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Symplectic methods for Hamiltonian systems with additive noise

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ABSTRACT. Stochastic systems, phase flows of which have integral invariants, are considered. Hamiltonian systems with additive noise being a wide class of such systems possess the property of preserving symplectic structure. For them, numerical methods preserving the symplectic structure are constructed. A special attention is paid to systems with separable Hamiltonians, to second order differential equations with additive noise, and to Hamiltonian systems with small additive noise.

1. INTRODUCTION

The problem of preserving integral invariants in numerical integration of deterministic differential equations is of great current interest (see, e.g., [15, 4, 14, 16, 17] and references therein). The phase flows of some classes of stochastic systems possess the property of phase-volume preservation or possess the more strong property of preserving symplectic structure (symplecticness) [3, 8, 1]. For instance, Hamiltonian equations with additive noise are a rather wide and important class of systems having these properties. In the present paper we construct special numerical methods which preserve integral invariants in such stochastic systems. However some issues are discussed for general systems.

Consider the Cauchy problem for system of stochastic differential equations (SDEs) in the sense of Stratonovich:

$$dX = a(t,X)dt + \sum_{r=1}^{m} b_r(t,X) \circ dw_r(t), \ X(t_0) = x, \tag{1.1}$$

where X, $a(t,x^1,\dots,x^d)$, $b_r(t,x^1,\dots,x^d)$ are d-dimensional column-vectors with the components X^i, a^i, b_r^i, $i=1,\dots,d$, and $w_r(t)$, $r=1,\dots,m$, are independent standard Wiener processes.

We suppose that all the coefficients of considered systems are sufficiently smooth functions defined for $(t,x) \in [t_0, t_0+T] \times R^d$ and they provide the property of extendability of solutions to the interval $[t_0, t_0+T]$ (some additional conditions on boundedness of partial derivatives of the coefficients in connection with considered methods are given in Section 3.2).

We denote by $X(t;t_0,x) = X(t;t_0,x^1,\dots,x^d)$, $t_0 \le t \le t_0+T$, the solution of the problem (1.1). A more detailed notation is $X(t;t_0,x;\omega)$, where ω is an elementary event. It is known that $X(t;t_0,x;\omega)$ is a phase flow (diffeomorphism) for almost every ω. See its properties in, e.g. [3, 6, 5, 8].

Let $D_0 \in R^d$ be a domain with finite volume. This domain may be random. We suppose that $D_0 = D_0(\omega)$ is independent of the Wiener processes $w_r(t)$, $t \in [t_0, t_0+T]$. The transformation $X(t;t_0,x;\omega)$ maps D_0 into the domain $D_t = D_t(\omega)$. The volume V_t of the domain D_t is equal to

$$V_t = \int_{D_t} dX^1 \dots dX^d = \int_{D_0} \left| \frac{D(X^1,\dots,X^d)}{D(x^1,\dots,x^d)} \right| dx^1 \dots dx^d. \tag{1.2}$$

Obviously, the volume-preserving condition consists in the equality

$$\left| \frac{D(X^1, \dots, X^d)}{D(x^1, \dots, x^d)} \right| = 1. \tag{1.3}$$

In Section 2.1, we find out a class of stochastic systems satisfying this condition (cf. [8, 1]).

Let us write a system of SDEs of even dimension $d = 2n$ in the form

$$dP = f(t, P, Q)dt + \sum_{r=1}^{m} \sigma_r(t, P, Q) \circ dw_r(t), \;\; P(t_0) = p, \tag{1.4}$$

$$dQ = g(t, P, Q)dt + \sum_{r=1}^{m} \gamma_r(t, P, Q) \circ dw_r(t), \;\; Q(t_0) = q.$$

Here P, Q, f, g, σ_r, γ_r are n-dimensional column-vectors.

Introduce the differential 2-form

$$\omega^2 = dp \wedge dq = dp^1 \wedge dq^1 + \dots + dp^n \wedge dq^n. \tag{1.5}$$

We are interested in systems (1.4) such that the transformation $(p, q) \mapsto (P, Q)$ preserves symplectic structure [2]:

$$dP \wedge dQ = dp \wedge dq, \tag{1.6}$$

i.e., when the sum of the oriented areas of projections onto the coordinate planes (p^1, q^1), ..., (p^n, q^n) is an integral invariant. As a consequence, all external powers of the 2-form

$$(\omega^2)^l = \underbrace{\omega^2 \wedge \dots \wedge \omega^2}_{l-times}$$

are invariant for such systems as well. The case $l = n$ gives preservation of phase volume.

Phase flows of deterministic Hamiltonian systems (i.e., when $\sigma_r = 0$, $\gamma_r = 0$, $r = 1, \dots, m$, and there is a function $H(t, p, q)$ such that $f^i = -\partial H/\partial q^i$, $g^i = \partial H/\partial p^i$, $i = 1, \dots, n$) are known to preserve symplectic structure. It turns out (see [3] and Section 2.2) that if there are functions $H(t, p, q)$, $H_r(t, p, q)$, $r = 1, \dots, m$, such that

$$f^i = -\partial H/\partial q^i, \quad g^i = \partial H/\partial p^i, \tag{1.7}$$

$$\sigma_r^i = -\partial H_r/\partial q^i, \quad \gamma_r^i = \partial H_r/\partial p^i, \quad i = 1, \dots, n, \quad r = 1, \dots, m,$$

then the phase flow of (1.4) preserves symplectic structure.

Let X_k, $k = 0, \dots, N$, $t_{k+1} - t_k = h_{k+1}$, $t_N = t_0 + T$:

$$X_0 = X(t_0), \;\; X_{k+1} = \bar{X}_{t_k, X_k}(t_{k+1}),$$

be a mean-square method for (1.1) based on the one-step mean-square approximation $\bar{X}_{t,x}(t+h) = \bar{X}(t+h; t, x)$. It is clear that a method preserves phase volume (said to be Liouvillian) if its one-step approximation satisfies the equality

$$\left| \frac{D(\bar{X}^1, \dots, \bar{X}^d)}{D(x^1, \dots, x^d)} \right| = 1. \tag{1.8}$$

Analogously, a method for (1.4) based on the one-step approximation $\bar{P} = \bar{P}(t+h; t, p, q)$, $\bar{Q} = \bar{Q}(t+h; t, p, q)$ preserves symplectic structure if

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq\,. \tag{1.9}$$

For a reader convenience, we give some auxiliary material on Hamiltonian methods for deterministic systems and on numerical integration of SDEs in Section 3.

In Section 4, we construct some implicit Hamiltonian methods of mean-square order 1 and 3/2 for general Hamiltonian systems with additive noise. We propose more effective methods for systems when the Hamiltonians have a special form. We consider the case of separable Hamiltonians $H(t,p,q) = V(p) + U(t,q)$ in Section 5 and the case of Hamiltonians $H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q)$ with M a constant, symmetric, invertible matrix in Section 6. Further, we pay a special attention to Hamiltonian systems with small additive noise. High-exactness methods for such systems are constructed in Section 7. Let us underline that all the derived methods are efficient with respect to simulation of the used random variables.

In the next paper we plan to propose mean-square Hamiltonian methods for Hamiltonian systems with multiplicative noise and to consider Liouvillian methods. We will also present some numerical experiments.

Here we construct mean-square methods. As is known (see, e.g. [9, 7, 18, 12]), when we simulate a stochastic system by the Monte Carlo technique, we can use weak approximations which are in many respects simpler than mean-square ones. Hamiltonian weak methods will be considered in a later publication.

2. Stochastic systems preserving phase volume and symplectic structure

2.1. **Preservation of phase volume.** We are going to find out a class of stochastic systems, which preserve phase volume, i.e., which satisfy the volume-preserving condition (1.3). To this end, we evaluate the Jacobian from (1.3). For a fixed j, the vector

$$Z := \frac{\partial X}{\partial x^j} = \left[\frac{\partial X^1}{\partial x^j}, \cdots, \frac{\partial X^d}{\partial x^j}\right]^\top$$

obeys the following system of linear SDEs

$$dZ^i = \sum_{k=1}^{d} \frac{\partial a^i}{\partial x^k}(t, X^1, \ldots, X^d) Z^k dt + \sum_{r=1}^{m}\sum_{k=1}^{d} \frac{\partial b^i_r}{\partial x^k}(t, X^1, \ldots, X^d) Z^k \circ dw_r, \tag{2.1}$$

$$Z^i(t_0) = \frac{\partial X^i}{\partial x^j}(t_0) = \delta_{ij} = \left\{ \begin{array}{l} 1,\ i = j \\ 0,\ i \neq j \end{array} \right. ,\ i = 1, \ldots, d,$$

where $[X^1(t;t_0,x), \ldots, X^d(t;t_0,x)]^\top$ is a solution to (1.1).

Introduce the fundamental matrix $\Phi(t)$ of solutions to (2.1) normalized at $t = t_0$:

$$\Phi(t) = \begin{bmatrix} \dfrac{\partial X^1}{\partial x^1} \cdots \dfrac{\partial X^1}{\partial x^d} \\ \vdots \; \cdots \; \vdots \\ \dfrac{\partial X^d}{\partial x^1} \cdots \dfrac{\partial X^d}{\partial x^d} \end{bmatrix}.$$

This matrix is a solution of the linear system

$$d\Phi = A\Phi dt + \sum_{r=1}^{m} B_r \Phi \circ dw_r(t), \; \Phi(t_0) = I. \tag{2.2}$$

Here A and B_r are matrices with the components $\dfrac{\partial a^i}{\partial x^j}(t, X^1(t;t_0,x), \ldots, X^d(t;t_0,x))$ and $\dfrac{\partial b_r^i}{\partial x^j}(t, X^1(t;t_0,x), \ldots, X^d(t;t_0,x))$ correspondingly. They depend on $x = (x^1, \ldots, x^d)^\top$ as on a parameter and depend on the previous behavior of the processes $w_r(s)$, $t_0 \le s \le t$, $r = 1, \ldots, m$. We denote by I the $d \times d$ unit matrix.

The formula for the determinant of solution to the linear matrix Stratonovich equation has the form (see, e.g., [8, 1])

$$\det \Phi(t) = \det \Phi(t_0) \cdot \exp \left\{ \int_{t_0}^{t} tr A \, ds + \int_{t_0}^{t} \sum_{r=1}^{m} tr B_r \circ dw_r(s) \right\}. \tag{2.3}$$

Since

$$\det \Phi(t) = \frac{D(X^1, \ldots, X^d)}{D(x^1, \ldots, x^d)} \quad \text{and} \quad \det \Phi(t_0) = 1,$$

the formulae (1.2), (2.3) and the condition (1.3) imply the following proposition.

Theorem 2.1. *The necessary and sufficient conditions for volume preservation by the phase flow of the Stratonovich system* (1.1) *consist in holding the equalities:*

$$\frac{\partial a^1(t,x)}{\partial x^1} + \cdots + \frac{\partial a^d(t,x)}{\partial x^d} = div \, a = 0, \tag{2.4}$$

$$\frac{\partial b_r^1(t,x)}{\partial x^1} + \cdots + \frac{\partial b_r^d(t,x)}{\partial x^d} = div \, b_r = 0, \; r = 1, \ldots, m. \tag{2.5}$$

Corollary 2.2. The necessary and sufficient conditions for volume preservation by the phase flow of the Ito system

$$dX = a(t,X)dt + \sum_{r=1}^{m} b_r(t,X) dw_r(t), \; X(t_0) = x, \tag{2.6}$$

consist in holding the equalities

$$div \, (a - \frac{1}{2} \sum_{r=1}^{m} \frac{\partial b_r}{\partial x} b_r) = 0. \tag{2.7}$$

$$div \, b_r = 0, \; r = 1, \ldots, m, \tag{2.8}$$

where $\dfrac{\partial b_r}{\partial x}$ is the matrix with the components $\dfrac{\partial b_r^i}{\partial x^j}(t, x^1, \ldots, x^d)$.

Let us recall that in the case of additive noise a system has the same form both in the sense of Ito and Stratonovich.

Corollary 2.3. If the phase flow of the deterministic system $dX = a(t, X)\, dt$ preserves volume, i.e., $div\, a = 0$, then the phase flow of the stochastic system with additive noise

$$dX = a(t, X)dt + \sum_{r=1}^{m} b_r(t) dw_r(t), \;\; X(t_0) = x, \tag{2.9}$$

preserves volume as well. In particular, Hamiltonian systems with additive noise preserve volume.

The Ito system

$$\frac{d^2X}{dt^2} = f(t, X)dt + \sum_{r=1}^{m} \sigma_r(t, X)\dot{w}_r(t) \tag{2.10}$$

gives another example of volume-preserving system. It has the following normal form

$$dX = Y dt \tag{2.11}$$

$$dY = f(t, X)\, dt + \sum_{r=1}^{m} \sigma_r(t, X)\, dw_r(t),$$

where X and Y are n-dimensional vectors, i.e., $d = 2n$.

Since

$$a = [y^1, \ldots, y^n, \; f^1, \ldots, f^n]^\top, \; b_r = [0, \ldots, 0, \; \sigma_r^1, \ldots, \sigma_r^n]^\top,$$

and σ_r^i are independent of $y^1, \ldots, y^n$, we get

$$div\, a = 0, \;\; div\, b_r = 0, \;\; \frac{\partial b_r}{\partial x} b_r = 0, \;\; r = 1, \ldots, m.$$

Thus, the phase flow of the system (2.11) preserves volume for any f and σ_r. We note that the system (2.11), which is with multiplicative noise, has the same form in the sense of Stratonovich.

A more general volume-preserving system has the form

$$dX = g(t, Y)dt + \sum_{r=1}^{m} \gamma_r(t)\, dw_r(t)$$

$$dY = f(t, X)\, dt + \sum_{r=1}^{m} \sigma_r(t, X)\, dw_r(t).$$

2.2. **Preservation of symplectic structure.** Consider the system (1.4). Our urgent aim is to indicate a class of stochastic systems, which preserve symplectic structure, i.e., which satisfy the condition (1.6).

Using the formula of change of variables in differential forms, we obtain

$$
\begin{aligned}
dP \wedge dQ &= dP^1 \wedge dQ^1 + \cdots + dP^n \wedge dQ^n \\
&= \sum_{k=1}^{n} \sum_{l=k+1}^{n} \sum_{i=1}^{n} \left(\frac{\partial P^i}{\partial p^k} \frac{\partial Q^i}{\partial p^l} - \frac{\partial P^i}{\partial p^l} \frac{\partial Q^i}{\partial p^k} \right) dp^k \wedge dp^l \\
&+ \sum_{k=1}^{n} \sum_{l=k+1}^{n} \sum_{i=1}^{n} \left(\frac{\partial P^i}{\partial q^k} \frac{\partial Q^i}{\partial q^l} - \frac{\partial P^i}{\partial q^l} \frac{\partial Q^i}{\partial q^k} \right) dq^k \wedge dq^l \\
&+ \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial P^i}{\partial p^k} \frac{\partial Q^i}{\partial q^l} - \frac{\partial P^i}{\partial q^l} \frac{\partial Q^i}{\partial p^k} \right) dp^k \wedge dq^l.
\end{aligned}
$$

Hence the phase flow of (1.4) preserves symplectic structure if and only if

$$
\sum_{i=1}^{n} \frac{D(P^i, Q^i)}{D(p^k, p^l)} = 0, \quad k \neq l, \tag{2.12}
$$

$$
\sum_{i=1}^{n} \frac{D(P^i, Q^i)}{D(q^k, q^l)} = 0, \quad k \neq l, \tag{2.13}
$$

and

$$
\sum_{i=1}^{n} \frac{D(P^i, Q^i)}{D(p^k, q^l)} = \delta_{kl}, \quad k, l = 1, \ldots, n. \tag{2.14}
$$

Introduce the notation

$$
P_p^{ik} = \frac{\partial P^i}{\partial p^k}, \quad P_q^{ik} = \frac{\partial P^i}{\partial q^k}, \quad Q_p^{ik} = \frac{\partial Q^i}{\partial p^k}, \quad Q_q^{ik} = \frac{\partial Q^i}{\partial q^k}.
$$

For a fixed k, we obtain that P_p^{ik}, Q_p^{ik} obey the following system of SDEs

(2.15)

$$
\begin{aligned}
dP_p^{ik} &= \sum_{\alpha=1}^{n} \left(\frac{\partial f^i}{\partial p^\alpha} P_p^{\alpha k} + \frac{\partial f^i}{\partial q^\alpha} Q_p^{\alpha k} \right) dt + \sum_{r=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial \sigma_r^i}{\partial p^\alpha} P_p^{\alpha k} + \frac{\partial \sigma_r^i}{\partial q^\alpha} Q_p^{\alpha k} \right) \circ dw_r, \ P_p^{ik}(t_0) = \delta_{ik}, \\
dQ_p^{ik} &= \sum_{\alpha=1}^{n} \left(\frac{\partial g^i}{\partial p^\alpha} P_p^{\alpha k} + \frac{\partial g^i}{\partial q^\alpha} Q_p^{\alpha k} \right) dt + \sum_{r=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial \gamma_r^i}{\partial p^\alpha} P_p^{\alpha k} + \frac{\partial \gamma_r^i}{\partial q^\alpha} Q_p^{\alpha k} \right) \circ dw_r, \ Q_p^{ik}(t_0) = 0, \\
& i = 1, \ldots, n.
\end{aligned}
$$

Analogously, for a fixed k, P_q^{ik}, Q_q^{ik} satisfy the system

(2.16)

$$
\begin{aligned}
dP_q^{ik} &= \sum_{\alpha=1}^{n} \left(\frac{\partial f^i}{\partial p^\alpha} P_q^{\alpha k} + \frac{\partial f^i}{\partial q^\alpha} Q_q^{\alpha k} \right) dt + \sum_{r=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial \sigma_r^i}{\partial p^\alpha} P_q^{\alpha k} + \frac{\partial \sigma_r^i}{\partial q^\alpha} Q_q^{\alpha k} \right) \circ dw_r, \ P_q^{ik}(t_0) = 0, \\
dQ_q^{ik} &= \sum_{\alpha=1}^{n} \left(\frac{\partial g^i}{\partial p^\alpha} P_q^{\alpha k} + \frac{\partial g^i}{\partial q^\alpha} Q_q^{\alpha k} \right) dt + \sum_{r=1}^{m} \sum_{\alpha=1}^{n} \left(\frac{\partial \gamma_r^i}{\partial p^\alpha} P_q^{\alpha k} + \frac{\partial \gamma_r^i}{\partial q^\alpha} Q_q^{\alpha k} \right) \circ dw_r, \ Q_q^{ik}(t_0) = \delta_{ik},
\end{aligned}
$$

$$i = 1, \dots, n.$$

The coefficients in (2.15) and (2.16) are calculated at (t, P, Q) with $P = P(t) = [P^1(t; t_0, p, q), \dots, P^n(t; t_0, p, q)]^\top$, $Q = Q(t) = [Q^1(t; t_0, p, q), \dots, Q^n(t; t_0, p, q)]^\top$ being a solution to (1.4).

Consider the condition (2.12). Clearly,

$$\frac{D(P^i(t_0), Q^i(t_0))}{D(p^k, p^l)} = \frac{D(p^i, q^i)}{D(p^k, p^l)} = 0.$$

Therefore, (2.12) is fulfilled iff

$$\sum_{i=1}^{n} d\frac{D(P^i(t), Q^i(t))}{D(p^k, p^l)} = 0. \tag{2.17}$$

Due to (2.15), we get

$$d\frac{\partial P^i}{\partial p^k}\frac{\partial Q^i}{\partial p^l} = dP_p^{ik}(t)Q_p^{il}(t)$$

$$= \sum_{\alpha=1}^{n}\left[\left(\frac{\partial f^i}{\partial p^\alpha}P_p^{\alpha k} + \frac{\partial f^i}{\partial q^\alpha}Q_p^{\alpha k}\right)Q_p^{il} + \left(\frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha l} + \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha l}\right)P_p^{ik}\right]dt$$

$$+\sum_{r=1}^{m}\sum_{\alpha=1}^{n}\left[\left(\frac{\partial \sigma_r^i}{\partial p^\alpha}P_p^{\alpha k} + \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_p^{\alpha k}\right)Q_p^{il} + \left(\frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha l} + \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha l}\right)P_p^{ik}\right] \circ dw_r\ .$$

Then (2.17) holds iff the following equalities take place:

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{n}\left(\frac{\partial f^i}{\partial p^\alpha}P_p^{\alpha k}Q_p^{il} + \frac{\partial f^i}{\partial q^\alpha}Q_p^{\alpha k}Q_p^{il} + \frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha l}P_p^{ik} + \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha l}P_p^{ik}\right.$$
$$\left.-\frac{\partial f^i}{\partial p^\alpha}P_p^{\alpha l}Q_p^{ik} - \frac{\partial f^i}{\partial q^\alpha}Q_p^{\alpha l}Q_p^{ik} - \frac{\partial g^i}{\partial p^\alpha}P_p^{\alpha k}P_p^{il} - \frac{\partial g^i}{\partial q^\alpha}Q_p^{\alpha k}P_p^{il}\right) = 0, \tag{2.18}$$

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{n}\left(\frac{\partial \sigma_r^i}{\partial p^\alpha}P_p^{\alpha k}Q_p^{il} + \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_p^{\alpha k}Q_p^{il} + \frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha l}P_p^{ik} + \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha l}P_p^{ik}\right.$$
$$\left.-\frac{\partial \sigma_r^i}{\partial p^\alpha}P_p^{\alpha l}Q_p^{ik} - \frac{\partial \sigma_r^i}{\partial q^\alpha}Q_p^{\alpha l}Q_p^{ik} - \frac{\partial \gamma_r^i}{\partial p^\alpha}P_p^{\alpha k}P_p^{il} - \frac{\partial \gamma_r^i}{\partial q^\alpha}Q_p^{\alpha k}P_p^{il}\right) = 0,\ \ r = 1, \dots, m. \tag{2.19}$$

It is not difficult to check that if the functions $f^i(t,p,q)$, $g^i(t,p,q)$ are such that

$$\frac{\partial f^i}{\partial p^\alpha} + \frac{\partial g^\alpha}{\partial q^i} = 0,\ \frac{\partial f^i}{\partial q^\alpha} = \frac{\partial f^\alpha}{\partial q^i},\ \frac{\partial g^i}{\partial p^\alpha} = \frac{\partial g^\alpha}{\partial p^i},\ \ i, \alpha = 1, \dots, n, \tag{2.20}$$

then (2.18) holds, and if the functions $\sigma_r^i(t,p,q)$, $\gamma_r^i(t,p,q)$, $r = 1, \dots, m$, are such that

$$\frac{\partial \sigma_r^i}{\partial p^\alpha} + \frac{\partial \gamma_r^\alpha}{\partial q^i} = 0,\ \frac{\partial \sigma_r^i}{\partial q^\alpha} = \frac{\partial \sigma_r^\alpha}{\partial q^i},\ \frac{\partial \gamma_r^i}{\partial p^\alpha} = \frac{\partial \gamma_r^\alpha}{\partial p^i},\ \ i, \alpha = 1, \dots, n, \tag{2.21}$$

then (2.19) holds. Thus, if the relations (2.20)-(2.21) take place, the condition (2.12) is fulfilled.

The condition (2.13) also holds when (2.20)-(2.21) are true. This can be proved analogously by using (2.16) instead of (2.15).

Now consider the condition (2.14). Clearly,

$$\sum_{i=1}^{n} \frac{D(P^i(t_0), Q^i(t_0))}{D(p^k, q^l)} = \sum_{i=1}^{n} \frac{D(p^i, q^i)}{D(p^k, q^l)} = \delta_{kl}.$$

Then the condition (2.14) is fulfilled iff $\sum_{i=1}^{n} d\frac{D(P^i(t), Q^i(t))}{D(p^k, q^l)} = 0$. Using the same arguments again, we prove that the relations (2.20)-(2.21) ensure this condition as well.

Finally, noting that the relations (1.7) imply (2.20)-(2.21), we obtain the following proposition (cf. [3]).

Theorem 2.4. *The phase flow of the system of SDEs*

$$dP^i = -\frac{\partial H}{\partial q^i}(t, P, Q)dt - \sum_{r=1}^{m} \frac{\partial H_r}{\partial q^i}(t, P, Q) \circ dw_r(t)$$

$$dQ^i = \frac{\partial H}{\partial p^i}(t, P, Q)dt + \sum_{r=1}^{m} \frac{\partial H_r}{\partial p^i}(t, P, Q) \circ dw_r(t), \quad i = 1, \dots, n,$$

with Hamiltonians $H(t, p, q)$, $H_r(t, p, q)$, $r = 1, \dots, m$, preserves symplectic structure.

Remark 2.1. It is also possible to prove this theorem using the necessary and sufficient condition of symplecticness (see [2]) which consists in

$$G^{\top} J G = J, \tag{2.22}$$

where $G = \partial(P(t), Q(t))/\partial(p, q)$ is the Jacobi matrix of the phase flow and J is the $2n \times 2n$ skew-symmetric matrix

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix},$$

O_n and I_n are zero and unit $n \times n$-matrices correspondingly.

Corollary 2.5. The phase flow of a Hamiltonian system with additive noise preserves symplectic structure.

Consider the system with colored noise

$$\begin{aligned} dP &= f(t, P, Q)dt + F(t, P, Q)Zdt, \ P(t_0) = p, \\ dQ &= g(t, P, Q)dt + G(t, P, Q)Zdt, \ Q(t_0) = q, \\ dZ &= \Gamma(t)Zdt + \sum_{r=1}^{m} \sigma_r(t)dw_r(t), \ Z(t_0) = z, \end{aligned} \tag{2.23}$$

where P, Q, f, and g are n-dimensional vectors, Z and $\sigma_r(t)$ are l-dimensional vectors, $\Gamma(t)$ is an $l \times l$ matrix, and $F(t, p, q)$ and $G(t, p, q)$ are $n \times l$ matrices.

It can be proved that the transformation $(p, q) \mapsto (P, Q)$ defined by (2.23) preserves symplectic structure if there are Hamiltonians $H(t, p, q)$ and $H_c(t, p, q; z)$ such that

$$\begin{aligned} f^i = -\partial H/\partial q^i, \quad g^i = \partial H/\partial p^i, \\ (Fz)^i = -\partial H_c/\partial q^i, \quad (Gz)^i = \partial H_c/\partial p^i, \quad i = 1, \dots, n. \end{aligned} \tag{2.24}$$

In particular, when the matrices F and G at the colored noise do not depend on p and q, the phase flow of (2.23) preserves symplectic structure if $f^i = -\partial H/\partial q^i$, $g^i = \partial H/\partial p^i$, $i = 1, \ldots, n$.

It is known [10] that specific features of a system with colored noise allow to obtain high-order mean-square methods. But we will not construct special symplectic methods for the Hamiltonian system with colored noise (2.23)-(2.24) in the present paper. This point will be considered elsewhere.

3. Auxiliary knowledge on numerical methods

For a reader convenience, we recall here some necessary formulae connected with symplectic numerical methods for ordinary deterministic differential equations and with numerical integration of SDEs, which we use in the next sections. Further details can be find in, e.g., [15] (symplectic methods) and in, e.g., [9, 7] (methods for SDEs).

3.1. **Hamiltonian methods for deterministic differential equations.** Let $H = H(t,p,q)$, p, $q \in R^n$, $t \in [t_0, t_0+T]$, be a sufficiently smooth function. Consider the Hamiltonian system of differential equations with Hamiltonian H :

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \quad i = 1, \ldots, n\,. \tag{3.1}$$

Introduce the n-dimensional vector-functions f and g:

$$f^i = -\frac{\partial H}{\partial q^i}, \quad g^i = \frac{\partial H}{\partial p^i}, \quad i = 1, \ldots, n\,.$$

3.1.1. *Symplectic Runge-Kutta methods for general Hamiltonian systems.* It is known [15, 14, 17] that in the general case symplectic Runge-Kutta (RK) methods are all implicit.

Two-parametric family of implicit symplectic methods is written as [17]

$$\begin{aligned} p_{k+1} &= p_k + hf(t_k + \beta h, \alpha p_{k+1} + (1-\alpha)p_k, (1-\alpha)q_{k+1} + \alpha q_k), \\ q_{k+1} &= q_k + hg(t_k + \beta h, \alpha p_{k+1} + (1-\alpha)p_k, (1-\alpha)q_{k+1} + \alpha q_k), \end{aligned} \tag{3.2}$$

where the parameters $\alpha, \beta \in [0,1]$.

For $\alpha = \beta = 1/2$ this method is of order 2, otherwise it is of order 1. For $\alpha \neq 1/2$, the p components are integrated by an RK formula and the q components with a different RK formula. The overall scheme is called a partitioned Runge-Kutta (PRK) method [15].

The one-parametric family of implicit second-order symplectic Runge-Kutta methods has the form [15, 17]

$$\begin{aligned} \mathcal{P}_1 &= p_k + \frac{\alpha}{2}hf(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1), \\ \mathcal{Q}_1 &= q_k + \frac{\alpha}{2}hg(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1), \end{aligned} \tag{3.3}$$

$$\mathcal{P}_2 = p_k + \alpha hf(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-\alpha}{2}hf(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2),$$

$$\mathcal{Q}_2 = q_k + \alpha h g(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-\alpha}{2} h g(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2),$$

$$p_{k+1} = p_k + h\left[\alpha f(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-\alpha) f(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)\right]$$

$$q_{k+1} = q_k + h\left[\alpha g(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-\alpha) g(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)\right].$$

Note that for $\alpha = 1$ this method coincides with one from (3.2) with $\alpha = \beta = 1/2$.

Some other symplectic Runge-Kutta methods for general Hamiltonian systems are attracted in Section 7.1.

3.1.2. *Explicit symplectic Runge-Kutta methods for Hamiltonian systems with separable Hamiltonians.* In comparison to the general case there are PRK methods that are explicit when the Hamiltonian $H(t,p,q)$ has the form

$$H(t,p,q) = V(p) + U(t,q). \tag{3.4}$$

Such Hamiltonians are called separable. We note that it is not difficult to consider a slightly more general separable Hamiltonian $H(t,p,q) = V(t,p) + U(t,q)$ but we restrict ourselves here and in Section 5 to the Hamiltonian (3.4).

The explicit PRK methods of the one-parametric family

$$\mathcal{Q} = q_k + \alpha h g(p_k), \;\; \mathcal{P} = p_k + h f(t_k + \alpha h, \mathcal{Q}) \tag{3.5}$$

$$q_{k+1} = \mathcal{Q} + (1-\alpha) h g(\mathcal{P}), \;\; p_{k+1} = \mathcal{P}, \;\; k = 0, \ldots, N-1,$$

preserve symplectic structure [17, 14, 15]. For $\alpha = 1/2$, the method is of order 2, otherwise it is of order 1. For $\alpha = 0$ and $\alpha = 1$, this method has the same form as the method (3.2), applied to a Hamiltonian system with H from (3.4), with $\alpha = 1, \; \beta = 0$ and $\alpha = 0, \; \beta = 1$ respectively.

The explicit fourth-order symplectic Runge-Kutta method is written as [15, p. 109]

$$\mathcal{P}_1 = p_k + h\frac{\varkappa}{2} f(t_k, q_k), \;\; \mathcal{Q}_1 = q_k + h\varkappa g(\mathcal{P}_1), \tag{3.6}$$

$$\mathcal{P}_2 = \mathcal{P}_1 + h\frac{1-\varkappa}{2} f(t_k + \varkappa h, \mathcal{Q}_1), \;\; \mathcal{Q}_2 = \mathcal{Q}_1 + h(1-2\varkappa) g(\mathcal{P}_2),$$

$$\mathcal{P}_3 = \mathcal{P}_2 + h\frac{1-\varkappa}{2} f(t_k + (1-\varkappa)h, \mathcal{Q}_2), \;\; \mathcal{Q}_3 = \mathcal{Q}_2 + h\varkappa g(\mathcal{P}_3)$$

$$\mathcal{P}_4 = \mathcal{P}_3 + h\frac{\varkappa}{2} f(t_k + h, \mathcal{Q}_3),$$

$$p_{k+1} = \mathcal{P}_4, \; q_{k+1} = \mathcal{Q}_3\;, \tag{3.7}$$

where $\varkappa = (2 + 2^{1/3} + 2^{-1/3})/3$.

The method (3.5) is used in Section 5. The method (3.6)-(3.7) is needed for Section 7.2.

3.1.3. *Explicit symplectic Runge-Kutta-Nyström methods.* A commonly occurring case of the separable Hamiltonian has $V(p) = \frac{1}{2}p^{\intercal}M^{-1}p$, with M a constant, symmetric, invertible matrix. A Hamiltonian system with such a Hamiltonian can be rewritten in the form of a second-order system and can be efficiently integrated by means of Runge-Kutta-Nyström (RKN) methods. As is known [16, 17, 15], each explicit, symplectic PRK method induces an explicit, symplectic RKN method.

The second-order RKN method induced by (3.5) with $\alpha = 1/2$ is written as [16, 17, 15]:

$$\mathcal{Q} = q_k + \frac{h}{2}M^{-1}p_k \tag{3.8}$$

$$p_{k+1} = p_k + hf(t_k + \frac{h}{2}, \mathcal{Q}),$$

$$q_{k+1} = q_k + hM^{-1}p_k + \frac{h^2}{2}M^{-1}f(t_k + \frac{h}{2}, \mathcal{Q}), \quad k = 0, \dots, N-1.$$

This scheme is called Störmer-Verlet method.

The explicit third-order RKN method

$$\mathcal{Q}_1 = q_k + \frac{7}{24}hM^{-1}p_k, \quad \mathcal{P}_1 = p_k + \frac{2}{3}hf(t_k + \frac{7h}{24}, \mathcal{Q}_1) \tag{3.9}$$

$$\mathcal{Q}_2 = \mathcal{Q}_1 + \frac{3}{4}hM^{-1}\mathcal{P}_1, \quad \mathcal{P}_2 = \mathcal{P}_1 - \frac{2}{3}hf(t_k + \frac{25h}{24}, \mathcal{Q}_2)$$

$$\mathcal{Q}_3 = \mathcal{Q}_2 - \frac{1}{24}hM^{-1}\mathcal{P}_2, \quad \mathcal{P}_3 = \mathcal{P}_2 + hf(t_k + h, \mathcal{Q}_3)$$

$$p_{k+1} = \mathcal{P}_3, \quad q_{k+1} = \mathcal{Q}_3, \quad k = 0, \dots, N-1, \tag{3.10}$$

preserves symplectic structure [13, 16, 17, 15].

The methods (3.8) and (3.9)-(3.10) are used in Section 6.

3.2. **Mean-square methods for SDEs.** In this section we recall some formulae of numerical mean-square methods for SDEs in the Ito sense

$$dX = a(t, X)dt + \sum_{r=1}^{m} b_r(t, X)dw_r(t), \; X(t_0) = X_0. \tag{3.11}$$

Note that in the case of additive noise (with which we mainly deal in the present paper) all the formulae remain true for the Stratonovich SDEs as well.

Consider mean-square approximations of the solution to the Ito system (3.11). A one-step mean-square approximation $\bar{X}_{t,x}(t+h)$, $t_0 \le t < t+h \le t_0 + T$, is constructed depending on t, x, h, and $\{w_1(\vartheta) - w_1(t), \dots, w_m(\vartheta) - w_m(t); \; t \le \vartheta \le t+h\}$. Using the one-step approximation, we recurrently obtain the approximation X_k, $k = 0, \dots, N$, $t_{k+1} - t_k = h_{k+1}$, $t_N = t_0 + T$:

$$X_0 = X(t_0), \; X_{k+1} = \bar{X}_{t_k, X_k}(t_{k+1}).$$

For simplicity, we will take $t_{k+1} - t_k = h = T/N$. Note that X_0 may be random.

Suppose the functions $a(t,x)$ and $b_r(t,x)$ are defined and continuous for $t \in [t_0, t_0+T]$, $x \in R^d$ and satisfy a uniform Lipschitz condition: for all $t \in [t_0, t_0+T]$, $x, y \in R^d$ there is a constant $L > 0$ such that

$$|a(t,x) - a(t,y)| + \sum_{r=1}^{m} |b_r(t,x) - b_r(t,y)| \le L\,|x-y|\,. \tag{3.12}$$

Theorem 3.1. (*see* [9]) *Suppose the one-step approximation $\bar{X}_{t,x}(t+h)$ has order of accuracy p_1 for the mathematical expectation of the deviation and order of accuracy p_2 for the mean-square deviation; more precisely, for arbitrary $t_0 \le t \le t_0 + T - h$, $x \in R^d$ the following inequalities hold:*

$$\left| E\left(X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right)\right| \le K \cdot (1+|x|^2)^{1/2} h^{p_1}, \tag{3.13}$$

$$\left[E\left|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right|^2\right]^{1/2} \le K \cdot (1+|x|^2)^{1/2} h^{p_2}. \tag{3.14}$$

Also, let

$$p_2 \ge \frac{1}{2},\; p_1 \ge p_2 + \frac{1}{2}. \tag{3.15}$$

Then for any N and $k = 0, \ldots, N$ the following inequality holds:

$$\left[E\left|X_{t_0,X_0}(t_k) - \bar{X}_{t_0,X_0}(t_k)\right|^2\right]^{1/2} \le K \cdot (1 + E|X_0|^2)^{1/2} h^{p_2 - 1/2}, \tag{3.16}$$

i.e., the mean-square order of accuracy of the method constructed using the one-step approximation $\bar{X}_{t,x}(t+h)$ is $p = p_2 - 1/2$.

We note that all constants K mentioned above, as well as the ones that will appear in the sequel, depend in the final analysis on the system (1.1) and the approximations only and do not depend on X_0 and h.

Let us assume that, in addition to (3.12), the functions $a(t,x)$ and $b_r(t,x)$ have partial derivatives with respect to t that grow at most as a linear function of x as $|x| \to \infty$ and that the derivatives $\dfrac{\partial a^i}{\partial x^j}$ and $\dfrac{\partial^2 a^i}{\partial x^j \partial x^k}$, $i, j, k = 1, \ldots, d$, are uniformly bounded.

It is known [9, 7] that under these assumptions the mean-square order of the Euler method is equal to $1/2$.

Let us recall the Euler method:

$$X_0 = X(t_0),\; X_{k+1} = X_k + \sum_{r=1}^{m} (b_r)_k\, \Delta_k w_r(h) + h a_k,\; k = 0, \ldots, N-1, \tag{3.17}$$

where a_k and $(b_r)_k$ are the coefficients a and b_r evaluated at the point (t_k, X_k) and $\Delta_k w_r(h) := w_r(t_k + h) - w_r(t_k)$.

For this method, we have $p_1 = 2$, $p_2 = 1$ in a general case of the system (3.11). In the case of system with additive noise (2.9) we have $p_1 = 2$, $p_2 = 3/2$, and the method's order is equal to 1 under the same smoothness and boundedness conditions on the coefficients.

The method of the mean-square order 3/2 for system with additive noise (2.9) has the form

$$X_0 = X(t_0),\ X_{k+1} = X_k + \sum_{r=1}^{m} (b_r)_k\, \Delta_k w_r(h) + h a_k \tag{3.18}$$
$$+ \sum_{r=1}^{m} (\Lambda_r a)_k\, (I_{r0})_k + \sum_{r=1}^{m} \left(\frac{\partial b_r}{\partial t}\right)_k (I_{0r})_k + h^2\, (La)_k /2.$$

Here

$$\Lambda_r = (b_r, \frac{\partial}{\partial x}),\ L = \frac{\partial}{\partial t} + (a, \frac{\partial}{\partial x}) + \frac{1}{2}\sum_{r=1}^{m}\sum_{i,j=1}^{d} b_r^i b_r^j \frac{\partial^2}{\partial x^i \partial x^j}\ , \tag{3.19}$$

$$(I_{r0})_k = \int_{t_k}^{t_{k+1}} (w_r(\vartheta) - w_r(t_k))\, d\vartheta,\ (I_{0r})_k = \int_{t_k}^{t_{k+1}} (\vartheta - t_k)\, dw_r(\vartheta). \tag{3.20}$$

The formula (3.18) has the random variables $\Delta_k w_r(h)$, $(I_{r0})_k$, $(I_{0r})_k$ joint distribution of which is Gaussian. They can be simulated at each step by $2m$ independent $N(0,1)$-distributed random variables ξ_{rk} and η_{rk}, $r = 0, \ldots, m$. As a result, the formula (3.18) takes the constructive form

$$X_{k+1} = X_k + h^{1/2}\sum_{r=1}^{m} (b_r)_k\, \xi_{rk} + h a_k + h^{3/2}\sum_{r=1}^{m} (\Lambda_r a)_k \cdot \left(\xi_{rk}/2 + \eta_{rk}/\sqrt{12}\right) \tag{3.21}$$
$$+ h^{3/2}\sum_{r=1}^{m} \left(\frac{\partial b_r}{\partial t}\right)_k \cdot \left(\xi_{rk}/2 - \eta_{rk}/\sqrt{12}\right) + h^2\, (La)_k /2.$$

A rigorous proof of the theorem about order of convergence for (3.18) rests on the following assumptions (see [9, 7]): the function $a(t,x)$ and all its first- and second-order partial derivatives, as well as the partial derivatives $\dfrac{\partial^3 a}{\partial t \partial x^i \partial x^j}$, $\dfrac{\partial^3 a}{\partial x^i \partial x^j \partial x^k}$, and $\dfrac{\partial^4 a}{\partial x^i \partial x^j \partial x^k \partial x^l}$ are continuous; the functions $b_r(t)$ are twice continuously differentiable; the first order partial derivatives with respect to x are uniformly bounded (so that a uniform Lipschitz condition is satisfied), while its remaining partial derivatives listed above, regarded as functions of x, grow at most as a linear function of $|x|$ as $|x| \to \infty$.

Note that in the sequel we shall not give analogous conditions on the coefficients which ensure corresponding orders of convergence for other methods. They consist in some conditions of smoothness and boundedness and can be restored using the general theory [9, 7]. At the same time we underline that these conditions are not necessary and the considered methods are applicable more widely. Also let us note that we shall use equalities with right hand side $O(h^p)$ instead of, for example, inequalities (3.13) or (3.14).

The following evident lemma will be useful below.

Lemma 3.1. *Let the one-step approximation $\bar{X}_{t,x}(t+h)$ satisfy the conditions of Theorem 3.1. And suppose that $\tilde{X}_{t,x}(t+h)$ is such that*

$$\left| E\left(\tilde{X}_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right)\right| = O(h^{p_1}), \tag{3.22}$$

$$\left[E\left|\tilde{X}_{t,x}(t+h)-\bar{X}_{t,x}(t+h)\right|^2\right]^{1/2}=O(h^{p_2}) \tag{3.23}$$

with the same p_1 and p_2. Then the method constructed using the one-step approximation $\tilde{X}_{t,x}(t+h)$ has the same mean-square order of accuracy as the method based on $\bar{X}_{t,x}(t+h)$, i.e., its order is equal to $p=p_2-1/2$.

4. Hamiltonian mean-square methods for general Hamiltonian systems with additive noise

In this section we consider the Hamiltonian system with additive noise (4.1)-(4.2) (recall once more that in the case of additive noise the Stratonovich form coincides with the Ito one)

$$dP=f(t,P,Q)dt+\sum_{r=1}^{m}\sigma_r(t)dw(t),\ P(t_0)=p, \tag{4.1}$$

$$dQ=g(t,P,Q)dt+\sum_{r=1}^{m}\gamma_r(t)dw(t),\ Q(t_0)=q,$$

$$f^i=-\partial H/\partial q^i,g^i=\partial H/\partial p^i,i=1,\dots,n, \tag{4.2}$$

where $P,\ Q,\ f,\ g,\ \sigma_r,\ \gamma_r$ are n-dimensional column-vectors, $w_r(t),\ r=1,\dots,m,$ are independent standard Wiener processes, and $H(t,p,q)$ is a Hamiltonian.

The phase flow of this system preserves symplectic structure (see Corollary 2.5).

4.1. First-order methods.

Consider the two-parametric family of implicit methods

$$\mathcal{P}=P_k+hf(t_k+\beta h,\alpha\mathcal{P}+(1-\alpha)P_k,(1-\alpha)\mathcal{Q}+\alpha Q_k), \tag{4.3}$$

$$\mathcal{Q}=Q_k+hg(t_k+\beta h,\alpha\mathcal{P}+(1-\alpha)P_k,(1-\alpha)\mathcal{Q}+\alpha Q_k),$$

$$P_{k+1}=\mathcal{P}+\sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r,\ \ Q_{k+1}=\mathcal{Q}+\sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r,\ k=0,\dots,N-1, \tag{4.4}$$

where $\Delta_k w_r(h):=w_r(t_k+h)-w_r(t_k)$ and the parameters $\alpha,\beta\in[0,1]$.

When $\sigma_r=0,\ \gamma_r=0,\ r=1,\dots,m$, this family coincides with the known family (3.2) of symplectic methods for deterministic Hamiltonian systems, and it can be considered as a generalization of (3.2) to the stochastic case.

The unique solvability of (4.3) with respect to $\mathcal{P},\ \mathcal{Q}$ for any $P_k,\ Q_k$ and sufficiently small h follows from the following lemma.

Lemma 4.1. *Let $F(x;c,s)$ be a continuous d-dimensional vector-function depending on $x\in R^d,\ c\in R^d$, and $s\in S$, where S is a set from an R^l. Suppose F has the first partial derivatives $\partial F^i/\partial x^j,\ i,j=1,\dots,d$, which are uniformly bounded in $R^d\times R^d\times S$. Then there is an $h_0>0$ such that the equation*

$$x=c+hF(x;c,s)+\nu \tag{4.5}$$

is uniquely solvable with respect to x *for* $0 < h \le h_0$ *and any* $c \in R^d$, $\nu \in R^d$, $s \in S$. *The solution of equation* (4.5) *can be found by the method of simple iteration with an arbitrary initial approximation.*

The proof of this lemma is not difficult and it is omitted.

The following lemma is true for system (4.1) with arbitrary f and g having bounded first derivatives (i.e., f and g may not obey the condition (4.2)).

Lemma 4.2. *The mean-square order of the methods* (4.3) – (4.4) *for the system* (4.1) *is equal to* 1.

Proof. Let us compare the one-step approximation of the Euler method (see (3.17))

$$\bar{P} = p + hf(t,p,q) + \sum_{r=1}^{m} \sigma_r(t)\Delta w_r,$$
$$\bar{Q} = q + hg(t,p,q) + \sum_{r=1}^{m} \gamma_r(t)\Delta w_r$$

with the one-step approximation $\tilde{P}$, $\tilde{Q}$ corresponding to (4.3)–(4.4)

$$\mathcal{P} = p + hf(t+\beta h, \alpha\mathcal{P} + (1-\alpha)p, (1-\alpha)\mathcal{Q} + \alpha q), \tag{4.6}$$
$$\mathcal{Q} = q + hg(t+\beta h, \alpha\mathcal{P} + (1-\alpha)p, (1-\alpha)\mathcal{Q} + \alpha q),$$

$$\tilde{P} = \mathcal{P} + \sum_{r=1}^{m} \sigma_r(t)\Delta w_r, \quad \tilde{Q} = \mathcal{Q} + \sum_{r=1}^{m} \gamma_r(t)\Delta w_r\ . \tag{4.7}$$

Clearly, the differences $\tilde{P} - \bar{P}$ and $\tilde{Q} - \bar{Q}$ are deterministic. And it is not difficult to show that

$$\left| E\left(\begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix} - \begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix} \right) \right| = \left| \begin{bmatrix} \tilde{P} - \bar{P} \\ \tilde{Q} - \bar{Q} \end{bmatrix} \right| = O(h^2),$$
$$\left(E\left| \begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix} - \begin{bmatrix} \bar{P} \\ \bar{Q} \end{bmatrix} \right|^2 \right)^{1/2} = \left(\left| \begin{bmatrix} \tilde{P} - \bar{P} \\ \tilde{Q} - \bar{Q} \end{bmatrix} \right|^2 \right)^{1/2} = O(h^2).$$

Then recalling that the Euler method has the first mean-square order of convergence for systems with additive noise and applying Lemma 3.1 (in this case the Euler method has $p_1 = 2$, $p_2 = 3/2$ (see Section 3.2)), we obtain that the method (4.3)–(4.4) is of the first mean-square order. □

As it has been marked in Introduction, the method based on a one-step approximation $\tilde{P} = \tilde{P}(t+h;t,p,q)$, $\tilde{Q} = \tilde{Q}(t+h;t,p,q)$ preserves symplectic structure if its one-step approximation satisfies

$$d\tilde{P} \wedge d\tilde{Q} = dp \wedge dq\,.$$

For the one-step approximation (4.6)–(4.7), we have $d\tilde{P} = d\mathcal{P}$, $d\tilde{Q} = d\mathcal{Q}$. Hence $d\tilde{P} \wedge d\tilde{Q} = d\mathcal{P} \wedge d\mathcal{Q}$. The relations for $\mathcal{P}$, $\mathcal{Q}$ coincide with ones for the one-step approximation corresponding to the symplectic method (3.2). Therefore, the method (4.3)–(4.4) is symplectic as well. From here and Lemma 4.2, we get the theorem.

Theorem 4.1. *The method* (4.3)–(4.4) *for the system* (4.1)–(4.2) *preserves symplectic structure and has the first mean-square order of convergence.*

Remark 4.1. Just as in the deterministic case, for $\alpha = 0$ and $\alpha = 1$ and arbitrary β the method (4.3)–(4.4) contains only one implicit relation.

Now consider another generalization of the family (3.2) to the Hamiltonian system (4.1):

$$
\begin{aligned}
P_{k+1} &= P_k + hf(t_k + \beta h, \alpha P_{k+1} + (1-\alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k) \\
&\quad + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r, \\
Q_{k+1} &= Q_k + hg(t_k + \beta h, \alpha P_{k+1} + (1-\alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k) \\
&\quad + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r, \quad k = 0, \dots, N-1,
\end{aligned}
\tag{4.8}
$$

with the parameters $\alpha, \beta \in [0, 1]$.

For sufficiently small h, the equations (4.8) are uniquely solvable with respect to P_{k+1}, Q_{k+1} due to Lemma 4.1.

Theorem 4.2. *The method* (4.8) *for the system* (4.1)–(4.2) *preserves symplectic structure and has the first mean-square order of convergence.*

Proof. Using Lemma 3.1, one can establish that the mean-square order of the method (4.8) is equal to 1.

Now we check symplecticness of the method. Let $\tilde{P}$, $\tilde{Q}$ be the one-step approximation corresponding to the method (4.8). Introduce

$$\hat{p} = p + \alpha \sum_{r=1}^{m} \sigma_r(t)\Delta w_r, \;\; \hat{q} = q + (1-\alpha) \sum_{r=1}^{m} \gamma_r(t)\Delta w_r$$

and

$$\hat{P} = \tilde{P} - (1-\alpha) \sum_{r=1}^{m} \sigma_r(t)\Delta w_r, \; \hat{Q} = \tilde{Q} - \alpha \sum_{r=1}^{m} \gamma_r(t)\Delta w_r \,.$$

We have

$$
\begin{aligned}
\hat{P} &= \hat{p} + hf(t + \beta h, \alpha \hat{P} + (1-\alpha)\hat{p}, (1-\alpha)\hat{Q} + \alpha \hat{q}), \\
\hat{Q} &= \hat{q} + hg(t + \beta h, \alpha \hat{P} + (1-\alpha)\hat{p}, (1-\alpha)\hat{Q} + \alpha \hat{q}).
\end{aligned}
$$

The relations for $\hat{P}$, $\hat{Q}$ coincide with the one-step approximation corresponding to the symplectic method (3.2). Therefore, $d\hat{P} \wedge d\hat{Q} = d\hat{p} \wedge d\hat{q}$. Further, it is obvious that $d\hat{P} \wedge d\hat{Q} = d\tilde{P} \wedge d\tilde{Q}$ and $d\hat{p} \wedge d\hat{q} = dp \wedge dq$. Consequently, $d\tilde{P} \wedge d\tilde{Q} = dp \wedge dq$, i.e., the method (4.8) is symplectic. □

4.2. **Methods of order 3/2.** The Taylor-type 3/2-order method (3.18) has the terms with derivatives: $\sum_{r=1}^{m} (\Lambda_r a)_k \, (I_{r0})_k$ and $\dfrac{h^2}{4} \sum_{r=1}^{m} \sum_{i,j=1}^{d} b_r^i b_r^j \dfrac{\partial^2 a}{\partial x^i \partial x^j}$. This is not appropriate for constructing a symplectic method for (4.1)-(4.2). To avoid using the derivatives, we will introduce new Runge-Kutta methods for systems with additive noise.

Consider the relations

$$\mathcal{P}_i = p + h\sum_{j=1}^{s} \alpha_{ij} f(t+c_j h, \mathcal{P}_j, \mathcal{Q}_j) + \varphi_i, \tag{4.9}$$

$$\mathcal{Q}_i = q + h\sum_{j=1}^{s} \alpha_{ij} g(t+c_j h, \mathcal{P}_j, \mathcal{Q}_j) + \psi_i,\ i = 1, \dots, s,$$

$$\bar{P} = p + h\sum_{i=1}^{s} \beta_i f(t+c_i h, \mathcal{P}_i, \mathcal{Q}_i) + \eta, \quad \bar{Q} = q + h\sum_{i=1}^{s} \beta_i g(t+c_i h, \mathcal{P}_i, \mathcal{Q}_i) + \zeta, \tag{4.10}$$

where φ_i, ψ_i, η, ζ do not depend on p and q, the parameters α_{ij} and β_i satisfy the condition

$$\beta_i \alpha_{ij} + \beta_j \alpha_{ji} - \beta_i \beta_j = 0, \ \ i, j = 1, \dots, s, \tag{4.11}$$

and c_i are arbitrary parameters.

The equations (4.9) are uniquely solvable with respect to $\mathcal{P}_i$, $\mathcal{Q}_i$, $i = 1, \dots, s$, for any p, q, φ_i, ψ_i, η, ζ and sufficiently small h due to Lemma 4.1.

For $\varphi_i = \psi_i = \eta = \zeta = 0$ the relations (4.9)–(4.10) coincide with a general form of s-stage Runge-Kutta methods for deterministic differential equations. It is known (see, e.g., Theorem 6.1 in [15]) that the symplectic condition $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$ holds for $\bar{P}$, $\bar{Q}$ from (4.9)–(4.10) with (4.11) and $\varphi_i = \psi_i = \eta = \zeta = 0$. Let us check the case of arbitrary φ_i, ψ_i, η, ζ.

Lemma 4.3. *The relations* (4.9) – (4.10) *with condition* (4.11) *preserve symplectic structure, i.e.,* $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$.

Proof. We generalize the proof of Theorem 6.1 from [15]. Denote for a while: $f_i = f(t+c_i h, \mathcal{P}_i, \mathcal{Q}_i)$, $g_i = g(t+c_i h, \mathcal{P}_i, \mathcal{Q}_i)$. Differentiate (4.9) and form the exterior products:

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq + h\sum_{i=1}^{s} \beta_i \, df_i \wedge dq + h\sum_{j=1}^{s} \beta_j \, dp \wedge dg_j + h^2 \sum_{i,j=1}^{s} \beta_i \beta_j \, df_i \wedge dg_j, \tag{4.12}$$

$$df_i \wedge d\mathcal{Q}_i = df_i \wedge dq + h\sum_{j=1}^{s} \alpha_{ij} \, df_i \wedge dg_j, \tag{4.13}$$

$$d\mathcal{P}_j \wedge dg_j = dp \wedge dg_j + h\sum_{i=1}^{s} \alpha_{ji} \, df_i \wedge dg_j. \tag{4.14}$$

Now using (4.13)-(4.14), find the expressions for $df_i \wedge dq$ and $dp \wedge dg_j$ and substitute them in (4.12):

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq + h\sum_{i=1}^{s} \beta_i \ (df_i \wedge d\mathcal{Q}_i + d\mathcal{P}_i \wedge dg_i) \tag{4.15}$$

$$+h^2 \sum_{i,j=1}^{s} \left(\beta_i \beta_j - \beta_i \alpha_{ij} - \beta_j \alpha_{ji}\right) \, df_i \wedge dg_j.$$

The last term in the right-hand side vanishes owing to (4.11).

Consider the second term in the right-hand side of (4.15). We have

$$df_i \wedge d\mathcal{Q}_i + d\mathcal{P}_i \wedge dg_i = \sum_{k=1}^{n} \left(df_i^k \wedge d\mathcal{Q}_i^k + d\mathcal{P}_i^k \wedge dg_i^k \right)$$

$$= \sum_{k,l=1}^{n} \left(\frac{\partial f_i^k}{\partial p^l} \, d\mathcal{P}_i^l \wedge d\mathcal{Q}_i^k + \frac{\partial f_i^k}{\partial q^l} \, d\mathcal{Q}_i^l \wedge d\mathcal{Q}_i^k + \frac{\partial g_i^k}{\partial p^l} \, d\mathcal{P}_i^k \wedge d\mathcal{P}_i^l + \frac{\partial g_i^k}{\partial q^l} \, d\mathcal{P}_i^k \wedge d\mathcal{Q}_i^l \right).$$

Taking into account the skew-symmetry of the wedge product and that f and g satisfy the condition (1.7) (see also (2.20)), it is not difficult to see that this expression vanishes. Returning to (4.15), we obtain $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$. $\square$

Remark 4.2. To prove this lemma, the condition (2.22) with $G = \partial(\bar{P}(t), \bar{Q}(t))/\partial(p, q)$ can be used as well (cf. [17]).

The next lemma is used in Theorem 4.3 for the Hamiltonian system (4.1)-(4.2). However the lemma is of great interest for arbitrary systems with additive noise as well (see Remark 4.3 below). So, we introduce the parametric family of one-step approximations for the system with additive noise (2.9):

$$\mathsf{X}_1 = x + \frac{\alpha}{2} h a(t + \frac{\alpha}{2} h, \mathsf{X}_1) + \sum_{r=1}^{m} b_r(t) \left(\lambda_1 J_{r0} + \mu_1 \Delta w_r \right), \tag{4.16}$$

$$\mathsf{X}_2 = x + \alpha h a(t + \frac{\alpha}{2} h, \mathsf{X}_1) + \frac{1-\alpha}{2} h a(t + \frac{1+\alpha}{2} h, \mathsf{X}_2) + \sum_{r=1}^{m} b_r(t) \left(\lambda_2 J_{r0} + \mu_2 \Delta w_r \right),$$

$$\bar{X} = x + h \left[\alpha a(t + \frac{\alpha}{2} h, \mathsf{X}_1) + (1-\alpha) a(t + \frac{1+\alpha}{2} h, \mathsf{X}_2) \right] + \sum_{r=1}^{m} b_r(t) \Delta w_r + \sum_{r=1}^{m} b_r'(t) I_{0r},$$

where

$$\Delta w_r := w_r(t+h) - w_r(t), \; I_{0r} := \int_t^{t+h} (\vartheta - t) \; dw_r(\vartheta), \; J_{r0} := \frac{1}{h} \int_t^{t+h} (w_r(\vartheta) - w_r(t)) \; d\vartheta,$$

and the parameters α, λ_1, λ_2, μ_1, μ_2 are such that

$$\alpha \lambda_1 + (1-\alpha) \lambda_2 = 1, \quad \alpha \mu_1 + (1-\alpha) \mu_2 = 0, \tag{4.17}$$

and

$$\alpha \left(\frac{\lambda_1^2}{3} + \lambda_1 \mu_1 + \mu_1^2 \right) + (1-\alpha) \left(\frac{\lambda_2^2}{3} + \lambda_2 \mu_2 + \mu_2^2 \right) = \frac{1}{2}. \tag{4.18}$$

For example, the following set of parameters satisfies (4.17)-(4.18):

$$\alpha = \frac{1}{2}, \; \lambda_1 = \lambda_2 = 1, \; \mu_1 = -\mu_2 = \frac{1}{\sqrt{6}}. \tag{4.19}$$

Note that the random variables Δw_r and J_{r0} are of the same mean-square order $O(h)$ (see (4.20)). Their combination helps us to compensate the derivatives indicated at the beginning of this subsection.

Lemma 4.4. *The method for the system with additive noise* (2.9) *based on the one-step approximation* (4.16) *with conditions* (4.17)–(4.18) *on its parameters is of the mean-square order of accuracy* 3/2.

Proof. Due to properties of the Wiener process and Ito integrals, we have

$$E\Delta w_i = 0,\ E\Delta w_i \Delta w_j = \delta_{ij} h,\ E\Delta w_i \Delta w_j \Delta w_k = 0,\ E\left(\Delta w_i\right)^4 = 3h^2,$$
$$EJ_{i0} = 0,\ \ EJ_{i0}J_{j0} = \delta_{ij}\frac{h}{3},\ EJ_{i0}J_{j0}J_{k0} = 0,\ E\left(J_{i0}\right)^4 = \frac{h^2}{3},$$
$$E\Delta w_i J_{j0} = \delta_{ij}\frac{h}{2},\ E\Delta w_i \Delta w_j J_{k0} = 0,\ E\Delta w_i J_{j0}J_{k0} = 0. \tag{4.20}$$

Let $\Delta \mathsf{X}_i := \mathsf{X}_i - x$, $i = 1, 2$. We have

$$|E\Delta \mathsf{X}_i| = O(h),\ E\left(\Delta \mathsf{X}_i\right)^{2l} = O(h^l),\ l = 1,2,3,4,\ i = 1,2\,,\ \left|E\left(\Delta \mathsf{X}_i\right)^3\right| = O(h^2). \tag{4.21}$$

Expand (4.16):

$$\Delta \mathsf{X}_1 = \frac{\alpha}{2}ha(t,x) + \sum_{r=1}^{m} b_r(t)\left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right) + \rho_1, \tag{4.22}$$

$$\Delta \mathsf{X}_2 = \frac{1+\alpha}{2}ha(t,x) + \sum_{r=1}^{m} b_r(t)\left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right) + \rho_2, \tag{4.23}$$

$$\begin{aligned}\bar{X} = x &+ \sum_{r=1}^{m} b_r(t)\Delta w_r + \sum_{r=1}^{m} b_r'(t) I_{0r} + ha(t,x) + \frac{h^2}{2}\frac{\partial a}{\partial t}(t,x) \\ &+ h\sum_{i=1}^{d}\frac{\partial a}{\partial x^i}(t,x)\cdot\left(\alpha\Delta \mathsf{X}_1^i + (1-\alpha)\Delta \mathsf{X}_2^i\right) \\ &+ \frac{h}{2}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i \partial x^j}(t,x)\cdot\left(\alpha\Delta \mathsf{X}_1^i\Delta \mathsf{X}_1^j + (1-\alpha)\Delta \mathsf{X}_2^i \Delta \mathsf{X}_2^j\right) + \bar{\rho}.\end{aligned} \tag{4.24}$$

Using (4.20)–(4.21), one can obtain

$$|E\rho_i| = O(h^2),\ \left|E\rho_i^l \Delta \mathsf{X}_i^k\right| = O(h^2),\ E\rho_i^2 = O(h^3) \tag{4.25}$$

and

$$|E\bar{\rho}| = O(h^3),\ E\bar{\rho}^2 = O(h^5). \tag{4.26}$$

Substituting (4.22)–(4.23) in (4.24) and using (4.17), we get

$$\begin{aligned}\bar{X} = x &+ \sum_{r=1}^{m} b_r \Delta w_r + \sum_{r=1}^{m} b_r' I_{0r} + ha + \frac{h^2}{2}\frac{\partial a}{\partial t} + \frac{h^2}{2}\sum_{i=1}^{d}\frac{\partial a}{\partial x^i}a^i \\ &+ h\sum_{r=1}^{m}\sum_{i=1}^{d} b_r^i \frac{\partial a}{\partial x^i} J_{r0} + \frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i \partial x^j} b_r^i b_r^j + R,\end{aligned} \tag{4.27}$$

$$R = \frac{h}{2}\sum_{r,l=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i \partial x^j} b_r^i b_l^j \cdot \left[\alpha\left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right)\left(\lambda_1 J_{l0} + \mu_1 \Delta w_l\right)\right.$$

$$+(1-\alpha)\left(\lambda_2 J_{r0}+\mu_2\Delta w_r\right)\left(\lambda_2 J_{l0}+\mu_2\Delta w_l\right)]-\frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{d}\frac{\partial^2 a}{\partial x^i\partial x^j}b_r^i b_r^j+\rho,$$

where the coefficients and their derivatives are calculated at (t,x) and ρ satisfies the same relations as $\bar{\rho}$ (see (4.26)).

The relations (4.20) and (4.18) imply

$$E[\alpha\left(\lambda_1 J_{r0}+\mu_1\Delta w_r\right)\left(\lambda_1 J_{l0}+\mu_1\Delta w_l\right) \tag{4.28}$$
$$+(1-\alpha)\left(\lambda_2 J_{r0}+\mu_2\Delta w_r\right)\left(\lambda_2 J_{l0}+\mu_2\Delta w_l\right)]=\frac{h}{2}\delta_{rl}\,.$$

Using the relations (4.20), (4.25)-(4.26), and (4.28), it is not difficult to get that

$$|ER|=O(h^3),\ \ \left(ER^2\right)^{1/2}=O(h^2). \tag{4.29}$$

Now denote by $\tilde{X}$ the one-step approximation corresponding to the method (3.18) which has the mean-square order of accuracy 3/2. Taking into account (4.27) and (4.29), we obtain

$$\left|E\left(\bar{X}-\tilde{X}\right)\right|=O(h^3),\ \left[E\left|\bar{X}-\tilde{X}\right|^2\right]^{1/2}=O(h^2).$$

Then according to Lemma 3.1, the method based on the one-step approximation $\bar{X}$ has the same mean-square order of accuracy as the method (3.18), i.e., its order is equal to 3/2. □

Remark 4.3. Doing in a similar way as the method (3.18) has been obtained, it is not difficult to construct new explicit Runge-Kutta methods of mean-square order 3/2 for an arbitrary system of differential equations with additive noise (2.9). For instance, we obtain the following explicit Runge-Kutta method of order 3/2 for (2.9) :

$$X_{k+1}=X_k+\sum_{r=1}^{m}b_r(t_k)\Delta_k w_r+\frac{h}{2}\left[a(t_k,X_k+\sum_{r=1}^{m}b_r(t_k)\cdot((J_{r0})_k+\frac{1}{\sqrt{6}}\Delta_k w_r))\right.$$
$$\left.+a(t_k+h,X_k+ha(t_k,X_k)+\sum_{r=1}^{m}b_r(t_k)\cdot((J_{r0})_k-\frac{1}{\sqrt{6}}\Delta_k w_r))\right]$$
$$+\sum_{r=1}^{m}b_r'(t_k)(I_{0r})_k\,,\ k=0,\dots,N-1.$$

Note that if we apply this method to (4.1)-(4.2) as well as any other explicit Runge-Kutta method, it will not preserve symplectic structure.

Now consider the parametric family of methods for the Hamiltonian system with additive noise (4.1):

$$\mathcal{P}_1=P_k+\frac{\alpha}{2}hf(t_k+\frac{\alpha}{2}h,\mathcal{P}_1,\mathcal{Q}_1)+\sum_{r=1}^{m}\sigma_r(t_k)\left(\lambda_1\left(J_{r0}\right)_k+\mu_1\Delta_k w_r\right), \tag{4.30}$$

$$\mathcal{Q}_1=Q_k+\frac{\alpha}{2}hg(t_k+\frac{\alpha}{2}h,\mathcal{P}_1,\mathcal{Q}_1)+\sum_{r=1}^{m}\gamma_r(t_k)\left(\lambda_1\left(J_{r0}\right)_k+\mu_1\Delta_k w_r\right),$$

$$\mathcal{P}_2 = P_k + \alpha h f(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-\alpha}{2} h f(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)$$
$$+ \sum_{r=1}^{m} \sigma_r(t_k) \left(\lambda_2 \left(J_{r0}\right)_k + \mu_2 \Delta_k w_r\right),$$
$$\mathcal{Q}_2 = Q_k + \alpha h g(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-\alpha}{2} h g(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)$$
$$+ \sum_{r=1}^{m} \gamma_r(t_k) \left(\lambda_2 \left(J_{r0}\right)_k + \mu_2 \Delta_k w_r\right),$$
$$P_{k+1} = P_k + h \left[\alpha f(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-\alpha) f(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)\right]$$
$$+ \sum_{r=1}^{m} \sigma_r(t_k) \Delta_k w_r + \sum_{r=1}^{m} \sigma_r'(t_k) \left(I_{0r}\right)_k,$$
$$Q_{k+1} = Q_k + h \left[\alpha g(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-\alpha) g(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)\right]$$
$$+ \sum_{r=1}^{m} \gamma_r(t_k) \Delta_k w_r + \sum_{r=1}^{m} \gamma_r'(t_k) \left(I_{0r}\right)_k,$$

where the parameters α, λ_1, λ_2, μ_1, μ_2 satisfy (4.17)–(4.18).

Let us note that the method (4.30) is reduced under $\sigma_r \equiv 0$, $\gamma_r \equiv 0$, $r = 1, \ldots, m$, to the well-known second-order symplectic Runge-Kutta method (3.3) for deterministic Hamiltonian systems (see, e.g., [15, p. 101]). Using the deterministic method (3.3) with $\alpha = 0$ (the midpoint rule), another implicit 3/2-order method for Hamiltonian systems with noise was proposed in [19]. But the method of [19] does not preserve symplectic structure.

The one-step approximation corresponding to this method is of the form (4.16). Therefore, due to Lemma 4.4, the method (4.30) is of the mean-square order 3/2. Moreover, this one-step approximation is of the form (4.9) with $s = 2$ and

$$\varphi_1 = \sum_{r=1}^{m} \sigma_r \left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right), \ \varphi_2 = \sum_{r=1}^{m} \sigma_r \left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right),$$
$$\psi_1 = \sum_{r=1}^{m} \gamma_r \left(\lambda_1 J_{r0} + \mu_1 \Delta w_r\right), \quad \psi_2 = \sum_{r=1}^{m} \gamma_r \left(\lambda_2 J_{r0} + \mu_2 \Delta w_r\right),$$
$$\eta = \sum_{r=1}^{m} \sigma_r \Delta w_r + \sum_{r=1}^{m} \sigma_r' I_{0r}, \quad \zeta = \sum_{r=1}^{m} \gamma_r \Delta w_r + \sum_{r=1}^{m} \gamma_r' I_{0r}$$

and

$$\alpha_{11} = \frac{\alpha}{2}, \ \alpha_{12} = 0, \ \alpha_{21} = \alpha, \ \alpha_{22} = \frac{1-\alpha}{2}, \ \beta_1 = \alpha, \ \beta_2 = 1 - \alpha, \ c_1 = \frac{\alpha}{2}, \ c_2 = \frac{1+\alpha}{2}.$$

This set of parameters α_{ij}, β_i, $i, j = 1, 2$, satisfies the conditions (4.11). Then due to Lemma 4.3, the method (4.30) is symplectic.

Thus we have obtained the following theorem.

Theorem 4.3. *Under conditions* (4.17)–(4.18) *on the parameters, the method* (4.30) *for system* (4.1)-(4.2) *preserves symplectic structure and has the mean-square order* 3/2 *of convergence.*

Remark 4.4. The method (4.30) can be rewritten in the constructive form as the formula (3.21) was obtained from (3.18) in Section 3.2.

Remark 4.5. All the methods in this Section are implicit. This corresponds to the fact that there are no explicit symplectic Runge-Kutta methods for general deterministic Hamiltonian systems (see e.g., [15, 14, 17] and also Section 3.1).

5. Hamiltonian mean-square methods in the case of separable Hamiltonian

In this section we consider the Hamiltonian system with additive noise (4.1), which Hamiltonian has the special structure

$$H(t,p,q) = V(p) + U(t,q). \tag{5.1}$$

Recall (see Subsection 3.1.2) that it is possible to consider a more general Hamiltonian $H(t,p,q) = V(t,p) + U(t,q)$. In mechanics V and U usually represent the kinetic and potential energy respectively. Hamiltonians of this form are called separable. When the Hamiltonian is separable, the system (4.1) takes the partitioned form

$$dP = f(t,Q)dt + \sum_{r=1}^{m} \sigma_r(t)dw_r(t), \;\; P(t_0) = p, \tag{5.2}$$

$$dQ = g(P)dt + \sum_{r=1}^{m} \gamma_r(t)dw_r(t), \;\; Q(t_0) = q,$$

where $f^i = -\partial U/\partial q^i$, $g^i = \partial V/\partial p^i$, $i = 1,\dots,n$.

Obviously, the implicit symplectic methods from the previous section can be applied to the partitioned system (5.2), and they take a more simple form in this case (we do not write them down here). We recall that there are no explicit symplectic RK methods for the system (4.1)-(4.2) with general Hamiltonian. However, for the partitioned system (5.2) it is possible to construct explicit symplectic methods just as in the deterministic case [17, 14, 15].

5.1. Explicit first-order methods.

On the basis of the family of deterministic PRK methods (3.5), we construct the family of explicit partitioned methods for stochastic system (5.2)

$$\mathcal{Q} = Q_k + \alpha h g(P_k) \tag{5.3}$$

$$\mathcal{P} = P_k + hf(t_k + \alpha h, \mathcal{Q})$$

$$Q_{k+1} = \mathcal{Q} + (1-\alpha)hg(\mathcal{P}) + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r$$

$$P_{k+1} = \mathcal{P} + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r, \quad k = 0,\dots,N-1.$$

Since the expressions for dP_{k+1}, dQ_{k+1} coincide with the ones corresponding to the deterministic symplectic method (3.5), the method (5.3) is symplectic. Further, by the same arguments as in the proof of Lemma 4.2, it is not difficult to show that the method (5.3) has the first mean-square order of accuracy. As a result, we obtain the theorem.

Theorem 5.1. *The explicit partitioned method* (5.3) *for the system* (5.2) *preserves symplectic structure and has the first mean-square order of convergence.*

Remark 5.1. For $\alpha = 0$ and $\alpha = 1$, the method (5.3) takes the same form as (4.3) with $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$ respectively.

Remark 5.2. By swapping the roles of p and q in the deterministic PRK method (3.5), we can propose the following symplectic method of the first mean-square order for the system (5.2):

$$\begin{gathered} \mathcal{P} = P_k + \alpha h f(t_k, Q_k) \\ \mathcal{Q} = Q_k + h g(\mathcal{P}) \\ P_{k+1} = \mathcal{P} + (1-\alpha) h f(t_{k+1}, \mathcal{Q}) + \sum_{r=1}^{m} \sigma_r(t_k) \Delta_k w_r \\ Q_{k+1} = \mathcal{Q} + \sum_{r=1}^{m} \gamma_r(t_k) \Delta_k w_r, \quad k = 0, \dots, N-1. \end{gathered} \tag{5.4}$$

For $\alpha = 0$ and $\alpha = 1$, this method takes the same form as (5.3) with $\alpha = 1$ and $\alpha = 0$ respectively.

Remark 5.3. In the special cases of $\alpha = 0$ and $\alpha = 1$ the methods (5.3) and (5.4) take a more simple form, in these cases they require evaluation of each of the coefficients f, g once per step only.

Remark 5.4. It is possible to propose other symplectic first-order methods for (5.2) on the basis of the deterministic PRK methods (3.5). For instance, the method

$$\begin{gathered} \mathcal{Q} = Q_k + \alpha h g(P_k) + \sum_{r=1}^{m} \gamma_r(t_k) \Delta_k w_r, \\ \mathcal{P} = P_k + h f(t_k + \alpha h, \mathcal{Q}) + \sum_{r=1}^{m} \sigma_r(t_k) \Delta_k w_r, \\ Q_{k+1} = \mathcal{Q} + (1-\alpha) h g(\mathcal{P}), \\ P_{k+1} = \mathcal{P}, \quad k = 0, \dots, N-1. \end{gathered} \tag{5.5}$$

is of the first mean-square order and symplectic.

5.2. **Explicit methods of order** $3/2$**.** On the basis of the deterministic second order PRK method (the method from the family (3.5) with $\alpha = 1/2$), we construct the explicit method for the stochastic system (5.2):

$$\begin{gathered} \mathcal{P}_1 = P_k, \quad \mathcal{Q}_1 = Q_k + \frac{h}{2} g(\mathcal{P}_1), \\ \mathcal{P}_2 = \mathcal{P}_1 + h f(t_k + \frac{h}{2}, \mathcal{Q}_1), \quad \mathcal{Q}_2 = \mathcal{Q}_1 + \frac{h}{2} g(\mathcal{P}_2), \end{gathered} \tag{5.6}$$

$$\mathcal{P}_3 = \mathcal{P}_2 + \sum_{r=1}^{m}\sum_{i=1}^{n} \gamma_r^i(t_k)\frac{\partial f}{\partial q^i}(t_k, \mathcal{Q}_2)(I_{r0})_k + \frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{n} \gamma_r^i(t_k)\gamma_r^j(t_k)\frac{\partial^2 f}{\partial q^i \partial q^j}(t_k, \mathcal{Q}_2),$$

$$\mathcal{Q}_3 = \mathcal{Q}_2,$$

$$\mathcal{P}_4 = \mathcal{P}_3 + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \sigma'_r(t_k)(I_{0r})_k,$$

$$\mathcal{Q}_4 = \mathcal{Q}_3 + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \gamma'_r(t_k)(I_{0r})_k + \sum_{r=1}^{m}\sum_{i=1}^{n} \sigma_r^i(t_k)\frac{\partial g}{\partial p^i}(\mathcal{P}_3)(I_{r0})_k$$

$$+\frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{n} \sigma_r^i(t_k)\sigma_r^j(t_k)\frac{\partial^2 g}{\partial p^i \partial p^j}(\mathcal{P}_3),$$

$$P_{k+1} = \mathcal{P}_4, \quad Q_{k+1} = \mathcal{Q}_4, \quad k = 0, \ldots, N-1,$$

where $(I_{r0})_k$ and $(I_{0r})_k$ are due to (3.20).

It is not difficult to prove that the transformation $P = p + F(t, q)$, $Q = q + S(t)$ preserves symplectic structure for any $S(t)$ and $F(t, q)$ such that $\partial F^i/\partial q^j = \partial F^j/\partial q^i$. Analogously the transformation $P = p + S(t)$, $Q = q + G(p)$ preserves symplectic structure for any $S(t)$ and $G(p)$ such that $\partial G^i/\partial p^j = \partial G^j/\partial p^i$. Using these facts and that the expressions for $\mathcal{P}_2$ and $\mathcal{Q}_2$ coincide with the ones corresponding to the deterministic symplectic method (the method (3.5) with $\alpha = 1/2$), it is not difficult to prove that the method (5.6) is symplectic.

Comparing the one-step approximation of the method (5.6) with the one-step approximation of the method (3.18) and applying Lemma 3.1, one can prove that the method (5.6) is of the mean-square order 3/2.

Thus, we obtain the theorem.

Theorem 5.2. *The explicit partitioned method* (5.6) *for the system* (5.2) *is of the mean-square order* 3/2 *and symplectic.*

Remark 5.5. The method (5.6) can be rewritten in the constructive form as the formula (3.21) was obtained from (3.18) in Section 3.2.

Remark 5.6. By swapping the roles of p and q in the deterministic method (3.5) with $\alpha = 1/2$, we obtain the following 3/2-order symplectic method for the system (5.2) :

$$\mathcal{Q}_1 = Q_k, \;\; \mathcal{P}_1 = P_k + \frac{h}{2} f(t_k, \mathcal{Q}_1), \tag{5.7}$$

$$\mathcal{Q}_2 = \mathcal{Q}_1 + hg(\mathcal{P}_1), \;\; \mathcal{P}_2 = \mathcal{P}_1 + \frac{h}{2} f(t_{k+1}, \mathcal{Q}_2),$$

$$\mathcal{Q}_3 = \mathcal{Q}_2,$$

$$\mathcal{P}_3 = \mathcal{P}_2 + \sum_{r=1}^{m}\sum_{i=1}^{n} \gamma_r^i(t_k)\frac{\partial f}{\partial q^i}(t_k, \mathcal{Q}_2)(I_{r0})_k + \frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{n} \gamma_r^i(t_k)\gamma_r^j(t_k)\frac{\partial^2 f}{\partial q^i \partial q^j}(t_k, \mathcal{Q}_2),$$

$$\mathcal{Q}_4 = \mathcal{Q}_3 + \sum_{r=1}^{m} \gamma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \gamma'_r(t_k)(I_{0r})_k + \sum_{r=1}^{m}\sum_{i=1}^{n} \sigma_r^i(t_k)\frac{\partial g}{\partial p^i}(\mathcal{P}_3)(I_{r0})_k$$

$$+\frac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{n} \sigma_r^i(t_k)\sigma_r^j(t_k)\frac{\partial^2 g}{\partial p^i \partial p^j}(\mathcal{P}_3),$$

$$\mathcal{P}_4 = \mathcal{P}_3 + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \sigma_r'(t_k)(I_{0r})_k,$$

$$P_{k+1} = \mathcal{P}_4, \quad Q_{k+1} = \mathcal{Q}_4, \quad k = 0, \ldots, N-1.$$

In Section 4.2 we propose the 3/2-order symplectic implicit Runge-Kutta method (4.30) for the general Hamiltonian system with additive noise (4.1)-(4.2). Of course, this implicit method can be applied to the system with separable Hamiltonian (5.2). In this section, using specificity of the system (5.2), we have obtained the 3/2-order symplectic explicit method (5.6), but it is not of a Runge-Kutta form. Our nearest aim is to construct a 3/2-order symplectic *explicit Runge-Kutta* method for (5.2).

To this end introduce the relations (cf. (4.9)–(4.10))

(5.8)
$$\mathcal{P}_i = p + h\sum_{j=1}^{s} \alpha_{ij} f(t + c_j h, \mathcal{Q}_j) + \varphi_i,$$

$$\mathcal{Q}_i = q + h\sum_{j=1}^{s} \hat{\alpha}_{ij} g(\mathcal{P}_j) + \psi_i, \ i = 1, \ldots, s,$$

(5.9)
$$\bar{P} = p + h\sum_{i=1}^{s} \beta_i f(t + c_i h, \mathcal{Q}_i) + \eta, \quad \bar{Q} = q + h\sum_{i=1}^{s} \hat{\beta}_i g(\mathcal{P}_i) + \zeta,$$

where φ_i, ψ_i, η, ζ do not depend on p and q, the parameters α_{ij}, $\hat{\alpha}_{ij}$, β_i and $\hat{\beta}_i$ satisfy the condition

(5.10)
$$\beta_i \hat{\alpha}_{ij} + \hat{\beta}_j \alpha_{ji} - \beta_i \hat{\beta}_j = 0, \ \ i, j = 1, \ldots, s,$$

and c_i are arbitrary parameters.

If $\varphi_i = \psi_i = \eta = \zeta = 0$, the relations (5.8)–(5.9) coincide with a general form of s-stage partitioned Runge-Kutta (PRK) methods for deterministic differential equations (see, e.g., [15, p. 34]). It is known [17, 15] that the symplectic condition holds for $\bar{P}$, $\bar{Q}$ from (5.8)–(5.9) with (5.10) in the case of $\varphi_i = \psi_i = \eta = \zeta = 0$. By a generalization of the proof of Theorem 6.2 from [15] (see also Lemma 4.3 of this paper), it is not difficult to get the following lemma.

Lemma 5.1. *The relations* (5.8) – (5.9) *with condition* (5.10) *preserve symplectic structure, i.e.,* $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$.

Introduce the parametric family of 2-stage explicit PRK methods for the system (5.2):

(5.11)
$$\mathcal{Q}_1 = Q_k + \sum_{r=1}^{m} \gamma_r(t_k)\left(\hat{\lambda}_1 (J_{r0})_k + \hat{\mu}_1 \Delta_k w_r\right)$$

$$\mathcal{P}_1 = P_k + h\beta_1 f(t_k + c_1 h, \mathcal{Q}_1) + \sum_{r=1}^{m} \sigma_r(t_k)\left(\lambda_1 (J_{r0})_k + \mu_1 \Delta_k w_r\right),$$

$$\mathcal{Q}_2 = Q_k + h\hat{\beta}_1 g(\mathcal{P}_1) + \sum_{r=1}^{m} \gamma_r(t_k)\left(\hat{\lambda}_2 (J_{r0})_k + \hat{\mu}_2 \Delta_k w_r\right),$$

$$\mathcal{P}_2 = P_k + h\sum_{i=1}^{2}\beta_i f(t_k + c_i h, \mathcal{Q}_i) + \sum_{r=1}^{m}\sigma_r(t_k)\left(\lambda_2 (J_{r0})_k + \mu_2\Delta_k w_r\right),$$

$$P_{k+1} = P_k + \sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m}\sigma_r'(t_k)(I_{0r})_k + h\sum_{i=1}^{2}\beta_i f(t_k + c_i h, \mathcal{Q}_i), \tag{5.12}$$

$$Q_{k+1} = Q_k + \sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m}\gamma_r'(t_k)(I_{0r})_k + h\sum_{i=1}^{2}\hat{\beta}_i g(\mathcal{P}_i),$$

where the parameters β_i, $\hat{\beta}_i$, c_i, λ_i, $\hat{\lambda}_i$, μ_i, $\hat{\mu}_i$, $i = 1, 2$, satisfy the conditions

$$\beta_1 + \beta_2 = 1,\ \hat{\beta}_1 + \hat{\beta}_2 = 1,\ \beta_2\hat{\beta}_1 = 1/2,\ c_1 = 0,\ c_2 = \hat{\beta}_1, \tag{5.13}$$

and

$$\begin{gathered}\beta_1\hat{\mu}_1 + \beta_2\hat{\mu}_2 = 0,\ \hat{\beta}_1\mu_1 + \hat{\beta}_2\mu_2 = 0,\\ \beta_1\hat{\lambda}_1 + \beta_2\hat{\lambda}_2 = 1,\ \hat{\beta}_1\lambda_1 + \hat{\beta}_2\lambda_2 = 1,\\ \beta_1\left(\frac{\hat{\lambda}_1^2}{3} + \hat{\lambda}_1\hat{\mu}_1 + \hat{\mu}_1^2\right) + \beta_2\left(\frac{\hat{\lambda}_2^2}{3} + \hat{\lambda}_2\hat{\mu}_2 + \hat{\mu}_2^2\right) = \frac{1}{2},\\ \hat{\beta}_1\left(\frac{\lambda_1^2}{3} + \lambda_1\mu_1 + \mu_1^2\right) + \hat{\beta}_2\left(\frac{\lambda_2^2}{3} + \lambda_2\mu_2 + \mu_2^2\right) = \frac{1}{2},\end{gathered} \tag{5.14}$$

and Δw_r, I_{0r}, J_{r0} are the same as ones defined after (4.16).

For example, the following set of parameters satisfies (5.13)-(5.14):

$$\begin{gathered}\beta_1 = \frac{1}{4},\ \beta_2 = \frac{3}{4},\ \hat{\beta}_1 = \frac{2}{3},\ \hat{\beta}_2 = \frac{1}{3},\\ \lambda_1 = \lambda_2 = \hat{\lambda}_1 = \hat{\lambda}_2 = 1,\ \mu_1 = \frac{1}{2\sqrt{3}},\ \mu_2 = -\frac{1}{\sqrt{3}},\ \hat{\mu}_1 = \frac{1}{\sqrt{2}},\ \hat{\mu}_2 = -\frac{1}{3\sqrt{2}}.\end{gathered} \tag{5.15}$$

Note that in the deterministic case (i.e., when $\sigma_r = 0$ and $\gamma_r = 0$, $r = 1, \ldots, m$) the family of methods (5.11)-(5.12) with conditions (5.13) on the parameters coincides with the family of 2-stage second-order deterministic PRK methods [15].

It is not difficult to see that the method (5.11)-(5.12) has the form of (5.8)-(5.9) and its parameters satisfy the condition (5.10). Then, Lemma 5.1 implies that this method preserves symplectic structure. Using ideas of the proof of Lemma 4.4, we establish that the method (5.11)-(5.12) with (5.13)-(5.14) is of the mean-square order 3/2. As a result, we get the following theorem.

Theorem 5.3. *Under conditions* (5.13) – (5.14) *on the parameters, the explicit PRK method* (5.11) – (5.12) *for system* (5.2) *preserves symplectic structure and has the mean-square order* 3/2 *of convergence.*

The method (5.11)-(5.12) can be rewritten in the constructive form as the formula (3.21) was obtained from (3.18) in Section 3.2.

Remark 5.7. Attracting other explicit deterministic second-order PRK methods from [15, 17], it is possible to construct other explicit symplectic methods of the order 3/2 for the system (5.2). For instance, by swapping the roles of p and q in the method (5.11) – (5.12), we can obtain another 3/2-order symplectic PRK method.

6. Hamiltonian methods in the case of additive noise and $H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q)$

Here we propose symplectic methods for the Hamiltonian system (5.2), when $\gamma_r(t) = 0$ and the separable Hamiltonian has the special form

$$H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q), \tag{6.1}$$

with M a constant, symmetric, invertible matrix (i.e., the kinetic energy $V(p)$ in (5.1) is equal to $\frac{1}{2}p^\top M^{-1}p$). In this case the system (5.2) reads (cf. (2.11)):

$$dP = f(t,Q)dt + \sum_{r=1}^{m} \sigma_r(t)dw_r(t),\ P(t_0) = p, \tag{6.2}$$

$$dQ = M^{-1}Pdt,\ Q(t_0) = q,$$

with

$$f^i = -\partial U/\partial q^i, i = 1,\dots,n. \tag{6.3}$$

This system can be written as a second-order differential equation with additive noise (cf. (2.10)):

$$\frac{d^2Q}{dt^2} = M^{-1}f(t,Q) + M^{-1}\sum_{r=1}^{m} \sigma_r(t)\dot{w}_r(t). \tag{6.4}$$

Clearly, the symplectic methods from Sections 4 and 5 can be applied to (6.2). Due to specific features of the system (6.2), these methods have a more simple form here. Moreover, one can prove that the methods (5.6) and (5.7) in application to (6.2)–(6.3) are of the mean-square order 2 (recall that these methods are of order 3/2 in the case of the more general system (5.2)). Besides, it turns out that it is possible to obtain a constructive method of the third accuracy order in the case of the system (6.2). In this section we restrict ourselves to explicit methods of orders 2 and 3.

6.1. Explicit methods of order 2.

On the basis of the deterministic second-order symplectic RKN method (3.8), we construct the method for the system (6.2)–(6.3):

$$\mathcal{Q} = Q_k + \frac{h}{2}M^{-1}P_k, \tag{6.5}$$

$$P_{k+1} = P_k + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + hf(t_k + \frac{h}{2}, \mathcal{Q}) + \sum_{r=1}^{m} \sigma_r'(t_k)(I_{0r})_k$$

$$Q_{k+1} = Q_k + hM^{-1}P_k + \sum_{r=1}^{m} M^{-1}\sigma_r(t_k)(I_{r0})_k + \frac{h^2}{2}M^{-1}f(t_k + \frac{h}{2}, \mathcal{Q}),\quad k = 0,\dots,N-1.$$

Theorem 6.1. *The explicit method* (6.5) *for the system* (6.2)$-$(6.3) *is of the mean-square order* 2 *and symplectic.*

Proof. Since the expressions for dP_{k+1}, dQ_{k+1} coincide with the ones corresponding to the deterministic symplectic RKN method (3.8), the method (6.5) is symplectic.

Now consider the mean-square order of convergence of the method (6.5). Denote by $\bar{P}$, $\bar{Q}$ the one-step approximation corresponding to this method. We have

$$\bar{P} = p + \sum_{r=1}^{m} \sigma_r(t)\Delta w_r + hf(t,q) + \sum_{r=1}^{m} \sigma_r'(t) I_{0r} \tag{6.6}$$

$$+\frac{h^2}{2}(\frac{\partial f}{\partial t}(t,q) + \sum_{i=1}^{n}(M^{-1}p)^i \frac{\partial f}{\partial q^i}(t,q)) + \rho_1,$$

$$\bar{Q} = q + hM^{-1}p + \sum_{r=1}^{m} M^{-1}\sigma_r(t) I_{r0} + \frac{h^2}{2}M^{-1}f(t,q) + \rho_2,$$

where ρ_1 and ρ_2 are deterministic and such that

$$|\rho_i| = O(h^3), \; i = 1, 2. \tag{6.7}$$

In the case of the system (6.2) the operators Λ_r and L (see (3.19)) take the form

$$\Lambda_r = (\sigma_r, \frac{\partial}{\partial p}), \; L = L_1 + L_2, \tag{6.8}$$

$$L_1 := \frac{\partial}{\partial t} + (f, \frac{\partial}{\partial p}) + (M^{-1}p, \frac{\partial}{\partial q}), \quad L_2 := \frac{1}{2}\sum_{r=1}^{m}\sum_{i,j=1}^{n} \sigma_r^i \sigma_r^j \frac{\partial^2}{\partial p^i \partial p^j}.$$

Due to specific features of the system (6.2), we get in particular that

$$\Lambda_r f = 0, \; \Lambda_r g = M^{-1}\sigma_r, \; L\sigma_r = \frac{d\sigma_r}{dt}, \; Lf = L_1 f, \; Lg = M^{-1}f, \tag{6.9}$$

$$\Lambda_i \Lambda_j g = 0, \; L\Lambda_r g = M^{-1}\frac{d\sigma_r}{dt}, \; L^2\sigma_r = \frac{d^2\sigma_r}{dt^2}, \; \Lambda_r L f = \Lambda_r L_1 f, \; \Lambda_r L g = 0,$$

$$L^2 f = L_1^2 f, \; L^2 g = L_1\left(M^{-1}f\right),$$

where $g = M^{-1}p$.

Let $P(s) = P(s;t,p,q)$, $Q(s) = Q(s;t,p,q)$, $s \geq t$, be a solution to (6.2). Using the Wagner-Platen expansion [9, 7] and (6.9), it is not difficult to obtain

$$R_1 := P(t+h) - \bar{P} = \sum_{r=1}^{m} \int_t^{t+h}\int_t^{s_1}\int_t^{s_2} \Lambda_r L_1 f(s_3, Q(s_3)) dw_r(s_3) ds_2 ds_1$$

$$+\sum_{r=1}^{m} \int_t^{t+h}\int_t^{s_1}\int_t^{s_2} \sigma_r''(s_3) ds_3 ds_2 dw_r(s_1) + \int_t^{t+h}\int_t^{s_1}\int_t^{s_2} L_1^2 f(s_3, Q(s_3)) ds_3 ds_2 ds_1 - \rho_1 \,,$$

$$R_2 := Q(t+h) - \bar{Q} = \sum_{r=1}^{m} \int_t^{t+h}\int_t^{s_1}\int_t^{s_2} M^{-1}\sigma_r'(s_3) ds_3 dw_r(s_2) ds_1$$

$$+\int_t^{t+h}\int_t^{s_1}\int_t^{s_2} L_1^2 M^{-1} P(s_3) ds_3 ds_2 ds_1 - \rho_2 \,.$$

By the properties of Ito integrals and (6.7), we get

$$|ER_i| = O(h^3),\ (ER_i^2)^{1/2} = O(h^{5/2}),\ i = 1, 2.$$

Then, Theorem 3.1 implies that the method (6.5) is of the second mean-square order. □

Remark 6.1. It is possible to simulate the normally distributed random vector $\int_{t_k}^{t_{k+1}} \sigma_r(s) dw_r(s)$ instead of $\sum_{r=1}^m \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^m \sigma_r'(t_k) \cdot (I_{0r})_k$ in (3.8).

Remark 6.2. By swapping the roles of p and q in the RKN method (3.8), we analogously construct another symplectic method of the mean-square order 2 for the system (6.2)–(6.3) (cf. (5.6)):

$$\mathcal{P} = P_k + \frac{h}{2} f(t_k, Q_k),\quad \mathcal{Q} = Q_k + hM^{-1}\mathcal{P}, \tag{6.10}$$

$$P_{k+1} = P_k + \sum_{r=1}^m \sigma_r(t_k)\Delta_k w_r + \frac{h}{2}\left[f(t_k, Q_k) + f(t_{k+1}, \mathcal{Q})\right] + \sum_{r=1}^m \sigma_r'(t_k) \cdot (I_{0r})_k$$

$$Q_{k+1} = Q_k + hM^{-1}\mathcal{P} + \sum_{r=1}^m M^{-1}\sigma_r(t_k)(I_{r0})_k,\quad k = 0, \ldots, N-1.$$

Remark 6.3. The methods (6.5) and (6.10) can be rewritten in the constructive form as the formula (3.21) was obtained from (3.18) in Section 3.2.

6.2. **Explicit methods of order** 3. Along with the integrals $(I_{0r})_k$ and $(I_{r0})_k$ (see (3.20) we introduce the Ito integrals

$$(I_{00r})_k := \frac{1}{2}\int_{t_k}^{t_{k+1}} (\vartheta - t_k)^2\ dw_r(\vartheta),\quad (I_{0r0})_k := \int_{t_k}^{t_{k+1}}\int_{t_k}^{\vartheta_1} (\vartheta_2 - t_k)\ dw_r(\vartheta_2)d\vartheta_1, \tag{6.11}$$

$$(I_{r00})_k := \int_{t_k}^{t_{k+1}}\int_{t_k}^{\vartheta_1} (w_r(\vartheta_2) - w_r(t_k))\ d\vartheta_2 d\vartheta_1,\ (J_r)_k = \int_{t_k}^{t_{k+1}} (\vartheta - t_k)(w_r(\vartheta) - w_r(t_k))d\vartheta.$$

Joint distribution of the random variables $\Delta_k w_r(h)$, $(I_{0r})_k$, $(I_{r0})_k$, $(I_{0r0})_k$, $(I_{r00})_k$, $(I_{00r})_k$ is Gaussian. They can be simulated at each step by $3m$ independent $N(0,1)$-distributed random variables ξ_{rk}, η_{rk}, and ζ_{rk}, $r = 0, \ldots, m$:

$$\Delta_k w_r = h^{1/2}\xi_{rk},\quad (I_{r0})_k = h^{3/2}(\eta_{rk}/\sqrt{3} + \xi_{rk})/2,\quad (I_{0r})_k = h\Delta_k w_r - (I_{r0})_k, \tag{6.12}$$

$$(J_r)_k = h^{5/2}(\xi_{rk}/3 + \eta_{rk}/(4\sqrt{3}) + \zeta_{rk}/(12\sqrt{5})),$$

$$(I_{r00})_k = h(I_{r0})_k - (J_r)_k,\quad (I_{0r0})_k = 2(J_r)_k - h(I_{r0})_k,\quad (I_{00r})_k = h^2\Delta_k w_r/2 - (J_r)_k.$$

Clearly, for $\sigma_r = 0$, $r = 1, \ldots, m$, the stochastic system (6.2) is reduced to the deterministic system

$$\frac{dq}{dt} = M^{-1}p,\ \frac{dp}{dt} = f(t, q). \tag{6.13}$$

The following lemma is true for system (6.2) with an arbitrary f (i.e., f may not obey the condition (6.3)).

Lemma 6.1. *Let* $\bar{q} = q + G(t,p,q;h)$, $\bar{p} = p + F(t,p,q;h)$ *be a one-step approximation of the third-order explicit method for the deterministic system* (6.13). *Suppose an* n*-dimensional (deterministic) variable* $\mathcal{Q} = \mathcal{Q}(t,p,q;h)$ *is such that*

$$|\mathcal{Q} - q| = O(h).$$

Then, the following method for the system (6.2)

$$(6.14)\qquad P_{k+1} = P_k + F(t,P_k,Q_k;h) + \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m} \sigma_r'(t_k)(I_{0r})_k$$
$$+ \sum_{r=1}^{m} \sigma_r''(t_k)(I_{00r})_k + \sum_{r=1}^{m}\sum_{i=1}^{n} (M^{-1}\sigma_r(t_k))^i \frac{\partial f}{\partial q^i}(t_k,\mathcal{Q}_k)(I_{r00})_k$$
$$Q_{k+1} = Q_k + G(t,P_k,Q_k;h) + \sum_{r=1}^{m} M^{-1}\sigma_r(t_k)(I_{r0})_k + \sum_{r=1}^{m} M^{-1}\sigma_r'(t_k)(I_{0r0})_k$$

is of the mean-square order 3.

Proof. Due to the assumption of the lemma, the functions F and G can be presented as (recall that the operator L_1 is defined in (6.8)):

$$(6.15)\qquad F(t,p,q;h) = hf(t,q) + \frac{h^2}{2}L_1 f(t,p,q) + \frac{h^3}{6}L_1^2 f(t,p,q) + \rho_1,$$
$$G(t,p,q;h) = hM^{-1}p + \frac{h^2}{2}M^{-1}f(t,q) + \frac{h^3}{6}M^{-1}L_1 f(t,p,q) + \rho_2,$$

where ρ_i are deterministic and

$$(6.16)\qquad |\rho_i| = O(h^4),\ i = 1,2.$$

Using the assumption on $\mathcal{Q}$, we obtain

$$(6.17)\qquad \sum_{r=1}^{m}\sum_{i=1}^{n}(M^{-1}\sigma_r)^i\frac{\partial f}{\partial q^i}(t,\mathcal{Q})I_{r00} = \sum_{r=1}^{m}\sum_{i=1}^{n}(M^{-1}\sigma_r)^i\frac{\partial f}{\partial q^i}(t,q)I_{r00} + \Delta,$$

where Δ is such that

$$(6.18)\qquad |E\Delta| = 0,\ \left[E\Delta^2\right]^{1/2} = O(h^{7/2}).$$

To continue the proof, one can use the Wagner-Platen expansion. However due to the specific form of system (6.2), it is more convenient to derive the corresponding expansion directly.

By the Ito formula we get for any sufficiently smooth function Φ :

$$(6.19)\qquad \Phi(s,P(s),Q(s)) = \Phi(t,p,q) + \int_t^s \left(\frac{\partial\Phi}{\partial s} + \sum_{i=1}^{n}\frac{\partial\Phi}{\partial p^i}f^i + \sum_{i=1}^{n}\frac{\partial\Phi}{\partial q^i}(M^{-1}P)^i\right.$$
$$\left.+\frac{1}{2}\sum_{r=1}^{m}\sum_{i,j=1}^{n}\frac{\partial^2\Phi}{\partial p^i\partial p^j}\sigma_r^i\sigma_r^j\right) ds_1 + \int_t^s \sum_{r=1}^{m}\sum_{i=1}^{n}\frac{\partial\Phi}{\partial p^i}\sigma_r^i dw_r(s_1),$$

where $P(s)=P(s;t,p,q)$, $Q(s)=Q(s;t,p,q)$, $s\geq t$, is the solution of (6.2) and the full notation for, e.g., $\partial\Phi/\partial s$ is $\frac{\partial\Phi}{\partial s}(s_1,P(s_1),Q(s_1))$.

We have

$$Q(t+h)=q+\int_t^{t+h}M^{-1}P(s_1)ds_1,$$

$$P(t+h)=p+\int_t^{t+h}f(s_1,Q(s_1))ds_1+\int_t^{t+h}\sum_{r=1}^m\sigma_r(s_1)dw_r(s_1).$$

Applying (6.19) to the integrands $M^{-1}P$ and $f(s,Q)$ in the above formula, we obtain (6.20)

$$Q(t+h)=q+hM^{-1}p+\int_t^{t+h}\int_t^{s_1}M^{-1}f(s_2,Q(s_2))ds_2ds_1+\int_t^{t+h}\int_t^{s_1}\sum_{r=1}^m M^{-1}\sigma_r(s_2)dw_r(s_2)ds_1,$$

$$P(t+h)=p+hf(t,q)+\int_t^{t+h}\int_t^{s_1}(\frac{\partial f}{\partial s}+\sum_{i=1}^n\frac{\partial f}{\partial q^i}(M^{-1}P)^i)ds_2ds_1+\int_t^{t+h}\sum_{r=1}^m\sigma_r(s_1)dw_r(s_1).$$

Here applying (6.19) to the integrands $M^{-1}f$ and $\partial f/\partial s+\sum_{i=1}^n\partial f/\partial q^i(M^{-1}P)^i$, we get

$$Q(t+h)=q+hM^{-1}p+\frac{h^2}{2}M^{-1}f(t,q) \tag{6.21}$$

$$+\int_t^{t+h}\int_t^{s_1}\int_t^{s_2}(M^{-1}\frac{\partial f}{\partial s}+\sum_{i=1}^n M^{-1}\frac{\partial f}{\partial q^i}(M^{-1}P)^i)ds_3ds_2ds_1+\int_t^{t+h}\int_t^{s_1}\sum_{r=1}^m M^{-1}\sigma_r(s_2)dw_r(s_2)ds_1,$$

$$P(t+h)=p+hf(t,q)+\frac{h^2}{2}(\frac{\partial f}{\partial s}(t,q)+\sum_{i=1}^n\frac{\partial f}{\partial q^i}(t,q)(M^{-1}p)^i)+\int_t^{t+h}\int_t^{s_1}\int_t^{s_2}(\frac{\partial^2 f}{\partial s^2}$$

$$+2\sum_{i=1}^n\frac{\partial^2 f}{\partial s\partial q^i}(M^{-1}P)^i+\sum_{i=1}^n\frac{\partial f}{\partial q^i}(M^{-1}f)^i+\sum_{i,j=1}^n\frac{\partial^2 f}{\partial q^i\partial q^j}(M^{-1}P)^i(M^{-1}P)^j)ds_3ds_2ds_1$$

$$+\int_t^{t+h}\int_t^{s_1}\int_t^{s_2}\sum_{r=1}^m\sum_{i=1}^n\frac{\partial f}{\partial q^i}(M^{-1}\sigma_r)^i dw_r(s_3)ds_2ds_1+\int_t^{t+h}\sum_{r=1}^m\sigma_r(s_1)dw_r(s_1).$$

Expanding again the integrands in (6.21) according to (6.19) and using the properties of Ito integrals, we obtain

$$Q(t+h)=q+hM^{-1}p+\frac{h^2}{2}M^{-1}f(t,q)+\frac{h^3}{6}M^{-1}L_1f(t,p,q) \tag{6.22}$$

$$+\int_t^{t+h}\int_t^{s}\sum_{r=1}^m M^{-1}\sigma_r(s_1)dw_r(s_1)ds+R_2,$$

$$P(t+h) = p + hf(t,q) + \frac{h^2}{2}L_1 f(t,p,q) + \frac{h^3}{6}L_1^2 f(t,p,q)$$
$$+\sum_{r=1}^{m}\sum_{i=1}^{n}\frac{\partial f}{\partial q^i}(t,q)(M^{-1}\sigma_r(t))^i I_{r00} + \int_t^{t+h}\sum_{r=1}^{m}\sigma_r(s)dw_r(s) + R_1,$$

where

$$|ER_i| = O(h^4),\ \left(ER_i^2\right)^{1/2} = O(h^{7/2}),\ i=1,2. \tag{6.23}$$

Denote by $\bar{P}$, $\bar{Q}$ the one-step approximation corresponding to the method (6.14). Taking into account (6.15), (6.17), and (6.22), it is not difficult to get that

$$\bar{R}_1 := P(t+h) - \bar{P},\ \bar{R}_2 := Q(t+h) - \bar{Q}$$

satisfy the relations (see (6.16), (6.18), and (6.23))

$$\left|E\bar{R}_i\right| = O(h^4),\ \left(E\bar{R}_i^2\right)^{1/2} = O(h^{7/2}),\ i=1,2.$$

Then Theorem 3.1 implies that the method (6.5) is of the third mean-square order. □

Remark 6.4. Lemma 6.1 can be generalized to the system

$$\frac{d^2Q}{dt^2} = M^{-1}f(t,Q) + \Gamma\frac{dQ}{dt} + M^{-1}\sum_{r=1}^{m}\sigma_r(t)\dot{w}_r(t),$$

where Γ is a constant matrix.

Using the deterministic third-order symplectic method (3.9)–(3.10), we obtain the following method for the system (6.2)–(6.3):

$$\mathcal{Q}_1 = Q_k + \frac{7}{24}hM^{-1}P_k,\ \ \mathcal{P}_1 = P_k + \frac{2}{3}hf(t_k + \frac{7h}{24}, \mathcal{Q}_1) \tag{6.24}$$
$$\mathcal{Q}_2 = \mathcal{Q}_1 + \frac{3}{4}hM^{-1}\mathcal{P}_1,\ \ \mathcal{P}_2 = \mathcal{P}_1 - \frac{2}{3}hf(t_k + \frac{25h}{24}, \mathcal{Q}_2)$$
$$\mathcal{Q}_3 = \mathcal{Q}_2 - \frac{1}{24}hM^{-1}\mathcal{P}_2,\ \ \mathcal{P}_3 = \mathcal{P}_2 + hf(t_k+h, \mathcal{Q}_3)$$

$$P_{k+1} = \mathcal{P}_3 + \sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r + \sum_{r=1}^{m}\sigma_r'(t_k)(I_{0r})_k \tag{6.25}$$
$$+\sum_{r=1}^{m}\sigma_r''(t_k)(I_{00r})_k + \sum_{r=1}^{m}\sum_{i=1}^{n}(M^{-1}\sigma_r(t_k))^i\frac{\partial f}{\partial q^i}(t_k,\mathcal{Q}_3)(I_{r00})_k,$$
$$Q_{k+1} = \mathcal{Q}_3 + \sum_{r=1}^{m}M^{-1}\sigma_r(t_k)(I_{r0})_k + \sum_{r=1}^{m}M^{-1}\sigma_r'(t_k)(I_{0r0})_k,\ \ k=0,\dots,N-1.$$

Theorem 6.2. *The explicit method* (6.24)–(6.25) *for the system* (6.2)–(6.3) *is symplectic and of the mean-square order* 3.

Proof. It is not difficult to check that $dP_{k+1}\wedge dQ_{k+1} = d\mathcal{P}_3\wedge d\mathcal{Q}_3$. The expression for $d\mathcal{P}_3\wedge d\mathcal{Q}_3$ coincides with the one corresponding to the deterministic symplectic RKN method (3.9)–(3.10). This implies that the method (6.24)–(6.25) is symplectic. By Lemma 6.1 we get that the method has the mean-square order 3. □

Remark 6.5. Using the formulae (6.12), the method (6.24) – (6.25) can be rewritten in the constructive form.

Remark 6.6. By other deterministic third-order symplectic methods (see, e.g. [16, 17, 15]), other symplectic methods of the mean-square order 3 for the system (6.2) – (6.3) can be constructed. For instance, by swapping the roles of p and q in the RKN method (3.9) – (3.10) or by using the adjoint of (3.9) – (3.10) [15, p. 108], the corresponding symplectic methods for (6.2) – (6.3) can easily be written down.

Remark 6.7. If the property of symplecticness is not required, it is possible to propose a more simple third-order method in comparison to (6.24) – (6.25). This can be done by taking a standard deterministic third-order Runge-Kutta method and putting $\mathcal{Q}_k = Q_k$ in (6.14).

7. Hamiltonian methods for Hamiltonian systems with small additive noise

An important instance of a stochastic system is given by a stochastic differential equation with small noise, since often fluctuations, which affect a dynamical system are sufficiently small. It was shown in [11] that mean-square methods adapted to systems with small noise can be more efficient than general methods. The errors of these methods are estimated in terms of products $h^i \varepsilon^j$, where h is the step-size of discretization and ε is a small parameter at noise. Usually, global error has the form $O(h^j + \varepsilon^k h^l)$, where $j > l,\ k > 0$. Thanks to the fact that the accuracy order of such methods is equal to a comparatively small l, they are not too complicated, while due to the large j and the small factor ε^k at h^l, their errors are fairly low. This allows us to construct effective (high-exactness) mean-square methods with low time-step order but which nevertheless have small errors.

In this section we apply the ideas of [11] to the Hamiltonian system with small additive noise (cf. (4.1)-(4.2)):

$$dP = f(t,P,Q)dt + \varepsilon \sum_{r=1}^{m} \sigma_r(t) dw_r(t),\ P(t_0) = p, \tag{7.1}$$

$$dQ = g(t,P,Q)dt + \varepsilon \sum_{r=1}^{m} \gamma_r(t) dw_r(t),\ Q(t_0) = q,$$

$$f^i = -\partial H/\partial q^i,\ g^i = \partial H/\partial p^i,\ \ i = 1,\dots,n, \tag{7.2}$$

where $\varepsilon > 0$ is a small parameter, $P,\ Q,\ f,\ g,\ \sigma_r,\ \gamma_r$ are n-dimensional column-vectors, $w_r(t)$, $r = 1,\dots,m$, are independent standard Wiener processes, and $H(t,p,q)$ is a Hamiltonian. The phase flow of this system preserves symplectic structure (see Corollary 2.5).

7.1. Systems with Hamiltonians of the general form.

First we note that the method (4.30) in application to the system with small noise (7.1)-(7.2) is of the order $O(h^2 + \varepsilon^2 h^{3/2})$ (cf. [11]). We can simplify (4.30) and obtain the following one-parametric family of methods for system (7.1)-(7.2):

$$\mathcal{P}_1 = P_k + \frac{\alpha}{2} h f(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1), \tag{7.3}$$

$$\mathcal{Q}_1 = Q_k + \frac{\alpha}{2}hg(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1)$$
$$\mathcal{P}_2 = P_k + \alpha hf(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-\alpha}{2}hf(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2),$$
$$\mathcal{Q}_2 = Q_k + \alpha hg(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-\alpha}{2}hg(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2),$$
$$P_{k+1} = P_k + \varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r + h[\alpha f(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-\alpha)f(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)],$$
$$Q_{k+1} = Q_k + \varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r + h[\alpha g(t_k + \frac{\alpha}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-\alpha)g(t_k + \frac{1+\alpha}{2}h, \mathcal{P}_2, \mathcal{Q}_2)].$$

Using [11], it is not difficult to prove the following theorem.

Theorem 7.1. *The implicit method* (7.3) *for system* (7.1) – (7.2) *is symplectic and its mean-square error is estimated as* $O(h^2 + \varepsilon h)$.

Now we are going to obtain a more accurate symplectic method for the system (7.1)-(7.2). To this end consider the implicit method

$$\mathcal{P}_1 = P_k + h\frac{\varkappa}{2}f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\left(\lambda_1(J_{r0})_k + \mu\Delta_k w_r\right), \tag{7.4}$$
$$\mathcal{Q}_1 = Q_k + h\frac{\varkappa}{2}g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\left(\lambda_1(J_{r0})_k + \mu\Delta_k w_r\right),$$
$$\mathcal{P}_2 = P_k + h[\varkappa f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-2\varkappa}{2}f(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2)],$$
$$\mathcal{Q}_2 = Q_k + h[\varkappa g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-2\varkappa}{2}g(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2)],$$
$$\mathcal{P}_3 = P_k + h[\varkappa f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)f(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \frac{\varkappa}{2}f(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)]$$
$$+\varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\left(\lambda_2(J_{r0})_k - \mu\Delta_k w_r\right),$$
$$\mathcal{Q}_3 = Q_k + h[\varkappa g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)g(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \frac{\varkappa}{2}g(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)]$$
$$+\varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\left(\lambda_2(J_{r0})_k - \mu\Delta_k w_r\right),$$

$$P_{k+1} = P_k + \varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r + \varepsilon\sum_{r=1}^{m}\sigma_r'(t_k)(I_{0r})_k \tag{7.5}$$
$$+h\left[\varkappa f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)f(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \varkappa f(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)\right],$$
$$Q_{k+1} = Q_k + \varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r + \varepsilon\sum_{r=1}^{m}\gamma_r'(t_k)(I_{0r})_k$$
$$+h\left[\varkappa g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)g(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \varkappa g(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)\right],$$

where $\Delta w_r, I_{0r},\ J_{r0}$ are defined after (4.16), the number $\varkappa$ is equal to

$$\varkappa = \frac{1}{3}(2 + 2^{1/3} + 2^{-1/3}),$$

and the parameters $\lambda_1,\ \lambda_2,\ \mu$ satisfy

$$\varkappa(\lambda_1 + \lambda_2) = 1, \tag{7.6}$$
$$\varkappa\left(\frac{\lambda_1^2 + \lambda_2^2}{3} + \lambda_1\mu - \lambda_2\mu + 2\mu^2\right) = \frac{1}{2}.$$

For example, the following set of parameters satisfies (7.6)

$$\lambda_1 = \lambda_2 = \frac{1}{2\varkappa},\ \mu = \frac{\sqrt{3\varkappa - 1}}{\sqrt{12}\,\varkappa}. \tag{7.7}$$

For sufficiently small h, the equations (7.4) are uniquely solvable due to Lemma 4.1.

Let us note that the method (7.4) – (7.5) is reduced under $\sigma_r \equiv 0,\ \gamma_r \equiv 0,\ r = 1, \ldots, m$, to the well-known fourth-order symplectic Runge-Kutta method for deterministic Hamiltonian systems (see, e.g., [15, p. 101]).

Theorem 7.2. *Under conditions* (7.6) *on the parameters, the implicit method* (7.4) – (7.5) *for system* (7.1) – (7.2) *is symplectic and its mean-square error is estimated as* $O(h^4 + \varepsilon h^2 + \varepsilon^2 h^{3/2})$.

Proof. The fact that the error of (7.4)-(7.5) is estimated as $O(h^4 + \varepsilon h^2 + \varepsilon^2 h^{3/2})$ follows from the arguments similar to the ones in the proof of Lemma 4.4 and a mean-square theorem from [11]. Further, this one-step approximation is of the form (4.9) with $s = 3$ and

$$\varphi_1 = \varepsilon \sum_{r=1}^{m} \sigma_r \left(\lambda_1 J_{r0} + \mu \Delta w_r\right),\ \varphi_2 = 0,\ \varphi_3 = \varepsilon \sum_{r=1}^{m} \sigma_r \left(\lambda_2 J_{r0} - \mu \Delta w_r\right),$$

$$\psi_1 = \varepsilon \sum_{r=1}^{m} \gamma_r \left(\lambda_1 J_{r0} + \mu \Delta w_r\right),\ \psi_2 = 0,\ \psi_3 = \varepsilon \sum_{r=1}^{m} \gamma_r \left(\lambda_2 J_{r0} - \mu \Delta w_r\right),$$

$$\eta = \varepsilon \sum_{r=1}^{m} \sigma_r \Delta w_r + \varepsilon \sum_{r=1}^{m} \sigma_r' I_{0r},\quad \zeta = \varepsilon \sum_{r=1}^{m} \gamma_r \Delta w_r + \varepsilon \sum_{r=1}^{m} \gamma_r' I_{0r}$$

and

$$\alpha_{11} = \frac{\varkappa}{2},\ \alpha_{12} = 0,\ \alpha_{13} = 0,\ \alpha_{21} = \varkappa,\ \alpha_{22} = \frac{1 - 2\varkappa}{2},\ \alpha_{23} = 0,$$
$$\alpha_{31} = \varkappa,\ \alpha_{32} = 1 - 2\varkappa,\ \alpha_{33} = \frac{\varkappa}{2},$$
$$\beta_1 = \varkappa,\ \beta_2 = 1 - 2\varkappa,\ \beta_3 = \varkappa,\ c_1 = \frac{\varkappa}{2},\ c_2 = \frac{1}{2},\ c_3 = \frac{2 - \varkappa}{2}.$$

This set of parameters $\alpha_{ij},\ \beta_i,\ i, j = 1, 2, 3$, satisfies the conditions (4.11). Then due to Lemma 4.3, the method (7.4)-(7.5) is symplectic. □

Remark 7.1. Using the formulae (6.12), the method (7.4) – (7.5) can be rewritten in the constructive form with respect to simulation of the used random variables.

Remark 7.2. By other deterministic fourth-order symplectic methods (see, e.g. [16, 17, 15]), other symplectic methods with the error $O(h^4+\varepsilon h^2+\varepsilon^2 h^{3/2})$ for the system (7.1) – (7.2) can be constructed.

7.2. **Systems with separable Hamiltonians.** Consider the system (7.1)-(7.2) with separable Hamiltonian (cf. (5.2))

$$dP = f(t,Q)dt + \varepsilon\sum_{r=1}^{m}\sigma_r(t)dw_r(t),\ P(t_0)=p, \tag{7.8}$$

$$dQ = g(P)dt + \varepsilon\sum_{r=1}^{m}\gamma_r(t)dw_r(t),\ Q(t_0)=q,$$

where $f^i=-\partial U/\partial q^i$, $g^i=\partial V/\partial p^i$, $i=1,\ldots,n$.

Obviously, the implicit methods of Section 7.1 can be used for solution of (7.8). Besides, we can propose explicit symplectic methods for the system (7.8) using methods of Section 5. For instance, the explicit methods (5.6), (5.7), and (5.11)-(5.12) in application to (7.8) have the order $O(h^2+\varepsilon^2 h^{3/2})$. Further, we can simplify these methods as we simplified (4.30) to obtain (7.3) above. As a result, we will get the explicit PRK methods of order $O(h^2+\varepsilon h)$ for system (7.8) (cf. Theorem 7.1).

To construct a high-exactness symplectic method, consider the parametric family of explicit PRK methods

$$\mathcal{Q}_1 = Q_k + \varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\left(\hat\lambda_1(J_{r0})_k+\hat\mu\Delta_k w_r\right), \tag{7.9}$$

$$\mathcal{P}_1 = P_k + h\frac{\varkappa}{2}f(t_k,\mathcal{Q}_1) + \varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\left(\lambda_1(J_{r0})_k+\mu\Delta_k w_r\right),$$

$$\mathcal{Q}_2 = Q_k + h\varkappa g(\mathcal{P}_1),$$

$$\mathcal{P}_2 = P_k + h\frac{\varkappa}{2}f(t_k,\mathcal{Q}_1) + h\frac{1-\varkappa}{2}f(t_k+\varkappa h,\mathcal{Q}_2),$$

$$\mathcal{Q}_3 = \mathcal{Q}_2 + h(1-2\varkappa)g(\mathcal{P}_2),$$

$$\mathcal{P}_3 = \mathcal{P}_2 + h\frac{1-\varkappa}{2}f(t_k+(1-\varkappa)h,\mathcal{Q}_3) + \varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\left(\lambda_2(J_{r0})_k-\mu\Delta_k w_r\right),$$

$$\mathcal{Q}_4 = \mathcal{Q}_3 + h\varkappa g(\mathcal{P}_3) + \varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\left(\hat\lambda_2(J_{r0})_k-\hat\mu\Delta_k w_r\right),$$

$$P_{k+1} = \mathcal{P}_2 + \varepsilon\sum_{r=1}^{m}\sigma_r(t_k)\Delta_k w_r + \varepsilon\sum_{r=1}^{m}\sigma_r'(t_k)(I_{0r})_k \tag{7.10}$$

$$+h\frac{1-\varkappa}{2}f(t_k+(1-\varkappa)h,\mathcal{Q}_3) + h\frac{\varkappa}{2}f(t_k+h,\mathcal{Q}_4),$$

$$Q_{k+1} = \mathcal{Q}_3 + \varepsilon\sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r + \varepsilon\sum_{r=1}^{m}\gamma_r'(t_k)(I_{0r})_k + h\varkappa g(\mathcal{P}_3),\quad k=0,\ldots,N-1,$$

where $\varkappa = (2 + 2^{1/3} + 2^{-1/3})/3$ and the parameters λ, $\hat{\lambda}$, μ, $\hat{\mu}$ satisfy the conditions

$$\frac{\varkappa}{2}(\hat{\lambda}_1 + \hat{\lambda}_2) = 1, \quad \varkappa(\lambda_1 + \lambda_2) = 1, \tag{7.11}$$

$$\frac{\varkappa}{2}\left(\frac{\hat{\lambda}_1^2 + \hat{\lambda}_2^2}{3} + \hat{\lambda}_1\hat{\mu} - \hat{\lambda}_2\hat{\mu} + 2\hat{\mu}^2\right) = \frac{1}{2},$$

$$\varkappa\left(\frac{\lambda_1^2 + \lambda_2^2}{3} + \lambda_1\mu - \lambda_2\mu + 2\mu^2\right) = \frac{1}{2}$$

(see the definition of $\Delta w_r, I_{0r}, J_{r0}$ in (4.16)).

For example, the following set of parameters satisfies (7.11):

$$\lambda_1 = \lambda_2 = \frac{1}{2\varkappa}, \ \hat{\lambda}_1 = \hat{\lambda}_2 = \frac{1}{\varkappa}, \tag{7.12}$$

$$\mu = \frac{\sqrt{3\varkappa - 1}}{\sqrt{12}\,\varkappa}, \ \hat{\mu} = \frac{\sqrt{3\varkappa - 2}}{\sqrt{6}\,\varkappa}.$$

Note that the method (7.9)-(7.10) is a generalization of the deterministic fourth-order symplectic PRK method (3.6)-(3.7) from [15, p. 109] to the stochastic case.

It is not difficult to see that the method (7.9)-(7.10) has the form of (5.8)-(5.9) and its parameters satisfy the condition (5.10). Then, Lemma 5.1 implies that this method preserves symplectic structure. Analogously to the proof of Theorem 7.2, we establish that the method (5.11)-(5.12) with (5.13)-(5.14) is of order $O(h^4 + \varepsilon h^2 + \varepsilon^2 h^{3/2})$. As a result, we get the theorem.

Theorem 7.3. *Under conditions* (7.11) *on the parameters, the explicit PRK method* (7.9) – (7.10) *for system* (7.8) *is symplectic and its mean-square error is estimated as* $O(h^4 + \varepsilon h^2 + \varepsilon^2 h^{3/2})$.

Using the formulae (6.12), the method (7.9)-(7.10) can be rewritten in the constructive form with respect to simulation of the used random variables.

7.3. **Systems with Hamiltonians** $H(t,p,q) = \frac{1}{2}p^\top M^{-1}p + U(t,q)$**.** Consider the special case of system (7.8) (cf. (6.2) and (6.4)):

$$dP = f(t,Q)dt + \varepsilon \sum_{r=1}^{m} \sigma_r(t)dw_r(t), \ P(t_0) = p, \tag{7.13}$$

$$dQ = M^{-1}Pdt, \ Q(t_0) = q,$$

with $f^i = -\partial U/\partial q^i, i = 1, \ldots, n$.

On the basis of the fourth-order deterministic PRK method (3.6)-(3.7), we construct the following method for the system (7.13):

$$\mathcal{P}_1 = P_k + h\frac{\varkappa}{2}f(t_k, Q_k), \quad \mathcal{Q}_1 = Q_k + h\varkappa M^{-1}\mathcal{P}_1, \tag{7.14}$$

$$\mathcal{P}_2 = \mathcal{P}_1 + h\frac{1-\varkappa}{2}f(t_k + \varkappa h, \mathcal{Q}_1), \quad \mathcal{Q}_2 = \mathcal{Q}_1 + h(1 - 2\varkappa)M^{-1}\mathcal{P}_2,$$

$$\mathcal{P}_3 = \mathcal{P}_2 + h\frac{1-\varkappa}{2}f(t_k + (1-\varkappa)h, \mathcal{Q}_2), \quad \mathcal{Q}_3 = \mathcal{Q}_2 + h\varkappa M^{-1}\mathcal{P}_3,$$

$$\mathcal{P}_4 = \mathcal{P}_3 + h\frac{\varkappa}{2} f(t_k + h, \mathcal{Q}_3),$$

$$P_{k+1} = \mathcal{P}_4 + \varepsilon \sum_{r=1}^{m} \sigma_r(t_k)\Delta_k w_r + \varepsilon \sum_{r=1}^{m} \sigma'_r(t_k)(I_{0r})_k \tag{7.15}$$

$$+\varepsilon \sum_{r=1}^{m} \sigma''_r(t_k)(I_{00r})_k + \varepsilon \sum_{r=1}^{m}\sum_{i=1}^{n} (M^{-1}\sigma_r(t_k))^i \frac{\partial f}{\partial q^i}(t_k, \mathcal{Q}_3)(I_{r00})_k,$$

$$Q_{k+1} = \mathcal{Q}_3 + \varepsilon \sum_{r=1}^{m} M^{-1}\sigma_r(t_k)(I_{r0})_k + \varepsilon \sum_{r=1}^{m} M^{-1}\sigma'_r(t_k)(I_{0r0})_k, \quad k = 0, \dots, N-1,$$

where $\varkappa = (2 + 2^{1/3} + 2^{-1/3})/3$.

Theorem 7.4. *The explicit method* (7.14) – (7.15) *for the system* (7.13) *is symplectic and its mean-square error is estimated as* $O(h^4 + \varepsilon h^3)$.

Proof. It is not difficult to check that $dP_{k+1} \wedge dQ_{k+1} = d\mathcal{P}_4 \wedge d\mathcal{Q}_3$. The expression for $d\mathcal{P}_4 \wedge d\mathcal{Q}_3$ coincides with the one corresponding to the deterministic symplectic method (3.6)-(3.7). This implies that the method (7.14)-(7.15) is symplectic. Using arguments similar to ones used in the proof of Lemma 6.1 and a mean-square convergence theorem from [11], we get that the method is of the mean-square order $O(h^4 + \varepsilon h^3)$. □

Using the formulae (6.12), the method (7.14)-(7.15) can be rewritten in the constructive form with respect to simulation of the used random variables.

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