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On the functional equations of dynamical theta functions I.

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ABSTRACT. The twisted geodesic flow of compact locally symmetric spaces of rank one gives rise to a series of meromorphic functions on the complex plane satisfying simple functional equations. These results are discussed as part of geometric quantization and index theory.

1. Introduction.

The present paper is part of a program (initiated in [10]) to understand dynamical zeta functions from the point of view of index theory.

The dynamical zeta functions of interest here are the generalized Selberg zeta functions Z_σ being canonically associated to certain twists σ of the geodesic flows of compact locally symmetric spaces X of negative curvature. The zeta function Z_σ is defined in terms of the classical mechanics (periods of closed orbits, Poincaré mappings and monodromy operators) of the corresponding σ–twisted geodesic flow. It is a holomorphic function in an open half–plane and admits a meromorphic continuation to the whole complex plane.

Moreover, the zeta function Z_σ satisfies a dynamical functional equation. Its formulation only refers

(1) to the spectrum of the geometric quantization of the Hamiltonian flow obtained from the σ–twisted geodesic flow by replacing the real–valued time variable t by the imaginary–valued time variable it and
(2) to the classical mechanics of the underlying σ–twisted geodesic flow in terms of canonically associated secondary characteristic classes.

For more details we refer to [10].

Now in the present paper we introduce the dynamical theta functions Θ_σ. In analogy to theta functions of Schrödinger operators the theta functions Θ_σ are defined by sets of quantum numbers of the geometric quantization of the corresponding σ–twisted geodesic flows. These quantum numbers also occur as zeros and poles of the corresponding zeta function Z_σ and we use this relation to define Θ_σ.

The theta functions Θ_σ should be regarded as spectral theoretical counterparts of the generalized Selberg zeta functions Z_σ. The theta functions Θ_σ turn out to be meromorphic functions on the complex plane and satisfy beautiful functional equations.

The definition of the theta functions is motivated also by some classical results in the theory of the Riemann zeta function.

2. Theta functions of Riemann surfaces. Cramér's theta function.

Let X be a compact Riemann surface of genus $g \geq 2$ regarded by uniformization as a two–dimensional hyperbolic manifold of constant negative curvature -1. Then the Laplacian $-\Delta_X$ is non–negative and has a discrete spectrum. We define the *theta function*

$$\Theta_X(t) = \sum_\mu m(\mu)e^{-t\mu}, \quad Re(t) > 0, \tag{2.1}$$

where μ runs over

(1) the non–negative roots of the non–negative eigenvalues of the shifted Laplacian $-\Delta_X - \frac{1}{4}$ and
(2) the purely imaginary roots of the eigenvalues of $-\Delta_X - \frac{1}{4}$ in $[-\frac{1}{4}, 0)$ with negative imaginary part.

Here $m(\mu)$ is the dimension of the eigenspace of $-\Delta_X - \frac{1}{4}$ for the eigenvalue μ^2.

In [4] Cartier and Voros proved the existence of a meromorphic continuation of Θ_X to the complex plane and discovered the following beautiful *functional equation* for Θ_X. Let

$$\Theta^d(t) \stackrel{\text{def}}{=} \frac{\cosh\frac{t}{2}}{2\sinh^2\frac{t}{2}}, \quad t \neq 0. \tag{2.2}$$

Then $\Theta^d(t) = \Theta^d(-t), \Theta^d$ extends to a meromorphic function on $\mathbb{C}$ and we have the functional equation

$$\Theta_X(t) + \Theta_X(-t) = (1-g)\left(\Theta^d(it) + \Theta^d(-it)\right). \tag{2.3}$$

In [4] the function $\Theta^d(t)$ is regarded as the theta function of the positive square–root of the operator

$$-\Delta_{S^2} + \tfrac{1}{4},$$

where $-\Delta_{S^2}$ is the Laplacian of the 2–sphere S^2 with respect to the metric of constant positive curvature $+1$.

In fact, since

$$\frac{\cosh t/2}{2\sinh^2 t/2} = -\frac{d}{dt}\left(\frac{1}{\sinh t/2}\right)$$

and

$$\frac{1}{\sinh t/2} = 2\sum_{n=0}^{\infty} e^{-t(n+\frac{1}{2})}$$

for $t > 0$, we have

$$\Theta^d(t) = \sum_{n=0}^{\infty} e^{-t(n+\frac{1}{2})}(2n+1) \tag{2.4}$$

for $t > 0$. On the other hand, the sequence

$$\left\{n + \tfrac{1}{2}, n \geq 0\right\}$$

coincides with the sequence of positive square–roots of the eigenvalues $n(n+1) + \frac{1}{4}, n \geq 0$ of the shifted Laplacian

$$-\Delta_{S^2} + \tfrac{1}{4}$$

and the dimension of the eigenspace of $-\Delta_{S^2}$ for the eigenvalue $n(n+1)$ is well–known to be $2n+1$.

By classical polar decomposition we have the formula

$$\begin{aligned} \Theta^d(t) &= 2\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(t-2\pi i n)^2} \\ &= \Theta^d_+(t) + \Theta^d_+(-t), \end{aligned} \tag{2.5}$$

where

$$\Theta_{+}^{d}(t) \stackrel{\text{def}}{=} \frac{1}{t^2} + 2\sum_{n=1}^{\infty}(-1)^n(t+2\pi in)^{-2}. \tag{2.6}$$

Thus $\Theta_{+}^{d}(it)$ extends to a meromorphic function on $\mathbb{C}$ with the property that all of its poles are double poles contained in the non-positive real line. More precisely, $\Theta_{+}^{d}(t)$ has poles only at the points $t = -2\pi n,\ n = 0, 1, 2, \ldots$.

Moreover, it turns out (see [4]) that the theta function $\Theta_X(t)$ has double poles at the double poles of $\Theta_{+}^{d}(it)$ and simple poles at the points in the set

$$\{\pm i\ell_c\},$$

where ℓ_c denotes the length of a (not necessarily prime) closed geodesic c in X. The residue of the pole in $\pm i\ell_c$ is determined by the Poincaré–mapping of the corresponding closed loop c.

In analogy to this one can interpret the set

$$\{2\pi n, n \in \mathbb{Z}\}$$

as the set of all lengths of the closed geodesics in S^2 (with respect to the metric of curvature $+1$). Note that *all* geodesics in S^2 are closed and have the *same* prime length 2π. In accordance with the general theory of the trace of the wave–operator (see [6]) the multiplicity of the poles of $\Theta^d(t)$ should be interpreted as

$$1 + \tfrac{1}{2}\ \text{dimension (space of all closed geodesics of } S^2).$$

In fact, the space of all closed geodesics of S^2 is a manifold of dimension 2.

Let us notice also that the coefficient $1-g$ in (2.3) coincides with the quotient

$$\chi(X)/\chi(S^2)$$

of the Euler characteristics of X and the compact dual symmetric space S^2.

To sum up, we see that there is a very precise relation between the singularities of Θ_X and the periods of closed geodesics in X *and* in S^2. By regarding the geodesics flows of X and S^2 as *real* and *imaginary* part of a flow with a *complex time* variable, the philosophy of these results is similar to the philosophy of the basic conjectures of Balian and Bloch (see [2]) on theta functions associated to real–analytic Schrödinger operators derived within the path–integral approach to quantum mechanics. Cartier and Voros regarded these results as a new example of a quantum tunnel effect.

Next let us briefly compare these results for Riemann surfaces with some classical results of H. Cramér and A.P. Guinand (see [5], [8]). Cramér introduced and studied a theta function using the non–real zeros of the Riemann zeta function

$$\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}, \quad Re(s) > 1.$$

ζ extends to a meromorphic function on $\mathbb{C}$ with a simple pole in $s = 1$ (and no other poles) and satisfies the well–known symmetric functional equation

$$\hat{\zeta}(1-s) = \hat{\zeta}(s),$$

where

$$\hat{\zeta}(s) \stackrel{\text{def}}{=} \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s).$$

Cramér defined, under the assumption that the Riemann hypotheses ($\zeta(\rho) = 0, \rho \notin \mathbb{R} \Rightarrow Re(\rho) = \frac{1}{2}$) is true, the theta function

$$\theta_R(s) \stackrel{\text{def}}{=} \sum_{Im\rho>0} e^{-(Im\rho)s}, \tag{2.7}$$

where the sum is over all zeros ρ of ζ_R on the critical line $Re(s) = \frac{1}{2}$ with positive imaginary part. θ_R is absolutely convergent and thus holomorphic for $Re(s) > 0$. Moreover, θ_R admits a meromorphic continuation to $\mathbb{C}\backslash(-\infty, 0]$ with simple poles only at $s = im \log p, m \neq 0, m \in \mathbb{Z}$ with residues

$$-\frac{\log p}{2\pi p^{|m|/2}}.$$

θ_R (defined for $\arg(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $s \neq 0$ by (2.7)) extends to a function on the logarithmic Riemann surface. The only additional poles (of the branches) of θ_R (as a function of s) are *simple poles* at $s = \pm 2\pi m$ with residues

$$(-1)^m(\arg s)/2\pi i$$

and *simple poles* at $s = -(2m+1)\pi$, $m = 0, 1, 2, \ldots$ with residues,

$$\tfrac{1}{2}(-1)^{m+1}$$

and a branch point at $s = 0$. Moreover, the function

$$\hat{\theta}_R(s) \stackrel{\text{def}}{=} \theta_R(s) + (4\pi \sin \tfrac{s}{2})^{-1} \log s \tag{2.8}$$

has a unique continuation to $\mathbb{C}\backslash 0$ and satisfies the functional equation

$$\hat{\theta}_R(s) + \hat{\theta}_R(-s) = 2\cos\tfrac{s}{2} - (4\cos\tfrac{s}{2})^{-1}. \tag{2.9}$$

(see [8], theorem 3).

Note that only the non–trivial Γ–factor $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ of ζ_R forces one to go to the logarithmic Riemann surface to establish a meromorphic continuation of θ_R.

In particular, we see that, in contrast to Θ_R, the behaviour of the theta function Θ_X of a compact Riemann surface X is even more simple and more beautiful.

3. The main ideas.

Now the purpose of the present paper is to formulate and discuss some far–reaching generalizations of the functional equation (2.3) in the framework of twisted geodesic flows of compact locally symmetric spaces X of negative curvature and even dimension. Detailed proofs will be given in part II.

Let us emphasize, first of all, that the existence of a meromorphic continuation of a theta function as well as the validity of a corresponding functional equation is a rather singular phenomenon and very strongly depends on the proper choice of the defining data. In particular, in the case of a Riemann surface X all the nice properties of Θ_X are *lost* if one replaces the shifted Laplacian $-\Delta_X - \frac{1}{4}$, for instance, by a different shifted Laplacian $-\Delta_X - c, c \neq \frac{1}{4}$. Thus a prerequisite of any generalization of (2.3) is a deeper understanding of the nature of a *nice* theta function (a theta function *with* a functional equation).

Now our general principle to obtain such theta functions is as follows. *Nice* theta functions are induced by the set of *quantum numbers* of a (possibly twisted) geodesic flow on the sphere bundle $S(X)$ of X. These quantum numbers are determined by the *geometric quantization* of the geodesic flow using the *stable foliation* of $S(X)$ as polarization. A basic result (see [10]) tells us that these quantum numbers *coincide* with the singularities (zeros and poles) of generalized Selberg zeta functions.

In the case of a Riemann surface X the zeros of the Selberg zeta function

$$Z_X(s) \stackrel{\text{def}}{=} \prod_c \prod_{n \geq 0} \left(1 - e^{-(s+n)\ell_c}\right), \quad Re(s) > 1 \tag{3.1}$$

are of the form

$$s = \tfrac{1}{2} \pm i\mu, \quad \mu^2 \in \operatorname{spec}(-\Delta_X - \tfrac{1}{4})$$

and

$$s = -N, \quad N \geq 0, \quad N \in \mathbb{Z}.$$

Here the *same* numbers μ as in the definition (2.1) of Θ_X occur, but there are *additional* zeros of Z_X at the non-positive integers. However, the latter zeros are only those of the Γ–factor $\Gamma_X(s)$ of $Z_X(s)$ (in sense of [10]).

By completing $Z_X(s)$ by the Γ–factor $\Gamma_X(s)$ defined by

$$\Gamma_X(1-s) \stackrel{\text{def}}{=} \Big(\prod_{n \geq 1}{}'(n-s)^{(2n-1)}\Big)^{\chi(X)},$$

where $\prod'$ denotes a zeta–regularized product (see [10]), one obtains a *completed Selberg zeta function*

$$\hat{Z}_X(s) = \Gamma_X(s) Z_X(s) \tag{3.2}$$

with a symmetric functional equation

$$\hat{Z}_X(1-s) = \hat{Z}_X(s) \tag{3.3}$$

and $\hat{Z}_X(s)$ has *no zeros* (and poles) outside the critical strip $Re(s) \in [0,1]$. In fact, the zeros of $\hat{Z}_X$ are all of the form

$$\tfrac{1}{2} \pm i\mu, \quad \mu^2 \in \operatorname{spec}(-\Delta_X - \tfrac{1}{4}), \quad \text{(Riemann hypotheses)},$$

and (up to the shift $\frac{1}{2}$ and the factor i) this set is the symmetrization of the set of complex numbers μ defining Θ_X. Note that there is also a more explicit formula for $\Gamma_X(s)$ in terms of the Barnes' double Γ–function (see [15]).

Now, in general, the idea is to use the singularities of the completed generalized Selberg zeta function $\hat{Z}_\sigma$ (defined below) in a similar way to define the theta function Θ_σ.

All but finitely many quantum numbers of the σ–twisted geodesic flow, used in the definition of Z_σ, can be related to the *spectrum* of principal series representations $\Pi_{\lambda,\sigma}$ in the canonically associated right–regular unitary representation, R_Γ, of the isometry group G of the universal cover Y of X on $L^2(\Gamma/G)$.

It is very important to emphasize that our theta functions Θ_σ are *not* canonically associated to a differential operator (at least in general). As a consequence, the possibly still existing relations between Θ_σ and the spectra of certain differential operators on X have to be regarded as being of a secondary nature, in contrast to the traditional point of view.

Now the next problem in generalizing (2.3) is to find the proper *substitute* for the right hand side of the functional equation. At first glance, (2.3) seems to suggest that it might be natural to generalize the relation between Θ^d, and the Laplacian of the compact dual symmetric space S^2.

In fact, this idea is one of the basic point of views behind Kurokawa's recent results (see [12]) on the Γ–factors of Selberg zeta functions. In the case of the untwisted geodesic flow ($\sigma = 1$) this idea works well (see also [3]), but in the general case it is misleading. In the appendix of [3] one can find a more detailed discussion of the relations between this idea and the point of view of the present paper.

Since the theta function Θ_σ itself is no longer canonically associated to the spectrum of an operator but canonically associated to a twisted geodesic flow (*real* time) on $S(X)$ (via the corresponding twisted Selberg zeta function) it seems to be more natural to look for a *dual* theta function Θ^d_σ in terms of the twisted geodesic flow (*imaging* time) related to the compact dual symmetric space Y^d.

Moreover, since all the *singularities* of the Selberg zeta function Z_σ entering into the definition of Θ_σ have a natural interpretation as *quantum numbers* of the twisted geodesic flow on $S(X)$ it would be very satisfying to have an analogous definition of Θ^d_σ in terms of *quantum numbers* of a twisted geodesic flow on $S(Y^d)$ as well. In fact, this turns out to be the case!

The underlying observation is very simple and can be described as follows. The complexification of the tangent bundle of the stable foliation of $S(X)$ can be regarded canonically as the *holomorphic* tangent bundle of the Kähler manifold

$$Y^d_{\text{geo}} = S(Y^d)/\Phi^d$$

of all orbits of the geodesic flow Φ^d on the sphere bundle $S(Y^d)$ of Y^d. This is a version of the unitary trick. In particular, we have a Φ^d–equivariant twisted Dolbeault complex on $S(Y^d)$ being *transversally elliptic* with respect to the orbits of the geodesic flow Φ^d on $S(Y^d)$. Since *all* geodesics in Y^d are closed and have the *same prime period* this complex is transversally elliptic with respect to the action of a one–dimensional torus. In particular, the proper analog of the set of quantum numbers obtained by geometric quantization of the geodesic flow on $S(X)$ now is the set of characters of the latter torus contributing to the distributional index (in sense of [1]) of the transversally elliptic twisted Dolbeault complex on $S(Y^d)$. These *characters* and their corresponding *indices* enter into the definition of our dual theta function Θ^d_σ.

In the case of a Riemann surface X this yields a new way to look at the theta function Θ^d. In fact, from the new point of view the sequence

$$\left\{n+\tfrac{1}{2}, n \in \mathbb{Z}\right\}$$

is the sequence of those (infinitesimal) characters of the torus $SO(2)$ (acting on the unit sphere bundle $S(S^2)$ of the compact dual symmetric space S^2 by the geodesic flow) that contribute to the distributional index of the $SO(2)$–transversally elliptic complex on $S(S^2) = SO(3)$ that projects under

$$S(S^2) = SO(3) \to S(S^2)/\Phi^d = SO(3)/SO(2) = S^2$$

to the Dolbeault complex associated to the canonical complex structure on S^2, shifted by $\frac{1}{2}$.

Moreover, the multiplicity $2n+1$ coincides with the multiplicity of the corresponding character χ_n in the distributional index of the transversally elliptic complex on $S(S^2)$ (= index of χ_n–twisted Dolbeault–cohomology on $S(S^2)/\Phi^d = S^2$) which by Borel–Weil–Bott theory is given by the dimension of an irreducible $SO(3)$–module of highest weight n.

Finally we have to find the proper meaning and corresponding generalization of the coefficient $(1-g)$ in (2.3). It turns out that in full generality this coefficient *coincides* with the proportionality factor known from the proportionality formulas (Hirzebruch proportionality) relating the index of a locally invariant elliptic differential operator on X to the index of a corresponding invariant differential operator on the compact dual space Y^d. This coefficient is equal to

$$\frac{\chi(X)}{\chi(Y^d)} = (-1)^{\dim(X)/2}\frac{\text{vol}\,(X)}{\text{vol}\,(Y^d)}, \tag{3.4}$$

where canonically compatible volumes are used. Thus the functional equation for a theta function Θ_σ also can be regarded as a proportionality theorem.

4. Definition of the dynamical theta functions in the general case. Formulation of the main results.

Now let us give the precise definitions concerning the theta functions. Let us begin with the definition of the functions Θ_σ.

Let $X = \Gamma\backslash G/K$ be a compact smooth quotient of the non–compact symmetric space $Y = G/K$ by a discrete subgroup Γ of G. Assume that the *rank* of G/K is *one.* Then the sectional curvature of the locally symmetric metric on X is strictly negative. The curvature is negatively constant iff X is the quotient of a real hyperbolic space. Moreover, let us assume that the dimension of X is *even.*

The theta functions depend on parameters in $\hat{M} \times \mathfrak{a}^*$, where $P = MAN$ is a fixed but arbitrary (minimal) parabolic subgroup of G.

We fix a representation $\sigma \in \hat{M}$.

Let $Z_\sigma(\lambda), \lambda \in \mathfrak{a}^*$ be the associated generalized Selberg zeta function. Recall that for $Re(\lambda) > h, \mathfrak{a}_0^* \ni h = 2\rho_0$ = topological entropy of the *abstract* geodesic flow Φ_a on $\Gamma\backslash G/M$, the zeta function Z_σ is defined by the doubly infinite product

$$Z_\sigma(\lambda) \stackrel{\text{def}}{=} \prod_c \prod_{N\geq 0} \det(1-\sigma(m_c)S^N(P_c^s)e^{-\lambda(X_c)}), \tag{4.1}$$

where c runs over the (prime) periodic orbits c of (prime) period $X_c \in \mathfrak{a}_0^+$, and P_c^s is the stable part of the Poincaré mapping of the closed loop c in the unit sphere bundle $S(X)$ (see [10], [7]).

Z_σ extends to a *meromorphic* function on the complex plane $\mathfrak{a}^*$. The extension satisfies the *dynamical functional equation*

$$\Gamma_\sigma(\lambda)Z_\sigma(\lambda) \cong \Gamma_\sigma(h-\lambda)Z_\sigma(h-\lambda) \tag{4.2}$$

(here $\cong$ means equality up to a holomorphic exponential polynomial), where $\Gamma_\sigma(\Lambda)$ is the corresponding Γ–factor (as defined in [10]). The singularities (zeros and poles) of $\Gamma_\sigma(\lambda)$ are contained in the real half line $(-\infty, \frac{h}{2}) = (-\infty, \rho_0)$. Moreover, the zeros and poles of Z_σ in the *critical strip*

$$\tfrac{h}{2} \leq Re(\lambda) \leq h$$

are either on the *critical line*

$$Re(\lambda) = \tfrac{h}{2}$$

(these are zeros being distributed symmetrically with respect to $\lambda = \frac{h}{2}$) or in the interval

$$(\tfrac{h}{2}, h]$$

where, in general, zeros *and* poles occur.

In particular, $\frac{h}{2} + i\lambda_0, \quad \lambda_0 \in \mathfrak{a}_0^*$, is a zero of Z_σ iff the multiplicity $N_\Gamma(\Pi_{\lambda_0,\sigma})$ of the unitary principal representation $\Pi_{\lambda_0,\sigma}$ in the unitary right regular representation, R_Γ, on $L^2(\Gamma\backslash G)$, being defined by

$$(R_\Gamma(g)u)(g') = u(\Gamma g'g), \quad u \in L^2(\Gamma\backslash G), \tag{4.3}$$

doesn't vanish. Moreover, for $\lambda_0 \neq 0$ its multiplicity is given by $N_\Gamma(\Pi_{\lambda_0,\sigma})$.

Now define

$$\Theta_\sigma(X) \stackrel{\text{def}}{=} \sum_{\mu\in D(Z_\sigma)^+} m(\mu)\exp(i\mu(X))\exp(-i\tfrac{h}{2}(X)) \tag{4.4}$$

for $X \in \mathfrak{a}_0^+$, where μ runs over the set $D(Z_\sigma)^+ \subset \mathfrak{a}^*$ of singularities of Z_σ in

$$\left[\tfrac{h}{2}, \tfrac{h}{2} + i\mathfrak{a}_0^+\right) \cup \left(\tfrac{h}{2}, h\right].$$

The multiplicity $m(\mu)$ of μ in (4.3) is given by the corresponding multiplicity of the singularity of Z_σ in $\lambda = \mu$ if $\mu \neq \frac{h}{2} = \rho_0$. Here we use the convention that zeros have *positive* (integral) multiplicity and poles have *negative* (integral) multiplicity.

The multiplicity of the singularity of Z_σ in $\lambda = \frac{h}{2}$ is even and we take half of its multiplicity as the coefficient of the exponential 1 in the definition of Θ_σ.

The infinite sum (4.3) naturally decomposes into an infinite sum running over the zeros of Z_σ on $Re(\lambda) = \frac{h}{2}$ and a finite sum running over the zeros and poles in $(\frac{h}{2}, h]$. Moreover, the multiplicity of $\mu = \frac{h}{2} + i\lambda, \quad \lambda > 0$ (λ being positive on $\mathfrak{a}_0^+$)

coincides with the multiplicity of the unitary principal series representation $\Pi_{\lambda,\sigma}$ in $L^2(\Gamma\backslash G)$. However, for the zeros and poles of Z_σ in $(\frac{h}{2}, h)$ there exist, in general, no analogous explicit description in terms of multiplicities of unitary representations in $L^2(\Gamma\backslash G)$.

According to the theory developed in [10] it is natural to think of the zeta function Z_σ as being associated to the (abstract) σ–twisted geodesic flow $\Phi_{\sigma,a}$ defined by the A–action

$$\Phi_{\sigma,a} : \Gamma\backslash G \times_M V_\sigma \ni (\Gamma g, v) \overset{a}{\mapsto} (\Gamma g a^{-1}, v) \in \Gamma\backslash G \times_M V_\sigma. \tag{4.5}$$

on the locally homogeneous vector bundle

$$\mathcal{V}_\sigma : \Gamma\backslash G \times_M V_\sigma \to \Gamma\backslash G/M$$

over the unit sphere bundle $S(X) \cong \Gamma\backslash G/M$ with typical fibre V_σ = representation space of $\sigma \in \hat{M}$. The σ–twisted geodesic flow is a lift of the usual (abstract) geodesic flow Φ_a on $\Gamma\backslash G/M$.

The analytical properties of Z_σ then are all reflected by a canonically associated cohomology theory on $S(X)$. In particular, all the singularities of Z_σ have a common cohomological meaning in the sense that they are characterized by the condition that a certain analytical Euler characteristic doesn't vanish. Moreover, these Euler characteristics coincide with the corresponding multiplicities of the singularities.

The cohomological nature of Z_σ also is the deeper reason for the validity of the functional equation of Θ_σ.

In fact, the functional equation for the dynamical theta function Θ_σ relates the sum

$$\Theta_\sigma(X) + \Theta_\sigma(-X)$$

to an analogous sum defined by a theta function Θ_σ^d of a similar *cohomological nature* being associated to the σ–twisted geodesic flow of the compact dual symmetric space $Y^d = G^d/K$. It is a special function with a close relation to Harish–Chandra's c–functions and, for shortness, it will be referred to as the theta function *dual* to Θ_σ.

Now let us give the definition of Θ_σ^d. In analogy to the identification of the sphere bundle $S(X)$ with the space $\Gamma\backslash G/M$ the sphere bundle $S(Y^d)$ of the compact dual symmetric space $Y^d = G^d/K$ will be identified (as a G^d–space) with G^d/M. Then the (abstract) geodesic flow Φ^d on G^d/M has the form

$$\Phi^d_{a^d} : G^d/M \ni g^d M \overset{a^d}{\mapsto} g^d(a^d)^{-1}M \in G^d/M. \tag{4.6}$$

Let us note that the compact group M and the 1–torus A^d may have a non–trivial intersection containing at most one non–trivial element (of order 2). The intersection $M \cap A^d$ is trivial if and only if $Y = G/K$ is a real hyperbolic space (of even dimension).

The next step in the process to define Θ_σ^d is to define the *complex structure* and the corresponding twisted Dolbeault complex on $Y^d_{\text{geo}} = G^d/MA^d$. Moreover, it is a crucial step to see the canonical relation of this structure to the hyperbolic structure of the geodesic flow of X on $S(X)$. As a convenient tool to discuss these structures we use again the group–theoretical models

$$\begin{aligned} \Gamma gM &\overset{a}{\mapsto} \Gamma g a^{-1} M \quad (\text{ right } A-\text{action}) \\ g^d M &\overset{a^d}{\mapsto} g^d (a^d)^{-1} M \,(\text{ right } A^d - \text{action}) \end{aligned} \tag{4.7}$$

of the geodesic flows of X and Y^d. In terms of the right A–action on $\Gamma\backslash G/M$ we obtain the following simple description of the hyperbolic (or Anosov) structure of the geodesic flow of X. The (real) tangent bundle $T(\Gamma\backslash G/M)$ of $\Gamma\backslash G/M$ admits a canonical smooth Anosov–deomposition

$$T = T^+ \oplus T^0 \oplus T^- \tag{4.8}$$

into flow–invariant integrable subbundles $T^\pm, T^0$. The geodesic flow (exponentially) expands and contracts tangent vectors in T^+ and T^-, respectively. T^0 is the one–dimensional tangent bundle to the orbits of the flow. Note that the subbundle $T^+ \oplus T^-$ is completely non–integrable in the sens that the commutators of its sections generate T.

The Anosov–decomposition of $T(\Gamma\backslash G/M)$ is locally homogeneous, i.e., it is induced by a corresponding G–invariant decomposition of the tangent bundle $T(G/M)$ of the sphere bundle G/M of $Y = G/K$. It follows that the Anosov–decomposition of $T(G/M)$ also admits a description in Lie–algebraic terms. In fact, the (real) Lie algebra $\mathfrak{g}_0$ of G has a $Ad(MA)$–invariant decomposition

$$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus (\mathfrak{m}_0 \oplus \mathfrak{a}_0) \oplus \mathfrak{n}_0^+ \tag{4.9}$$

into the Lie algebras $\mathfrak{n}_0^\pm$ and $\mathfrak{m}_0 \oplus \mathfrak{a}_0$. Here the (at most) step 2 nilpotent Lie algebras $\mathfrak{n}_0^+$ and $\mathfrak{n}_0^-$ can be constructed as the direct sum of the eigenspaces of $ad(X)$ for $X \in \mathfrak{a}_0^+$ acting on $\mathfrak{g}_0$ for positive and negative eigenvalues, respectively. Moreover, $ad(X)$ centralizes the subalgebra $\mathfrak{m}_0 \oplus \mathfrak{a}_0$ of $\mathfrak{g}_0$. let $N^\pm \subset G$ be the Lie subgroups with Lie algebras $\mathfrak{n}_0^\pm$. Then the tangent space $T^+_{\Gamma eM}(\Gamma\backslash G/M)(T^-_{\Gamma eM}(\Gamma\backslash G/M))$ coincides with the tangent space to the N^+– orbit (N^-– orbit) through ΓeM.

Next we regard the complexification of the decomposition (4.9) as a decomposition of the complexification of the *real* Lie algebra

$$\mathfrak{g}_0^d = k_0 \oplus i p_0,$$

i.e.,

$$(\mathfrak{g}_0^d)_{\mathbb{C}} = \mathfrak{n}^- \oplus (\mathfrak{m} \oplus \mathfrak{a}) \oplus \mathfrak{n}^+, \tag{4.10}$$

where $\mathfrak{n}^\pm = (\mathfrak{n}_0^\pm)_{\mathbb{C}}$ and $\mathfrak{m} = (\mathfrak{m}_0)_{\mathbb{C}}, \mathfrak{a} = (\mathfrak{a}_0)_{\mathbb{C}} = (\mathfrak{a}_0^d)_{\mathbb{C}} \quad (\mathfrak{a}_0^d = i\mathfrak{a}_0)$.

It follows that by regarding the complex Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ as the subspaces

$$T_{\underline{e}}^{(1,0)}(G^d/MA^d) \quad \text{and} \quad T_{\underline{e}}^{(0,1)}(G^d/MA^d)$$

of holomorphic and antiholomorphic tangent vectors (in $\underline{e} = eMA^d$) in the complexified tangent space $T_{\underline{e}}(G^d/MA^d)_{\mathbb{C}}$, respectively, a G^d–invariant complex structure on $Y^d_{\text{geo}} = G^d/MA^d$ is defined.

Moreover, the (infinitesimal) Cartan involution Θ (of $\mathfrak{g}_0$) yields an isomorphism

$$\Theta : \mathfrak{n}_0^+ \to \mathfrak{n}_0^-$$

and it is easy to check that the *involution* on $T(G^d/MA^d)_{\mathbb{C}}$ induced by Θ yields an isomorphism

$$T^{(1,0)}(G^d/MA^d) \to T^{(0,1)}(G^d/MA^d)$$

which is nothing else than complex conjugation In fact, if we write $X \in \mathfrak{g}_0$ in the form

$$\tfrac{1}{2}(X + \Theta X) - \tfrac{i}{2}(iX - i\Theta X) \in k_0 \oplus i(ip_0)$$

then complex conjugation with respect to the real form $\mathfrak{g}_0^d = k_0 \oplus ip_0$ yields

$$\tfrac{1}{2}(X + \Theta X) + \tfrac{i}{2}(iX - i\theta X) = \Theta X.$$

Y^d_{geo} can be regarded also as a symplectic manifold obtained by Marsden–Weinstein–reduction from the canonical symplectic phase space

$$T(Y^d)\backslash 0 \simeq T^*(Y^d)\backslash 0,$$

the isomorphism $\simeq$ being induced by the invariant Riemannian metric. The space Y^d_{geo} is, in fact, a Kähler manifold. Moreover, $T^{1,0}$ and $T^{0,1}$ are two *transversal complex Lagrangian* subbundles of $T(Y^d_{\text{geo}})_{\mathbb{C}}$.

The latter observation corresponds to the fact that the (real) tangent bundle of the stable and the unstable foliations of $S(X)$ are real Lagrangian subbundles of $T^+(S(X)) \oplus T^-(S(X))$.

Now set $T^{(1,0)}(S(Y^d)) = \pi_0^*(T^{(1,0)}(Y^d_{\text{geo}}))$ and $T^{(0,1)}(S(Y^d)) = \pi_0^+(T^{(0,1)}(Y^d_{\text{geo}}))$, where

$$\pi_0 : S(Y^d) \to Y^d_{\text{geo}}$$

is the canonical projection.

Then the A^d–invariant decomposition

$$T_{\mathbb{C}}(S(Y^d)) = T^{(1,0)}(S(Y^d)) \oplus T^0_{\mathbb{C}}(S(Y^d)) \oplus T^{(0,1)}(S(Y^d)), \tag{4.11}$$

where T^0 is the line bundle formed by the tangent vectors to the orbits of the geodesic flow Φ^d of Y^d defines an A^d–invariant complex structure transversally to the foliation by the orbits of the geodesic flow Φ^d. (4.11) serves as the structure *dual* to the hyperbolic structure of the geodesic flow of X.

In addition to the A–invariant hyperbolic structure on $S(X)$ and the corresponding A^d–invariant transversally complex structure on $S(Y^d)$ we need on both phase spaces $S(X)$ and $S(Y^d)$ certain compatible vector bundles. In fact, each representation $\sigma \in \hat{M}$ induces the locally homogeneous vector bundles

$$\mathcal{V}_\sigma : \Gamma\backslash G \times_M V_\sigma \to \Gamma\backslash G/M \tag{4.12}$$

and the homogeneous vector bundle

$$\mathcal{V}_\sigma^d : G^d \times_M V_\sigma \to G^d/M. \tag{4.13}$$

Moreover, the A–action on $\Gamma\backslash/M$ as well as the A^d–action on G^d/M lift to actions on $\mathcal{V}_\sigma$ and $\mathcal{V}_\sigma^d$, respectively.

Next we have to consider the *distributional index* of the A^d–transversally elliptic Dolbeault complex formed by the differential forms of type $(0,*)$ on G^d/M with coefficients in $\mathcal{V}_\sigma^d$. Recall that the A^d–transversal invariant complex structure on $S(Y^d)$ is given by

$$T_{eM}^{(0,1)}(S(Y^d)) \cong \mathfrak{n}^- \text{ and } T_{eM}^{(1,0)}(S(Y^d)) \cong \mathfrak{n}^+.$$

Let $\Omega^{(0,q)}(S(Y^d))$ be the space of smooth sections of the bundle $\Lambda^q((T^{(0,1)})^*(S(Y^d))) \cong G^d \times_M (\Lambda^q(\mathfrak{n}^-)^*)$. Sections in $\Omega^{(0,q)}(S(Y^d))$ will be regarded also as differential forms on $S(Y^d)$ annihilating sections of $T_{\mathbb{C}}^0(S(Y^d)) \oplus T^{(1,0)}(S(Y^d))$.

More generally, let

$$\Omega^{(0,q)}(\mathcal{V}_\sigma^d) = \Omega^{(0,q)}(S(Y^d), \mathcal{V}_\sigma^d) \tag{4.14}$$

be the space of smooth sections of the bundle

$$\Lambda^q((T^{(0,1)})^*(S(Y^d) \otimes \mathcal{V}_\sigma^d) \cong G^d \times_M (\Lambda^q(\mathfrak{n}^-)^* \otimes V_\sigma). \tag{4.15}$$

The spaces $\Omega^{(0,*)}(\mathcal{V}_\sigma^d)$ from an A^d–transversally elliptic complex

$$C^d(\mathfrak{n}^-;\sigma) : 0 \to \Omega^{(0,0)}(\mathcal{V}_\sigma^d) \xrightarrow{\bar\partial} \Omega^{(0,1)}(\mathcal{V}_\sigma^d) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{(0,n-1)}(\mathcal{V}_\sigma^d) \to 0, \tag{4.16}$$

$n = \dim(X) = \dim(Y^d)$.

The compactness of A^d allows us to define the *index*

$$\operatorname{ind}(\mathfrak{n}^-;\sigma) = \operatorname{ind}(C^d(\mathfrak{n}^-;\sigma))$$

of the complex $C^d(\mathfrak{n}^-;\sigma)$ as a distribution in $C^{-\infty}(A^d)$ (see [1]). In the sense of distributions $\operatorname{ind}(\mathfrak{n}^-,\sigma)$ admits a Fourier decomposition

$$\operatorname{ind}(\mathfrak{n}^-;\sigma)(a^d) = \sum_{\chi^d \in \hat{A}^d} \operatorname{ind}(\mathfrak{n}^-;\chi^d,\sigma)\chi^d(a^d) \tag{4.17}$$

with *integral* coefficients

$$\operatorname{ind}(\mathfrak{n}^-;\chi^d,\sigma) \in \mathbb{Z}.$$

The integer $\operatorname{ind}(\mathfrak{n}^-;\chi^d,\sigma)$ is the multiplicity of the A^d–character χ^d in the index $\operatorname{ind}(\mathfrak{n}^-;\sigma)$. On the other hand, characters of A^d with a non–trivial contribution in $\operatorname{ind}(\mathfrak{n}^-;\sigma)$ are only those in the subset $D_\sigma \subset \hat{A}^d$ of characters χ^d which satisfy the condition that the representation

$$\chi^d \otimes \sigma$$

of $A^d \times M$ is the lift of a representation of the product $A^d M$. Moreover, for $\chi^d \in D_\sigma$ the index

$$\mathrm{ind}(\mathfrak{n}^-; \chi^d, \sigma)$$

also can be identified with the index of the Dolbeault complex $C^d(\mathfrak{n}^-; \chi^d, \sigma)$ of differential forms of type $(0, *)$ on Y^d_{geo} with coefficients in the homogeneous vector bundle

$$G^d \times_{MA^d} (\mathbb{C}(\chi^d) \otimes V_\sigma) \to G^d / MA^d = Y^d_{\mathrm{geo}} \tag{4.18}$$

on the Kähler manifold Y^d_{geo}.

Now let $D^+_\sigma \subset D_\sigma$ be the subset of those characters $\chi^d \in D_\sigma \subseteq \hat{A}^d$ satisfying the condition (positivity)

$$\chi > \tfrac{h}{2} = \rho_0 \tag{4.19}$$

on $\mathfrak{a}_0^+$, where $\chi \in \mathfrak{a}_0^*$ is the real–valued linear form on $\mathfrak{a}_0$ induced by χ^d via

$$\exp i\chi(X) = \chi^d(\exp(iX)) \tag{4.20}$$

for $X \in \mathfrak{a}_0$.

Now we define for $X \in \mathfrak{a}_0^+$ the *dual theta function* Θ^d_σ by the absolutely convergent series

$$\Theta^d_\sigma(X) \stackrel{\mathrm{def}}{=} -\sum_{\chi^d \in D^+_\sigma} \mathrm{ind}(\mathfrak{n}^-; \overline{\chi^d}, \sigma) \exp(-\chi(X)) \exp(\tfrac{h}{2}(X)) \tag{4.21}$$

where $\overline{\chi^d}$ is the complex–conjugate character to χ^d. Note that $\overline{\chi^d}$ belongs to D_σ, too.

The first observation is that the function Θ^d_σ, defined for $Re(X) \in \mathfrak{a}_0^+$, extends to a meromorphic function on the complex plane $\mathfrak{a}$. More precisely, we have

Theorem 4.1. *Θ^d_σ admits a meromorphic continuation to $\mathfrak{a}$. The only poles of Θ^d_σ are contained in the set of periods $\in i\mathfrak{a}_0$ of the geodesic flow*

$$\exp(i\mathfrak{a}_0) \times G^d/M \to G^d/M.$$

Moreover, the continuation of Θ^d_σ satisfies the functional equation

$$\Theta^d_\sigma(X) = \Theta^d_\sigma(-X). \tag{4.22}$$

The main technical tool in our proof of Theorem 3.1. is an identity which is also of independent interest.

Proposition 4.1. *(Residue formula). There exists a constant c_0 (independent of σ) such that on the open Weyl–chamber $\mathfrak{a}_0^+$*

$$c_0 \int_{\mathfrak{a}_0^*} e^{-i\lambda(X)} |c_\sigma(\lambda)|^{-2} d\lambda = \sum_{\chi^d \in D_0^+} ind(\mathfrak{n}^-; \overline{\chi^d}, \sigma) \exp(-(\chi - \rho)(X)) \tag{4.23}$$

as an identity of (regular) distributions. Here $c_\sigma(\lambda)$ is Harish–Chandra's c–function for the M–type σ.

The function $|c_\sigma(\lambda)|^{-2}$ is the Plancherel density of the unitary principal series representations $\Pi_{\lambda,\sigma}$ of G which are (parabolically) induced by the product of a (unitary) character of $\mathfrak{a}_0$ and the unitary representation $\sigma \in \hat{M}$. Recall that the function $|c_\sigma(\lambda)|^{-2}$ can be explicity written as the product of a polynonial (depending on σ) and either a hyperbolic tangent or a hyperbolic cotangent (see [13]). Moreover, $|c_\sigma(\lambda)|^{-2}$ is an *even* function in λ.

By the theory of Borel–Weil–Bott the index $\mathrm{ind}(\mathfrak{n}^-;\chi^d,\sigma), \chi^d \in D_\sigma$ either coincides (up to a sign) with the dimension of an irreducible representation of the compact Lie group G^d or vanishes. Moreover, in the former case at most one of the cohomology groups of $C^d(\mathfrak{n}^-;\chi^d,\sigma)$ is non–trivial and the natural action of G^d on it turns it into an irreducible module.

Now we are able to formulate the main results.

Theorem 4.2. *(i) Θ_σ (defined on $\mathfrak{a}_0^+$ by (4.3)) extends to a meromorphic function on the complex plane $\mathfrak{a}$.*

(ii) The poles of Θ_σ are contained in the union of the set

$$0 \cup \pm i Per^+_{\mathbb{R}},$$

where $Per^+_{\mathbb{R}} \subset \mathfrak{a}_0^+$ is the set of (real) periods of the geodesic flow

$$\exp(\mathfrak{a}_0^+) \times \Gamma\backslash G/M \to \Gamma\backslash G/M,$$

and the set

$$iPer^+_I \subset \mathfrak{a}_0^- = -\mathfrak{a}_0^+,$$

where $Per^+_I \subset i\mathfrak{a}_0^+$ is the set of (imaginary) periods of the geodesic flow

$$\exp(i\mathfrak{a}_0^+) \times G^d/M \to G^d/M.$$

(iii) The continued theta functions Θ_σ and Θ^d_σ satisfy the functional equation

$$\begin{aligned} \Theta_\sigma(X) + \Theta_\sigma(-X) &= \frac{\chi(X)}{\chi(Y^d)}(\Theta^d_\sigma(iX) + \Theta^d_\sigma(-iX)) \qquad (4.24) \\ &= 2\frac{\chi(X)}{\chi(Y^d)}\Theta^d_\sigma(iX). \end{aligned}$$

It is an easy exercise to claim that (4.24) is, in fact, a generalization of the functional equation (2.3).

In the untwisted case $\sigma = 1$ the theta functions $\Theta_X = \Theta_1$ and $\Theta_{Y^d} = \Theta^d_1$ have the following equivalent descriptions.

$$\Theta_X(t) = \sum_\mu m(\mu)\exp(-\mu t), \quad t > 0, \qquad (4.25)$$

where $\mu \in \mathbb{C}$ is in

$$[0,\infty) \cup (0, -ic_0]$$

and satisfies

$$\mu^2 + c_0^2 \in \mathrm{spec}(-\Delta_X).$$

Here $c_0 \in \mathbb{R}$ is the infimum of the essential spectrum of the Laplacian $-\Delta_Y$ on the universal cover Y of X and $-\Delta_X$ is the positive Laplacian (on functions) on X. The multiplicity $m(\mu)$ coincides with the dimension of the eigenspace $E(\mu^2+c_0^2)$ of $-\Delta_X$ for the eigenvalue $\mu^2 + c_0^2$. (4.25) is a consequence of the well-known description of the zeros of the spherical Selberg zeta function $Z_X = Z_1$ (see [7]).

Moreover, the spherical dual theta function Θ_{Y^d} is given by

$$\Theta_{Y^d}(t) = \sum_{\mu^d} m(\mu^d) \exp(-\mu^d t), \quad t > 0, \tag{4.26}$$

where the sum runs over all positive $\mu^d \in \mathbb{R}$ such that

$$(\mu^d)^2 - c_0^2 \in \operatorname{spec}(-\Delta_{Y^d})$$

and $m(\mu^d)$ is the dimension of the corresponding eigenspace of $-\Delta_{Y^d} + c_0^2$, $-\Delta_{Y^d}$ being the Laplacian for the invariant metric on Y^d with the opposite signs of the sectional curvatures. This is a consequence of Helgason's characterization of the spherical representations of G^d (i.e., those occuring as irreducible subrepresentations in $L^2(Y^d) = L^2(G^d/K)$) (see [9]) and the Borel–Weil–Bott theory identifying the indices in (4.21) as dimensions of representations. For more details see the appendix of [3].

The functional equation then is

$$\Theta_X(t) + \Theta_X(-t) = \frac{\chi(X)}{\chi(Y^d)}(\Theta_{Y^d}(it) + \Theta_{Y^d}(-it)), \quad t \in \mathbb{C}. \tag{4.27}$$

[3] contains an independent proof of (4.27) resting on the observation that the wave kernels of $(-\Delta_Y - c_0^2)^{\frac{1}{2}}$ and $(-\Delta_{Y^d} + c_0^2)^{\frac{1}{2}}$ are related by some sort of analytic continuation. On the other hand, our proof of theorem 4.2. (even in the special case $\sigma = 1$) rests on the functional equations satisfied by the Selberg zeta functions which, in turn, are consequences of the natural compatibility relations for the characters of (unitary) representations of G on both types (compact and non–compact) of Cartan subgroups of G. In fact, the functional equation (4.27) can be reformulated as a functional equation relating the theta functions which are canonically associated to each type of Cartan subgroups. This will be the starting point of a generalization of the present theory to higher rank spaces in part III.

(4.27) might suggest to look for more general *pairs* of geometric operators defining theta functions that satisfy similar functional equations.

As we already mentioned in section 2 this is not a good perspective. For instance, no analogous duality is known for the theta functions defined by (shifted) Laplacians on differential forms on X and Y^d (see, however, the discussion on Θ_R at the end of the paper).

The coefficients (of the exponentials) in the definition (4.21) of Θ_σ^d are given by the indices (Euler characteristics) of twisted Dolbeault complexes on Y^d_{geo}. In analogy to this the multiplicities of the exponentials $\exp(i\mu - i\frac{h}{2})$ contributing to Θ_σ (see (4.3)) also can be interpreted as Euler characteristics of certain complexes on X_{geo} = space of all geodesics in X. Of course, this only can be done in a formal sense since X_{geo} is *not* a manifold. However, by using the hyperbolic structure of

$S(X)$ and the vector bundle $\mathcal{V}_\sigma$ on $S(X)$ an A–equivariant complex on $S(X)$ can be constructed with the property that a given A–character contributes to the A–equivariant Euler characteristics according to its multiplicity in Θ_σ. This is a basic principle in the cohomological theory of the zeta functions Z_σ (as developed in [10]).

In other words, both sides of the functional equation (4.27) have cohomological interpretations which are, in some sense, dual to each other. Thus (4.27) very much resembles the proportionality formulas in elliptic index theory (Hirzebruch proportionality).

The functional equation (4.27) can be regarded also as a *spectral counterpart* to the dynamical functional equation satisfied by the zeta function Z_σ (see [10]). In fact, one of the forms of the dynamical functional equation of Z_σ (equivalent to (4.2)) relates the quotient

$$Z_\sigma(\lambda)/Z_\sigma(h-\lambda)$$

to a similar quotient

$$\left(Z^d_\sigma(\lambda)/Z^d_\sigma(h-\lambda)\right)^{\chi(X)/\chi(Y^d)},$$

where the *dual zeta function* Z^d_σ is a regularized determinant constructed from the data $(\chi^d, \mathrm{ind}(\mathfrak{n}^-;\chi^d,\sigma)), \chi^d \in D_\sigma$. In later papers we will show that, while for compact higher rank spaces X there is, in general, no theory of Selberg type zeta functions, there still exist many theta functions satisfying functional equations similar to those discussed here!

Note that the functional equation (4.24) is completely determined only by constructions using the hyperbolic structure and the dual complex structure. In particular, notions from harmonic analysis are not needed for its formulations.

Next recall that the Ruelle zeta function

$$Z_R(\lambda) \stackrel{\mathrm{def}}{=} \prod_c (1-e^{-\lambda(X_c)}), \quad \lambda \in \mathfrak{a}^*, \quad Re(\lambda) > h \tag{4.28}$$

of the geodesic flow of $X = \Gamma\backslash G/K$ (the product runs over closed oriented geodesics in X) can be written as a complicated alternating product of generalized Selberg zeta functions. Since, by definition, the divisor of the singularities of the zeta function Z_σ determines the theta function Θ_σ it is natural to expect a functional equation of the form (4.24) for a theta function which is defined by using the divisor of the Ruelle zeta function Z_R. In fact, the resulting functional equation is very beautiful and, again, should be regarded as a spectral counterpart to the dynamical functional equation for Z_R (see [10]).

Let $D(Z_R) \subset \mathfrak{a}^* \simeq \mathbb{C}$ denote the divisor of zeros and poles λ (with multiplicities $m(\lambda)$) of the Ruelle zeta function Z_R. We define

$$D(Z_R)^+ \subset D(Z_R) \tag{4.29}$$

as the subset of all $\lambda \in D(Z_R)$ such that

$$Im(\lambda) > 0.$$

Then

$$D(Z_R) = D(Z_R)^+ \cup (-D(Z_R)^+) \cup D(Z_R)^0,$$

where

$$D(Z_R)^0 = \{\lambda \in D(Z_R), \quad Im(\lambda) = 0\}$$

and $\lambda \in D(Z_R)^+$, $Re(\lambda) > 0$, implies $-\overline{\lambda} \in D(Z_R)^+$, $Re(-\overline{\lambda}) < 0$. Moreover, for $\lambda \in D(Z_R)^+$ with the property $Re(\lambda) > 0$ we have the identity

$$m(-\overline{\lambda}) = -m(\lambda)$$

for the multiplicities. The zeros and poles in $D(Z_R)^+$ are to be found on the union of finitely many vertical lines (parallel to the imaginary axis), also called the *critical lines* of Z_R.

Now define

$$\begin{aligned}\Theta_R(X) &\stackrel{\text{def}}{=} \Big(\sum_{\lambda \in D(Z_R)^+} m(\lambda)\exp(i\lambda(X))\Big) + \Big(\sum_{\substack{\lambda \in D(Z_R)^0 \\ Re(\lambda)>0}} m(\lambda)\exp(i\lambda(X))\Big) \\ &= \Big(\sum_{\substack{\lambda \in D(Z_R)^+ \\ Re(\lambda)>0}} m(\lambda)\exp(i\lambda(X))\Big) - \Big(\sum_{\substack{\lambda \in D(Z_R)^+ \\ Re(\lambda)>0}} m(\lambda)\exp(-i\overline{\lambda}(X))\Big) \\ &\quad + \Big(\sum_{\substack{\lambda \in D(Z_R) \\ Im(\lambda)=0,\ Re(\lambda)>0}} m(\lambda)\exp(i\lambda(X))\Big)\end{aligned} \tag{4.30}$$

for $X \in \mathfrak{a}_0^+$.

Theorem 4.3. *Let the situation be as in Theorem 4.2.*
(i) Θ_R extends to a meromorphic function on the complex plane $\mathfrak{a}$.
(ii) The continued theta function satisfies the functional equation

$$\Theta_R(X) - \Theta_R(-X) = \frac{\chi(X)}{\chi(Y^d)}(\Theta_R^d(iX) - \Theta_R^d(-iX)) \tag{4.31}$$

on $\mathfrak{a}$. Here

$$\Theta_R^d(X) \stackrel{def}{=} \frac{\chi(Y_{geo}^d)}{2}\coth(\mu_0(X)/2), \tag{4.32}$$

where $\mu_0 \in \mathfrak{a}_0^+, \mu_0 > 0$ (on $\mathfrak{a}_0^+$) is characterized by the property

$$\mu_0(X_0^+) = 2\pi$$

for the prime period iX_0^+ of the geodesic flow

$$\exp(i\mathfrak{a}_0^+) \times \ G^d/M \to G^d/M$$

of Y^d.

Since $-\Theta_R^d(-X) = \Theta_R^d(X)$ the functional equation (4.31) also can be written in the form

$$\Theta_R(X) - \Theta_R(-X) = \frac{\chi(X)}{\chi(Y^d)}\chi(Y^d_{\text{geo}}) \ \coth(\mu_0(X)/2). \tag{4.33}$$

Note also that the integer

$$\frac{\chi(X)}{\chi(Y^d)}\chi(Y^d_{\text{geo}})$$

coincides with the multiplicity of the central singularity of Z_R^{-1} in $\lambda = 0$. Moreover, there is an intrinsic formula for this integer only in terms of a curvature tensor associated to the hyperbolic structure of $S(X)$ (see [11]).

The theta function Θ_R^d is canonically associated to the geodesic flow of Y^d since

$$\Theta_R^d(X) = \frac{\chi(Y^d_{\text{geo}})}{2} + \sum_{N=1}^{\infty} \chi(Y^d_{\text{geo}}) \exp(-N\mu_0(X))$$

and the coefficient $\chi(Y^d_{\text{geo}})$ coincides with the indices of the twisted de Rham complexes of differential forms on Y^d_{geo} with values in the line bundles on $Y^d_{\text{geo}} = G^d/MA^d$ which are induced by the characters $\exp(N\mu_0)$, $N \in \mathbb{Z}$, of A^d.

On the other hand, we know from [10] that the multiplicities of all elements in $D(Z_R)$ are given by (analytical) Euler characteristics of certain complexes on $S(X)$ which are, in some sense, dual to the twisted de Rham complexes on Y^d_{geo}.

Whereas the definition of Θ_R (by the divisor of the Ruelle zeta function Z_R) is quite analogous to the definition of the theta function θ_R of Cramér (by the zeros of the Riemann zeta function) Theorem 4.3. shows, in particular, that the corresponding functional equations seriously *differ* from each other.

In the case of a compact Riemann surface X (of genus $g \geq 2$) we obtain the functional equation

$$\Theta_R(t) - \Theta_R(-t) = \chi(X)\coth(\tfrac{it}{2})$$

for Θ_R directly from the functional equation for Θ_X. In fact, we have

$$\Theta_R(t) = e^{\frac{i}{2}t}(\Theta_X(t) + \sum_{0 \leq \mu < \frac{1}{2}} m(i\mu)e^{-i\mu t}) - e^{-\frac{i}{2}t}(\Theta_X(t) - \sum_{0 \leq \mu < \frac{1}{2}} m(-i\mu)e^{i\mu t})$$

and thus we obtain (using $m(i\mu) = m(-i\mu)$)

$$\begin{aligned}
\Theta_R(t) - \Theta_R(-t) &= (\Theta_X(t) + \Theta_X(-t))(e^{\frac{i}{2}t} - e^{-\frac{i}{2}t}) \\
&= 2(1-g)\Theta^d(it)(e^{\frac{i}{2}t} - e^{-\frac{i}{2}t}) \\
&= \chi(X) \ \frac{\cosh(it/2)}{2\sinh^2(it/2)} \ 2\sinh(it/2) \\
&= \chi(X) \ \coth(\tfrac{it}{2}).
\end{aligned}$$

In the case of a compact hyperbolic space $X = \Gamma\backslash H^n_{\mathbb{R}}$ (n even) it is interesting to make the definition of Θ_R more explicit. On X we use the metric of constant negative curvature -1. Then the critical lines of

$$Z_R(\lambda) = \prod_c (1 - e^{-\lambda \ell_c}), \quad \lambda \in \mathbb{C}, \quad Re(\lambda) > n-1,$$

are

$$Re(\lambda) = \tfrac{n-1}{2} - p, \quad p = 0, 1, \ldots, n-1,$$

each containing an infinite set of singularities (zeros or poles) of Z_R.

More precisely, the multiplicity of the singularity of Z_R in

$$\tfrac{n-1}{2} - p + i\mu, \quad \mu \in \mathbb{R}, \quad 0 \le p \le \tfrac{n}{2} - 1$$

is given by

$$(-1)^p \dim_{\mathbb{C}} \; \{\omega \in C^\infty(\Lambda^p(T^*(X))),\ \delta_p\omega = 0, -\Delta_p\omega = (\mu^2 + (\tfrac{n-1}{2} - p)^2)\omega\}$$

if $\mu \neq 0$, whereas the multiplicity of the singularity of Z_R in $\lambda = \frac{n-1}{2} - p$ is twice the corresponding signed dimension. Here $\Delta_p \overset{\text{def}}{=} d_{p-1}\delta_p + \delta_{p+1}d_p$.

λ with $Im(\lambda) > 0$ is a singularity iff $\overline{\lambda}$ is a singularity and the multiplicities of λ and $\overline{\lambda}$ coincide.

λ with $Im(\lambda) > 0$ is a zero (pole) of Z_R iff $-\lambda$ is a pole (zero) and

$$|m(\lambda)| = |m(-\lambda)|.$$

On the real line we have a slightly more complicated behaviour. To define Θ_R we only need a description of the zeros and poles in $(0, n-1]$.

(1) In $\lambda = n-1$ there is a zero of multiplicity 1.

(2) In each open interval $(0, n-1-2p) = (\frac{n-1}{2} - p - (\frac{n-1}{2} - p), \frac{n-1}{2} - p + (\frac{n-1}{2} - p))$ there are finitly many singularities of the form

$$(\tfrac{n-1}{2} - p) \pm \lambda, \quad \lambda \in (0, \tfrac{n-1}{2} - p)$$

of multiplicity

$$(-1)^p \dim_{\mathbb{C}} \{\omega \in C^\infty(\Lambda^p(T^*(X))),\ \delta_p\omega = 0, -\Delta_p\omega = (-\lambda^2 + (\tfrac{n-1}{2} - p)^2)\omega\}.$$

(3) In $\lambda = n - 1 - 2p$, $p = 0, \ldots, \frac{n}{2} - 1$ there is a singularity of multiplicity

$$b_0 - b_1 \pm \cdots + (-1)^p b_p, \quad b_p = b_p(X) = p^{\text{th}} \text{ Betti number}.$$

This description shows *how* the spectra of *all* the operators $-\Delta_p$ on the differential forms on X *contribute* to Θ_R.

It would be interesting to have an analogous description for all other compact rank 1 spaces.

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