| \frac{\partial u_1^n}{\partial x_2} \right|^2 + \cdots + \left| \frac{\partial r_3^n}{\partial x_1} \right|^2 + \left| \frac{\partial r_3^n}{\partial x_2} \right|^2 \right) dx_1\, dx_2 = 1 \quad \forall\, n \in I\!N \,, \tag{3.20}$$

such that

$$\mathcal{K}(\bar{u}^n, \bar{r}^n) \to 0 \quad \text{for} \quad n \to \infty \,. \tag{3.21}$$

In view of (3.20), we can assume without loss of generality that $\bar{u}^n \to \bar{u}$ and $\bar{r}^n \to \bar{r}$ weakly in $V(\omega)$ and, by compact imbedding, strongly in $L^2(\omega)^3$. The weak lower semicontinuity of the quadratic form yields that

$$\lim_{n\to\infty} \mathcal{K}(\bar{u}^n, \bar{r}^n) \;\geq\; \mathcal{K}(\bar{u}, \bar{r}) \;\geq\; 0\,, \tag{3.22}$$

and we can infer from (3.22), (3.21) and **Lemma 3.2** that $u_i = 0$, $r_i = 0$, a. e. in ω, $i = \overline{1,3}$.

However, from (3.19) and (3.20), and since $0 < \varepsilon < 1$, we can infer that

$$\begin{aligned}
\mathcal{K}(\bar{u}^n, \bar{r}^n) \;\;&\geq\;\; \frac{2\,\hat{c}\,\varepsilon^3}{3} \int_{\omega} \sum_{i=1}^{3} \left\{ \left| \frac{\partial u_i^n}{\partial x_1} \right|^2 + \left| \frac{\partial u_i^n}{\partial x_2} \right|^2 + \left| \frac{\partial r_i^n}{\partial x_1} \right|^2 + \left| \frac{\partial r_i^n}{\partial x_2} \right|^2 \right\} dx_1\, dx_2 \\
&\quad - \hat{C}\,\varepsilon \int_{\omega} \sum_{i,j=1}^{3} (r_{3j}^n)^2\, (h_{3j}^0)^2 \sqrt{1 + p_1^2 + p_2^2}\; dx_1\, dx_2 \\
&= \;\; \frac{2\,\hat{c}\,\varepsilon^3}{3} \;-\; \hat{C}\,\varepsilon \int_{\omega} \sum_{i,j=1}^{3} (r_i^n)^2\, (h_{3j}^0)^2 \sqrt{1 + p_1^2 + p_2^2}\, dx_1 dx_2\,.
\end{aligned} \tag{3.23}$$

The strong convergence of $\bar{r}^n$ in $L^2(\omega)^3$ allows to pass to the limit as $n \to \infty$ in (3.23), whence we arrive at a contradiction. This concludes the proof of the lemma. $\square$

Remark 3.1 The coercivity constants of $\mathcal{K}$ have the form from (3.19), with the last term (containing the r_i, $i = \overline{1,3}$) just neglected.

Proof. of Theorem 2.1 We use the form (2.30) of $B(\hat{u}, \hat{v})$ and (3.1), (3.3). We estimate the expression

$$\begin{aligned}
A \;\; := \;\; & \int_{\Omega} \Big[\Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) h_{11} + \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) h_{21} + r_1 h_{31} \Big]^2 \, |\det J(\bar{x})|\, d\bar{x} \\
& - \int_{\Omega} \Big[\Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) h_{11}^0 + \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) h_{21}^0 + r_1 h_{31}^0 \Big]^2 \sqrt{1 + p_1^2 + p_2^2}\; d\bar{x}\,.
\end{aligned}$$

From the way we will prove an advantageous estimate for A it will become clear that similar estimates can be obtained for all the other terms occurring in $B(\hat{u},\hat{u})$, and therefore we will be able to employ **Lemma 3.4** to get the desired coercivity conclusion for $B(\hat{u},\hat{u})$. Now let

$$
\begin{aligned}
M \;&:=\; \Big(\frac{\partial u_1}{\partial x_1} + x_3\frac{\partial r_1}{\partial x_1}\Big)h_{11} + \Big(\frac{\partial u_1}{\partial x_2} + x_3\frac{\partial r_1}{\partial x_2}\Big)h_{21} + r_1 h_{31}\,,\\
\tilde{M} \;&:=\; \Big(\frac{\partial u_1}{\partial x_1} + x_3\frac{\partial r_1}{\partial x_1}\Big)\tilde{h}_{11} + \Big(\frac{\partial u_1}{\partial x_2} + x_3\frac{\partial r_1}{\partial x_2}\Big)\tilde{h}_{21} + r_1 \tilde{h}_{31}\,,\\
M_0 \;&:=\; \Big(\frac{\partial u_1}{\partial x_1} + x_3\frac{\partial r_1}{\partial x_1}\Big)h^0_{11} + \Big(\frac{\partial u_1}{\partial x_2} + x_3\frac{\partial r_1}{\partial x_2}\Big)h^0_{21} + r_1 h^0_{31}\,.
\end{aligned}
\tag{3.24}
$$

Then

$$
\begin{aligned}
A \;&=\; \int_\Omega M^2\,|\det J(\bar{x})|\,d\bar{x} \;-\; \int_\Omega M_0^2\sqrt{1+p_1^2+p_2^2}\;d\bar{x}\\
&=\; \left(\int_\Omega M^2\,|\det J(\bar{x})|\,d\bar{x} \;-\; \int_\Omega \frac{\tilde{M}^2}{\sqrt{1+p_1^2+p_2^2}}\;d\bar{x}\right)\\
&\quad + \left(\int_\Omega \frac{\tilde{M}^2}{\sqrt{1+p_1^2+p_2^2}}\;d\bar{x} - \int_\Omega M_0^2\,\sqrt{1+p_1^2+p_2^2}\;d\bar{x}\right)\\
&=:\; A_1 \;+\; A_2\,,
\end{aligned}
\tag{3.25}
$$

with obvious meaning of A_1, A_2. We have, by (2.17) - (2.20),

$$
\begin{aligned}
\frac{\tilde{h}_{11}}{\sqrt{1+p_1^2+p_2^2}} - h^0_{11} \;&=\; \frac{x_3}{1+p_1^2+p_2^2}\,\Big(\frac{\partial n_2}{\partial x_2} + p_2\frac{\partial n_3}{\partial x_2}\Big)\,,\\
\frac{\tilde{h}_{21}}{\sqrt{1+p_1^2+p_2^2}} - h^0_{21} \;&=\; \frac{-\,x_3}{1+p_1^2+p_2^2}\,\Big(\frac{\partial n_2}{\partial x_1} + p_2\frac{\partial n_3}{\partial x_1}\Big)\,,\\
\frac{\tilde{h}_{31}}{\sqrt{1+p_1^2+p_2^2}} - h^0_{31} \;&=\; \frac{x_3}{\sqrt{1+p_1^2+p_2^2}}\,\Big[\frac{\partial n_2}{\partial x_1}\Big(p_2 + x_3\frac{\partial n_3}{\partial x_2}\Big)\\
&\qquad - \frac{\partial n_2}{\partial x_2}\Big(p_1 + x_3\frac{\partial n_3}{\partial x_1}\Big) \;-\; \frac{\partial n_3}{\partial x_1}\Big]\;.
\end{aligned}
\tag{3.26}
$$

Using (3.24) and (3.26), we find that

$$
\begin{aligned}
A_2 \; = \; \int\limits_{\Omega} \Bigg\{ & \Big(\frac{\partial u_1}{\partial x_1} + x_3 \frac{\partial r_1}{\partial x_1} \Big) \Big(\frac{\partial n_2}{\partial x_2} + p_2 \frac{\partial n_3}{\partial x_2} \Big) \; - \; \Big(\frac{\partial u_1}{\partial x_2} + x_3 \frac{\partial r_1}{\partial x_2} \Big) \Big(p_2 \frac{\partial n_3}{\partial x_1} + \frac{\partial n_2}{\partial x_1} \Big) \\
& + r_1 \sqrt{1 + p_1^2 + p_2^2} \; \Big[\frac{\partial n_2}{\partial x_1} \Big(p_2 + x_3 \frac{\partial n_3}{\partial x_2} \Big) - \frac{\partial n_2}{\partial x_2} \Big(p_1 + x_3 \frac{\partial n_3}{\partial x_1} \Big) - \frac{\partial n_3}{\partial x_1} \Big] \Bigg\} \\
& \cdot \frac{x_3}{1 + p_1^2 + p_2^2} \; \Big(\tilde{M} \; + \; M_0 \sqrt{1 + p_1^2 + p_2^2} \Big) \; d\bar{x} \, . \qquad (3.27)
\end{aligned}
$$

From this expression, and from the definitions of $\tilde{M}$, M_0 , it is clear that A_2 is of the form

$$
A_2 \; = \; \int\limits_{\Omega} \Big[x_3 \, X(x_1, x_2) \; + \; x_3^2 \, Y(x_1, x_2) \; + \; x_3^3 \, Z(x_1, x_2) \Big] \, d\bar{x} \, , \qquad (3.28)
$$

where X , Y , Z are quadratic polynomials of the variables $\frac{\partial u_1}{\partial x_1}$, $\frac{\partial u_1}{\partial x_2}$, $\frac{\partial r_1}{\partial x_1}$, $\frac{\partial r_1}{\partial x_2}$, and r_1 , whose coefficients all belong to $L^\infty(\omega)$ since $p \in W^{2,\infty}(\omega)$. The terms with odd powers of x_3 vanish after integration with respect to x_3 , and thus we only have to examine the expression

$$
L \; := \; \int\limits_{\Omega} x_3^2 \, Y(x_1, x_2) \, d\bar{x} \; = \; \frac{2 \, \varepsilon^3}{3} \int\limits_{\omega} Y(x_1, x_2) \, dx_1 \, dx_2 \, . \qquad (3.29)
$$

It is clear that $Y(x_1, x_2)$ is formed from the summation of terms that appear when terms in A_2 without the factor x_3 are multiplied by terms having the factor x_3 . From the definition of $\tilde{M}$ and M_0 , and from inspecting (3.27), we find that

$$
\begin{aligned}
L \; = \; \frac{2 \, \varepsilon^3}{3} \int\limits_{\omega} \Bigg\{ & r_1^2 \; y^{(1)}(x_1, x_2) \; + \; \sum_{i,j=1}^{2} \frac{\partial u_1}{\partial x_i} \, \frac{\partial r_1}{\partial x_j} \; y_{ij}^{(2)}(x_1, x_2) \\
& + \sum_{i=1}^{2} r_1 \; \Big(\frac{\partial u_1}{\partial x_i} \, y_i^{(3)}(x_1, x_2) \; + \; \frac{\partial r_1}{\partial x_i} \; y_i^{(4)}(x_1, x_2) \Big) \Bigg\} \; dx_1 \, dx_2 \, ,
\end{aligned}
$$

$$(3.30)$$

where all the coefficient functions $y^{(1)}$, $y_{ij}^{(2)}$, $y_i^{(3)}$, and $y_i^{(4)}$, respectively, are known to be bounded in $L^\infty(\omega)$, since $p \in W^{2,\infty}(\omega)$. We thus can estimate, using Young's inequality and the fact that $0 < \varepsilon < 1$,

$$\begin{aligned} |L| &\le \frac{2\,\hat{C}_1\,\varepsilon^3}{3} \int\limits_{\omega} \Bigg\{ r_1^2 + \sum_{i,j=1}^{2} \left|\frac{\partial u_1}{\partial x_i}\right| \left|\frac{\partial r_1}{\partial x_j}\right| \\ &\qquad\qquad + \sum_{i=1}^{2} |r_1| \left(\left|\frac{\partial u_1}{\partial x_i}\right| + \left|\frac{\partial r_1}{\partial x_i}\right| \right) \Bigg\}\, dx_1\, dx_2 \\ &\le \hat{C}_2\,\varepsilon^2 \sum_{i=1}^{3} |u_i|^2_{H^1(\omega)} + \hat{C}_2\,\varepsilon^4 \sum_{i=1}^{3} |r_i|^2_{H^1(\omega)} + \hat{C}_2\,\varepsilon^2 \sum_{i=1}^{3} |r_i|^2_{L^2(\omega)}, \end{aligned} \tag{3.31}$$

with constants $\hat{C}_1 > 0$, $\hat{C}_2 > 0$ that only depend on the $L^\infty(\omega)$ – norms of the functions $y^{(1)}$, $y^{(2)}_{ij}$, $y^{(3)}_i$, and $y^{(4)}_i$.

By comparing this inequality with (3.19) and **Remark 3.1**, we see that L is dominated by $\mathcal{K}(\bar{u}, \bar{r})$, provided that $\varepsilon > 0$ is sufficiently small in comparison with the (a priori known) constant $\hat{C}_2$.

It remains to estimate A_1. Note that, owing to (3.24), and in view of (2.17) to (2.20), we have $\tilde{M} = M \cdot \det J(\bar{x})$, and hence it follows from (2.14), (2.15) that

$$\begin{aligned} A_1 &= \int_\Omega \tilde{M}^2 \left(\frac{1}{\det J(\bar{x})} - \frac{1}{\sqrt{1 + p_1^2 + p_2^2}} \right) d\bar{x} \\ &= -\int\limits_\Omega \tilde{M}^2\, x_3\, \frac{\dfrac{\partial n_1}{\partial x_1} + \dfrac{\partial n_2}{\partial x_2} + x_3 \left(\dfrac{\partial n_1}{\partial x_1}\dfrac{\partial n_2}{\partial x_2} - \dfrac{\partial n_1}{\partial x_2}\dfrac{\partial n_2}{\partial x_1} \right)}{\det J(\bar{x})}\, d\bar{x}\,. \end{aligned} \tag{3.32}$$

Next, we perform a Taylor expansion of the function $\varphi(x_3) := 1/\det J(x_1, x_2, x_3)$ around $x_3 = 0$. We easily find that

$$\frac{1}{\det J(x_1, x_2, x_3)} = \frac{1 - x_3 \left(\frac{\partial n_1}{\partial x_1}(x_1, x_2) + \frac{\partial n_2}{\partial x_2}(x_1, x_2) \right) + x_3^2\, \alpha(x_1, x_2, x_3)}{\sqrt{1 + p_1^2(x_1, x_2) + p_2^2(x_1, x_2)}} \tag{3.33}$$

with some function $\alpha \in L^\infty(\Omega)$ whose $L^\infty(\Omega)$ - norm is bounded from above by a constant that only depends on $\|p\|_{W^{2,\infty}(\omega)}$.

We now can argue as follows: the first two terms in (3.33) can be combined with the remaining ones occurring in A_1, and we can explicitly integrate and estimate them as in the case of L. Again, they are dominated by $\mathcal{K}(\bar{u}, \bar{r})$ provided that $\varepsilon > 0$ is small enough. The remaining term from (3.33), which depends in a complicated way on x_1, x_2, x_3, is of order x_3^2, and direct estimates can be performed in combination with the other factors in A_1 to see that it is also dominated by $\mathcal{K}(\bar{u}, \bar{r})$.

We are now in the position to conclude the proof of the assertion: indeed, from the method of estimation used above for A it is apparent that similar computations and estimates can be carried out for all the other terms occurring in $B(\hat{u},\hat{u})$. Since these estimations are straightforward (while quite lengthy), we do not present them in detail, here. It turns out that all the occurring differences are dominated by $\mathcal{K}(\bar{u},\bar{r})$ provided that $\varepsilon > 0$ is sufficiently small. Consequently, $B(\hat{u},\hat{u})$ inherits the coercivity of $\mathcal{K}$. This ends the proof of the theorem. □

Remark 3.2 **Theorem 2.1** and its proof remain valid if the shell $\hat{\Omega}$ is of nonconstant thickness, as long as the thickness remains bounded from below by $\varepsilon > 0$. Adequate regularity assumptions on $\partial\hat{\Omega}$ have to be imposed.

Remark 3.3 It is obvious from the proof of **Theorem 2.1** that the coercivity constant of the bilinear form B is of the order ε^3, and ε must be small for its validity. This explains the well-known instability appearing in numerical computations for shells.

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