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Inverse scattering for periodic structures: Stability of polygonal interfaces

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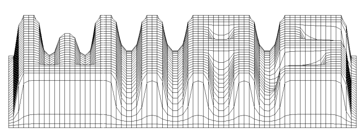

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We consider the two–dimensional TE and TM diffraction problems for a time harmonic plane wave incident on a periodic grating structure. An inverse diffraction problem is to determine the grating profile from measured reflected and transmitted waves away from the structure. We present a new approach to this problem which is based on the material derivative with respect to the variation of the dielectric coefficient. This leads to local stability estimates in the case of interfaces with corner points.

1. Introduction

Consider scattering of electromagnetic waves by a diffraction grating periodic in the x_1 variable and constant in the x_3 variable. More specifically we assume that the three–dimensional space is filled with two different materials having dielectric constants ϵ^+ in the region G^+ above S and ϵ^- in the region G^- below S, where the interface S is 2π–periodic in x_1 direction. The magnetic permeability is assumed to be constant ($\mu = 1$) throughout. Suppose further that a plane wave of the form

$$v_I = \exp(i\alpha x_1 - i\beta x_2)\,, \quad (\alpha, \beta) := \omega(\epsilon^+)^{1/2}(\sin\theta, \cos\theta)$$

is incident to S (from G^+), where ω is the frequency and $\theta \in (-\pi/2, \pi/2)$ the angle of incidence.

Then the total field may be decomposed into a linear combination of two polarizations: transverse electric (TE) polarization where the electric field is transverse to the (x_1, x_2) plane and transverse magnetic (TM) polarization where the magnetic field is transverse to the (x_1, x_2) plane. In either case of polarization, the electromagnetic wave propagation which is governed by the time harmonic Maxwell equations can be determined from a single scalar quantity $v = v(x_1, x_2)$ (the x_3 component of the total electric or magnetic field).

The function v satisfies the Helmholtz equation $(\Delta + k^2)v = 0$ for TE polarization, and Maxwell's equations simplify as $\nabla \cdot (k^{-2}\nabla v) + v = 0$ in the TM case, where $k = k^\pm = \omega(\epsilon^\pm)^{1/2}$ in $G^\pm$. Moreover, v satisfies radiation conditions as $x_2 \to \pm\infty$ and is α quasi–periodic in x_1: $v(x_1 + 2\pi, x_2) = \exp(2\pi\alpha i)v(x_1, x_2)$. For TE polarization v and its normal derivative $\partial_\nu v$ have to cross the interface continuously, whereas in TM polarization $k^{-2}\partial_\nu v$ has to be continuous; cf. the monograph [16] for more details. The corresponding variational formulations of these transmission problems will be presented in Section 2.

An inverse diffraction problem may be formulated as follows: given the incident field, determine the interface S from measured reflected and transmitted fields, say at $x_2 = \pm b$, b large. In applications, it is impossible to make exact measurements. Stability is crucial in the practical reconstruction of profiles since it contains necessary information to determine to what extend the data can be trusted.

In the present paper, we study the local stability of this problem for both the TE and the TM case. Suppose S_h is a small perturbation of the interface S such that the Hausdorff distance $d(S, S_h)$ is of order h as $h \to 0$, and denote by v and v_h the electromagnetic fields of the corresponding scattering problems. We are interested in proving Lipschitz type estimates

$$d(S, S_h) \le C\,(\,|(v - v_h)(\,\cdot\,, b)| + |(v - v_h)(\,\cdot\,, -b)|\,) \tag{1.1}$$

in a suitable norm $|\cdot|$; see Section 3 for a precise formulation.

For smooth surfaces S and S_h such estimates were first obtained by Bao and Friedman; see [2] where also the more general case of two material interfaces has been considered.

Earlier related local stability results are known for the inverse conductivity problem [3], even in the case of only piecewise smooth interfaces [4].

To prove (1.1) for polygonal interfaces, we employ the concept of the material derivative (instead of the usual domain derivative); see Section 4. This approach allows to treat more general perturbations of non–smooth interfaces than those considered in [4] and enables us to handle the rather strong singularities occurring in the solutions of TM diffraction problems at corner points of the grating profile. Sections 5 and 6 are devoted to the proof of the stability estimates.

2. Direct diffraction problems

The TE and TM transmission problems admit variational formulations in a bounded periodic cell in $\mathbf{R}^2$, enforcing implicitly the transmission and radiation conditions (cf. [1], [7], [8]). Introduce two artificial boundaries $\Gamma^\pm = \{x_2 = \pm b\}$ lying above resp. below the grating profile S, and denote by Ω the rectangle $(0,2\pi)\times(-b,b)$. Since we consider solutions v for which $u := \exp(-i\alpha x_1)v$ is 2π–periodic in x_1, the diffraction problems can be transformed to variational problems for u in the set Ω.

In TE polarization u satisfies the equation

$$\Delta_\alpha u + k^2 u = 0$$

where we use the notation

$$\nabla_\alpha = \nabla + i(\alpha, 0)\ , \quad \Delta_\alpha = \nabla_\alpha \cdot \nabla_\alpha = \Delta + 2i\alpha\partial_1 - \alpha^2\,.$$

The radiation conditions are equivalent to the nonlocal boundary conditions

$$\partial_\nu u|_{\Gamma^+} = -T_\alpha^+ u - 2i\beta\exp(-i\beta b)\ , \quad \partial_\nu u|_{\Gamma^-} = -T_\alpha^- u\ ,$$

where $T_\alpha^\pm$ is the periodic pseudodifferential operator (of order 1)

$$(T_\alpha^\pm v)(x_1) := -\sum_{n\in\mathbf{Z}} i\beta_n^\pm \hat{v}_n e^{inx_1},\ \hat{v}_n = (2\pi)^{-1}\int_0^{2\pi} v(x_1)e^{-inx_1}\,dx_1 \tag{2.1}$$

and the coefficients $\beta_n^\pm = \beta_n^\pm(\alpha)$ are defined by

$$\beta_n^\pm(\alpha) := |(k^\pm)^2 - (n+\alpha)^2|^{1/2}\, e^{i\gamma_n^\pm/2}$$

with

$$\gamma_n^\pm = \arg((k^\pm)^2 - (n+\alpha)^2)\,, \quad 0 \le \gamma_n^\pm < 2\pi\,.$$

The operator $T_\alpha^\pm$ is continuous from $H_p^s(\Gamma^\pm)$ to $H_p^{s-1}(\Gamma^\pm)$ where $H_p^s(\Gamma^\pm)$ denotes the trace space of $H_p^{s+1/2}(\Omega)$, the Sobolev space of functions on Ω which are 2π–periodic in x_1. Integration by parts then leads to the variational formulation for the TE diffraction problem:

$$\begin{aligned} B_{TE}(k;u,\varphi) &:= \int_\Omega (\nabla_\alpha u\cdot\overline{\nabla_\alpha\varphi} - k^2\, u\,\bar\varphi) + \int_{\Gamma^+}(T_\alpha^+ u)\,\bar\varphi + \int_{\Gamma^-}(T_\alpha^- u)\,\bar\varphi \\ &= -2i\beta e^{-i\beta b}\int_{\Gamma^+}\bar\varphi\,, \quad \forall\varphi\in H_p^1(\Omega)\,. \end{aligned} \tag{2.2}$$

Analogously, the TM diffraction problem

$$\begin{aligned} \nabla_\alpha \cdot (k^{-2} \nabla_\alpha u) + u &= 0 \quad \text{in} \quad \Omega\,, \\ T_\alpha^+ u + \partial_\nu u &= -2i\beta \exp(-i\beta b) \quad \text{on} \quad \Gamma^+\,, \\ T_\alpha^- u + \partial_\nu u &= 0 \quad \text{on} \quad \Gamma^- \end{aligned}$$

admits the variational formulation

$$\begin{aligned} B_{TM}(k; u, \varphi) :=& \int_\Omega \Big(\frac{1}{k^2} \nabla_\alpha u \cdot \overline{\nabla_\alpha \varphi} - u\,\bar\varphi \Big) + \frac{1}{(k^+)^2} \int_{\Gamma^+} (T_\alpha^+ u)\,\bar\varphi + \frac{1}{(k^-)^2} \int_{\Gamma^-} (T_\alpha^- u)\,\bar\varphi \\ =& -\frac{2i\beta\, e^{-i\beta b}}{(k^+)^2} \int_{\Gamma^+} \bar\varphi\,, \quad \forall\, \varphi \in H_p^1(\Omega)\,. \end{aligned} \tag{2.3}$$

We will assume throughout that the refractive index satisfies

$$k^+ > 0\,, \quad \text{Re } k^- > 0\,, \quad \text{Im } k^- \geq 0\,. \tag{2.4}$$

Then the sesquilinear forms B_{TE} and B_{TM} are strongly elliptic, i.e., after multiplication by some complex number they are coercive modulo compact operators on $H_p^1(\Omega)$. This leads to existence and uniqueness results for the variational equations (2.2) and (2.3); see [1], [6], [7], [8]. In particular, the TE and TM diffraction problems are uniquely solvable for all but a sequence of frequencies ω_j, $\omega_j \to \infty$, and the solution is unique for all frequencies if Im $k^- > 0$.

While the solution to the TE problem is sufficiently smooth ($u \in H_p^2(\Omega)$), the TM solution may have singularities at the corner points of the grating profile. More precisely, near a corner O of S with angle δ, one has $u = r^\lambda f + g$, where r denotes the distance to O, the exponent λ with $0 <$ Re $\lambda < 1$ is the solution with minimal positive real part of the equation

$$\left(\frac{\sin{(\pi - \delta)\lambda}}{\sin \pi\lambda} \right)^2 = \left(\frac{(k^+)^2 + (k^-)^2}{(k^+)^2 - (k^-)^2} \right)^2 \,, \tag{2.5}$$

and f, g are certain smoother functions. Note that Re $\lambda \in (1/2, 1)$ if k^- is real, whereas Re λ may become arbitrarily close to 0 for Im $k^- > 0$. A detailed regularity theory of the TM problem can be found in [8]; see also [11] for the more general case of diffraction by a time harmonic oblique incident plane wave.

3. Stability estimates for the inverse problem

As above, let S be the profile curve dividing the rectangle $\Omega = (0, 2\pi) \times (-b, b)$ into the two subregions $\Omega^\pm$ of refractive index $k^\pm$, and let $f(x_1) = (f_1(x_1), f_2(x_1))$, $0 \leq x_1 \leq 2\pi$, be a parametric representation of S. We shall assume in the following that S is a curved polygon of class $C^{1,1}$, i.e., the derivative of f is Lipschitz continuous with the exception of a finite number of corner points (with angles different from 0 and 2π). Consider a family of perturbed interfaces

$$S_h = \{ f(x_1) + h g(x_1) : 0 \leq x_1 \leq 2\pi \}, \quad 0 < h \leq h_0, \tag{3.1}$$

where the function $g = (g_1, g_2)$ is 2π–periodic, Lipschitz and satisfies the condition

$$(g \cdot \nu)(x_1) \neq 0 \quad \text{a.e. on } [0, 2\pi]. \tag{3.2}$$

Here $\nu = (\nu_1, \nu_2)$ denotes the normal to S pointing from Ω^+ to Ω^-. Clearly, S_h converges to S in the Hausdorff distance d and (3.2) implies that

$$C_1 h \leq d(S, S_h) \leq C_2 h \quad \text{as} \quad h \to 0, \quad C_1, C_2 > 0. \tag{3.3}$$

Set $k_h = k^+$ above S_h, $k_h = k^-$ below S_h, and consider the corresponding perturbed TE and TM problems

$$B_{TE}(k_h; u_h, \varphi) = -2i\beta e^{-i\beta b} \int_{\Gamma^+} \bar{\varphi}\,, \quad \forall \varphi \in H_p^1(\Omega)\,, \tag{3.4}$$

$$B_{TM}(k_h; u_h, \varphi) = -\frac{2i\beta\, e^{-i\beta b}}{(k^+)^2} \int_{\Gamma^+} \bar{\varphi}\,, \quad \forall\, \varphi \in H_p^1(\Omega)\,. \tag{3.5}$$

We will always assume that the original problems (2.2) and (2.3) have a unique solution $u \in H_p^1(\Omega)$. Then, as it was proved in [9], the perturbed problems (3.4) and (3.5) are uniquely solvable in $H_p^1(\Omega)$ for any sufficiently small $h > 0$. This is also a special case of the more general result on conical diffraction in [10].

We are now ready to state our results on the local stability of the inverse diffraction problems.

Theorem 3.1. *Let S be a curved polygon of class $C^{1,1}$. Assume (3.1), (3.2), and suppose that $\nu_2(x) \leq 0$ a.e. on S if k^- is real. Then in the TE case the estimate*

$$d(S, S_h) \leq C \|u - u_h\|_{H_p^{1/2}(\Gamma^+)} \tag{3.6}$$

holds, where C is a constant independent of h.

Theorem 3.2. *The conditions of the preceding theorem imply the stability estimate*

$$d(S, S_h) \leq C\{\|u - u_h\|_{H_p^{1/2}(\Gamma^+)} + \|u - u_h\|_{H_p^{1/2}(\Gamma^-)}\} \tag{3.7}$$

for TM diffraction.

The results indicate that for small h, if the measurements are $O(h)$ close to the true scattered fields in the $H^{1/2}$ norm, then S_h is $O(h)$ close to the true profile in the Hausdorff distance. The stability properties (3.6) and (3.7) will be established in Sections 5 and 6, respectively. It is possible to extend them to the slightly more general case that the interfaces S_h are parameterized by $f + hg_h$, where the Lipschitz functions g_h satisfy

$$|g_h'(x_1)| \leq C \quad \text{a.e.}, \quad g_h(x_1) \to g(x_1) \quad \text{uniformly as} \quad h \to 0$$

and g fulfils condition (3.2). Related results for smooth profiles were obtained in [2]; see also [3], [4] for the inverse conductivity problem.

At present we do not know whether the above theorems hold if (3.2) is replaced by the less restrictive condition $g \cdot \nu \not\equiv 0$, though [2] presents some results along this direction for TM diffraction and smooth interfaces. However, in our opinion, the proof there contains a gap since a solution of the homogeneous Helmholtz equation in $\Omega^\pm$, which has vanishing tangential and normal derivatives on some open subset of S, is (in contrast to Laplace's equation) not necessarily constant on the whole domain $\Omega^\pm$.

4. The material derivative

Let χ be an infinitely smooth cut–off function in $\mathbf{R}^2$, 2π–periodic in x_1, and such that $0 \le \chi \le 1$, $\chi = 1$ in some neighbourhood of the interface S and $\chi = 0$ outside a somewhat larger neighbourhood. With χ and the perturbed interfaces (3.1) we associate a family of Lipschitz diffeomorphisms of the strip $\Pi = \mathbf{R} \times (-b, b)$ onto itself,

$$\Phi_h(x) := x + h\chi(x)g(x_1), \quad 0 < h \le h_0, \tag{4.1}$$

where h_0 is sufficiently small. Note that $\Phi_h(S) = S_h$, $\Phi_h = \mathrm{id}$ outside some neighbourhood of S, and the Jacobian Φ'_h of (4.1) satisfies $\Phi'_h(x) = \mathrm{id} + O(h)$ uniformly in $x \in \mathbf{R}^2$. It is now easy to check that Φ_h is indeed a diffeomorphism of Π onto itself:

Fix y_0 and let x_0 be a solution of the equation $\Phi_h(x) = y_0$, which is equivalent to

$$F_h(x) := x - \Phi'_h(x_0)^{-1}(\Phi_h(x) - y_0) = x. \tag{4.2}$$

Since the Jacobian of F_h,

$$F'_h(x) := \mathrm{id} - \Phi'_h(x_0)^{-1}\Phi'_h(x),$$

satisfies $F'_h(x) = O(h)$ uniformly in $\mathbf{R}^2$, the mapping (4.2) is contracting on each disk of centre x_0 if h is sufficiently small. Hence the mapping Φ_h is globally one–to–one, which finishes the proof.

For a fixed cut–off function χ (and the corresponding diffeomorphism (4.1)), we now define the material derivative u_χ by

$$\lim_{h\to 0} h^{-1}(u_h \circ \Phi_h - u)\,, \tag{4.3}$$

where the limit is understood in the sense of $H^1_p(\Omega)$; compare Lemma 4.1 below. Here u denotes the solution of the TE problem (2.2) resp. the TM problem (2.3) and u_h is the solution of the perturbed problem (3.4) resp. (3.5). To define the limit (4.3) correctly, we have to consider u_h as a function given on the "curved" rectangle $\Omega_h = \Phi_h(\Omega)$ whose lateral boundaries are slightly perturbed segments; note that integration over Ω in the corresponding variational formulations can be replaced by that over Ω_h since Φ_h is 2π–periodic in x_1.

The function u_χ has the advantage that it is in general "less singular" at corner points of the interface than the usual domain derivative $\lim h^{-1}(u_h - u)$, $h \to 0$. The material derivative approach (we refer to [20] for an introduction) has recently successfully been used to derive effective formulas for the derivatives of far field pattern with respect to small perturbations of non–smooth boundaries or interfaces; see [8], [9] for problems in diffractive optics and [5] for some acoustic scattering problems.

The following two lemmas, taken from [9], are crucial for establishing our local stability estimates. They are also special cases of the more general results in [10].

Lemma 4.1. *The solution $u_h \in H^1_p(\Omega_h)$ of the problem (3.4) resp. (3.5) takes the form*

$$u_h \circ \Phi_h = u + hu_\chi + h^2 u_{2,h}, \tag{4.4}$$

where the remainder term satisfies

$$\|u_{2,h}\|_{H^1_p(\Omega)} \le C \quad \textit{for} \quad 0 < h \le h_0.$$

Moreover, in the TE case the material derivative $u_\chi \in H_p^1(\Omega)$ *solves the equation*

$$B_{TE}(k; u_\chi, \varphi) = C_{TE}(\chi; u, \varphi)\,, \quad \forall \varphi \in H_p^1(\Omega)\,, \tag{4.5}$$

where the sesquilinear form on the right–hand side is given by

$$\begin{aligned} C_{TE}(\chi; u, \varphi) &= \int_\Omega k^2(\partial_1(\chi g_1) + \partial_2(\chi g_2))u\,\overline{\varphi} \\ &+ \int_\Omega \{\partial_1(\chi g_1)(\partial_1 u\,\overline{\partial_1\varphi} - \partial_2 u\,\overline{\partial_2\varphi} - \alpha u\overline{\varphi}) + \partial_2(\chi g_2)(\partial_2 u\,\overline{\partial_2\varphi} - \partial_{1,\alpha} u\,\overline{\partial_{1,\alpha}\varphi})\} \\ &+ \int_\Omega \{\partial_1(\chi g_2)(\partial_{1,\alpha} u\,\overline{\partial_2\varphi} + \partial_2 u\,\overline{\partial_{1,\alpha}\varphi}) + \partial_2(\chi g_1)(\partial_1 u\,\overline{\partial_2\varphi} + \partial_2 u\,\overline{\partial_1\varphi})\}. \end{aligned}$$

For TM polarization, u_χ *is the* $H_p^1(\Omega)$ *solution of the problem*

$$B_{TM}(k; u_\chi, \varphi) = C_{TM}(\chi; u, \varphi)\,, \quad \forall \varphi \in H_p^1(\Omega)\,, \tag{4.6}$$

where the sesquilinear form C_{TM} *is defined by*

$$\begin{aligned} C_{TM}(\chi; u, \varphi) &= \int_\Omega (\partial_1(\chi g_1) + \partial_2(\chi g_2))u\,\overline{\varphi} \\ &+ \int_\Omega \Big\{\frac{\partial_1(\chi g_1)}{k^2}(\partial_1 u\,\overline{\partial_1\varphi} - \partial_2 u\,\overline{\partial_2\varphi} - \alpha u\overline{\varphi}) + \frac{\partial_2(\chi g_2)}{k^2}(\partial_2 u\,\overline{\partial_2\varphi} - \partial_{1,\alpha} u\,\overline{\partial_{1,\alpha}\varphi})\Big\} \\ &+ \int_\Omega \Big\{\frac{\partial_1(\chi g_2)}{k^2}(\partial_{1,\alpha} u\,\overline{\partial_2\varphi} + \partial_2 u\,\overline{\partial_{1,\alpha}\varphi}) + \frac{\partial_2(\chi g_1)}{k^2}(\partial_1 u\,\overline{\partial_2\varphi} + \partial_2 u\,\overline{\partial_1\varphi})\Big\}. \end{aligned}$$

Here we have used the notation $\partial_j = \partial/\partial x_j$ and $\partial_{1,\alpha} = \partial_1 + i\alpha$. For the proof of Lemma 4.1, it is in fact not necessary to assume (as in [9], [10]) that Φ_h is a diffeomorphism of Ω onto itself. As we have noticed above, it is sufficient to work with the diffeomorphism $\Phi_h : \Omega \to \Omega_h$.

The second lemma shows that, under additional assumptions on the test functions φ, the domain integrals of C_{TE} and C_{TM} can partly be transformed to integrals over the interface S. This follows by repeated application of Green's formula; see [9]. Let $\tau = (-\nu_2, \nu_1)$ be the tangential vector to S, and introduce the weighted normal and tangential derivatives

$$\partial_{\nu,\alpha} = \nu_1\partial_{1,\alpha} + \nu_2\partial_2\ ,\ \partial_{\tau,\alpha} = -\nu_2\partial_{1,\alpha} + \nu_1\partial_2\ .$$

Furthermore $[v]_S$ stands for the jump $v|_S^+ - v|_S^-$ across S, where $v|_S^\pm$ represents the limit as the interface is approached from the region $\Omega^\pm$.

Lemma 4.2. *For all* $\varphi \in H_p^2(\Omega)$

$$\begin{aligned} C_{TE}(\chi; u, \varphi) = &-\int_\Omega (\chi g_1\partial_1 u + \chi g_2\partial_2 u)\ \overline{(\Delta_\alpha\varphi + \overline{k}^2\varphi)} \\ &+ \int_S (g\cdot\nu)\ [k^2]_S\, u\overline{\varphi}\,, \end{aligned} \tag{4.7}$$

and for all $\varphi \in H_p^1(\Omega)$ *with* $\varphi|_{\Omega^\pm} \in H_p^2(\Omega^\pm)$ *and* $[(1/\overline{k}^2)\partial_{\nu,\alpha}\varphi]_S = 0$*, we have*

$$\begin{aligned} C_{TM}(\chi; u, \varphi) = &-\int_\Omega (\chi g_1\partial_1 u + \chi g_2\partial_2 u)\ \overline{\Big(\nabla_\alpha\cdot\frac{1}{\overline{k}^2}\nabla_\alpha\varphi + \varphi\Big)} \\ &+ \int_S (g\cdot\nu)\ \Big[\frac{1}{k^2}(\partial_{\nu,\alpha} u\ \overline{\partial_{\nu,\alpha}\varphi} - \partial_{\tau,\alpha} u\ \overline{\partial_{\tau,\alpha}\varphi})\Big]_S\,. \end{aligned} \tag{4.8}$$

5. Proof of Theorem 3.1

To verify the stability property (3.6), we shall apply the relations (4.5) and (4.7) for a sequence of cut–off functions χ with support shrinking to the interface S, i.e. $d(S, \operatorname{supp}\chi) \to 0$. The proof is essentially based on the following lemma.

Lemma 5.1. *In the TE case the material derivative u_χ satisfies*

$$\|u_\chi\|_{L^2(\Omega)} \le C \quad as \quad d(S, \operatorname{supp}\chi) \to 0\,, \tag{5.1}$$

where C is a constant independent of χ.

Proof: Let $f \in L^2(\Omega)$ and consider the adjoint TE diffraction problem

$$B_{TE}(k; \varphi, v) = \int_\Omega \varphi \overline{f}\,, \quad \forall \varphi \in H_p^1(\Omega)\,. \tag{5.2}$$

Since (2.2) is assumed to be uniquely solvable, the problem (5.2) has a unique solution $v \in H_p^2(\Omega)$ which satisfies the estimate

$$\|v\|_{H_p^2(\Omega)} \le C\ \|f\|_{L^2(\Omega)}$$

with a constant C independent of f; see [7] or [8]. In particular, equation (5.2) with the right–hand side $f = u_\chi$ has a unique solution v_χ with the bound

$$\|v_\chi\|_{H_p^2(\Omega)} \le C\ \|u_\chi\|_{L^2(\Omega)}\,, \tag{5.3}$$

uniformly in χ. Moreover, by Lemma 4.1 we have the equation

$$B_{TE}(k; u_\chi, v_\chi) = C_{TE}(\chi; u, v_\chi)\ ,$$

and in view of the relation (4.7) (with $\varphi = v_\chi$) the right–hand side can be uniformly bounded as

$$|C_{TE}(\chi; u, v_\chi)| \le C\ \|u\|_{H_p^1(\Omega)}\ \|v_\chi\|_{H_p^2(\Omega)}\,.$$

Together with (5.3), this implies the inequality

$$\int_\Omega |u_\chi|^2 = |B_{TE}(k; u_\chi, v_\chi)| \le C\ \|u\|_{H_p^1(\Omega)}\ \|u_\chi\|_{L^2(\Omega)}\,,$$

which finishes the proof of (5.1). ∎

To prove Theorem 3.1 by contradiction, we assume that estimate (3.6) is not true. Then we have, upon using (3.3),

$$\|h^{-1}(u_h - u)\|_{H_p^{1/2}(\Gamma^+)} \to 0\,, \quad h \to 0\,,$$

and (4.4) gives

$$u_\chi|_{\Gamma^+} = 0\,, \quad \partial_\nu u_\chi|_{\Gamma^+} = -T_\alpha^+ u_\chi = 0$$

for any cut–off function χ. Then u_χ, which solves the homogeneous Helmholtz equation in $\Omega^+ \backslash \operatorname{supp}\chi$, vanishes in that domain by Holmgren's theorem. Furthermore, Lemma 5.1 implies that u_χ converges weakly to some element u_0 in $L^2(\Omega)$ as $d(S, \operatorname{supp}\chi) \to 0$ (more precisely, for some sequence χ_n). Hence $u_0 = 0$ in Ω^+.

Next we show that u_0 coincides with the (unique) solution $u^* \in H_p^1(\Omega)$ of the problem

$$B_{TE}(k; u^*, \psi) = \int_S (g\cdot\nu)\ [k^2]_S\ u\,\overline{\psi}\ , \quad \forall \psi \in H_p^1(\Omega)\,. \tag{5.4}$$

Note that the right–hand side of (5.4) generates a continuous linear functional on $H_p^1(\Omega)$. If ψ is chosen as the solution of the adjoint problem (5.2), we obtain using (4.5)

$$\int_\Omega u_\chi \overline{f} = B_{TE}(k; u_\chi, \psi) = C_{TE}(\chi; u, \psi) \tag{5.5}$$

From (4.7) we see that the right–hand side of (5.5) tends to $\int_S (g\cdot\nu)\,[k^2]_S\, u\,\overline{\psi}$ for $d(S, \operatorname{supp}\chi) \to 0$, whereas the left–hand side converges to $\int_\Omega u_0\,\overline{f}$. Moreover, the left–hand side of (5.4) takes the form $\int_\Omega u^*\,\overline{f}$, and therefore we get

$$\int_\Omega (u_0 - u^*)\,\overline{f} = 0 \quad \text{for any} \quad f \in L^2(\Omega)\,,$$

hence $u^* = u_0$ in Ω and $u^* = 0$ in Ω^+.

Now we observe that $u^*|_{\Omega^-} \in H_p^1(\Omega^-)$ is a solution of the Dirichlet problem

$$\begin{aligned} &\Delta_\alpha v + (k^-)^2 v = 0 \quad \text{in} \quad \Omega^-\,, \\ &v|_S = 0\,, \quad \partial_\nu v|_{\Gamma^-} + T_\alpha^- v = 0\,. \end{aligned} \tag{5.6}$$

Then it follows from Lemma 5.2 below that $u^* = 0$ in Ω^-. Consequently, $u^* = 0$ in Ω and (5.4) implies the relation

$$\int_S (g\cdot\nu)\, u\,\varphi = 0\,, \quad \forall \varphi \in H_p^1(\Omega)\,.$$

Employing Gagliardo's trace lemma and condition (3.2), we then obtain $u = 0$ on S, hence $u = 0$ in Ω by Lemma 5.2, which is a contradiction to (2.2). This finishes the proof of Theorem 3.1.

Remark. It can be proved that the solution u^* of problem (5.4) is just the domain derivative $\lim h^{-1}(u_h - u)$, $h \to 0$, where the limit is understood in the sense of $H_p^1(\Omega)$.

We now present the required uniqueness result for the Dirichlet problem which is well known, at least in the case of smooth boundaries (see [13], [6]). Nevertheless we include a proof since the arguments can be used in Section 6 to establish a uniqueness result for the Neumann problem.

Lemma 5.2. *Let S be a curved polygon of class $C^{1,1}$, and assume that either* $\operatorname{Im}\ k^- > 0$, *or $k^- > 0$ and $\nu_2(x) \le 0$ a.e. on S. Then any solution $v \in H_p^1(\Omega^-)$ of the Dirichlet problem (5.6) vanishes on Ω^-.*

Proof: If $\operatorname{Im}\ k^- > 0$, then a simple partial integration argument yields $\int_{\Omega^-} |v|^2 = 0$, hence $v \equiv 0$; compare Lemma 3.1 in [8]. Let $k^- > 0$. Integrating by parts we obtain

$$\begin{aligned} 0 &= 2\,\mathrm{Re} \int_{\Omega^-} (\Delta_\alpha v + (k^-)^2 v)\, \partial_2 \overline{v} \\ &= \int_{\partial\Omega^-} \left(\partial_{\nu,\alpha} v\ \partial_2 \overline{v} + \partial_{\tau,\alpha} v\ \overline{\partial_{1,\alpha} v} + \nu_2 (k^-)^2 |v|^2\right). \end{aligned} \tag{5.7}$$

Note that the integrals in (5.7) are well defined since v has H^s regularity on the polygonal domain Ω^- for some $s > 3/2$ (see e.g. [12]), which implies that $\nabla v|_S \in L^2(S)$.

On the straight line Γ^-, the integrand on the right–hand side of (5.7) takes the form

$$A := -|\partial_2 v|^2 + |\partial_{1,\alpha} v|^2 - (k^-)^2|v|^2 \,,$$

and we show next that the corresponding integral vanishes. Since v satisfies the relation

$$\int\limits_{\Omega^-} (\nabla_\alpha v \cdot \overline{\nabla_\alpha v} - (k^-)^2\, |v|^2) + \int\limits_{\Gamma^-} (T_\alpha^- v)\, \overline{v} = 0 \,,$$

the integral (cf. (2.1))

$$\int\limits_{\Gamma^-} (T_\alpha^- v)\, \overline{v} = -i \sum_{n\in\mathbf{Z}} \beta_n^-\, |\hat{v}_n|^2$$

is real. Therefore $\hat{v}_n = 0$ if $\beta_n^- = ((k^-)^2 - (n+\alpha)^2)^{1/2} > 0$. Thus for $x_2 \leq -b$ the function v admits the Rayleigh expansion

$$v(x) = \sum a_n \exp(inx_1 - i\beta_n^- x_2)\,, \quad a_n \in \mathbf{C}$$

where the sum is taken over all indices $n \in \mathbf{Z}$ such that $\beta_n^- = i|\beta_n^-|$. This implies

$$\int\limits_{\Gamma^-} A = 2\pi \sum \big(- |\beta_n^-|^2 + (n+\alpha)^2 - (k^-)^2\big)\, |a_n|^2 \, \exp(-2b|\beta_n^-|\,) = 0 \,.$$

From (5.7) we now obtain the equality

$$\int\limits_S (\partial_{\nu,\alpha} v \; \partial_2 \overline{v} + \partial_{\tau,\alpha} v \; \overline{\partial_{1,\alpha} v} + \nu_2\, (k^-)^2\, |v|^2) = 0 \,. \tag{5.8}$$

Using the boundary condition $v|_S = 0$, (5.8) leads to

$$\int\limits_S \partial_{\nu,\alpha} v \; \partial_2 \overline{v} = \int\limits_S (\partial_{\nu,\alpha} v + i\alpha\nu_1 v)\, (\nu_2 \partial_\nu \overline{v} + \nu_1 \partial_\tau \overline{v}) = \int\limits_S \nu_2\, |\partial_\nu v|^2 = 0 \,.$$

Hence $v = \partial_\nu v = 0$ on an open subset of S, which gives the result. ∎

6. Proof of Theorem 3.2

To prove the stability estimate (3.7), we shall proceed as in Section 5. However, because of the strong singularities of the solution to problem (2.3) at interface corners, more effort is needed to derive an analogue of Lemma 5.1. Therefore we begin with some considerations about the Fredholm property of the TM diffraction problem.

Consider the adjoint variational problem

$$B_{TM}(k; \varphi, v) = \int\limits_\Omega \varphi \overline{f} \,, \quad \forall \varphi \in H_p^1(\Omega) \,, \tag{6.1}$$

or equivalently, the transmission problem

$$\Delta_\alpha v + (\overline{k^\pm})^2 v = f \quad \text{in} \quad \Omega^\pm \,,$$

$$\Big[\frac{1}{\overline{k}^2} \partial_{\nu,\alpha} v\Big]_S = 0 \,, \quad \partial_\nu v|_{\Gamma^\pm} + (T_\alpha^\pm)^* v = 0 \,, \tag{6.2}$$

where $(T_\alpha^\pm)^*$ denotes the adjoint of the boundary operators (2.1). Denote by $r = r(x)$ the distance of x to the (finite) set of corner points of the polygonal interface S, and introduce for $\varrho \geq 0$ the weighted L^2 space $Y^\varrho(\Omega)$ with norm

$$\|v\|_{Y^\varrho(\Omega)} = \|r^\varrho v\|_{L^2(\Omega)}$$

and the weighted Sobolev space

$$X^{\varrho}(\Omega) := \{u \in H^1_p(\Omega) : \|r^{\varrho}\partial_1^i\partial_2^j u\|_{L^2(\Omega^\pm)} < \infty\,, \quad i+j=2\}$$

equipped with the canonical norm.

Since (2.3) is uniquely solvable by assumption, the adjoint problem (6.1) (or (6.2)) generates an invertible continuous linear operator $\mathcal{B} : H^1_p(\Omega) \to H^{-1}_p(\Omega)$. Moreover, $\mathcal{B}$ is a bounded operator with trivial kernel from $X^{\varrho}(\Omega)$ into $Y^{\varrho}(\Omega)$, and the next lemma presents a result on its Fredholm property.

Lemma 6.1. *If $\varrho > 0$ is sufficiently small, then $\mathcal{B}$ is an injective Fredholm operator of $X^{\varrho}(\Omega)$ into $Y^{\varrho}(\Omega)$.*

Proof: We can apply well known techniques for elliptic boundary value problems in polygonal domains. Let $f \in Y^{\varrho}(\Omega)$. If U is a subdomain of Ω not containing a corner point of S, then standard elliptic estimates for transmission problems (see e.g. [19], [18]) imply that $v \in H^2(U \cap \Omega^\pm)$ for any solution v of (6.2).

Let O be a corner point of S and let χ_o be a smooth cut–off function with support in a small neighbourhood of O. Then the regularity of $\chi_o v$ can be studied using Kondratiev's method of local Mellin transformation [14]. We refer to [11] for the specific case of TM diffraction, which leads to an eigenvalue problem for a system of ordinary differential equations. The eigenvalues λ of that problem are given by the roots of the transcendental equation (2.5), where δ is the angle at O seen from Ω^+.

In particular, adapting the general approach of [14] or [15] (for weighted Sobolev spaces with nonhomogeneous norms) to our special case, we obtain that $\chi_o v \in X^{\varrho}(\Omega)$ if (2.5) has no root on the "critical line" Re $\lambda = \varrho - 1$ and the right–hand side f of (6.2) satisfies a finite number of solvability conditions on $Y^{\varrho}(\Omega)$. Then, for any sufficiently small $\varrho > 0$, those critical lines can be avoided for each corner point of S, and $\mathcal{B}(X^{\varrho}(\Omega)) \subset Y^{\varrho}(\Omega)$ is a closed subspace of finite codimension which gives the result. ∎

Note that for small ϱ the operator $\mathcal{B}$ does not map $X^{\varrho}(\Omega)$ onto $Y^{\varrho}(\Omega)$, in general. Recall from Section 2 that the solution of (6.2) may have a singularity of order r^λ where Re λ is close to zero. Nevertheless the above result will be sufficient for our purpose; see the proof of Lemma 6.2 below.

To prove Theorem 3.2, we argue by contradiction and suppose that the estimate (3.7) is not valid. Then we obtain using (3.3)

$$\|h^{-1}(u_h - u)\|_{H^{1/2}_p(\Gamma^\pm)} \to 0\,, \quad h \to 0\,, \tag{6.3}$$

which together with (4.4) gives

$$u_\chi|_{\Gamma^\pm} = 0\,, \quad \partial_\nu u_\chi|_{\Gamma^\pm} = -T^\pm_\alpha u_\chi = 0\,.$$

Hence the material derivative u_χ vanishes in $\Omega \backslash \operatorname{supp} \chi$, where χ is an arbitrary cut–off function with support around the interface S. The following lemma is the analogue of Lemma 5.1 and establishes the uniform boundedness of u_χ.

Lemma 6.2. *For sufficiently small $\varrho > 0$ there exists a constant C not depending on χ such that*

$$\int_\Omega r(x)^{-2\varrho}|u_\chi(x)|^2 \le C \quad \textit{as} \quad d(S, \operatorname{supp}\chi) \to 0\,. \tag{6.4}$$

P r o o f : We begin with a discussion of the solvability of equation (6.1), or equivalently $\mathcal{B}v = f$, where $f \in Y^{\varrho}(\Omega)$ and $\varrho > 0$ is sufficiently small. By Lemma 6.1 we have the direct topological sum

$$\mathcal{B}(X^{\varrho}(\Omega)) \oplus N = Y^{\varrho}(\Omega) \tag{6.5}$$

with some finite dimensional space N. Since the linear set $\mathcal{M}$ of all infinitely smooth functions with support in $\Omega \backslash S$ is dense in $Y^{\varrho}(\Omega)$, we can assume that N is the span of certain functions $f_j \in \mathcal{M}$, $j = 1, ..., q = \dim N$; see e.g. [17, Chap.1, Lemma 2.2]. Choosing a corresponding biorthogonal system $\{\psi_j\}$ of continuous linear functionals on $Y^{\varrho}(\Omega)$, we conclude from (6.5) and Lemma 6.1 that the modified equation

$$\mathcal{B}v = f - \sum_{j=1}^{q} \psi_j(f)\, f_j\,, \quad \text{with} \quad \operatorname{supp} f_j \cap S = \emptyset\,, \quad j = 1, ..., q\,,$$

has always a unique solution $v \in X^{\varrho}(\Omega)$, which satisfies the estimate

$$\|v\|_{X^{\varrho}(\Omega)} \leq C \|f\|_{Y^{\varrho}(\Omega)}\,,$$

where C is independent of f.

We now apply the above considerations to the problem (6.1) with right–hand sides

$$f = f_{\chi} := r^{-2\varrho} u_{\chi} - \sum_{j=1}^{q} \psi_j(r^{-2\varrho} u_{\chi})\, f_j \tag{6.6}$$

and obtain unique solutions $v_{\chi} \in X^{\varrho}(\Omega)$ satisfying the uniform bound

$$\|v_{\chi}\|_{X^{\varrho}(\Omega)} \leq C \|r^{-2\varrho} u_{\chi}\|_{Y^{\varrho}(\Omega)} \leq C \|r^{-\varrho} u_{\chi}\|_{L^2(\Omega)}\,. \tag{6.7}$$

Note that $r^{-2\varrho} u_{\chi} \in Y^{\varrho}(\Omega)$ for small ϱ since $u_{\chi} \in H^1_p(\Omega)$, hence $f_{\chi} \in Y^{\varrho}(\Omega)$. On the other hand, by Lemma 4.1 we have

$$\int_{\Omega} u_{\chi}\, \overline{f_{\chi}} = B_{TM}(k; u_{\chi}, v_{\chi}) = C_{TM}(\chi; u, v_{\chi})\,. \tag{6.8}$$

From (4.8) we see that the last term can be uniformly bounded as

$$\begin{aligned} |C_{TM}(\chi; u, v_{\chi})| &\leq C\, (\, \|u\|_{L^2(\Omega)} + \|r^{-\varrho} \nabla u\|_{L^2(\Omega)}\,)\, \|v_{\chi}\|_{X^{\varrho}(\Omega)} \\ &\leq C \|v_{\chi}\|_{X^{\varrho}(\Omega)}\,. \end{aligned} \tag{6.9}$$

Note that the solution u to (2.3) satisfies $r^{-\varrho} \nabla u \in L^2(\Omega)$ for small $\varrho > 0$ (cf. [8, Sec. 3.3]. Then (6.9) is obvious for the domain integral in (4.8). To estimate the interface integral, we make use of the inequalities

$$\|r^{\varrho} \nabla_{\alpha} v_{\chi}|_S^{\pm}\|_{H_p^{1/2}(S)} \leq C\, \|v_{\chi}\|_{X^{\varrho}(\Omega)}\,,$$

$$\|r^{-\varrho} \nabla_{\alpha} u|_S^{\pm}\|_{H_p^{-1/2}(S)} \leq C\, (\, \|u\|_{L^2(\Omega)} + \|r^{-\varrho} \nabla u\|_{L^2(\Omega)}\,)\,.$$

The first estimate is a consequence of the trace theorem for weighted Sobolev spaces (see [15]). The proof of the second bound is analogous to that of the well known version for $\varrho = 0$ (see e.g. [12, Chap. 1]), using Green's formula and the fact that u satisfies homogeneous Helmholtz equations in $\Omega^{\pm}$.

Now we observe that if $d(S, \operatorname{supp}\chi)$ is sufficiently small then $\operatorname{supp} u_\chi \cap \operatorname{supp} f_j = \emptyset$ for all j, so that the left–hand side of (6.8) takes the form $\int\limits_\Omega r^{-2\varrho}|u_\chi|^2$; compare (6.6). Combining this with (6.7) and (6.9) gives the uniform estimate

$$\|r^{-\varrho}u_\chi\|^2_{L^2(\Omega)} = \int\limits_\Omega r^{-2\varrho}|u_\chi|^2 \le C\|r^{-\varrho}u_\chi\|_{L^2(\Omega)}\,. \quad \blacksquare$$

Remark. It can be proved (even without the assumption that u_χ vanishes outside a small neighbourhood of the interface S) that estimate (6.4) holds with $\varrho = 1/2$ if k^- is real; see also [4] where the domain derivative approach was applied to the inverse conductivity problem in polygonal domains. However, in the case $\operatorname{Im} k^- > 0$ where stronger singularities may occur at corners of S, the mentioned property of the material derivative u_χ (following from (6.3)) was crucial for establishing its uniform boundedness.

We now continue the proof of Theorem 3.2. Choose an arbitrary element $\varphi \in H^1_p(\Omega)$ such that $\varphi|_{\Omega^\pm} \in H^2_p(\Omega^\pm)$ and $[(1/\overline{k}^2)\,\partial_{\nu,\alpha}\varphi]_S = 0$. An integration by parts gives

$$B_{TM}(k; u_\chi, \varphi) = -\int\limits_\Omega u_\chi\,\overline{\Big(\nabla_\alpha\cdot\frac{1}{\overline{k}^2}\nabla_\alpha\varphi + \varphi\Big)}$$

since $u_\chi = 0$ in $\Omega\backslash\operatorname{supp}\chi$. Thus from (4.6) and (4.8) we conclude that

$$\begin{aligned}\int\limits_\Omega u_\chi\,\overline{\Big(\nabla_\alpha\cdot\frac{1}{\overline{k}^2}\nabla_\alpha\varphi + \varphi\Big)} &= \int\limits_\Omega (\chi g_1\partial_1 u + \chi g_2\partial_2 u)\,\overline{\Big(\nabla_\alpha\cdot\frac{1}{\overline{k}^2}\nabla_\alpha\varphi + \varphi\Big)} \\ &\quad -\int\limits_S (g\cdot\nu)\,\Big[\frac{1}{k^2}(\partial_{\nu,\alpha}u\,\overline{\partial_{\nu,\alpha}\varphi} - \partial_{\tau,\alpha}u\,\overline{\partial_{\tau,\alpha}\varphi})\Big]_S\,. \end{aligned} \tag{6.10}$$

Furthermore, since u_χ is uniformly bounded in $L^2(\Omega)$ by Lemma 6.2, we observe that u_χ converges weakly to 0 in $L^2(\Omega)$ as $d(S, \operatorname{supp}\chi) \to 0$. Therefore (6.10) implies that

$$\int\limits_S (g\cdot\nu)\,\Big[\frac{1}{k^2}(\partial_{\nu,\alpha}u\,\overline{\partial_{\nu,\alpha}\varphi} - \partial_{\tau,\alpha}u\,\overline{\partial_{\tau,\alpha}\varphi})\Big]_S = 0\,. \tag{6.11}$$

Applying the trace theorem for polygonal domains (see [12, Thm. 1.5.2.1]), for any $\psi \in H^{1/2}_p(S)$ we find a function $\varphi \in H^1_p(\Omega)$ such that $\varphi|_{\Omega^\pm} \in H^2_p(\Omega^\pm)$ and

$$\varphi|^\pm_S = 0\,, \quad \partial_\nu\varphi|^-_S = \overline{\psi}\,, \quad [(1/\overline{k}^2)\,\partial_\nu\varphi]_S = 0\,.$$

Then it follows from (6.11) that

$$\int\limits_S (g\cdot\nu)\,\Big[\frac{1}{k^2}(\partial_{\nu,\alpha}u\,\overline{\partial_{\nu,\alpha}\varphi}\Big]_S = \int\limits_S (g\cdot\nu)\,\Big(\frac{(k^+)^2-(k^-)^2}{(k^-)^4}\Big)\,\partial_{\nu,\alpha}u|^-_S\,\psi = 0$$

for any $\psi \in H^{1/2}_p(S)$, hence $\partial_{\nu,\alpha}u|^-_S = 0$ by condition (3.2). Consequently, $u \in H^1_p(\Omega)$ solves the homogeneous Neumann problem

$$\begin{aligned} \Delta_\alpha u + (k^-)^2 u = 0 \quad \text{in} \quad \Omega^-\,, \\ \partial_{\nu,\alpha}u|_S = 0\,, \quad \partial_\nu u|_{\Gamma^-} + T^-_\alpha u = 0\,. \end{aligned} \tag{6.12}$$

If $\operatorname{Im} k^- > 0$ then integration by parts easily leads to $u \equiv 0$ in Ω^-, hence $u \equiv 0$ in Ω, which is a contradiction proving estimate (3.7) in this case.

Finally, let $k^- > 0$ and let ζ be an arbitrary smooth function on $\overline{\Omega}$. Since $u \in H^s_p(\Omega^\pm)$ for some $s > 3/2$ (see [8, Sec. 3.3]), the relations (4.8), (6.10) and (6.11) may be extended to the case $\varphi = \zeta u$ giving

$$\int_S \sigma\, \partial_{\tau,\alpha} u\, \overline{\partial_{\tau,\alpha}(\zeta u)} = 0\,, \quad \text{with} \quad \sigma := [1/k^2]_S\, g \cdot \nu\,. \tag{6.13}$$

Now we proceed as in the proof of Corollary 3.4 in [3]. For any $\varepsilon > 0$, choose a smooth cut–off function ζ_ε such that

$$\zeta_\varepsilon = 1 \quad \text{on} \quad S \cap \{\sigma > \varepsilon\}\,, \quad \zeta_\varepsilon = 0 \quad \text{on} \quad S \cap \{\sigma \le 0\}\,, \quad |\nabla \zeta_\varepsilon| \le C/\varepsilon \quad \text{on} \quad \overline{\Omega}\,.$$

Taking $\zeta = \zeta_\varepsilon$ in (6.13) we get

$$\int_{S \cap \{\sigma > \varepsilon\}} \sigma\, |\partial_{\tau,\alpha} u|^2 = - \int_{S \cap \{0 < \sigma < \varepsilon\}} (\sigma \zeta_\varepsilon\, |\partial_{\tau,\alpha} u|^2 + \sigma \overline{u}\, \partial_{\tau,\alpha} u\, \partial_\tau \zeta_\varepsilon)\,.$$

Since $\nabla u \in L^2(S)$ and meas $\{0 < \sigma < \varepsilon\} \to 0$, the last integral tends to 0 as $\varepsilon \to 0$. Hence $\partial_{\tau,\alpha} u = 0$ on $S \cap \{\sigma > 0\}$ by condition (3.2), and one verifies analogously that $\partial_{\tau,\alpha} u$ vanishes on $S \cap \{\sigma < 0\}$.

Consequently, u is a solution of the homogeneous Neumann problem (6.12) satisfying the additional condition $\partial_{\tau,\alpha} u|_S = 0$, and in that case an analogue of Lemma 5.2 holds. Indeed, repeating the arguments leading to relation (5.8), we obtain

$$0 = \int_S (\, \partial_{\nu,\alpha} u\, \partial_2 \overline{u} + \partial_{\tau,\alpha} u\, \overline{\partial_{1,\alpha} u} + \nu_2\, (k^-)^2\, |u|^2\,) = \int_S \nu_2\, (k^-)^2\, |u|^2\,.$$

Hence $u = \partial_\nu u = 0$ on an open subset of S, which gives $u = 0$ in Ω. This contradiction finishes the proof of Theorem 3.2.

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