STRONG CLUMPING OF SUPER-BROWNIAN MOTION IN A STABLE CATALYTIC MEDIUM

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A typical feature of the long time behaviour of continuous super-Brownian motion in a stable catalytic medium is the development of large mass clumps or clusters at spatially rare sites. We describe this phenomenon by means of a functional limit law under renormalisation. The limiting process is a Poisson point field of mass clumps with no spatial motion component and with infinite variance. The mass of each cluster evolves independently according to a continuous process trapped at mass zero, which we describe explicitly by means of a Brownian snake construction in a random medium. We also determine the survival probability and asymptotic size of the clumps.

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1. Introduction

1.1. **Motivation.** Models of particle movement and branching in random media have been widely studied in the last twenty years. A class which received particular interest are models of measure-valued processes where, heuristically speaking, the individual branching rates of the moving particles depend on the amount of contact between the particle, called the *reactant*, and a singular random medium, called the *catalyst*. In dimension one even very thin catalysts, for example point catalysts, can be considered. A particularly natural choice of a catalytic medium are *stable* random measures Γ on $\mathbb{R}$ of index $0 < \gamma < 1$, which are the prototypes of a singular catalyst with infinite asymptotic density, see formula (3) below, and [6, Subsections 1.3–1.4] for further motivation for this choice of catalytic medium. A rather general one-dimensional model combining super-stable motions of the reactant particles with possibly moving random catalysts, covering the case of the stable medium Γ was developed in [6, 7]. For an up-to-date introduction to catalytic super-Brownian motion, we refer to [9].

Recent research on super-Brownian motion with a stable catalytic medium lead to several interesting results; we restrict our attention to the case of a Brownian moving reactant, which branches with finite variance in the presence of a non-moving stable catalyst in $\mathbb{R}$. In this case, starting from a finite initial mass, the compact support property was shown in [11], and finite time extinction in [10], see [16] for a quick route. Already in [6], in the case of an infinite initial measure, the long-term clumping behaviour of the reactant was exposed in a mass-time-space rescaling limit theorem. It states that at a fixed macroscopic time t the suitably mass-space-rescaled clumps form a random measure with *independent increments*, see [6, Theorem 1.9.4]. But it could not be settled, see [6, p.251], whether or not the clumps are macroscopically spatially isolated, that is whether the limiting measure is carried by a Poisson point field on $\mathbb{R}$ as known in the constant medium case [5].

The main motivation for the present paper was to attack this problem. We show that in fact the clumps *are* isolated, that is, the limiting measure is a homogeneous Poisson point field of mass clumps, Theorem 1 (ii) below. This is achieved by a refinement of a method of good and bad historical reactant paths, which was developed in [10] and goes back to [12, 15, 22].

Beyond this problem, we describe the mass of the rescaled clumps as a process in macroscopic time. For this purpose we provide a *functional* limit approach, Theorem 1 (i) below, which shows convergence of the rescaled processes on a path space of *continuous* measure-valued processes. The time evolution of these masses is described in terms of exit measures of a Brownian snake in a random medium with a motion process featuring the inverse of the collision local times of the reactant paths with the medium, see Theorem 5 (ii) below. Whereas the clumps of the original process have finite variance given the medium, this property is lost in the limit, a remarkable property conjectured in [6, p.253]. In fact, the clump sizes of the limit have probability tails of index $1 + \gamma < 2$, see Theorem 10 (iii) below. This is in contrast to the constant medium case studied in [5] and due to the fact that the stable catalyst does not have locally finite expectations, so that the atomic catalyst sizes are highly fluctuating. We also determine the (macroscopic) survival probability of clumps, see Theorem 10 (ii).

A main tool for the functional limit theorem is the representation of both the catalytic super-Brownian motion and the limit process in terms of exit measures of a *Brownian snake in the stable medium* Γ. The use of exit measures and subordination for the historical particles to describe general branching mechanisms goes back to [1], though the present paper seems to be the first instance where this approach is used to deal with the case of catalytic, space-dependent, branching.

Revealing the even macroscopically isolated nature of the clumps embeds the present investigation in the realm of the concept of *intermittency.* Roughly speaking, intermittency means in our context that in the long time the catalytic superprocess exhibits a spatially extremely irregular structure consisting of islands of high mass peaks, which are located at great distance from each other. See for instance [17, 18] or [21] for other work in this direction.

1.2. **Statement of the main results.**

1.2.1. *Super-Brownian motion in a stable catalytic medium: preliminaries.* Let $\mathcal{M}(\mathbb{R})$ denote the space of all locally finite measures on $\mathbb{R}$, equipped with the vague topology generated by the mappings $\varphi \mapsto \langle \mu, \varphi \rangle$, for all $\varphi : \mathbb{R} \to [0, \infty)$ continuous with compact support. Here and throughout the paper we use both notations $\langle \mu, \varphi \rangle$ and $\int_{\mathbb{R}} \varphi \, d\mu$ to denote integrals. There is a sequence $\{\varphi_n : n \in \mathbb{N}\}$ of such functions such that

$$d(\mu, \nu) := \sum_{n=1}^{\infty} 2^{-n} \Big(\big| \langle \mu, \varphi_n \rangle - \langle \nu, \varphi_n \rangle \big| \wedge 1 \Big), \text{ for } \mu, \nu \in \mathcal{M}(\mathbb{R}), \tag{1}$$

defines a metric, which makes $\mathcal{M}(\mathbb{R})$ Polish.

Define Φ to be the set of all continuous functions $\varphi : \mathbb{R} \to [0, \infty)$ such that there are constants $a, b > 0$ with $\varphi(x) \le a \exp(-bx^2)$ for all $x \in \mathbb{R}$. For all measure-valued processes in this paper we choose the state space to be the space of tempered measures

$$\mathcal{M}_{\text{tem}} := \mathcal{M}_{\text{tem}}(\mathbb{R}) := \Big\{ \mu \in \mathcal{M}(\mathbb{R}) : \ \langle \mu, \varphi \rangle < \infty \ \text{ for all } \ \varphi \in \Phi \Big\}. \tag{2}$$

Note that in particular the Lebesgue measure ℓ belongs to $\mathcal{M}_{\text{tem}}$. We let $\mathcal{M}_{\text{tem}} \subseteq \mathcal{M}(\mathbb{R})$ inherit the vague topology of $\mathcal{M}(\mathbb{R})$.

Suppose that Γ is a *stable random measure* on $\mathbb{R}$ of index $0 < \gamma < 1$, i.e. for every measurable $\varphi : \mathbb{R} \to [0, \infty)$ we have

$$\mathsf{E}\big\{ \exp\langle \Gamma, -\varphi \rangle \big\} = \exp\Big(- \int_{\mathbb{R}} \varphi(x)^{\gamma} \, dx \Big). \tag{3}$$

Almost surely, Γ belongs to $\mathcal{M}_{\text{tem}}$. This follows from the fact that the integral on the right hand side of (3) is always finite for $\varphi \in \Phi$. Moreover, Γ is almost surely a purely atomic measure with atoms densely located in $\mathbb{R}$.

The measure-valued processes under consideration may be considered as random variables with values in the space $C((0, \infty), \mathcal{M}_{\text{tem}})$ of continuous functions $\nu : (0, \infty) \to \mathcal{M}_{\text{tem}}$, where for topological reasons it is sometimes convenient to exclude the time $t = 0$. We endow this space with the topology of *uniform convergence on compact intervals*, which is induced by the metric

$$\mathbf{d}(\mu, \nu) := \sum_{n=1}^{\infty} 2^{-n} \sup_{1/n \le t \le n} d\big(\mu(t), \nu(t)\big), \text{ for } \mu, \nu \in C\big((0, \infty), \mathcal{M}_{\text{tem}}\big), \tag{4}$$

and is easily seen to be Polish.

Let $X := X[\Gamma] := \{X_t : t \ge 0\}$ denote the continuous super-Brownian motion in $\mathbb{R}$ in the catalytic random medium Γ. Throughout the paper we refer to probabilities and expectations with respect to the random medium Γ with letters P and E and to the probabilities and expectations of the process with given medium Γ by $\mathbb{P}^{\Gamma}$ and $\mathbb{E}^{\Gamma}$, sometimes with a subscript indicating the starting measure. With this convention, for *given* Γ, the process $X = X[\Gamma]$ is the continuous, time-homogeneous Markov process with Laplace transition functionals

$$\mathbb{E}^{\Gamma} \big\{ \exp\langle X_t, -\varphi \rangle \, \big| \, X_s = \mu \big\} = \exp\langle \mu, -V^{\Gamma}_{t-s} \varphi \rangle, \ \text{ for } t > s \ge 0, \tag{5}$$

where $\mu \in \mathcal{M}_{\text{tem}}$, $\varphi \in \Phi$, and $V^\Gamma \varphi = \{V_t^\Gamma \varphi(x) : t \geq 0,\, x \in \mathbb{R}\}$ is the unique nonnegative solution of the equation

$$V_t^\Gamma \varphi(y) = S_t \varphi(y) - 2 \int_0^t ds \int_{\mathbb{R}} p_s(x-y) \left[V_{t-s}^\Gamma \varphi(x)\right]^2 \Gamma(dx), \quad \text{for } t \geq 0,\, y \in \mathbb{R}. \tag{6}$$

Here p denotes the standard heat kernel in $\mathbb{R}$, and $S = \{S_t : t \geq 0\}$ the heat flow semigroup defined by $S_t \varphi(y) = \int_{\mathbb{R}} p_t(x-y)\varphi(x)\, dx$. The nonlinear semigroup $V^\Gamma = \{V_t^\Gamma : t \geq 0\}$ operates in Φ. The interpretation of the process X as a process whose reactant particles branch at site $x \in \mathbb{R}$ with rate $2\,\Gamma(dx)$ corresponds to the fact that, loosely speaking, given Γ, the function $v = V^\Gamma \varphi$ solves the symbolic partial differential equation

$$\frac{\partial}{\partial t} v = \frac{1}{2} \frac{\partial^2}{\partial x^2} v - 2\,\Gamma v^2, \quad \text{with initial condition } v\big|_{t=0} = \varphi. \tag{7}$$

Existence and uniqueness of nonnegative solutions V^Γ of (6) were established in [7], X was constructed as a Markov process in [6, Section 2], and its continuity follows from [8, Corollary 2 (p.257), Proposition 12 (p.230), and Theorem 1(b) (p.235)], even in a stronger topology. Note also that in the case of a finite starting measure the total mass process $\|X\| := \{\|X_t\| : t \geq 0\}$ is a continuous martingale [8, Proposition 3, p.236].

We stress the fact that we always use a *quenched* approach in dealing with the model $X = X[\Gamma]$ in the random medium Γ: First the catalyst Γ is sampled, and then the reactant process $X[\Gamma]$ is run, given the catalyst Γ. In particular, the law $\mathbb{P}^\Gamma$ of the reactant is random, and the randomness is inherited from the law P of Γ.

1.2.2. *Strong clumping of catalytic super-Brownian motion.* Define the *scaling index*

$$\eta := (\gamma + 1)/(2\gamma) \tag{8}$$

and observe that this number is larger than 1. For every $k > 0$ we introduce the renormalised measure-valued process $X^k = X^k[\Gamma] = \{X_t^k : t \geq 0\}$ by

$$X_t^k(B) := k^{-\eta} X_{kt}(k^\eta B), \text{ for } B \subseteq \mathbb{R} \text{ Borel},\ t \geq 0. \tag{9}$$

The next theorem summarises the main results obtained before the present paper.

Theorem 0 (Results of [6, Theorem 1.9.4]).

(i) Convergence. *Starting $X = X[\Gamma]$ in the Lebesgue measure $X_0 = \ell$, for every* fixed t *there is a random variable X_t^∞ defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P}_\ell)$, which is independent of the medium Γ, such that, in P–probability, the following weak convergence of probability measures on $\mathcal{M}_{\text{tem}}$ holds:*

$$\lim_{k \uparrow \infty} \mathbb{P}_\ell^\Gamma\{X_t^k \in \bullet\} = \mathbf{P}_\ell\{X_t^\infty \in \bullet\}. \tag{10}$$

(ii) Characterization of the limit. *For every bounded, measurable function $\varphi : \mathbb{R} \to [0, \infty)$ let*

$$U^\Gamma \varphi = \{U_r^\Gamma \varphi(x) : r \geq 0,\ x \in \mathbb{R}\} \tag{11}$$

be the nonnegative solution of the equation

$$U_r^\Gamma \varphi(y) = S_r \varphi(y) - 2 \int_0^r ds \int_{\mathbb{R}} p_s(x-y) \left[U_{r-s}^\Gamma \varphi(x)\right]^2 \Gamma(dx), \text{ for } r \geq 0,\, y \in \mathbb{R}, \tag{12}$$

which is constructed in [7, Theorem 2.14]. *Then the Laplace functional of X_t^∞ satisfies*

$$\mathbf{E}_\ell\{\exp(-\theta X_t^\infty(A))\} = \exp(-\ell(A)\, \mathsf{E} U_t^\Gamma \theta(0)), \text{ for } A \subseteq \mathbb{R} \text{ Borel},\ \theta \geq 0. \tag{13}$$

(iii) Properties of the limit. $\mathbf{P}_\ell$*–almost surely,* X_t^∞ *is non-degenerate and has independent, stationary increments. The scaling procedure is* persistent *in the sense that* $\mathbf{E}_\ell X_t^\infty = \ell$.

The important feature of this result is that the non-degenerate limit is obtained by a different, stronger scaling as in the classical case of a constant medium [5], hence the qualitative nature of the clumping behaviour is different.

Crucial questions about the spatial structure of the limit measures X_t^∞ were left open in [6]. A question of particular interest in this realm was posed in [6, p.251]: the problem is whether or not the X_t^∞ are compound Poisson point fields on $\mathbb{R}$, i.e. whether on the macroscopic level the clumps are spatially separated. Our first main result answers this question in the affirmative. Our second aim in this paper is to give a full description of the spatial and temporal evolution of the field of clumps at a macroscopic level. This requires, as a first step, a functional limit theorem. This question was not investigated in [6] and is particularly interesting, as the limit process turns out to be non-Markovian and continuous. Here is the precise statement:

Theorem 1 (Main result)**.** *Let* X *be the continuous super-Brownian motion in the stable random medium* Γ *started with* $X_0 = \ell$, *and* $X^k = X^k[\Gamma] = \{X_t^k : t > 0\}$, *for* $k > 0$, *the renormalised processes defined in* (9).

(i) Functional limit theorem. *In* P*–probability, the random laws of the renormalised processes* $X^k[\Gamma]$ *converge weakly on the function space* $C((0,\infty), \mathcal{M}_{\text{tem}})$ *as* $k \uparrow \infty$ *to the law of a limit process* $X^\infty = \{X_t^\infty : t > 0\}$ *defined on some probability space* $(\Omega, \mathcal{A}, \mathbf{P}_\ell)$, *which is independent of the sampled medium* Γ, *and is started in* $X_0^\infty = \ell$.

(ii) Compound Poisson structure. *Let* X^∞ *be the limit process of part* (i)*. Then, for each time* $t > 0$, *the state* X_t^∞ *of the limit process is a compound Poisson point field, i.e. a random discrete measure on* $\mathbb{R}$ *with atoms located in the points of a Poisson point field and with independent identically distributed atomic weights. The temporal development of* X^∞ *is as follows: Almost surely, the atoms do not move in space, no new atoms are born, but each atom dies in finite time.*

Remark 2 (The role of $t = 0$)**.** If $Z = \{Z_t : t > 0\}$ is a random variable in $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$, and Z_0 a random variable in $\mathcal{M}_{\text{tem}}$, we say that Z is *started in* Z_0, provided that Z_t converges to Z_0 in law, as $t \downarrow 0$. In the functional limit theorem we get convergence of the processes X^k started in $X_0^k = \ell$ (to X^∞ started in $X_0^\infty = \ell$). This requirement can be relaxed slightly but not completely omitted (see Remark 15 below). Nevertheless it is possible to construct the process X^∞ canonically for *any* starting measure $X_0^\infty = \mu \in \mathcal{M}_{\text{tem}}$, see Theorem 5 or Corollary 9 below, leaving *open* the question of sample path continuity of X^∞ at time $t = 0$. Moreover, part (ii) of the theorem holds for the process X^∞ started in any measure $X_0^\infty = \mu \in \mathcal{M}_{\text{tem}}$. ◇

Remark 3 (Continuous limit process)**.** Note that convergence in our functional limit theorem holds in the strong sense of convergence of laws on the space $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$ of continuous $\mathcal{M}_{\text{tem}}(\mathbb{R})$–valued paths. In particular, the limit process X^∞ is *continuous* on $(0,\infty)$ as well. ◇

1.2.3. *The crossing property.* An interesting path property of the unscaled process X, which enters in the proof of the functional limit theorem and may be of independent interest, is the following *crossing property*, which is reminiscent of the compact support property investigated in [11]. For the precise formulation, denote by $\ell_{(a,b)}$ for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ the restriction of Lebesgue measure ℓ on $\mathbb{R}$ to the open interval (a, b). We show that for the catalytic super-Brownian motion X started with $\ell_{(0,\infty)}$ the amount of total mass at a time which has travelled across the origin to the nonpositive halfline is bounded in time.

Theorem 4 (Crossing property). *Suppose that $\{X_t : t \geq 0\}$ is the catalytic super-Brownian motion in the stable medium Γ and $X_0 = \ell_{(0,\infty)}$. Then, for P–almost every Γ,*

$$\sup_{t\geq 0} X_t\left((-\infty,0]\right) < \infty, \quad \mathbb{P}^\Gamma\text{–almost surely.} \tag{14}$$

1.2.4. *Snake representations of X and X^∞.* As a further major tool in the proof of Theorem 1 we construct representations of both the original process X and the limiting process X^∞ in terms of exit measures of a Brownian snake in the stable medium Γ. This makes the limit process explicit and permits a comparison of the two processes X and X^∞. As this is also of independent interest, we present the results here.

The idea of using the path-valued process or Brownian snake to represent classical super-Brownian motion is due to Le Gall and has since been generalised to various other types of superprocesses. Bertoin, Le Gall and Le Jan have extended this technique to represent super-processes with more general, but not space-dependent, branching mechanisms. Roughly speaking, they use individual time-changes for each particle, which allow to pass from one branching mechanism to a different one by subordination on the particle level. References are [19, 20] for the first explicit snake construction and [1] for the extension.

In the present paper we extend this idea to our particular case of a *space-dependent* branching mechanism — recall that in rough terms the branching rate at site x is given by $2\,\Gamma(dx)$. To formulate the result we briefly introduce the basic notation of the Brownian snake $w = w[\Gamma]$ in our random medium case, and its excursion measures $\mathbb{N}_x^\Gamma$, both in the quenched situation of a fixed sample Γ of the random medium. More details can be found in Subsection 2.1 below.

To describe our approach, let us first look at a generic reactant particle, which moves along a Brownian path $W = \{W(t) : t \geq 0\}$ in $\mathbb{R}$ until its death. Of course, the motion process could as well be described by the two-dimensional Markov process $t \mapsto \big(t, W(t)\big)$ with phase space $D := [0,\infty) \times \mathbb{R}$. The branching of the reactant particle, however, is governed by its *collision local time* $L_{[\Gamma,W]}$ with the medium Γ, defined by

$$L_{[\Gamma,W]}(r) = \int_{\mathbb{R}} \Gamma(dy) L^y(r), \quad \text{for } r \geq 0, \tag{15}$$

where $r \mapsto L^y(r)$ is the continuous local time of W at level $y \in \mathbb{R}$. $L_{[\Gamma,W]}$ is an nondecreasing continuous additive functional of Brownian motion W. As the positions of the atoms of Γ are dense in $\mathbb{R}$, it is easy to see that $L_{[\Gamma,W]}$ is (strictly) increasing.

We use the continuous inverse function $L^{-1}_{[\Gamma,W]}$ of $L_{[\Gamma,W]}$ to introduce a new time scale for the reactant particle on which its collision local time grows linearly. More precisely, instead of $t \mapsto \big(t, W(t)\big)$, we define a time-homogeneous continuous Markov process $\xi := \{\xi_r : r \geq 0\}$ with values in $D = [0,\infty) \times \mathbb{R}$ by

$$\xi_r := \left(L^{-1}_{[\Gamma,W]}(r),\, W \circ L^{-1}_{[\Gamma,W]}(r)\right), \quad \text{for } r \geq 0, \tag{16}$$

where W is a Brownian motion started in x. The first component of this process can be interpreted as the new individual clock of the Brownian reactant particle, travelling in the medium Γ, and the second component as its position along the new time scale. For all $t > 0$, define the *first exit time*

$$\tau_t := \tau_t(\xi) := \inf\big\{r > 0 : \xi_r \notin [0,t) \times \mathbb{R}\big\} \tag{17}$$

of the path ξ from the domain $D^t := [0,t) \times \mathbb{R}$. At time τ_t the process ξ is in the state $\big(t, W(t)\big)$, and the reactant particle has accumulated the collision local time

$$L_{[\Gamma,W]}(t) = \tau_t \tag{18}$$

and is placed in $W(t)$. Less formally, a single generic reactant particle of X_t may be represented by a path ξ stopped at the random time τ_t.

The Brownian snake can be interpreted as a natural parametrisation of the collection of all reactant particles in the range of X, where each particle is represented by a stopped path. For this purpose, define the *set of stopped paths* by

$$\mathfrak{P} := \Big\{ f \in C\big([0,\infty), D\big) : \text{ there exists } \zeta \geq 0 \text{ with } f(r) = f(\zeta) \text{ for all } r \geq \zeta \Big\}. \tag{19}$$

With every $f \in \mathfrak{P}$ we can associate the *lifetime* $\zeta = \zeta(f)$, which is the minimal $\zeta \geq 0$ such that the path f is constant on $[\zeta, \infty)$. We equip $\mathfrak{P}$ with the following metric: for $f, f' \in \mathfrak{P}$ let ζ, ζ' the associated lifetimes and

$$\mathfrak{d}(f, f') := \big|f(0) - f'(0)\big| + |\zeta - \zeta'| + \int_0^{\zeta \wedge \zeta'} \Big(\sup_{x \in [0,u]} \big|f(x) - f'(x)\big| \wedge 1 \Big)\, du. \tag{20}$$

The *Brownian snake* w with start in $(0,x) \in D$ and motion process ξ is a certain continuous strong Markov process $w : [0,\infty) \to \mathfrak{P}$ whose state space is the set of all stopped paths $f \in \mathfrak{P}$ with $f(0) = (0,x)$. Brownian snakes with general Markov processes as motion process were constructed in [1].

With every $\mathfrak{P}$-valued Markov process we can associate the *lifetime process* $\zeta : [0,\infty) \to [0,\infty)$ defined by $\zeta_s := \zeta(w_s)$. For the Brownian snake w, the lifetime process ζ is by definition a reflected Brownian motion. Moreover, given ζ, two paths w_{s_1} and w_{s_2}, $s_1 < s_2$, agree up to time $m := \min_{[s_1,s_2]} \zeta$, and the two continuations $\{w_{s_1}(m+r) : 0 \leq r \leq \zeta_{s_1} - m\}$ and $\{w_{s_2}(m+r) : 0 \leq r \leq \zeta_{s_2} - m\}$ with fixed starting point $w_{s_1}(m) = w_{s_2}(m)$ are independent (see also Figure 1 below). Heuristically, if $m = 0$ the particles represented by w_{s_1} and w_{s_2} belong to different families, whereas if $m > 0$ and $s \in [s_1, s_2]$ satisfies $\zeta_s = m$, the path w_s represents the last common ancestor of w_{s_1} and w_{s_2}.

The constant path $f \in \mathfrak{P}$ given by $f(r) = (0,x)$ for all $r \geq 0$ is a regular recurrent point for the Markov process w. Indeed, this follows immediately from the fact that the lifetime process is a reflected Brownian motion. Hence we can define $\mathbb{N}_x^\Gamma$ to be the suitably normalised *excursion measure* of the Brownian snake w from the constant path $f = (0,x)$; see e.g. [3] for the excursion theory of Markov processes. Every sample of such an *excursion from* $(0,x)$ is a continuous path-valued function $w : [0,\sigma] \to \mathfrak{P}$ for some finite $\sigma = \sigma(w) > 0$, the *length* of the excursion, such that $w_0 = w_\sigma$ is the constant path remaining in $(0,x)$, and w_s is not constant for all $s \in (0,\sigma)$. Then $\mathbb{N}_x^\Gamma$ is a σ-finite measure on the set

$$\mathfrak{W} := \Big\{ w \in C\big([0,\sigma], \mathfrak{P}\big) : \text{ for some } \sigma > 0 \Big\} \tag{21}$$

of path-valued functions. Although it is stretching the usual terminology a bit, we use the words sample and process also in the case of the underlying non-probability measure $\mathbb{N}_x^\Gamma$.

With each excursion $w : [0,\sigma] \to \mathfrak{P}$ we can again associate the *lifetime process* $\zeta : [0,\sigma] \to [0,\infty)$ by letting $\zeta_s := \zeta(w_s)$, which under $\mathbb{N}_x^\Gamma$ is a Brownian excursion from 0. Heuristically, an excursion w represents the whole family tree of an reactant particle, which at time 0 is located at x.

Following [1] or [14], we can define, for every $t > 0$, the *exit local time at level* t of an excursion $w \in \mathfrak{W}$ as the process $L^t[w] := L^t := \{L^t_s : s \in [0, \sigma]\}$ such that

$$L^t_s := L^t_s[w] := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s 1_{\{\tau_t(w_u) < \zeta_u < \tau_t(w_u) + \varepsilon\}} \, du, \tag{22}$$

for every s, $\mathbb{N}^{\Gamma}_x$–almost surely, where τ_t is defined as in (17). The *total exit local time at level* $t > 0$ *of an excursion* w is $L^t_\sigma[w] := L^t_{\sigma[w]}[w]$. Note that the measure associated with the monotone function $s \mapsto L^t_s[w]$ is supported by those s where $\tau_t(w_s) = \zeta_s$ and recall that exactly those paths w_s represent particles of X_t. The *exit measure at level* $t > 0$ is the measure $Z^t := Z^t[w]$ on $\partial D^t := \{t\} \times \mathbb{R}$ defined by

$$\langle Z^t[w], \varphi \rangle := \int_0^\sigma \varphi\big(w_s(\zeta_s)\big) \, L^t_\bullet(ds), \quad \text{for } \varphi : \partial D^t \to [0, \infty) \text{ measurable}, \tag{23}$$

where the integral is a Stieltjes integral with respect to the nondecreasing function $s \mapsto L^t_s[w]$. Slightly abusing notation, for fixed $t > 0$, we can identify ∂D^t with $\mathbb{R}$, that is, we can consider $Z^t[w]$ and φ as a measure respectively function on $\mathbb{R}$. Such identifications will often be used in the following. The measure $Z^t[w]$ can be interpreted as the spatial distribution of the descendants at time t of a reactant particle, which at time 0 is located as x. The quantity $L^t_\sigma[w]$ describes the mass of the totally produced reactant progeny of this particle at time t.

We now have the means to describe both X and X^∞ in terms of the excursion measures $\mathbb{N}^{\Gamma}_x$. In the case of a measure $\mu \in \mathcal{M}_{\text{tem}}$ different from Lebesgue measure, (27) below should be understood as the natural definition of X^∞ with $X^\infty_0 = \mu$.

Theorem 5 (Snake representations). *Let $\mu \in \mathcal{M}_{\text{tem}}$ be an arbitrary starting measure.*

(i) Representation of X. *Given Γ, let $\Pi = \Pi[\Gamma]$ be a Poisson point field on the space $\mathfrak{W} \times \mathbb{R}$ with intensity measure $\pi = \pi[\Gamma]$ defined by*

$$\pi(dw\, dx) := \int_{\mathbb{R}} \mathbb{N}^{\Gamma}_y(dw) \otimes \delta_y(dx) \, \mu(dy). \tag{24}$$

Then, for P–almost all Γ, the superprocess $X = X[\Gamma]$ with $X_0 = \mu$ can be represented as

$$\langle X_t, \varphi \rangle = \int_{\mathfrak{W} \times \mathbb{R}} \langle Z^t[w], \varphi \rangle \, \Pi(dw \times \mathbb{R}), \tag{25}$$

for all $t > 0$ and $\varphi : \mathbb{R} \to [0, \infty)$ measurable.

(ii) Representation of X^∞. *Let Π^∞ be a Poisson point field on $\mathfrak{W} \times \mathbb{R}$ with intensity measure*

$$\pi^\infty(dw\, dx) := \Big(\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^{\Upsilon}_0(dw) \, \mathsf{P}(d\Upsilon) \Big) \otimes \mu(dx). \tag{26}$$

Then the limit process X^∞ with $X^\infty_0 = \mu$ has the representation

$$\langle X^\infty_t, \varphi \rangle = \int_{\mathfrak{W} \times \mathbb{R}} L^t_\sigma[w] \varphi(x) \, \Pi^\infty(dw\, dx), \tag{27}$$

for all $t > 0$ and $\varphi : \mathbb{R} \to [0, \infty)$ measurable.

Remark 6 (Only dependence on the marginal measure)**.** Note that only the marginal measures $\Pi(dw \times \mathbb{R})$ enter in the representation (25). We have included the space coordinate into the definition of the Poisson point field Π in order to simplify a comparison of the intensity measures (24) and (26). ◇

Remark 7 (Continuous versions of $Z[w]$ and $L_\sigma[w]$)**.** As the process X has a continuous version, there is a continuous version of the process $Z[w] = \{Z^t[w] : t > 0\}$ of exit measures as well. We may henceforth assume that $Z[w]$ under $\mathbb{N}_x^\Gamma$ is this continuous version. Similarly, from the continuity of the total mass process $\|X\|$ in the case of a finite starting measure, we can see that also the process $L_\sigma[w] = \{L_\sigma^t[w] : t > 0\}$ has a continuous version, which we henceforth use. $\diamond$

Remark 8 (A finiteness property)**.** From the representation (27) it can be seen easily, that X^∞ has the compound Poisson structure stated in Theorem 1 (ii) if and only if the intensity measure π^∞ in (26) satisfies

$$\int_{\mathcal{M}_{\mathrm{tem}}} \mathbb{N}_0^\Upsilon\{w : L_\sigma^t[w] > 0\}\, \mathsf{P}(d\Upsilon) < \infty, \quad \text{for } t > 0. \tag{28}$$

This *finiteness property* of π^∞ implies that after an arbitrarily small, positive amount t of macroscopic time, locally only finitely many macroscopic clumps exist. Moreover, together with the continuity of $L_\sigma[w]$ this also implies the continuity of X^∞ in the representation (27). $\diamond$

The snake representations enable us to make a comparison of X and X^∞, and, moreover, draw a revealing heuristic picture of the limit process X^∞. For both processes, mass is initially spread on $\mathbb{R}$ according to μ. In the case of the original process X, starting from each infinitesimal small mass point $\mu(dx)$ a potential family of reactant particle is evolving, whereas in the case of the limit process X^∞ at each infinitesimal small mass point $\mu(dx)$ a potential macroscopic clump can be created. After an arbitrarily small, positive amount of time locally only finitely many families of X survive and, similarly, after an arbitrarily small, positive amount of *macroscopic* time locally only finitely macroscopic clumps survive in X^∞. The further development of the *total mass* of the offspring progeny of any reactant particle in X or of any macroscopic clump in X^∞ is in both cases governed by the laws of $t \mapsto L_\sigma^t[w]$ under the excursion measure $\mathbb{N}_\bullet^\Gamma$ in the random medium Γ.

There are however a number of *significant differences*:

- In X each particle family uses the excursion measures $\mathbb{N}_x^\Gamma$ for the same sample Γ (though around different places x). The clumps of X^∞ however are based on the samples w of the measure $\mathsf{E}\mathbb{N}_0^\Upsilon(dw)$ which is independent of the position x of the clump and of the medium sample Γ. For each individual clump the sample w is in fact the result of a two stage experiment: First Υ is sampled with the law P of the stable medium, and then w is sampled according to the law $\mathbb{N}_0^\Upsilon(dw)$.
- Whereas the reactant particle families of X have a spatial spread and their motion component is visible, this is not the case with the clumps of X^∞. Macroscopic clumps are mass points, which remain at their original spatial position, only their mass is variable. Indeed, whereas the full measure $Z^t[w]$ enters into (25), only $L_\sigma^t[w]$ enters into formula (27) and the spatial structure of $Z^t[w]$ is suppressed. This in particular leads to the loss of the Markov property in the limit process X^∞. Heuristically speaking, the clumps have a hidden *micro-life*, governing the branching behaviour, but invisible from the outside, since the excursion measure $\mathbb{N}_0^\Upsilon$ in the random medium Υ is used in the annealed sense $\mathsf{E}\mathbb{N}_0^\Upsilon$ only.

From the representation (27) of the limit process X^∞, we easily get the Laplace functionals of its finite dimensional marginals — note that the result is consistent with the representation of the one-dimensional marginals mentioned already in (13).

Corollary 9 (Finite dimensional distributions). *The finite dimensional marginals of the limit process X^∞ with starting measure $X_0^\infty = \mu$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$, are determined by*

$$\mathbf{E}_\mu \Big\{ \exp\Big(-\sum_{i=1}^n \langle X_{t_i}^\infty, \varphi_i \rangle \Big) \Big\} = \exp\Big(-\int_{\mathbb{R}} \mathsf{E} U^\Gamma_{t_1,\dots,t_n} \big[\varphi_1(x), \dots, \varphi_n(x) \big] (0)\, \mu(dx) \Big), \tag{29}$$

for $0 \le t_1 \le \ldots \le t_n$ and measurable $\varphi_1, \dots, \varphi_n : \mathbb{R} \to [0, \infty)$ for $n \ge 1$. Here $U^\Gamma_{t_1}[a_1] := U^\Gamma_{t_1} a_1$ is taken from (12) *with constant function $\varphi = a_1 \ge 0$, and $U^\Gamma_{t_1,\dots,t_n}[a_1, \dots, a_n]$ is defined inductively: for $n \ge 2$,*

$$U^\Gamma_{t_1,\dots,t_n}[a_1, \dots, a_n] = U^\Gamma_{t_1} \Big[a_1 + U^\Gamma_{t_2-t_1,\dots,t_n-t_1}[a_2, \dots, a_n] \Big], \tag{30}$$

for all $0 \le t_1 \le \ldots \le t_n$ and $a_1, \dots, a_n \ge 0$.

1.2.5. *Further properties of the limit process X^∞.* To round out the picture of the limit model we describe the major indices related to the survival probability and the tail behaviour of the mass clumps on the macroscopic level. Recall the index $\gamma \in (0, 1)$ of our stable medium Γ, and the scaling index η introduced in (8).

Theorem 10 (Properties of the limit process X^∞). *Let X^∞ be the limit process started in ℓ.*

(i) Self-similarity. *X^∞ satisfies, for every $k > 0$,*

$$X_t^\infty(B) = k^{-\eta} X_{kt}^\infty(k^\eta B) \ \text{ in distribution, for } \ t \ge 0 \text{ and } B \subseteq \mathbb{R} \text{ Borel.}$$

(ii) Survival probability. *The ratio of the intensities $\lambda(t)$ and $\lambda(s)$ of the Poisson point fields carrying the clumps at various macroscopic times $t > s > 0$, respectively, satisfies*

$$\lambda(s)/\lambda(t) = \big(t/s \big)^\eta.$$

Hence, denoting by $\gimel_s(t)$ the mass at the macroscopic time t of a clump at time s, the survival probability of $\gimel_s(t)$ is given by

$$\mathbf{P}_\ell \big\{ \gimel_s(t) > 0 \big\} = (s/t)^\eta, \ \text{ for all } \ t > s > 0.$$

Moreover, we have, for all $t > s > 0$,

$$\gimel_t(t) = \Big(\frac{t}{s} \Big)^\eta \gimel_s(s) \ \text{ in distribution.}$$

(iii) Clump size tails. *The tail behaviour of the clump size $\gimel_t(t)$ is governed by*

$$\mathbf{P}_\ell \big\{ \gimel_t(t) > a \big\} \approx t^{\eta(\gamma+1)}\, a^{-\gamma-1} \ \text{ as } a \uparrow \infty.$$

Here $\approx$ means that the ratio of the quantities involved is bounded away from zero and infinity as $a \uparrow \infty$ by constants independent of $t > 0$.

Remark 11 (Infinite variance). It is quite remarkable, that the clump size is heavy-tailed, in particular it has *infinite variance*. The latter fact was conjectured in [6, Subsection 1.14]. $\diamond$

Remark 12 (Open problem). Note that the intensity $\lambda(t)$ of the carrying Poisson point field at time $t > 0$ occurring in (ii) is positive, but it is *open* to determine its exact value. $\diamond$

1.3. **Outline of the paper.** Here we indicate the further structure of the paper, give a guideline to where various parts of the proofs can be found, and briefly review the main methods of proof.

Section 2 is devoted to those aspects of the paper related to the Brownian snake construction in a random medium. In Subsection 2.1 we establish the Brownian snake representation of super-Brownian motion X in the catalytic medium Γ [Theorem 5 (i)]. Subsection 2.2 contains the proofs of the Laplace functionals in Corollary 9 and the description of X^∞ in Theorem 1 (ii), which both rely on the definition of X^∞ in terms of its snake representation [Theorem 5 (ii)]. Both snake representations are used in Subsection 2.3, together with Birkhoff's individual ergodic theorem, to prove the functional limit theorem. The proof also relies on two further steps of independent interest, whose proofs are deferred to Section 3: the finiteness property (28) of the intensity measure π^∞, and the crossing property, Theorem 4.

Section 3 concerns the aspects of proof related to the method of good and bad paths. In Subsections 3.1 and 3.2 we formulate a quantitative extension of this method. The key step is to give an upper bound on the survival probability of the catalytic super-Brownian motion with a finite starting mass in terms of a quantitative characteristic of the random medium Γ. This is then applied in Subsection 3.3 to prove the crossing property and in 3.4 to verify the finiteness statement (28) and thus derive the compound Poisson structure of the limit process X^∞. We also like to point out that our approach to the method of good and bad paths (other than the approach of [10]) does not rely on the compact support property of catalytic super-Brownian motion established in [11], and conversely seems to be a good starting point for an independent, new probabilistic proof of the compact support property.

Section 4 deals with the more analytical proof techniques. We first investigate the time evolution of the mass of the clumps in our limit model. The calculations of the Poisson intensities and survival probabilities stated in Theorem 10 exploit the natural scaling invariance of the limit process together with the Poisson carrier structure, see Subsection 4.1. The calculation of the tail behaviour in Theorem 10 is based on a Feynman-Kac representation of the solutions of the log-Laplace equation (6), provided in Subsection 4.2 and a simple version of the Tauberian Theorem of Bingham and Doney [2, Theorem 8.16].

2. The Brownian snake approach in the case of a catalytic medium

In this section we prove the snake representations, Theorem 5, and the functional limit theorem, Theorem 1 (i).

2.1. **The Brownian snake representation of catalytic super-Brownian motion.** We now formalize the construction of the Brownian snake and verify the snake representation of X, Theorem 5 (i).

As announced, we first take a fixed sample of the catalytic medium Γ. Recall that $L^{-1}_{[\Gamma,W]}$ denotes the inverse function of the collision local time $L_{[\Gamma,W]}$ of a Brownian path W with Γ, which was introduced in (15).

The continuous time-homogeneous Markov process $\xi = \{\xi_r : r \geq 0\}$ on $D = [0,\infty) \times \mathbb{R}$ with start in $(a,x) \in D$ is defined by

$$\xi_r = \big(a + L^{-1}_{[\Gamma,W]}(r), W \circ L^{-1}_{[\Gamma,W]}(r)\big), \quad \text{for } r \geq 0, \tag{31}$$

where W is a Brownian motion started in $x \in \mathbb{R}$. Let $P_{(a,x)}$ denote the law of ξ started at time $t = 0$ in (a,x) and, for $b \geq 0$, denote by $P^b_{(a,x)}$ the law of the related stopped paths $\{\xi_{r\wedge b} : r \geq 0\}$.

We now briefly describe the definition of the Brownian snake with motion process ξ, following the construction of the Brownian snake for an arbitrary continuous Markovian motion process in [20]. Consider a stopped path $f \in \mathfrak{P}$ with lifetime $\zeta(f) > 0$ and such that $f(0) = (0, x)$ as introduced around (19).

If $0 \le a \le \zeta(f)$ and $b \ge a$ we define $\mathbf{Q}_{a,b}(f, d\tilde{f})$ to be the unique probability measure on $\mathfrak{P}$ such that

- $\mathbf{Q}_{a,b}(f, d\tilde{f})$–almost surely $\tilde{f}(r) = f(r)$, for all $r \in [0, a]$,
- the law under $\mathbf{Q}_{a,b}(f, d\tilde{f})$ of $\{\tilde{f}(a + r) : r \ge 0\}$ is the law of $\{\xi_r : r \ge 0\}$ under $P^{b-a}_{f(a)}$.

This transition can be thought of as follows. From its endpoint $\zeta(f)$, the path f is erased backwards in the original time until the absolute time a, and then renewed according to the random motion process ξ, but stopped at the absolute time b. In particular, $\mathbf{Q}_{0,b}(f, d\tilde{f}) = P^b_{(0,x)}(d\tilde{f})$. By convention we also let $\mathbf{Q}_{0,b}(x, d\tilde{f}) := P^b_{(0,x)}(d\tilde{f})$.

The parameters a, b entering into the transition laws $\mathbf{Q}_{a,b}$ are used to control erasing and renewal of the paths. In snake constructions, these parameters are determined continuously by a stochastic process, for the Brownian snake this role is played by a reflected Brownian motion. To be more precise, for $r, s \ge 0$, denote by $\vartheta^r_s(da\, db)$ the joint distribution of the pair $\big(\min_{\tilde{s}\in[0,s]} |B_{\tilde{s}}|, |B_s|\big)$, where $B = \{B_t : t \ge 0\}$ is a Brownian motion on $\mathbb{R}$ with $B_0 = r$. Note that $\vartheta^r_s\big\{(a, b) \in [0, r] \times \mathbb{R} : a \le b\big\} \equiv 1$.

The *Brownian snake with motion process* ξ and start in $(0, x)$ is defined to be the time-homogeneous continuous strong Markov process $w = \{w_s : s \ge 0\}$ whose transition kernels are given by

$$\text{(32)} \quad \mathbf{Q}_s(f, d\tilde{f}) = \int_0^\infty \int_0^\infty \vartheta^{\zeta(f)}_s(da\, db)\, \mathbf{Q}_{a,b}(f, d\tilde{f}), \quad \text{for } s \ge 0 \text{ and } f \in \mathfrak{P} \text{ with } f(0) = (0, x),$$

see [1, Proposition 5]. Recall that the *lifetime process* $\zeta = \{\zeta_s : s \ge 0\}$ is defined by $\zeta_s = \zeta(w_s)$. Under the law of w determined by the transition kernels $\mathbf{Q}_s$, $s \ge 0$, the lifetime process ζ is a reflected Brownian motion, just by construction.

To interpret the dynamics of the snake w, observe that if $s_1 < s_2$ the path w_{s_2} is obtained from w_{s_1} by erasing from its endpoint ζ_{s_1} down to the absolute time $m := \min_{[s_1,s_2]} \zeta$ and adding an independent tip of length $\zeta_{s_2} - m$ at the end. Figure 1 tries to visualise this. The paths w_{s_1} and w_{s_2}, which are stopped versions of ξ, have to be chosen to be identical on the time interval $[0, m]$ (indicated by the thick lines), but to be independent of the intervals $[m, \zeta_{s_1}]$ and $[m, \zeta_{s_2}]$, respectively, except the common starting point $w_{s_1}(m) = w_{s_2}(m)$. In particular, if $m = 0$, a new path is created, starting again in $(0, x)$. In this case the paths do not have a common part, which means that the reactant particles they represent do not have a common ancestor. This can also interpreted in the sense that the *excursions* from $(0, x)$ of the Markov process w correspond to different families of particles.

To be more precise, note that the constant path $(0, x)$ is a regular recurrent point of the Markov process w. Denote by $\mathbb{N}^\Gamma_x$ the *excursion measure* $\mathbb{N}^\Gamma_x$ of w from this path, which is a σ-finite measure on the space $\mathfrak{W}$ defined in (21). Again we can associate with every excursion $w : [0, \sigma] \to \mathfrak{P}$ a lifetime process $\zeta(w) := \zeta := \{\zeta_s : s \in [0, \sigma]\}$. Observe that under the measure $\mathbb{N}^\Gamma_x$ the process ζ is a Brownian excursion. Under $\mathbb{N}^\Gamma_x$, every excursion $w : [0, \sigma] \to \mathfrak{P}$ has a finite life length $\sigma(w) = \sigma > 0$, and $\zeta_s > 0$ on $(0, \sigma)$. As usual, $\mathbb{N}^\Gamma_x$ is normalised such that

$$\text{(33)} \qquad \mathbb{N}^\Gamma_x\Big\{ \sup_{s\in[0,\sigma]} \zeta_s > \varepsilon \Big\} = \frac{1}{2\varepsilon}\,.$$

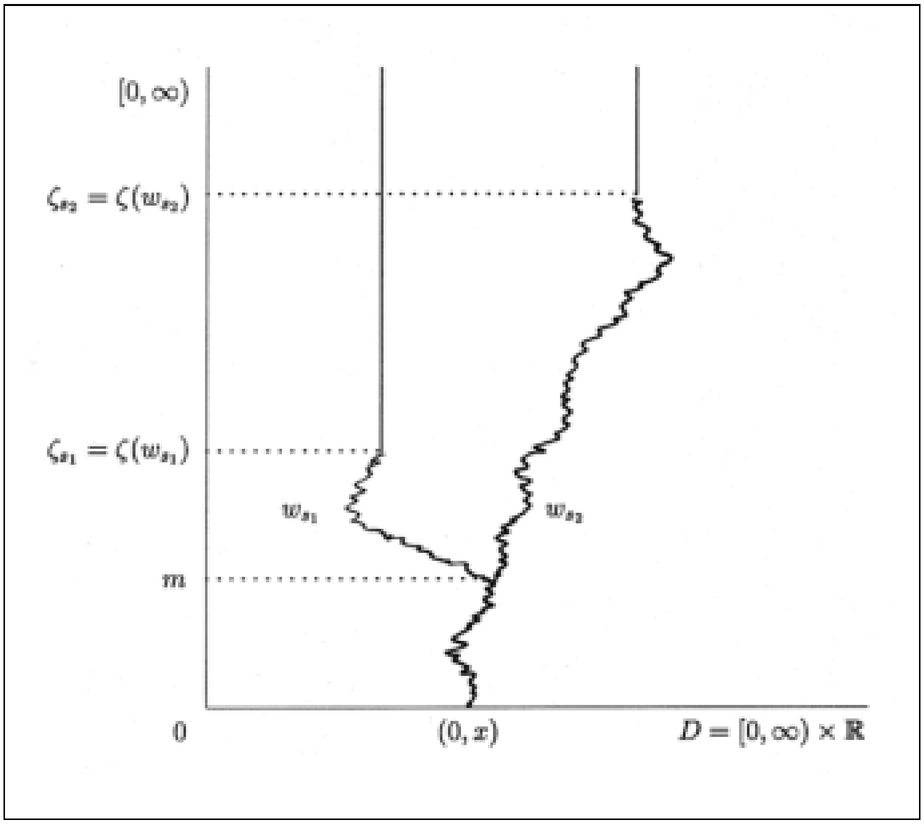

FIGURE 1. Erasing the path w_{s_1} to renew it to w_{s_2}

At this point it is worth looking back at the definition of the intensity measures $\pi = \pi[\Gamma]$ and π^{∞} in (24) respectively (26) and noting that (33) implies that both are in fact σ-finite measures, as needed for the definition of the Poisson point fields.

Proof of Theorem 5 (i). Recall that a sample Γ is fixed. For each measure $\mu \in \mathcal{M}_{\text{tem}}$ we consider a Poisson point field Π with intensity measure π as in (24), defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P}^{\Gamma}_{\mu})$. We have to verify that the process X *defined* on this space by (25) is indeed a catalytic super-Brownian motion in the medium Γ, started in μ.

For $\varphi \in \Phi$, $t > 0$, and $x \in \mathbb{R}$, using $\mu = \delta_x$, let

$$U_t\varphi\,(x) := \mathbb{E}^{\Gamma}_{\delta_x}\bigl\{\exp\langle X_t, -\varphi\rangle\bigr\}. \tag{34}$$

It suffices to verify the following two points:

(a) $U\varphi := \bigl\{U_t\varphi\,(x) :\ t > 0,\ x \in \mathbb{R}\bigr\}$ defines a nonnegative solution of equation (6).
(b) For all $0 \le h < t$ and $\varphi \in \Phi$,

$$\mathbb{E}^{\Gamma}_{\mu}\Bigl\{\exp\langle X_t\,, -\varphi\rangle \Bigm| X_u\,,\, u \le h\Bigr\} = \exp\langle U_{t-h}\varphi, -X_h\rangle.$$

Fix $\varphi \in \Phi$ and $t > 0$ for the remaining proof. In order to give the *proof of* (a) we need the following facts concerning the exit measures $Z^t[w]$ of (23) under the excursion measures, see [1, Proposition 6]. For $x \in \mathbb{R}$ and $0 \le s < t$, define

$$u_t(s,x) := \int_{\mathfrak{W}} \mathbb{N}_x^\Gamma(dw)\Big(1 - \exp\big\langle Z^{t-s}[w], -\varphi\big\rangle\Big), \tag{35}$$

then u_t satisfies the equation

$$u_t(0,x) = E_{(0,x)}\big\{\varphi(\xi_{\tau_t})1_{\{\tau_t<\infty\}}\big\} - 2E_{(0,x)}\Big\{\int_0^{\tau_t}\big[u_t(\xi_s)\big]^2\,ds\Big\}, \quad \text{for } x \in \mathbb{R}, \tag{36}$$

with $\tau_t = \tau_t(\xi)$ from (17), using $\xi_{\tau_t} \in \partial D^t = \{t\} \times \mathbb{R}$ and again the identification of ∂D^t and $\mathbb{R}$. On the other hand, by the Laplace functional formula for Poisson point fields, from (34) and (25) we have

$$U_t\varphi\,(x) = \int_{\mathfrak{W}} \mathbb{N}_x^\Gamma(dw)\Big(1 - \exp\big\langle Z^{t}[w], -\varphi\big\rangle\Big). \tag{37}$$

Consequently, $u_t(s,x) = U_{t-s}\varphi\,(x)$. Then (35) and (36) show that

$$U_t\varphi\,(x) = E_{(0,x)}\big\{\varphi(\xi_{\tau_t})1_{\{\tau_t<\infty\}}\big\} - 2E_{(0,x)}\Big\{\int_0^{\tau_t}\big[u_t(\xi_s)\big]^2\,ds\Big\}. \tag{38}$$

Recalling that ξ with law $P_{(0,x)}$ can by definition be expressed by a Brownian motion W starting at time 0 from x, whose law we denote by $\mathcal{P}_{0,x}$, and that at time $\tau_t = L_{[\Gamma,W]}(t)$ the process ξ is in the state $\big(t, W(t)\big)$, which is identified with $W(t)$, formula line (38) can be rewritten as

$$U_t\varphi\,(x) = \mathcal{E}_{0,x}\left\{\varphi\big(W(t)\big)\right\} - 2\,\mathcal{E}_{0,x}\Big\{\int_0^{L_{[\Gamma,W]}(t)}\Big[U_{t-L^{-1}_{[\Gamma,W]}(s)}\varphi\left(W \circ L^{-1}_{[\Gamma,W]}(s)\right)\Big]^2\,ds\Big\}.$$

We now substitute s for $L^{-1}_{[\Gamma,W]}(s)$ in the second summand. Thus

$$U_t\varphi\,(x) = \mathcal{E}_{0,x}\left\{\varphi\big(W(t)\big)\right\} - 2\,\mathcal{E}_{0,x}\Big\{\int_0^{t}\left[U_{t-s}\varphi\left(W(s)\right)\right]^2\,dL_{[\Gamma,W]}(s)\Big\}. \tag{39}$$

But this is a probabilistic representation of (6), that is, we have proved (a).

Now we give the *proof of* (b). Recall that $t > 0$ and $\varphi \in \Phi$ are fixed. By the definition (25) of X, it obeys the branching property, hence it clearly suffices to consider finite starting measures μ. The main ingredient is the *special Markov property* of the exit measures $Z^t[w]$ of the Brownian snake w with respect to its recursion measures $\mathbb{N}_x^\Gamma$. Instead of giving the most general statement of this property, we just quote the special case we need, which follows directly from the formulation in [1, Proposition 7]. In our case, the special Markov property states that, for all $0 < h < t$ and $x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{N}_x^\Gamma\Big\{\exp\big(-\langle Z^t,\varphi\rangle\big)\,\Big|\,Z^u, u \le h\Big\} \\ &= \exp\Big[-\int_{\mathbb{R}} Z^h(dz)\int_{\mathfrak{W}}\mathbb{N}_z^\Gamma(dw)\Big(1-\exp\big\langle Z^{t-h}[w],-\varphi\big\rangle\Big)\Big]. \end{aligned} \tag{40}$$

By (37) and (34), the right hand side of (40) equals $\exp\big\langle Z^h, -U^\Gamma_{t-h}\varphi\big\rangle$, and we infer that with respect to $\mathbb{N}_x^\Gamma$ the conditional distribution of $\langle Z^t,\varphi\rangle$ given $\{Z^u : u \le h\}$ is equal to the distribution of $\langle X_{t-h}\,,\varphi\rangle$ for the starting measure Z^h. Now $\langle X_t\,,\varphi\rangle$ can be written as the sum of a Poissonian number of random variables $\langle Z_i^t,\varphi\rangle$, where Z_i are independent with distribution $\int_{\mathbb{R}}\mathbb{N}_x^\Gamma\big\{Z[w] \in \bullet \;\big|\; Z^h[w] \ne 0\big\}\,\mu(dx)$. Hence, the conditional distribution of $\langle X_t\,,\varphi\rangle$ given $\{X_u\,,\,u \le h\}$ is the sum of independent samples of $\langle X_{t-h}\,,\varphi\rangle$ with starting measures adding up to X_h. Hence (b) follows from the branching property, and this finishes the proof of Theorem 5 (i). □

2.2. **The Brownian snake representation of the limit process.** We now assume that an arbitrary starting measure $\mu \in \mathcal{M}_{\text{tem}}$ is fixed and a Poisson point field Π^∞ with intensity measure π^∞ as in (26) is defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$. Recall from (33) that π^∞ is σ-finite and hence the Poisson point field is well-defined. We *define* the process X^∞ on this space by (27). In this subsection we show, using results of Section 3 below, that this process has the properties claimed in Theorem 1 (ii) and Corollary 9. The proof of the convergence $X^k \to X^\infty$ is deferred to Subsection 2.3 below, which then completes the proof of Theorem 5 (ii).

Proof of Corollary 9. Consider $\mu, t_1, \dots, t_n$ and $\varphi_1, \dots, \varphi_n$ as in the corollary. By the definition (27) of X^∞, recalling the formula for the Laplace functional of a Poisson point field,

$$\mathbf{E}_\mu\Big\{\exp\Big(-\sum_{i=1}^n \langle X^\infty_{t_i}, \varphi_i\rangle\Big)\Big\} = \mathbf{E}_\mu\Big\{\exp\Big(-\int_{\mathfrak{W}\times\mathbb{R}} \sum_{i=1}^n L^{t_i}_\sigma[w]\, \varphi_i(x)\, \Pi^\infty(dw\, dx)\Big)\Big\}$$

$$= \exp\bigg(\int_{\mathbb{R}}\int_{\mathcal{M}_{\text{tem}}}\int_{\mathfrak{W}} \Big(\exp\Big(-\sum_{i=1}^n L^{t_i}_\sigma[w]\,\varphi_i(x)\Big) - 1\Big)\, \mathbb{N}^\Upsilon_0(dw)\, \mathsf{P}(d\Upsilon)\, \mu(dx)\bigg). \tag{41}$$

The total mass process of $\{X_t : t \geq 0\}$ started in a finite measure $X_0 = \nu$ has, as is easily seen by induction using (5), the Laplace transform

$$\mathbb{E}^\Gamma\Big\{\exp\Big(-\sum_{i=1}^n c_i \|X_{t_i}\|\Big)\Big\} = \exp\Big(-\int_{\mathbb{R}} U^\Gamma_{t_1,\dots,t_n}[c_1, \dots, c_n]\,(z)\, \nu(dz)\Big) \tag{42}$$

with $U^\Gamma_{t_1,\dots,t_n}[c_1, \dots, c_n]$ from (30). On the other hand, by the snake representation (25) of X, this Laplace transform can also be written as

$$\mathbb{E}^\Gamma\Big\{\exp\Big(-\sum_{i=1}^n c_i \|X_{t_i}\|\Big)\Big\} = \exp\bigg(\int_{\mathbb{R}}\int_{\mathfrak{W}} \Big(\exp\Big(-\sum_{i=1}^n c_i L^{t_i}_\sigma[w]\Big) - 1\Big)\, \mathbb{N}^\Gamma_x(dw)\, \nu(dx)\bigg). \tag{43}$$

Comparing (42) and (43) as well as taking expectations with respect to the medium Γ,

$$\mathsf{E}\int_{\mathbb{R}}\int_{\mathfrak{W}} \Big(\exp\Big(-\sum_{i=1}^n c_i L^{t_i}_\sigma[w]\Big) - 1\Big)\, \mathbb{N}^\Gamma_x(dw)\, \nu(dx) = -\int_{\mathbb{R}} \mathsf{E} U^\Gamma_{t_1,\dots,t_n}[c_1, \dots, c_n]\,(z)\, \nu(dz). \tag{44}$$

Specializing to $\nu = \delta_0$ gives

$$\int_{\mathcal{M}_{\text{tem}}}\int_{\mathfrak{W}} \Big(\exp\Big(-\sum_{i=1}^n c_i L^{t_i}_\sigma[w]\Big) - 1\Big)\, \mathbb{N}^\Upsilon_0(dw)\, \mathsf{P}(d\Upsilon) = -\mathsf{E} U^\Gamma_{t_1,\dots,t_n}[c_1, \dots, c_n]\,(0). \tag{45}$$

Plugging this into (41) yields the formula stated in Corollary 9. □

Proof of Theorem 1 (ii) [subject to the proof of (28), which will be given in Section 3 below.] We still allow an arbitrary starting measure μ, cf. Remark 2. From the definition (27) of X^∞ in terms of the Poisson point field Π^∞ it is clear that, for every $t > 0$, the measure X^∞_t is supported by the points of a Poisson point field on $\mathbb{R}$ with intensity measure

$$\Big(\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^\Upsilon_0\big\{w : L^t_\sigma[w] > 0\big\}\, \mathsf{P}(d\Upsilon)\Big)\, \mu(dx). \tag{46}$$

By the finiteness property (28) the factor in front of the measure μ is *finite*, say $c > 0$. Moreover, the masses of the atoms at these locations are independent with common distribution

$$\frac{1}{c}\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^\Upsilon_0\big\{L^t_\sigma[w] \in \bullet\,\big\}\, \mathsf{P}(d\Upsilon). \tag{47}$$

This establishes the compound Poisson property.

Suppose that $I \subseteq \mathbb{R}$ is a bounded interval and $t > 0$. Again by (28), $\mathbf{P}_\mu$–almost surely, the point field Π^∞ restricted to the set $\{(w,x) \in \mathfrak{W} \times I : L^t_\sigma[w] > 0\}$ is supported by finitely many points in $\mathfrak{W} \times I$, say

$$(w_1, x_1), \ldots, (w_n, x_n), \quad \text{with } x_1 \le \ldots \le x_n . \tag{48}$$

For every $s \ge t$, the measure X^∞_s is supported by the set $\{x_i : 1 \le i \le n, \, L^s_\sigma[w_i] > 0\}$. Hence atoms cannot move in space. To show that no new atoms can be born it would suffice to show that zero is an absorbing state for the process $s \mapsto L^s_\sigma[w_i]$. However, it is easier to argue via the Laplace transform of Corollary 9. Indeed, for all $s, t > 0$,

$$\begin{aligned} \mathbf{P}_\mu\{X^\infty_t(I) = 0\} &= \lim_{\theta \uparrow \infty} \exp\big(-\mu(I)\, \mathsf{E} U^\Gamma_t \theta\,(0)\big) = \lim_{\theta \uparrow \infty} \exp\big(-\mu(I)\, \mathsf{E} U^\Gamma_t [\theta + U^\Gamma_s \theta]\,(0)\big) \\ &= \mathbf{P}_\mu\{X^\infty_t(I) = 0 \text{ and } X^\infty_{t+s}(I) = 0\} \end{aligned} \tag{49}$$

since $0 \le U^\Gamma_s \theta \le \theta$ and by monotonicity. This shows that zero is an absorbing state for $t \mapsto X^\infty_t(I)$ and hence also for $t \mapsto L^t_\sigma[w]$.

Finally, for the proof that macroscopic clumps have almost surely finite lifetime, it suffices to show that, for every bounded interval I,

$$\lim_{t \to \infty} \mathbf{P}_\mu\{X^\infty_t(I) = 0\} = \lim_{t \to \infty} \lim_{\theta \uparrow \infty} \exp\big(-\mu(I)\, \mathsf{E} U^\Gamma_t \theta\,(0)\big) = 1 . \tag{50}$$

This does not depend on the starting measure μ, so that we can assume $\mu = \ell$. In this case the result follows directly from the stronger statement $\mathbf{P}_\ell\{\beth_s(t) > 0\} = (s/t)^\eta$, which we prove in Subsection 4.1 below. $\square$

2.3. **The functional limit theorem.** In this section we prove the weak convergence in P–probability of the random distributions of $X^k[\Gamma]$, as $k \uparrow \infty$, which was claimed in Theorem 1 (i). For this purpose we also rescale the catalytic medium, but with a *different* spatial scaling, namely

$$\Gamma^k(\,\bullet\,) := k^{1/(2\gamma)} \Gamma\big(\,\bullet\, / \sqrt{k}\big) \ \text{ for } k > 0. \tag{51}$$

But note that by self-similarity the rescaled stable medium Γ^k has the same distribution as Γ. Our strategy is to look at the distributions of the renormalised process $X^k[\Gamma^k] = \{X^k_t[\Gamma^k] : t \ge 0\}$ with changing medium Γ^k (instead of Γ), and show, using the representation of Theorem 5 (i), P–almost surely(!) the weak convergence of the random distributions of $X^k[\Gamma^k]$. This clearly implies weak convergence in P–probability of the random distributions of the rescaled processes $X^k[\Gamma]$ in the unscaled medium.

We start by looking at the case of the constant test function $\varphi \equiv 1$. i.e. at the total mass process $t \mapsto \|X^k_t\|$, and start X with the restricted Lebesgue measure, $X^k_0 = \ell_{(a,b)}$ for $a < b$ real. The following proposition is the core of our proof of the functional limit theorem. We equip the space $C((0,\infty), \mathbb{R})$ with the Polish topology of uniform convergence on compact intervals, which matches the earlier definition of the topology on $C((0,\infty), \mathcal{M}_{\text{tem}})$.

Proposition 13 (Total mass process). *Fix real numbers $a < b$.*

(i) Convergence. P*–almost surely, the random laws of the renormalised total mass processes $\big\|X^k[\Gamma^k]\big\| = \{\|X^k_t[\Gamma^k]\| : t > 0\}$ with $X^k_0[\Gamma^k] = \ell_{(a,b)}$ converge weakly on the path space $C\big((0,\infty), \mathbb{R}\big)$ as $k \uparrow \infty$ to the law of a limit process $\{X^\infty_t(a,b) : t > 0\}$.*

(ii) Identification of the limit. *Let $\Pi^{\infty}_{a,b}$ be a Poisson point field on $\mathfrak{W}$ with intensity measure*

$$\pi^{\infty}_{a,b}(dw) := (b-a)\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^{\Upsilon}_0(dw)\,\mathsf{P}(d\Upsilon).$$

Then the limit process satisfies

$$X^{\infty}_t(a,b) = \int_{\mathfrak{W}} L^t_{\sigma}[w]\,\Pi^{\infty}_{a,b}(dw), \quad \textit{for } t>0, \tag{52}$$

in particular, it is independent of the medium Γ.

Proof. Fix $a<b$. To begin with, we infer from (33) that $\pi^{\infty}_{a,b}$ is σ-finite and hence the Poisson point field $\Pi^{\infty}_{a,b}$ is well-defined. We can thus assume that the process $X^{\infty}(a,b) := \{X^{\infty}_t(a,b) : t>0\}$ is *defined* by (52), and our aim is to show that P–almost surely the processes $\|X^k[\Gamma^k]\|$ with $X^k_0[\Gamma^k] = \ell_{(a,b)}$ converge in law on $C\big((0,\infty),\mathbb{R}\big)$ to $X^{\infty}(a,b)$ as $k\uparrow\infty$.

The first step is to derive a representation of $\|X^k[\Gamma^k]\|$ as a k-independent functional of a Poisson point field, with k-dependent intensity measure. To do this fix $k>0$ and the medium Γ throughout the first step. From the Brownian snake representation of Theorem 5 (i) we infer that

$$\big\|X^k_t[\Gamma^k]\big\| = k^{-\eta}\int_{\mathfrak{W}} L^{kt}_{\sigma}[w]\,\Pi(dw), \quad \text{for } t>0, \tag{53}$$

where $\Pi = \Pi[\Gamma^k]$ is a Poisson point field on $\mathfrak{W}$ with intensity measure $\int_{k^{\eta}a}^{k^{\eta}b}\mathbb{N}^{\Gamma^k}_x\,dx$. As the total exit local time $L^{kt}[w]$ of a snake excursion w does not depend on the second component of the motion process ξ, we can equivalently use the intensity measure

$$\int_{k^{\eta}a}^{k^{\eta}b} \mathbb{N}^{T^x\Gamma^k}_0\,dx. \tag{54}$$

We now show that

$$\begin{aligned}&\text{the distributions of } \big\{k^{-\eta}L^{kt}_{\sigma}[w] : t>0\big\} \text{ under } \mathbb{N}^{T^x\Gamma^k}_0 \text{ and}\\ &\text{of } \big\{L^t_{\sigma}[w] : t>0\big\} \text{ under } k^{-\eta}\mathbb{N}^{T^{x/\sqrt{k}}\Gamma}_0 \text{ coincide.}\end{aligned} \tag{55}$$

Indeed, a Brownian scaling of time and space yields for the collision local times

$$L_{[\Gamma^k,x+W]}(kt) = k^{\eta}L_{[\Gamma,x/\sqrt{k}+W^k]}(t), \quad \text{for } t>0, \tag{56}$$

where W^k is defined by $W^k_t = (1/\sqrt{k})W_{tk}$, for $t\geq 0$. We now define a scaling $\mathfrak{W}\to\mathfrak{W}$ mapping w to w^k in such a way that

- the lifetime process ζ^k of w^k is given by $r\mapsto \zeta^k_r = k^{-\eta}\zeta_{k^{2\eta}r}$,
- the motion process of w^k is ξ^k given as

$$r\mapsto \xi^k_r = \Big(L^{-1}_{[\Gamma,x/\sqrt{k}+W^k]}(r),\, W^k\circ L^{-1}_{[\Gamma,x/\sqrt{k}+W^k]}(r)\Big). \tag{57}$$

Hence, if w has the distribution $\mathbb{N}^{T^x\Gamma^k}_0$, then w^k has the distribution $k^{-\eta}\mathbb{N}^{T^{x/\sqrt{k}}\Gamma}_0$. Note that $\sigma(w^k) = k^{-2\eta}\sigma(w)$ is the length of the excursion w^k. For the stopping times τ_t we obtain from the formula lines (56), (57) and (18) the relation

$$\tau_{kt}(w_u) = k^{\eta}\tau_t\big(w^k_{k^{-2\eta}u}\big), \quad \text{for all } u\in[0,\sigma],\ t>0,\ w\in\mathfrak{W}. \tag{58}$$

Looking at the total exit local times $L^{kt}_{\sigma(w)}[w]$ and using (22) and (58) and substitutions $v = k^{-2\eta}u$ and $\delta = k^{-\eta}\varepsilon$ gives, for all $t > 0$,

$$
\begin{aligned}
L^{kt}_{\sigma(w)}[w] &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{\sigma} 1_{\{\tau_{kt}(w_u)<\,\zeta_u\,<\tau_{kt}(w_u)+\varepsilon\}}\,du \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{\sigma} 1_{\{k^{\eta}\tau_t(w^k_{k^{-2\eta}u})<\,\zeta_u\,<k^{\eta}\tau_t(w^k_{k^{-2\eta}u})+\varepsilon\}}\,du \\
&= k^{2\eta}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{k^{-2\eta}\sigma} 1_{\{\tau_t(w^k_v)<k^{-\eta}\zeta(k^{2\eta}v)<\tau_t(w^k_v)+k^{-\eta}\varepsilon\}}\,dv \\
&= k^{\eta}\lim_{\delta\downarrow 0} \frac{1}{\delta}\int_0^{\sigma(w^k)} 1_{\{\tau_t(w^k_v)<\zeta^k(v)<\tau_t(w^k_v)+\delta\}}\,dv = k^{\eta} L^{t}_{\sigma(w^k)}[w^k].
\end{aligned}
\tag{59}
$$

Hence, (55) is proved, and from (53) and (55) we get the representation

$$
\big\|X^k_t[\Gamma^k]\big\| = \int_{\mathfrak{W}} L^t_{\sigma}[w]\,\Pi^k_{a,b}(dw), \quad \text{for } t>0, \tag{60}
$$

where $\Pi^k_{a,b} = \Pi^k_{a,b}[\Gamma]$ is a Poisson point field on $\mathfrak{W}$ with intensity measure

$$
\pi^k_{a,b} = \pi^k_{a,b}[\Gamma] := k^{-\eta}\int_{k^{\eta}a}^{k^{\eta}b} \mathbb{N}_0^{\mathcal{T}^{x/\sqrt{k}}\Gamma}\,dx = k^{-\rho}\int_{k^{\rho}a}^{k^{\rho}b} \mathbb{N}_0^{\mathcal{T}^{x}\Gamma}\,dx, \tag{61}
$$

with $\rho := \eta - 1/2 > 0$. This finishes the first step in the proof.

Comparing (52) and (60) we note that $\big\|X^k[\Gamma^k]\big\|$ and the right hand side in (52) are defined by the same functional of a Poisson point field on $\mathfrak{W}$, of course with different intensity measures.

To do the second step in the proof and show that P-almost surely the processes $\big\|X^k[\Gamma^k]\big\|$ converge in law on $C\big((0,\infty),\mathbb{R}\big)$ to $X^{\infty}(a,b)$, one has to show, by definition of the topology on $C\big((0,\infty),\mathbb{R}\big)$, that for every compact set $I\subset(0,\infty)$, P–almost surely, the processes $\big\{\|X^k_t[\Gamma^k]\| : t\in I\big\}$ converge in law on the space $C(I,\mathbb{R})$ with the uniform topology to $\big\{X^{\infty}_t(a,b) : t\in I\big\}$. Clearly, it suffices to show this for compact sets of the form $I=[1/n,n]$, so fix an arbitrary positive integer n.

Abbreviate $C_n := C([1/n,n],\mathbb{R})$ and, for $w\in\mathfrak{W}$, let $L_n[w]\in C_n$ denote the function defined by $L_n[w](s) := L^s_{\sigma}[w]$ for all $s\in[1/n,n]$. By *Birkhoff's Individual Ergodic Theorem* applied to the group of spatial shifts acting ergodically on the stable random measure Γ we obtain, for each measurable $F : C_n\to[0,\infty)$, P–almost surely,

$$
\lim_{k\uparrow\infty} k^{-\rho}\int_{k^{\rho}a}^{k^{\rho}b}\int_{\mathfrak{W}} F\big(L_n[w]\big)\,\mathbb{N}_0^{\mathcal{T}^x\Gamma}(dw)\,dx = (b-a)\int_{\mathcal{M}_{\text{tem}}}\mathsf{P}(d\Upsilon)\int_{\mathfrak{W}} F\big(L_n[w]\big)\,\mathbb{N}_0^{\Upsilon}(dw). \tag{62}
$$

Define random measures μ_k on C_n by

$$
\mu_k(B) := k^{-\rho}\int_{k^{\rho}a}^{k^{\rho}b}\int_{\mathfrak{W}} 1_{B\setminus\{0\}}\big(L_n[w]\big)\,\mathbb{N}_0^{\mathcal{T}^x\Gamma}(dw)\,dx, \quad \text{for } B\subseteq C_n \text{ Borel}. \tag{63}
$$

Define, similarly, a measure μ on C_n by

$$
\mu(B) := (b-a)\int_{\mathcal{M}_{\text{tem}}}\mathsf{P}(d\Upsilon)\int_{\mathfrak{W}} 1_{B\setminus\{0\}}\big(L_n[w]\big)\,\mathbb{N}_0^{\Upsilon}(dw), \quad \text{for } B\subseteq C_n \text{ Borel}. \tag{64}
$$

Note that, by (28), μ_k and μ are finite measures since we did not allow them to have mass at the zero function in C_n .

As the space C_n is Polish, there is a countable family $\{F_k : C_n \to [0,\infty) : k \geq 1\}$ of continuous and bounded functions, which are convergence determining for the weak convergence of finite measures on C_n. This fact together with (62) implies that P–almost surely the measures μ_k converge weakly to the measure μ on C_n.

Formula lines (60) and (61) together state that $\big\|X^k[\Gamma^k]\big\| = \big\{\|X_t^k[\Gamma^k]\| : t \in [1/n, n]\big\}$ is equal in law to the sum of all functions in C_n in the support of a Poisson point field with intensity measure μ_k. By the finiteness of μ and elementary properties of Poisson point fields we infer that P–almost surely this sum converges in distribution to the sum of all functions in the support of a Poisson point field in C_n with intensity measure μ. In other words, P–almost surely, in distribution on the space C_n with the topology of uniform convergence,

$$\lim_{k\uparrow\infty} \big\|X^k[\Gamma_k]\big\| = \lim_{k\uparrow\infty} \int_{\mathfrak{W}} L_n[w]\, \Pi^k_{a,b}(dw) = \int_{\mathfrak{W}} L_n[w]\, \Pi^\infty_{a,b}(dw). \tag{65}$$

This finishes the proof of the second step and thus proves the statements (i) and (ii) in the proposition. □

In order to be able to deal with the real-valued processes $t \mapsto X_t^k(a,b)$, started in $X_0 = \ell$, we use the crossing property, Theorem 4 (which is proved in Section 3 below) to derive the following corollary.

Corollary 14 (No mass transport on macroscopic scales). *Let (a,b) be a bounded interval and consider the rescaled processes $\{X_t^k : t > 0\}$ with $X_0^k = \ell_{(-\infty,a)}$ or $X_0^k = \ell_{(b,\infty)}$. Then, in P–probability, the processes $\big\{X_t^k(a,b) : t > 0\big\}$ converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the zero function as $k \uparrow \infty$.*

Proof. By translation and (if needed) reflection, we see that it is equivalent to show that, in P–probability, the processes

$$\big\{X_t^k(a-b,0) : t > 0\big\}, \text{ for } X_0 = \ell_{(0,\infty)}, \tag{66}$$

converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the zero function. Now observe that

$$\sup_{t\geq 0} X_t^k(a-b,0) = \sup_{t\geq 0} k^{-\eta} X_{kt}\left(k^\eta(a-b),0\right) \leq k^{-\eta} \sup_{t\geq 0} X_t(-\infty,0] \xrightarrow[k\uparrow\infty]{} 0, \tag{67}$$

$\mathbb{P}^\Gamma$–almost surely, for P–almost all Γ, by Theorem 4. This finishes the proof. □

We now have the means to complete the *proof of Theorem* 1 subject to the proof of the crossing property, Theorem 4, and the finite mass property (28). We start the processes X in $X_0 = \ell$ and show the convergence of the rescaled processes X^k to the process X^∞ defined by (27) with starting mass $X_0^\infty = \ell$.

Let again $a < b$. Given Γ, by the branching property, $\{X_t^k(a,b) : t \geq 0\}$ is the sum of independent processes started in $X_0^k = \ell_{(a,b)}$, $\ell_{(b,\infty)}$ and $\ell_{(-\infty,a)}$. Combining the total mass convergence, Proposition 13 (i), and Corollary 14 we see that, in P–probability, the processes $\big\{X_t^k(a,b) : t \geq 0\big\}$ converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the limit process $\big\{X_t^\infty(a,b) : t \geq 0\big\}$, which is described in Proposition 13 (ii) and coincides, of course, with the limit process applied to the interval (a,b).

It remains to lift the result from the indicator functions $1_{(a,b)}$ to any continuous function $\varphi\colon \mathbb{R} \to [0,\infty)$ with bounded support, say the support is contained in (a,b). Let $\delta > 0$. We choose step functions (i.e. linear combinations of indicator functions on bounded, open intervals) $g, h\colon (a,b) \to [0,\infty)$ with $h \le \varphi \le g$ and $\|g-h\|_\infty < \delta$. Then, for all positive integers n,

$$\sup_{1/n \le t \le n} \langle X_t^k, h\rangle \le \sup_{1/n \le t \le n} \langle X_t^k, \varphi\rangle \le \sup_{1/n \le t \le n} \langle X_t^k, g\rangle. \tag{68}$$

In P–probability, the left and right hand side as well as $\sup_{1/n \le t \le n} \langle X_t^k, g-h\rangle$ converge in distribution as $k \uparrow \infty$ to

$$\sup_{1/n \le t \le n} \langle X_t^\infty, h\rangle \le \sup_{1/n \le t \le n} \langle X_t^\infty, g\rangle \quad \text{and} \quad \sup_{1/n \le t \le n} \langle X_t^\infty, g-h\rangle, \text{ respectively.} \tag{69}$$

Moreover, the latter term is bounded by $\delta \sup_{1/n \le t \le n} X_t^\infty(a,b)$. As, by Proposition 13 (i), the process $X^\infty(a,b)$ has almost surely continuous paths, this can be made arbitrarily small by choice of δ. Recalling the definition of the metric from (1), we see that this implies convergence of the processes X^k on $C\big((0,\infty), \mathcal{M}(\mathbb{R})\big)$. But the states of the limit process are again in $\mathcal{M}_{\text{tem}}$, because

$$\mathbf{E}_\ell \langle X_t^\infty, \varphi\rangle = \langle \ell, \varphi\rangle < \infty, \text{ for all } \varphi \in \Phi \text{ and } t > 0, \tag{70}$$

recall Theorem 0 (iii), and path continuity. This finishes the proof. □

Remark 15 (Other starting measures). It is possible to start the process X with k–dependent initial measures $X_0 = \mu^{(k)}$ on $\mathbb{R}$ such that $X_0^k \equiv \mu$, where μ is a sufficiently *diffuse* measure in $\mathcal{M}_{\text{tem}}$. For example it is sufficient to require that μ has a continuous density g with the property that, for some constants $a, b \ge 0$,

$$\lim_{x \uparrow \infty} \int_x^\infty \big|g(y) - a\big|\, dy = \lim_{x \downarrow -\infty} \int_{-\infty}^x \big|g(y) - b\big|\, dy = 0. \tag{71}$$

To show this, observe that one can extend the convergence easily from the case $X_0^k = \ell$ to $X_0^k = \ell_{(a,b)}$. By uniform approximation from above and below one can then get convergence for all starting measures satisfying (71). However, the functional limit law does *not* hold without any condition for the scaled starting measure μ. Starting, for example, with the counting measure $X_0^k \equiv \sum_{z \in \mathbb{Z}} \delta_z$ does *not* lead to a limit process that is independent of Γ. ◇

3. The method of good and bad paths

In Subsection 3.1 and 3.2 we formulate a quantitative approach to the method of good and bad paths extending a recent result of [10]. The main result of this part, Theorem 19, enables us to prove the crossing property of Theorem 4, in Subsection 3.3, and the compound Poisson property of Theorem 1 (ii), in Subsection 3.4.

3.1. Regularity of the catalytic medium.

In this subsection we introduce a characteristic quantity $N(\Gamma)$, which measures the regularity of the particular sample Γ of the catalyst. This is used, in Subsection 3.2 below, to formulate an upper bound on the survival property of the superprocess $X = X[\Gamma]$ either at a fixed time or of an associated stopped measure at a Brownian stopping time.

Recall from (3) the definition of the stable random measure Γ of index $0 < \gamma < 1$. For every $n \ge 1$ we denote by $\pi_n = \pi_n[\Gamma] \subseteq \mathbb{R}$ the set of spatial positions of the atoms of Γ whose weights are in $[2^{-n}, 2^{-n+1})$. Then π_n is the support of a homogeneous Poisson point field on $\mathbb{R}$ with intensity

$$I_n := 2^{\gamma n} c_\gamma\,, \quad \text{where } c_\gamma := \frac{1 - 2^{-\gamma}}{\gamma \displaystyle\int_0^\infty \frac{1 - \mathrm{e}^{-r}}{r^{1+\gamma}}\, dr}\,, \tag{72}$$

see e.g. [7].

Definition 16 (p–perfect Γ samples)**.** Fix a value $\beta \in (0, \gamma \log 2)$ once and forever. Given a positive integer m and a positive real k, we denote by $A(m,k)$ the event that the maximal connected component of $[-k,k] \setminus \pi_m$ is shorter than $\Delta_m := \mathrm{e}^{-\beta m}$. We call a Γ sample *p–perfect* if it satisfies $A(m,k)$ for all k and $m \geq p$. $\diamond$

The following lemma is adapted from results of [10].

Lemma 17 (Large gaps in $\pi_n(\Gamma)$)**.** *Fix $0 < \varrho < \gamma \log 2 - \beta$ and suppose that $k(n)$ is an increasing sequence of positive reals with $\log\log k(n) = o(n)$ as $n \uparrow \infty$. Then there are constants $c, d > 0$ such that, for every $N \geq 1$,*

$$\mathsf{P}\Big\{ \textit{there is an } n \geq N \textit{ such that } A\big(n, k(n)\big) \textit{ fails} \Big\} \leq c \exp\big(- d\, \mathrm{e}^{\varrho N} \big). \tag{73}$$

Consequently, neighbouring atoms of Γ in $\big[-k(n),\, k(n)\big]$ of weight about 2^{-n} are more than Δ_n away only with an exponentially small probability.

Proof. We write $J(n)$ for the number of points in $[-1/2, 1/2] \cap \pi_n$, then $J(n)$ has a Poisson distribution with parameter $a(n) := c_\gamma 2^{\gamma n}$. Denote x_0 the largest point of π_n left of $[-1/2, 1/2]$, moreover $x_1 \leq \ldots \leq x_{J(n)}$ the points in $[-1/2, 1/2] \cap \pi_n$, and finally $x_{J(n)+1}$ the smallest point to the right of the interval. Define the distances $y_k := x_{k+1} - x_k$ for $k = 0, \ldots, J(n)$. The y_k are independent exponentially distributed random variables with parameter $a(n)$. Following the arguments in [10, Subsection 5.2] we find that there are constants $c_1, c_2, c_3 > 0$ such that

$$\mathsf{P}\Big\{ J(n) + 1 > 2\, a(n) \Big\} \leq \exp(-c_1 2^{\gamma n}), \tag{74}$$

and, writing $\sigma := \gamma \log 2 - \beta > 0$,

$$\mathsf{P}\Big\{ \max_{0 \leq i \leq 2\, a(n)} y_i > \mathrm{e}^{-\beta n} \Big\} \leq c_2\, \mathrm{e}^{n \gamma \log 2} \exp(-c_3\, \mathrm{e}^{\sigma n}). \tag{75}$$

From this we infer that, for some $c_4, c_5 > 0$,

$$\mathsf{P}\Big\{ A(n, 1/2) \text{ fails} \Big\} \leq c_4 \exp(-c_5\, \mathrm{e}^{n\sigma}). \tag{76}$$

This is invariant under shifts of the interval $(-1/2, 1/2)$ and thus, by adding up the events, we obtain for some c_6 , $c_7 > 0$, and all $n \geq 1$,

$$\mathsf{P}\Big\{ A\big(n, k(n)\big) \text{ fails } \Big\} \leq c_4 k(n) \exp\big(- c_5\, \mathrm{e}^{\sigma n} \big) \leq c_6 \exp\big(- c_7\, \mathrm{e}^{\varrho n} \big). \tag{77}$$

Finally, we can add up the events $A\big(n, k(n)\big)$ over all $n \geq N$ and find a suitable c and $d := c_7$, such that the statement (73) holds. $\square$

Definition 18 (Characteristic $N(\Gamma)$)**.** Fix $k(n) := \exp\exp\sqrt{n}$ for $n \geq 1$ and define the *characteristic* $N(\Gamma)$ of the sampled medium Γ by

$$N(\Gamma) := \min \big\{ N \geq 1 :\, A\big(n, k(n)\big) \text{ holds for all } n \geq N \big\}\,. \tag{78}$$

Using the Borel-Cantelli Lemma and Lemma 17, one can see that $N(\Gamma)$ is a well-defined integer, P–almost surely. $\diamond$

3.2. **An upper bound on the survival probability.** In this subsection we determine an upper bound on the survival probability of a super-Brownian motion in a fixed catalytic medium sample Γ with characteristic $N(\Gamma)$. The approach taken here is similar to the key technique of [10], in particular we also work in a historical setting and use a decomposition into good and bad reactant paths, but we have to make a more quantitative approach. Moreover, our argument does not rely on the compact support property of $X[\Gamma]$. The most crucial point is that we work out explicitly how the upper bound of the survival probability depends on $N(\Gamma)$.

For the further development, we presuppose the reader is familiar with basic ideas and the formalism of historical catalytic super-Brownian motion. Here we closely follow the presentation of [10]. Denote by $Y = \{Y_t : t \geq 0\}$ the *historical super-Brownian motion* in the catalytic medium Γ with starting measure $\mu \in \mathcal{M}_{\text{tem}}$ defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P}^{\Gamma}_{\mu})$; if there is no danger of confusion we omit Γ or μ from the notation, that is, we write simply $\mathbb{P}$ and $\mathbb{E}$. At the same time we use, for any finite Brownian stopping time T, Dynkin's stopped measure Y_T and the pre-T σ-field $\mathcal{G}(T)$ for the historical superprocess as introduced in [14] and reviewed in [10, Subsection 3.2]. We suppose that $W = \{W_s : s \geq t\}$ under the distribution $\mathcal{P}_{t,x}$ is a Brownian path started at $W_t = x$.

The central result of this subsection is the following theorem. For its formulation we extend the definition (15) of the collision local time $L_{[W,\Gamma]}(s)$ for $s \geq t$ to Brownian paths W distributed according to $\mathcal{P}_{t,x}$ meaning the collision local time of W and Γ on the time interval $[t, s]$. We introduce two formal hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ on an increasing sequence $T_n \uparrow T < \infty$ of Brownian stopping times with $T_0 = 0$. Suppose positive integers d and p, a sequence of positive thresholds l_n and small reals $\varepsilon_1, \varepsilon_2 > 0$ are given.

$\mathbf{H_1}$**:** The sequence of stopping times satisfies the *hypothesis* $\mathbf{H_1}(d, \varepsilon_1)$ *in* $x \in \mathbb{R}$ if

$$\sum_{n=0}^{\infty} 2^{n+1} \mathcal{P}_{0,x}\Big\{ \sup_{T_n \leq s \leq T_{n+1}} |W_s| > k(n+d) \Big\} \leq \varepsilon_1 . \tag{79}$$

$\mathbf{H_2}$**:** For every nonnegative integer n, define the set $B(n)$ of *bad paths* for the catalytic medium Γ on the random time interval $[T_n, T_{n+1}]$ as the set of paths W satisfying

$$\int_{T_n}^{T_{n+1}} L_{[W,\Gamma]}(ds) < l_n . \tag{80}$$

The sequence of stopping times satisfies the *hypothesis* $\mathbf{H_2}(p, l_n, \varepsilon_2)$ *in* $x \in \mathbb{R}$ if, in every p–perfect medium Γ, we have for $x_0 = 0$ and any x_n in the range of $W(T_n)$,

$$\sum_{n=0}^{\infty} \mathcal{P}_{T_n, x_n} \{ W \in B(n) \} \leq \varepsilon_2 . \tag{81}$$

Here is the announced result on the survival probability.

Theorem 19 (Upper bound on survival probability)**.** *Suppose $T_0 = 0$ and $T_n \uparrow T < \infty$ is a sequence of Brownian stopping times, μ a unit starting mass for Y, supported by a compact interval I and $\varepsilon > 0$ is fixed. Suppose further there are nonnegative integers m and d and a sequence of thresholds $l_n > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{2^{-m-n}}{l_n} \leq \frac{\varepsilon}{3} , \tag{82}$$

and the hypotheses $\mathbf{H_1}\big(d,(\varepsilon/3)2^m\big)$ *and* $\mathbf{H_2}(m+d,l_n\,,\varepsilon/6)$ *are satisfied in any* $x\in I$. *Then, for* P*–almost every* Γ *with* $N(\Gamma)\le m$ *and for the stopped measure* Y_T*, we have*

$$\mathbb{P}^{\Gamma}_{2^{-m}\mu}\{Y_T\neq 0\}\le\varepsilon. \tag{83}$$

The remainder of this subsection is devoted to the proof of this theorem. We work with a fixed catalyst Γ with characteristic $N=N(\Gamma)$ and use the notation from the theorem, additionally abbreviating $M_n:=2^{-m-n}$ for the fixed m. We start with a simple lemma taken from [10, Subsection 3.4].

Lemma 20 (Extinction by partitioning)**.** *Define events* $A_n\;:=\;\big\{\|Y_{T_n}\|\le M_n\big\}$ *and* $A\;:=\;\cap_{n=1}^{\infty}A_n$. *Then, for every* $\nu\in\mathcal{M}_{\text{tem}}(\mathbb{R})$*, we have* $\mathbb{P}^{\Gamma}_{\nu}$*–almost surely on* A *that* $Y_T=0$.

Proof. By Markov's inequality, for each $n\ge 1$ and arbitrary $\zeta>0$,

$$\mathbb{P}^{\Gamma}_{\nu}\Big(\big\{\|Y_T\|>\zeta\big\}\cap A\Big)\le\zeta^{-1}\,\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_T\|\big\} \tag{84}$$

and, by the special Markov property,

$$\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_T\|\big\}=\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\mathbb{E}^{\Gamma}_{Y_{T_n}}\{\|Y_T\|\}\big\}=\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_{T_n}\|\big\}\le M_n\,. \tag{85}$$

As n was arbitrary, we infer that

$$\mathbb{P}^{\Gamma}_{\nu}\Big(\big\{\|Y_T\|>\zeta\big\}\cap A\Big)=0, \tag{86}$$

and as ζ can be made arbitrarily small, we get the statement. □

The idea now is to divide the set of paths W alive at time T_{n+1} in two classes: the good paths, which have accumulated a large amount of collision local time in the time interval $[T_n\,,T_{n+1}]$, and the bad paths, which have not. Formally, define the set $E(n)$ of *good paths* on the interval $[T_n\,,T_{n+1}]$ with respect to the medium Γ as the set of paths W, which are not bad, i.e. where inequality (80) fails. For the good paths we use the comparison with the survival probability in Feller's branching diffusion from [10, Proposition 12].

Lemma 21 (Comparison with Feller's branching diffusion)**.** *For all* $n\ \ge\ 0$*, and every* ν *in* $\mathcal{M}_{\text{tem}}(\mathbb{R})$*, we have, for* P*–almost every* Γ,

$$\mathbb{P}^{\Gamma}_{\nu}\Big\{Y_{T_{n+1}}\big(E(n)\big)>0\;\Big|\;\mathcal{G}(T_n)\Big\}\le\frac{\|Y_{T_n}\|}{l_n}\,. \tag{87}$$

The lemma above takes care of the good paths, and it remains to show, that there are not too many bad paths. To show this we use $\mathbf{H_1}$ and $\mathbf{H_2}$. There are two possible reasons, why a path could be bad on $[T_n\,,T_{n+1}]$ for the medium Γ, namely the occurrence of one of the following two disjoint events.

- **Event** $B_1(n)$**:** the set of paths, that leave the interval $\big[-k(d{+}n),\,k(d{+}n)\big]$ during $[T_n\,,T_{n+1}]$ and thus enter an area where we have no control over the atoms,
- **Event** $B_2(n)$**:** the set of paths, that stay inside the interval $\big[-k(d+n),\,k(d+n)\big]$ but for which the collision local time accumulated during $[T_n\,,T_{n+1}]$ is below the threshold l_n.

Note that in case of event $B_2(n)$, if $n\ge N(\Gamma)$, the path stays in an area, where the medium coincides with an $(N+d)$–perfect medium. It is clear that we have the decompositions

$$\operatorname{supp}Y_{T_{n+1}}\subseteq E(n)\cup B(n)\quad\text{and}\quad B(n)\subseteq B_1(n)\cup B_2(n). \tag{88}$$

The following lemma provides estimates for the extinction probability of the bad paths.

Lemma 22 (Mass of bad paths). *Consider the historical superprocess Y with the starting measure $\nu = 2^{-m}\mu$. Then, for P–almost every Γ with $N(\Gamma) \le m$,*

$$\text{(i)}\ \sum_{n=0}^{\infty} \mathbb{P}_\nu^\Gamma\Big\{ Y_{T_{n+1}}\big(B_1(n)\big) \ge M_{n+1} \Big\} \le \frac{\varepsilon}{3},$$
$$\text{(ii)}\ \sum_{n=0}^{\infty} \mathbb{P}_\nu^\Gamma\Big\{ Y_{T_{n+1}}\big(B_2(n)\big) \ge M_{n+1} \,\Big|\, \|Y_{T_n}\| \le M_n \Big\} \le \frac{\varepsilon}{3}.$$

Proof. The proof is based on the expectation formula for the historical mass on a set B of stopped paths $W : [0, T_{n+1}] \to \mathbb{R}$. If B depends only on $\big\{W(s) : s \ge T_n\big\}$, we have

$$\mathbb{E}_\nu^\Gamma\left\{ Y_{T_{n+1}}(B) \,\big|\, Y_{T_n} \right\} = \int_{\mathbb{R}} \mathcal{P}_{T_n, \tilde W(T_n)}\{W \in B\}\, Y_{T_n}(d\tilde W), \tag{89}$$

see e.g. [10, (37)]. From Markov's inequality, the expectation formula and (79), we infer,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}_\nu^\Gamma\left\{ Y_{T_{n+1}}\big(B_1(n)\big) \ge M_{n+1} \right\} &\le \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \mathbb{E}_\nu^\Gamma\Big\{ Y_{T_{n+1}}\big(B_1(n)\big)\Big\} \\ &= \sum_{n=0}^{\infty} M_{n+1}^{-1} 2^{-m}\, \mathcal{P}_{0,x}\Big\{ \sup_{T_n \le s \le T_{n+1}} |W_s| > k(d+n) \Big\} \le \frac{\varepsilon}{3}, \end{aligned} \tag{90}$$

which is (i). Note that every path $W \notin B_1(n)$ spends the time $[T_n, T_{n+1}]$ inside a compact interval in which the medium Γ coincides with an $(m+d)$–perfect medium $\bar\Gamma$. Hence we can use the bound in (81) together with Markov's inequality, the special Markov property and the expectation formula (89) to see that,

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathbb{P}_\nu^\Gamma\Big\{ Y_{T_{n+1}}\big(B_2(n)\big) \ge M_{n+1} \,\Big|\, \|Y_{T_n}\| \le M_n \Big\} \\ &\quad \le \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \mathbb{E}_\nu^{\bar\Gamma}\left\{ \mathbb{E}^{\bar\Gamma}\Big\{ Y_{T_{n+1}}\big(B_2(n)\big) \,\Big|\, Y_{T_n} \Big\} \,\Big|\, \|Y_{T_n}\| \le M_n \right\} \\ &\quad \le \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \mathbb{E}_\nu^{\bar\Gamma}\Big\{ \int \mathcal{P}_{T_n, \tilde W(T_n)}\big\{W \in B_2(n)\big\}\, dY_{T_n}(\tilde W) \,\Big|\, \|Y_{T_n}\| \le M_n \Big\} \le \frac{\varepsilon}{3}, \end{aligned}$$

which is (ii). This ends the proof of the lemma. □

Now recall Lemma 20 and in particular the definition of the sets A_n and A. Lemmas 21 and 22 provide the ingredients we need to bound $\mathbb{P}_\nu^\Gamma(A^c)$ by ε, for the starting measure $\nu = 2^{-m}\mu$. Note that the event A_0 has probability one. Hence, for P–almost all Γ with $N(\Gamma) \le m$,

$$\text{(91a)} \qquad \mathbb{P}_\nu^\Gamma(A^c) = \sum_{n=0}^\infty \mathbb{P}_\nu^\Gamma\Big(A_0 \cap \cdots \cap A_n \cap A_{n+1}^c\Big)$$

$$\text{(91b)} \qquad \le \sum_{n=0}^\infty \mathbb{P}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_1(n)\big) \ge M_{n+1}\Big\}$$

$$\text{(91c)} \qquad + \sum_{n=0}^\infty \mathbb{P}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \ge M_{n+1}\;\Big|\; \|Y_{T_n}\| \le M_n\Big\}$$

$$\text{(91d)} \qquad + \sum_{n=0}^\infty \mathbb{P}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(E(n)\big) \ge M_{n+1}\;\Big|\; \|Y_{T_n}\| \le M_n\Big\}.$$

Now, by Lemma 22, the series in (91b) and (91c) are each bounded by $\varepsilon/3$. By Lemma 21, we obtain for (91d), using (82),

$$\text{(92)} \qquad \sum_{n=0}^\infty \mathbb{P}_\nu^\Gamma\Big\{Y_{T_{n+1}}\big(E(n)\big) \ge M_{n+1}\;\Big|\; \|Y_{T_n}\| \le M_n\Big\} \le \sum_{n=0}^\infty \frac{M_n}{l_n} \le \frac{\varepsilon}{3}\,.$$

Hence $\mathbb{P}_\nu^\Gamma(A^c) \le \varepsilon$ and, by Lemma 20, this implies the statement of the theorem. □

3.3. **The crossing property.** In this section we prove Theorem 4. The following lemma constitutes the main step in the proof.

Lemma 23 (Decay of crossing probability)**.** *There is an integer m depending only on the characteristic $N(\Gamma)$ of the catalytic medium Γ, such that, for all sufficiently large x,*

$$\text{(93)} \qquad \mathbb{P}^\Gamma_{2^{-m}\delta_x}\Big\{\int_0^\infty X_t(-\infty,0]\,dt > 0\Big\} \le \frac{1}{x^{3/2}}\,.$$

The lemma is proved by choosing the right ingredients for the use of the survival probability bound of Theorem 19. As a preparation for the proof, define the Brownian stopping time T to be the first hitting time of level 0, then the event we are interested in is the survival of Y_T. We denote

$$\text{(94)} \qquad d := d(x) := \Big[\log\Big(1+\frac{5}{2}\log x\Big)\Big]^2.$$

Observe that $d(x)$ is growing slower than logarithmically as $x \uparrow \infty$. We say that x is *sufficiently large* if $x \ge 1$ and

$$\text{(95)} \qquad d(x) \le \frac{\log x}{4(\log 2 + \beta)}\,.$$

Fix x sufficiently large, let $\mu = \delta_x$ and $\varepsilon = 1/x^{3/2}$. To define the remaining quantities for Theorem 19, we first leave the integer parameter m open and define T_n and l_n in terms of m. Recall that $\alpha < \beta < 2\alpha$ and define

$$\text{(96)} \qquad d_n := \frac{\mathrm{e}^{(\alpha-\beta)n}}{1-\mathrm{e}^{\alpha-\beta}} = \frac{1}{\varepsilon(m)}\exp\big((\alpha-\beta)(n+m)\big), \quad \text{where } \varepsilon(m) := \frac{\mathrm{e}^{(\alpha-\beta)m}}{1-\mathrm{e}^{\alpha-\beta}}\,.$$

Define barriers

$$\text{(97)} \qquad x_0 = x \quad\text{and}\quad x_{n+1} = x_n - x d_n\,.$$

As $\sum_{n=1}^{\infty} d_n = 1$, we have $x_n \downarrow 0$, and we can define an increasing sequence of hitting times

$$T_n := \inf\{t > 0 : B(t) = x_n\}, \tag{98}$$

so that $T = \lim_{n\uparrow\infty} T_n$. Finally define

$$l_n := x^{3/2}\, m\, n^2\, 2^{-n-m}. \tag{99}$$

Lemma 23 follows from Theorem 19 if we verify (82) and the conditions $\mathbf{H_1}\big(d, (\varepsilon/3)2^m\big)$ and $\mathbf{H_2}(m+d, l_n, \varepsilon/6)$ in x for $\varepsilon = 1/x^{3/2}$ and a suitable m, which we choose at the end of the proof. The next lemma prepares the verification of hypothesis $\mathbf{H_1}$.

Lemma 24 (Escape probability)**.** *For all integers $n, m \geq 0$, the events*

$$D_0 := D_0(n) := \Big\{W : \sup_{t\in[T_n, T_{n+1}]} |W_t| > k(d+n+m)\Big\} \tag{100}$$

have probability

$$\mathcal{P}_{T_n, x_n}\{W \in D_0(n)\} \leq \frac{2}{x^{3/2}} \exp(-\exp\sqrt{n+m}\,). \tag{101}$$

Proof. To begin with, note that d is chosen in such a way that, for all $n \geq 0$,

$$x^{5/2} \leq \frac{\exp\exp\sqrt{d+n}}{\exp\exp\sqrt{n}} = \frac{k(d+n)}{k(n)}. \tag{102}$$

A Brownian path starting in x_n is in D_0 if and only if it hits level $k(d+n+m)$ before x_{n+1}. The probability of this event is

$$\begin{aligned} \mathcal{P}_{T_n, x}\{W \in D_0(n)\} &\leq \frac{x d_n}{k(d+n+m) - x_{n+1}} \\ &\leq \frac{2x}{k(d+n+m)} \leq \frac{2}{x^{3/2} k(n+m)}, \end{aligned} \tag{103}$$

where we have used (102), $d_n \leq 1$ and $x_{n+1} \leq k(d+n+m)/2$. □

We observe that hypothesis $\mathbf{H_1}\big(d, (\varepsilon/3)2^m\big)$ is verified in x if m is chosen such that

$$\sum_{n=0}^{\infty} 2^{n+2} \exp\big(-\exp\sqrt{n+m}\,\big) \leq \frac{2^m}{3}, \tag{104}$$

however, there will be other constraints on m coming from hypothesis $\mathbf{H_2}$ and we turn to the verification of this hypothesis now. For this purpose, let m be arbitrary and fix an $(m+d)$–perfect medium Γ. Recall $\alpha < \beta < 2\alpha$ and fix $\beta < \delta < 2\alpha$. Let

$$a_n := x^{7/4}\, \mathrm{e}^{\delta n}\, m\, \mathrm{e}^{\beta m}. \tag{105}$$

To estimate the probability from above that a path W is bad in $[T_n, T_{n+1}]$ it suffices to consider a special hitting strategy: As the medium is $(m+d)$–perfect we can select, for every $l \geq m$, atoms of mass in $[2^{-l-d}, 2^{-l-d+1})$ in such a way that all neighbouring atoms have distance in $[\Delta_{l+d}/2, 3\Delta_{l+d}/2]$, we call this set of atoms $\tilde{\pi}_l = \tilde{\pi}_l(x)$. During $[T_n, T_{n+1}]$ we only count collisions with the atoms $\tilde{\pi}_{n+m}$. If a path is bad on $[T_n, T_{n+1}]$, i.e. if (80) holds, this must be due to one of the following two events.

Event $D_1(n)$**:** the set of paths such that during the time interval $[T_n, T_{n+1}]$ the number of collisions with the catalytic atoms of $\tilde{\pi}_{n+m}$ is too small, say less than a_n,

Event $D_2(n)$**:** the set of paths such that it takes more than a_n visits to the chosen atoms before the collision local time exceeds the threshold l_n.

Note that $B(n) \subseteq D_1(n) \cup D_2(n)$ and hence we have to check the probability that a single Brownian path encounters one of these two events. Note, that the constants $\theta > 0$ and $C_0, \dots, C_3$ in the following two lemmas only depend on our fixed choices α, β, δ and nothing else.

Lemma 25 (Probability of Event $D_1(n)$)**.** *We define a sequence* $\{\kappa_j\}$ *of Brownian stopping times by* $\kappa_0 := T_n$ *and, for* $j \geq 1$,

$$\kappa_j := \inf\Big\{ t \geq \kappa_{j-1} : W_t \text{ hits } \tilde{\pi}_{n+m} \setminus \{W_{\kappa_{j-1}}\}\Big\} \tag{106}$$

and denote

$$K_n := \max\{j : \kappa_j \leq T_{n+1}\}. \tag{107}$$

There is a $\theta > 0$ *and constants* $C_0, C_1 > 0$ *such that, for all integers* $n \geq 0$*, the events*

$$D_1 := D_1(n) := \{W : K_n \leq a_n\} \tag{108}$$

have probability

$$\mathcal{P}_{T_n, x_n}\big\{W \in D_1(n)\big\} \leq C_0 \exp\big(-C_1 a_n^\theta\big). \tag{109}$$

Proof. The indicated probability is bounded above by the probability that a simple random walk S_n defined on a probability space $(\Omega, \mathcal{A}, P)$ needs less than $a_n = x^{7/4} m\, \mathrm{e}^{\delta n + \beta m}$ steps to cross the level

$$\frac{x d_n}{(1/2)\Delta_{d+n+m}} \geq \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}. \tag{110}$$

By the reflection principle,

$$P\Big\{\max_{1 \leq k \leq a_n} S_k > \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}\Big\} \leq 2P\Big\{S_{a_n} \geq \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}\Big\}. \tag{111}$$

We fix a $0 < \theta_0 < 1/2$ such that $\delta(1-\theta_0) \leq \alpha$ and $(7/4)(1-\theta_0) \leq 1$. By the refinement of Cramér's Theorem given in [13, (3.7.1)] for fixed $\theta_0 < \theta_1 < 1/2$ and $c = 2(1-\mathrm{e}^{\alpha-\beta})$ there is a constant $C_0 > 0$ such that, for all integers k,

$$P\Big\{S_k \geq c k^{1-\theta_0}\Big\} \leq C_0 \exp\Big(-k^{1-2\theta_1}(c^2/2)\Big). \tag{112}$$

Our choice of θ_0 and c is such that

$$\frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)} \geq c a_n^{1-\theta_0}. \tag{113}$$

Hence we can use (112) and put $\theta := 1 - 2\theta_1 > 0$ to get

$$P\Big\{S_{a_n} > \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}\Big\} \leq C_0 \exp\Big(-a_n^\theta (c^2/2)\Big), \tag{114}$$

which is the required estimate with $C_1 := 2(1-\mathrm{e}^{\alpha-\beta})^2$. □

Lemma 26 (Probability of Event $D_2(n)$)**.** *For a Brownian path* W *started in* x_n *we denote by* $\{y_i = W_{\kappa_i} : 1 \leq i \leq a_n\}$ *the sequence of atoms in* $\tilde{\pi}_{n+m}$ *hit by the path. Define*

$$L_i := \int_{\kappa_i}^{\kappa_{i+1}} L^{y_i}(ds). \tag{115}$$

There are constants $C_2 > 0$ and $C_3 > 0$ such that, for all integers $n \geq 0$, the events

$$D_2(n) := \Big\{ W : \sum_{i=1}^{a_n} 2^{-(d+n+m)} L_i < l_n \Big\} \tag{116}$$

have probability

$$\mathcal{P}_{T_n, x_n}\big\{ W \in D_2(n) \big\} \leq C_2 \, \exp\big(-C_3 a_n \big). \tag{117}$$

Proof. By scaling, $\{L_i\}$ is bounded below by a sequence $\{\tilde{L}_i\}$ of independent, identically distributed positive random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ such that the distribution of $\tilde{L}_i/\Delta_{n+m+d}$ is independent of d, n and m. Hence,

$$\begin{aligned} \mathcal{P}_{T_n, x}\big\{ W \in D_2(n) \big\} &\leq P\Big\{ \sum_{j=1}^{a_n} 2^{-n-m-d} \tilde{L}_j < l_n \Big\} \\ &\leq P\Big\{ \frac{1}{a_n} \sum_{j=1}^{a_n} \frac{\tilde{L}_j}{\Delta_{n+m+d}} < \frac{2^{n+m+d} l_n}{a_n \Delta_{n+m+d}} \Big\}. \end{aligned} \tag{118}$$

Note that, for all $n, m \in \mathbb{N}$ and $x > 0$,

$$\frac{2^{n+m+d} l_n}{a_n \Delta_{n+m+d}} = \frac{n^2 2^d \, e^{\beta d}}{e^{(\delta-\beta)n} x^{1/4}} \leq n^2 e^{(\beta-\delta)n} \longrightarrow 0, \text{ as } n \uparrow \infty, \tag{119}$$

using that $2^d e^{\beta d} \leq x^{1/4}$ by (95). Hence, by Cramér's Theorem, the right hand side in (118) is bounded above by $C_2 \exp(-C_3 a_n)$, for suitable constants $C_2, C_3 > 0$. □

Completion of the proof of Lemma 23. It is now time to choose the value of m large enough such that $m \geq N(\Gamma)$ and the following set of conditions is satisfied, for $\varepsilon = 1/x^{3/2}$:

$$\text{(120a)} \quad \sum_{n=0}^{\infty} 2^{n+2} \exp\big(-\exp \sqrt{n+m} \,\big) \leq \frac{2^m}{3}, \qquad \text{(120b)} \quad \sum_{n=0}^{\infty} C_0 \exp\big(-C_1 \, a_n^{\theta} \big) \leq \frac{\varepsilon}{12},$$

$$\text{(120c)} \quad \sum_{n=0}^{\infty} C_2 \exp(-C_3 \, a_n) \leq \frac{\varepsilon}{12}, \qquad \text{(120d)} \quad \sum_{n=1}^{\infty} \frac{1}{mn^2} \leq \frac{1}{3}.$$

Note that a_n is a multiple of $x^{7/4}$ and hence m can be chosen independently of x. We have already seen in (104) that (120a) implies $\mathbf{H_1}\big(d, (\varepsilon/3)2^m\big)$. Moreover, (120b) and (120c) together with Lemmas 25 and 26 imply condition $\mathbf{H_2}(m+d, l_n, \varepsilon/6)$ and, finally (120d) is (82). Hence Lemma 23 follows from Theorem 19. □

Completion of the proof of the crossing property, Theorem 4. We use the branching property and the result of Lemma 23 to see that, for sufficiently large integers x,

$$\begin{aligned} \mathbb{P}^{\Gamma}_{2^{-m}\ell[x,x+1]}\Big\{ \int_0^{\infty} X_t(-\infty, 0] \, dt = 0 \Big\} &= \exp\bigg(\int_x^{x+1} \log \mathbb{P}^{\Gamma}_{2^{-m}\delta_y}\Big\{ \int_0^{\infty} X_t(-\infty, 0] \, dt = 0 \Big\} \, dy \bigg) \\ &\geq \exp\Big(\int_x^{x+1} \log\big(1 - y^{-3/2} \big) \, dy \Big) \geq 1 - \frac{1}{x^{3/2}}. \end{aligned} \tag{121}$$

Hence $\mathbb{P}^{\Gamma}_{2^{-m}\ell[x,x+1]}\{\int_0^{\infty} X_t(-\infty, 0] \, dt > 0\}$ is summable over all positive integers x. Hence, by the Borel-Cantelli Lemma, there exists some random integer $K > 0$ such that the process X

started in $2^{-m}\ell_{[K,\infty]}$ has the property $\int_0^\infty X_t(-\infty,0]\,dt = 0$. Repeating this argument 2^m times independently and taking the maximal value of K one can see that it remains to argue that

$$\sup_{t\geq 0} X_t(-\infty,0] < \infty, \text{ for } X_0 = \ell_{[0,K]}\,. \tag{122}$$

Because the process X with finite starting measure with compact support has the finite time extinction property, by[10, Theorem 6], there is a random time $T > 0$ such $X_t = 0$ for all $t \geq T$. By continuity of the process, finally, $X_t(-\infty,0]$ is bounded on $[0,T]$ and we are done. □

3.4. **The compound Poisson property.** In this section we prove Theorem 1 (ii). The following lemma constitutes the main step in the proof.

Lemma 27 (Lower bound for extinction). *For every time $t > 0$ there is a constant $\theta = \theta(t) > 0$, such that for P–almost all Γ,*

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\big\{X_t = 0\big\} \geq \theta^{(2^{N(\Gamma)})}. \tag{123}$$

To prove Lemma 27 by application of Theorem 19 we proceed similarly as in [10]. We fix t, leave the integer parameter m open for a while and define deterministic times T_n and thresholds l_n in terms of m. We first let

$$\varepsilon(m) := \Big(\frac{2}{t}\sum_{n=m}^{\infty} \mathrm{e}^{(\alpha-\beta)n}\Big)^{1/3}. \tag{124}$$

We then define $m_n := [\mathrm{e}^{\alpha(n+m)}/\varepsilon(m)]$ and $s_n := \mathrm{e}^{-\beta(n+m)}/\varepsilon(m)^2$ (here $[\bullet]$ denotes the integer part). Put

$$T_0 := 0 \text{ and } T_{n+1} := T_n + 2m_n s_n\,. \tag{125}$$

Note that $t \geq T := \lim_{n\uparrow\infty} T_n$. Finally define

$$\bar{l}_n := m_n\sqrt{s_n}\,2^{-n}. \tag{126}$$

We later define l_n to be a constant multiple of $\bar{l}_n$. Lemma 27 follows from Theorem 19 if we verify (82) and the hypotheses $\mathbf{H_1}(d,(\varepsilon/3)2^m)$ and $\mathbf{H_2}(m+d,l_n,\varepsilon/6)$ in all $x \in [0,1]$ for $d = 0$, $\varepsilon = 1/2$ and a suitable integer m, which we choose at the end of the proof. Indeed, define $\theta := (1/2)^{(2^m)} > 0$. By the branching property and Theorem 19 we obtain, for $M = m + N(\Gamma)$,

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\big\{X_t = 0\big\} \geq \Big(\mathbb{P}^{\Gamma}_{\ell_{[0,1]2^{-M}}}\big\{Y_t = 0\big\}\Big)^{(2^M)} \geq (1/2)^{(2^M)} = \theta^{(2^{N(\Gamma)})}, \tag{127}$$

which is the statement of Lemma 27.

To prepare the verification of the hypotheses we formulate three lemmas. The constants $C_0,\dots,C_3$ in these lemmas depend only on the fixed values of α and β. The first lemma is the main ingredient in the verification of hypothesis $\mathbf{H_1}$.

Lemma 28 (Escape probability). *There is a constant $C_0 > 0$ such that, for all starting points $x \in [0,1]$ and all integers $n, m \geq 0$, the events*

$$\overline{D_0} := \overline{D_0}(n) := \Big\{W : \sup_{s\in[T_n,T_{n+1}]} |W_s| > k(n+m)\Big\} \tag{128}$$

have probability

$$\mathcal{P}_{0,x}\big\{W \in \overline{D_0}(n)\big\} \leq C_0 \exp\big(-\exp\sqrt{n+m}\,\big). \tag{129}$$

Proof. Recall that the random variable $\sup_{0\le s\le t}|W_s|$ has finite first moment. Hence, using Markov's inequality, there is a constant $C_0>0$, depending only on t, such that

$$\mathcal{P}_{0,x}\Big\{\sup_{T_n\le s\le T_{n+1}}|W_s|>k(n+m)\Big\}\le \mathcal{P}_{0,x}\Big\{\sup_{0\le s\le t}|W_s|>k(n+m)\Big\}\le C_0\exp(-\exp\sqrt{n+m}\,)$$

for all $x\in[0,1]$ and $n,m\ge 0$. □

We conclude that $\mathbf{H_1}\big(d,(\varepsilon/3)2^m\big)$ holds if m is chosen such that

$$\sum_{n=0}^{\infty} C_0\,2^{n+1}\exp\big(-\exp\sqrt{n+m}\,\big)\le\frac{1}{6}\,2^m. \tag{130}$$

Again further restrictions on m follow from the verification of $\mathbf{H_2}$. For this purpose let Γ be an m–perfect medium and define the set $\bar\pi_{n+m}$ consisting of the atoms of Γ with mass in $[2^{-n-m},2^{-n-m+1})$. Note that the neighbouring pairs of atoms in $\bar\pi_{n+m}$ are no further than Δ_{n+m} apart. For our estimate we consider on the interval $[T_n,T_{n+1}]$ only the collisions with the atoms of $\bar\pi_{n+m}$. In fact, we can even restrict our view to a selection of collisions chosen according to a special strategy of [10], which is based on our choice of the sequences $s_n>0$ of small times and m_n of positive integers. Heuristically, on the interval $[T_n,T_{n+1}]$ the strategy suggests to wait until the Brownian particle hits the first atom of $\bar\pi_{n+m}$, count the collision local time with this particular atom for s_n time units and then wait for the next collision with $\bar\pi_{n+m}$. This procedure is iterated until m_n atoms are visited. A visualisation of this procedure can be found in [16, Figure 5]. The path W is good on $[T_n,T_{n+1}]$ unless one of the following two events has taken place.

Event $\overline{D_1}(n)$: the set of paths for which in our strategy the waiting times between the collisions with the catalytic atoms of $\bar\pi_{n+m}$ are too long, so that we have less than m_n visits during the time interval $[T_n,T_{n+1}]$,

Event $\overline{D_2}(n)$: the set of paths for which the collision local time accumulated during m_n visits of the path to the chosen atoms is below the threshold l_n.

Lemma 29 (Probability of Event $\overline{D_1}(n)$)**.** *We define a sequence $\{\kappa_n\}$ of Brownian stopping times by $\kappa_0:=T_n$ and, for $j\ge 1$,*

$$\kappa_j:=\bar\kappa_j+s_n\,,\ \text{where } \bar\kappa_j:=\inf\big\{s\ge\kappa_{j-1}:\,W_s\text{ hits }\bar\pi_{j+m}\big\}\,, \tag{131}$$

and denote waiting times by $H_m:=\bar\kappa_m-\kappa_{m-1}$. There is a constant $C_1>0$ such that, for all starting points x and integers $n\ge 0$, the events

$$\overline{D_1}(n):=\Big\{W:\,\sum_{j=1}^{m_n}H_j\ge m_n s_n\Big\} \tag{132}$$

have probability

$$\mathcal{P}_{T_n,x}\big\{W\in\overline{D_1}(n)\big\}\le C_1^{-1}m_n\exp\Big(-\frac{C_1 s_n}{\Delta_{n+m}^2}\Big). \tag{133}$$

Proof. This is estimate (93) in [10]. □

Lemma 30 (Probability of Event $\overline{D_2}(n)$)**.** *There are constants $C_2>0$ and $C_3>0$ with the property that, for all starting points x and integers $n\ge 0$, the events*

$$\overline{D_2}(n):=\Big\{W:\,\int_{T_n}^{\kappa_{m_n}}L_{[W,\Gamma]}(dr)<C_2\,\bar l_n\Big\} \tag{134}$$

have probability

$$\mathcal{P}_{T_n,x}\big\{W \in \overline{D_2}(n)\big\} \le \exp\big(-2C_3 m_n\big). \tag{135}$$

Proof. This is estimate (99) in [10]. □

Completion of the proof of Lemma 27. It is now clear that with the threshold value $l_n = C_2\,\bar{l}_n$ we have the decomposition

$$B(n) \subseteq \overline{D_1}(n) \cup \overline{D_2}(n). \tag{136}$$

Having provided the estimates for the probability of a path being bad, it is now time to make precise the value of m. We choose m large enough such that the following set of conditions is satisfied

$$\sum_{n=0}^{\infty} C_0\, 2^{n+1} \exp\big(-\exp\sqrt{n+m}\,\big) \;\le\; \frac{1}{6}\, 2^m, \tag{137a}$$

$$\sum_{n=m}^{\infty} \frac{1}{C_1}\, \frac{\mathrm{e}^{\alpha n}}{\varepsilon(m)}\, \exp\Big(-C_1\, \frac{\mathrm{e}^{\beta n}}{\varepsilon(m)^2}\Big) \;\le\; \frac{1}{12}\,, \tag{137b}$$

$$\sum_{n=m}^{\infty} \exp\Big(-2C_3\, \frac{\mathrm{e}^{\alpha n}}{\varepsilon(m)}\Big) \;\le\; \frac{1}{12}\,, \tag{137c}$$

$$\frac{1}{C_2}\, 2^{-m}\, \varepsilon(m)^2\, \mathrm{e}^{(\beta/2-\alpha)m} \sum_{n=0}^{\infty} \mathrm{e}^{(\beta/2-\alpha)n} \;\le\; \frac{1}{6}\,. \tag{137d}$$

Note that it is possible to find such an m: For (137a) this is trivial, for (137c) this is due to the fact that $\varepsilon(m) \downarrow 0$ and for (137d) note that $\alpha > \beta/2$. For (137b) it suffices to check that, for $a, b \ge 1$, the function $x \mapsto (a/x)\exp(-b/x^2)$ is increasing on the interval $(0,1)$. Hence hypothesis $\mathbf{H_1}(0, 2^m/6)$ holds by (137a), see(130), and $\mathbf{H_2}(m, l_n, 1/12)$ holds by (137b), (137c) together with Lemmas 29 and 30. Finally, (137d) is (82). This finishes the proof. □

Completion of the proof of the compound Poisson property, Theorem 1 (ii). Fixing $t > 0$ and a starting measure $\ell_{(a,b)}$, for (a,b) an interval of unit length, we have to show that the measure

$$\pi^\infty(dw\,dx) = \int_a^b \int_{\mathcal{M}_{\text{tem}}} \mathbb{N}_0^\Gamma(dw) \otimes \delta_y(dx)\, \mathsf{P}(d\Gamma)\, dy \tag{138}$$

is finite on the set

$$S = \Big\{(w,x) \in \mathfrak{W} \times (a,b) :\, L_\sigma^t[w] > 0\Big\}. \tag{139}$$

Then the snake representation Theorem 5 (ii) of the limit process describes a compound Poisson point field on (a,b) with underlying Poisson intensity $\lambda(t) := \pi^\infty(S)$. To prove finiteness of $\lambda(t)$ we have to show that the following expression is finite

$$\lambda(t) = \mathsf{E}\Big\{\mathbb{N}_0^\Gamma\big\{w :\, L_\sigma^t[w] > 0\big\}\Big\} = \mathsf{E}\Big\{\int_0^1 \mathbb{N}_x^\Gamma\big\{w :\, L_\sigma^t[w] > 0\big\}\, dx\Big\}, \tag{140}$$

where we have used the fact that the distribution of $L_\sigma^t[w]$ under $\mathbb{N}_x^\Gamma$ is independent of x. To interpret the integrand on the right hand side of (140) recall the snake representation in Theorem 5 (i). The process X started in $X_0 = \ell_{[0,1]}$ has become extinct at time t if and only if a Poisson point field with intensity $\int_0^1 \mathbb{N}_x^\Gamma \left\{w : L^t[w] > 0\right\} dx$ has the value 0, the probability of this event is

$$\exp\Big(-\int_0^1 \mathbb{N}_x^\Gamma \{w : L_\sigma^t > 0\}\, dx\Big). \tag{141}$$

This reduces our task to showing that

$$\mathsf{E}\Big\{-\log \mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\}\Big\} < \infty. \tag{142}$$

We now apply Fubini's Theorem to rewrite

$$\begin{aligned} \mathsf{E}\Big\{-\log \mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\}\Big\} &= \int_0^\infty \mathsf{P}\Big\{-\log \mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} > a\Big\}\, da \\ &= \int_0^\infty \mathsf{P}\Big\{\mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\}\, da. \end{aligned} \tag{143}$$

Hence our problem can be formulated as

$$\int_0^\infty \mathsf{P}\Big\{\mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\}\, da < \infty. \tag{144}$$

Here comes the *key idea* of our proof: With respect to the random medium Γ the event

$$\Big\{\mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\} \tag{145}$$

can only occur if Γ has unusually low density, or equivalently, if the points in the Poisson point fields π_n introduced before (72) are unusually far apart. This can be expressed in terms of the quantity $N(\Gamma)$. In fact, by Lemma 27,

$$\mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} \ge \exp\Big((\log\theta) 2^{N(\Gamma)}\Big), \tag{146}$$

and hence, the latter event implies

$$(\log\theta)2^{N(\Gamma)} < -a \iff N(\Gamma) > \frac{1}{\log 2}\, \log\big(-a/\log\theta\big)\,. \tag{147}$$

We can now use the estimate (73) obtained in the large gaps lemma, Lemma 17, for the quantity $N(\Gamma)$,

$$\begin{aligned} &\int_0^\infty \mathsf{P}\Big\{\mathbb{P}^\Gamma_{\ell_{[0,1]}}\{X_t = 0\} < \mathrm{e}^{-a}\Big\}\, da \\ &\le \int_0^\infty \mathsf{P}\Big\{N(\Gamma) > \frac{1}{\log 2}\, \log\big(-a/\log\theta\big)\Big\}\, da \\ &\le c\int_0^\infty \exp\Big(-d\exp\Big[(\varrho/\log 2)\log\big(-a/\log\theta\big)\Big]\Big)\, da \\ &\le c_0 + c_1\int_0^\infty \exp\big(-c_2 a^{c_3}\big)\, da < \infty, \end{aligned} \tag{148}$$

using suitable constants $c_0, c_1, c_2, c_3 > 0$. This proves (144), and Theorem 1 (ii) is established. □

4. Properties of the macroscopic clumps

In this section we prove the various parts of Theorem 10. Part (i) and, perhaps surprisingly, part (ii) can be obtained by soft arguments, whereas part (iii) requires a new approach based on a Feynman-Kac formula for the solutions of (6).

4.1. **The extinction probability of the clumps.** In this subsection we prove Theorem 10 (i) and (ii). From the definition of the renormalised processes (9) we infer, for all $k, l > 0$ and $t \geq 0$,

$$X_t^{kl}(B) = k^{-\eta} l^{-\eta} X_{(kt)l}\big(l^\eta(k^\eta B)\big), \qquad \text{for } B \subseteq \mathbb{R} \text{ Borel.} \tag{149}$$

Chosing any continuity set $B \subset \mathbb{R}$ and letting $l \uparrow \infty$ we obtain the self-similarity property

$$X_t^\infty(B) = k^{-\eta} X_{kt}^\infty(k^\eta B) \quad \text{ in distribution,} \tag{150}$$

first for all continuity sets $B \subset \mathbb{R}$ and then, by approximation, for all Borel sets $B \subset \mathbb{R}$. This proves Theorem 10 (i) and is also the key to part (ii). By the compound Poisson structure of X_t^∞ the Laplace functionals have the form

$$\mathbf{E}_\ell\Big\{\exp\big(-\theta X_t^\infty(0,a)\big)\Big\} = \exp\Big(- \lambda(t)\, a\big(1 - \Lambda^t(\theta)\big)\Big), \tag{151}$$

where $\lambda(t)$ is the intensity of the Poisson point field underlying the compound Poisson point field and Λ^t is the Laplace functional of the weights of an atom. Using (150) one obtains

$$\begin{aligned} \exp\Big(- \lambda(t)\, a\big(1 - \Lambda^t(\theta)\big)\Big) &= \mathbf{E}_\ell\Big\{\exp\big(-\theta X_t^\infty(0,a)\big)\Big\} = \mathbf{E}_\ell\Big\{\exp\big(-\theta k^{-\eta} X_{kt}^\infty(0,k^\eta a)\big)\Big\} \\ &= \exp\Big(- \lambda(kt) k^\eta a\big(1 - \Lambda^{kt}(\theta k^{-\eta})\big)\Big). \end{aligned} \tag{152}$$

We infer that $\lambda(t) = k^\eta \lambda(kt)$ and $\Lambda^t(\theta) = \Lambda^{kt}(\theta k^{-\eta})$. The former expression gives us the decay of the intensity $\lambda(t) = t^{-\eta}\lambda(1)$ of the Poisson point field, the latter yields the equality in distribution of $\beth_t(t)$ and $(t/s)^\eta \beth_s(s)$. Using $\lambda(s)\mathbf{P}_\ell\big\{\beth_s(t) > 0\big\} = \lambda(t)$ for $t > s$, we infer that the survival probabilities of the clumps satisfy

$$\mathbf{P}_\ell\big\{\beth_s(t) > 0\big\} = \Big(\frac{s}{t}\Big)^\eta \text{ and } \mathbf{P}_\ell\big\{X_t^\infty(0,a) > 0\big\} \sim \frac{\lambda(1)\, a}{t^\eta}, \text{ as } a \uparrow \infty, \tag{153}$$

where the latter form is obtained by conditioning on the number of clumps in an interval. □

4.2. **The tail behaviour of the clump size.** This subsection is devoted to the proof of Theorem 10 (iii). We first note that it suffices to give the proof for a fixed value of t, because the particular dependence on t, which is claimed in Theorem 10 (iii), already follows from the self-similarity of the process $\big\{\beth_t(t) : t > 0\big\}$ proved in Subsection 4.1.

We use the Feynman-Kac representation of the solutions $U\theta := U^\Gamma\theta$ of (12),

$$U_t\theta\,(y) = \theta - 2\int_0^t ds \int_{\mathbb{R}} p_s(x-y)\,\big[U_{t-s}\theta\,(x)\big]^2\,\Gamma(dx), \tag{154}$$

in order to obtain the tail asymptotics of the mass clumps. Recall that this equation can also be written probabilistically as

$$U_t\theta\,(y) = \theta - 2\,\mathcal{E}_y\Big\{\int_0^t \big[U_{t-s}\theta\,(W_s)\big]^2\, L_{[\Gamma,W]}(ds)\Big\}, \tag{155}$$

where $\mathcal{E}_{s,y}$ is used to indicate expectation with respect to a Brownian motion W started at time s in y, $\mathcal{E}_y := \mathcal{E}_{0,y}$, and $L_{[\Gamma,W]}$ is the collision local time between Γ and W, as defined in (15).

Lemma 31 (Feynman-Kac-representation). *For each fixed* Γ *the family* $U = \big\{U_t\theta\,(y) : t \geq 0,\, y \in \mathbb{R}\big\}$ *is a solution of* (154) *if and only if it is a solution of*

$$U_t\theta\,(y) = \theta\,\mathcal{E}_y\Big\{\exp\Big(- 2\int_0^t U_{t-s}\theta\,(W_s)\, L_{[W,\Gamma]}(ds)\Big)\Big\}, \quad \textit{for } t \geq 0, \ \ y \in \mathbb{R}. \tag{156}$$

Proof. Fix Γ. We first show that every solution of (156) solves (154). It suffices to consider $\theta = 1$, as the dependence on θ is simple. Let $U = \{U_t(y) : t \geq 0,\ y \in \mathbb{R}\}$ be a bounded solution of equation (156), which is continuous in t. By the fundamental theorem of calculus,

$$\begin{aligned} &\exp\Big[-2\int_0^t U_{t-r}(W_r)\, L_{[W,\Gamma]}(dr)\Big] \\ &= 1 - 2\int_0^t U_{t-s}(W_s) \exp\Big[-2\int_s^t U_{t-r}(W_r)\, dL_{[W,\Gamma]}(r)\Big]\, L_{[W,\Gamma]}(ds). \end{aligned} \tag{157}$$

Taking expectations,

$$U_t(y) - 1 = -2\,\mathcal{E}_y\Big\{\int_0^t U_{t-s}(W_s) \exp\Big[-2\int_s^t U_{t-r}(W_r)\, dL_{[W,\Gamma]}(r)\Big]\, L_{[W,\Gamma]}(ds)\Big\}. \tag{158}$$

The Markov property (and a glance at the definition of Stieltjes integrals) allows us to continue this with

$$\begin{aligned} &= -2\,\mathcal{E}_y\Big\{\int_0^t U_{t-s}(W_s)\, \mathcal{E}_{W_s}\Big\{\exp\Big(-2\int_0^{t-s} U_{t-s-r}(W_r)\, dL_{[W,\Gamma]}(r)\Big)\Big\}\, L_{[W,\Gamma]}(ds)\Big\} \\ &= -2\,\mathcal{E}_y\Big\{\int_0^t \{U_{t-s}(W_s)\}^2\; L_{[W,\Gamma]}(ds)\Big\} = -2\int_{\mathbb{R}} \Gamma(dx)\int_0^t p_s(y-x)\,\big[U_{t-s}(x)\big]^2\, ds, \end{aligned} \tag{159}$$

which is the formula we had to prove.

To show conversely that every solution $U_t(y)$ of (154) solves (156), we start with the formula

$$\begin{aligned} &2\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big)\Big[U_{t-s}(W_s) - 1\Big] U_{t-s}(W_s)\, L_{[\Gamma,W]}(ds) \\ &= -\int_0^t \big[U_{t-s}(W_s) - 1\big]\; d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big)\Big). \end{aligned} \tag{160}$$

We take the expectation, use (155), apply the Markov property as before, and finally use Fubini's Theorem to see

$$\begin{aligned} &\mathcal{E}_y\Big\{2\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big)\,\big[U_{t-s}(W_s)\big]^2\; L_{[\Gamma,W]}(ds)\Big\} \\ &\quad - 2\,\mathcal{E}_y\Big\{\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big) U_{t-s}(W_s)\, L_{[\Gamma,W]}(ds)\Big\} \\ &= -\mathcal{E}_y\Big\{\int_0^t \mathcal{E}_{s,W_s}\Big\{-2\int_s^t \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\} d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big)\Big)\Big\} \\ &= \mathcal{E}_y\Big\{2\int_0^t \int_s^t \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\, d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big)\Big)\Big\} \\ &= \mathcal{E}_y\Big\{2\int_0^t \int_0^v d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big)\Big)\,\big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\} \\ &= \mathcal{E}_y\Big\{2\int_0^t \Big[\exp\Big(-2\int_0^v U_{t-r}(W_r)\, L_{[\Gamma,W]}(dr)\Big) - 1\Big]\,\big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\}. \end{aligned}$$

From this and (155) we infer that

$$\mathcal{E}_y\Big\{2\int_0^t \exp\Big(-2\int_0^s U_{t-v}(W_v)\,L_{[\Gamma,W]}(dv)\Big)U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds)\Big\} \tag{161}$$
$$= \mathcal{E}_y\Big\{2\int_0^t \big[U_{t-r}(W_r)\big]^2\,L_{[\Gamma,W]}(dr)\Big\} = 1 - U_t(y).$$

By the fundamental theorem once more, expression (161) equals

$$1 + \mathcal{E}_y\Big\{-\exp\Big(-2\int_0^t U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big\}, \tag{162}$$

from which (156) follows. □

We now aim for upper and lower bounds of $\mathsf{E}U_t\theta\,(0)$, which give the tail asymptotic of the clump sizes by means of a Tauberian theorem. From (12) one immediately sees that $U_t\theta\,(y) \le \theta$ and hence $\mathsf{E}U_t\theta\,(0) \le \theta$. Now plugging the estimate into (12) leads only to a trivial lower bound for $\mathsf{E}U_t\theta\,(y)$. A better lower bound is obtained by means of the Feynman-Kac representation.

Lemma 32 (Asymptotic behaviour of $\mathsf{E}U_t\theta$)**.** *For every $t > 0$ there are positive, finite constants $C_1 = C_1(t)$ and $C_2 = C_2(t)$, such that*

$$\theta - C_1(t)\,\theta^{\gamma+1} \le \mathsf{E}U_t\theta\,(0) \le \theta - C_2(t)\,\theta^{\gamma+1}, \ \textit{for all } \theta \in (0,1). \tag{163}$$

Proof. We fix $t > 0$. Plugging $U_t\theta\,(y) \le \theta$ into the Feynman-Kac-representation (156) yields,

$$U_t\theta\,(y) \ge \theta\,\mathcal{E}_y\Big\{\exp\big(-\theta L_{[W,\Gamma]}(t)\big)\Big\} \ \text{ for all } y \in \mathbb{R}, \tag{164}$$

Taking expectation with respect to the medium and using the Laplace functional formula (3) for stable random measures gives

$$\begin{aligned} \mathsf{E}U_t\theta\,(0) \ &\ge\ \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta L_{[W,\Gamma]}(t)\Big)\Big\} \ =\ \theta\,\mathcal{E}_0\exp\Big(-2^\gamma\theta^\gamma\int_{\mathbb{R}} L^x(t)^\gamma\,dx\Big) \\ &\ge\ \theta\Big(1 - 2^\gamma\theta^\gamma\,\mathcal{E}_0\int_{\mathbb{R}} L^x(t)^\gamma\,dx\Big) \ \ =\ \theta - \theta^{\gamma+1}2^\gamma\int_{\mathbb{R}}\mathcal{E}_0\left\{L^x(t)^\gamma\right\}\,dx. \end{aligned} \tag{165}$$

We now show that

$$C_1(t) := 2^\gamma\int_{\mathbb{R}}\mathcal{E}_0\big\{L^x(t)^\gamma\big\}\,dx = t^{\gamma\eta}\,\frac{\gamma}{\gamma+1}\,\frac{2^{\gamma(\eta+1)}}{\sqrt{\pi}}\,G\Big(\frac{\gamma}{2}\Big) < \infty, \tag{166}$$

where G denotes the Gamma function. Indeed, for $x, y > 0$ the density function of $L^x(t)$ at y is given by $\sqrt{2/\pi t}\,\exp\big(-(x+y)^2/2t\big)$. Hence

$$\int_{\mathbb{R}}\mathcal{E}_0\big\{L^x(t)^\gamma\big\}\,dx = \sqrt{\frac{8}{\pi t}}\int_0^\infty y^\gamma\int_y^\infty \exp(-z^2/2t)\,dz\,dy \tag{167}$$
$$= \sqrt{\frac{8}{\pi t}}\int_0^\infty \frac{y^{\gamma+1}}{\gamma+1}\exp(-y^2/2t)\,dy,$$

using integration by parts. One can get the result by substituting $x = y^2/2t$ and recalling the definition of the Gamma function G.

To obtain the upper bound fix $t > 0$. We use (164) in (156) and obtain

$$U_t\theta\,(y) \le \theta\,\mathcal{E}_y\Big\{\exp\Big(-2\,\theta\int_0^t \mathcal{E}_{W_s}\Big\{\exp\Big(-2\,\theta L_{[\tilde{W},\Gamma]}(t-s)\Big)\Big\}\,L_{[W,\Gamma]}(ds)\Big)\Big\}. \tag{168}$$

Next we write $\Gamma = \sum_{j=1}^{\infty} a_j \delta_{x_j}$ and use Jensen's inequality,

$$
\begin{aligned}
\mathsf{E} U_t \theta(0) &\le \theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \int_0^t L_{[W,\Gamma]}(ds) \, \mathcal{E}_{W_s} \{ \exp\big(-2\theta L_{[\tilde{W},\Gamma]}(t-s) \big) \} \Big) \Big\} \\
&\le \theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{i=1}^{\infty} a_i \int_0^t L^{x_i}(ds) \, \mathcal{E}_{x_i} \{ \exp\big(-2\theta \sum_{j=1}^{\infty} a_j L^{x_j}(t-s) \big) \} \Big) \Big\} \qquad (169) \\
&\le \theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{i=1}^{\infty} a_i \int_0^t L^{x_i}(ds) \, \Big[\exp\big(-2\theta \sum_{j=1}^{\infty} a_j \, \mathcal{E}_{x_i} \{ L^{x_j}(t-s) \} \big) \Big] \Big) \Big\}.
\end{aligned}
$$

Using monotonicity we can continue the estimate with

$$
\begin{aligned}
&\theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{i=1}^{\infty} a_i \int_0^t L^{x_i}(ds) \, \Big[\exp\big(-2\theta \sum_{j=1}^{\infty} a_j \, \mathcal{E}_{x_i} \{ L^{x_j}(t-s) \} \big) \Big] \Big) \Big\} \\
&\qquad \le \theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{i=1}^{\infty} a_i \, L^{x_i}(t) \Big[\exp\big(-2\theta \sum_{j=1}^{\infty} a_j \, \mathcal{E}_{x_i} \{ L^{x_j}(t) \} \big) \Big] \Big) \Big\} \qquad (170) \\
&\qquad \le \theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{|x_i| \le 1} a_i \, L^{x_i}(t) \Big[\exp\big(-2\theta \sum_{j=1}^{\infty} a_j \, \mathcal{E}_{x_i} \{ L^{x_j}(t) \} \big) \Big] \Big) \Big\}.
\end{aligned}
$$

Now we split the sum in the innermost exponential into the sum over the atoms x_j inside, respectively outside, the unit ball. Recall that

$$
\mathcal{E}_{x_i} \big\{ L^{x_j}(t) \big\} = f(t, x_i - x_j) \ \text{ for } \ f(t,x) := \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\Big(-\frac{x^2}{2s} \Big) \, ds. \qquad (171)
$$

For the atoms x_j inside the unit ball we use the estimate

$$
\mathcal{E}_{x_i} \big\{ L^{x_j}(t) \big\} \le c_0 := \sqrt{\frac{2t}{\pi}}, \qquad (172)
$$

which gives

$$
\begin{aligned}
&\theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{|x_i| \le 1} a_i \, L^{x_i}(t) \Big[\exp\big(-2\theta \sum_{j=1}^{\infty} a_j \, \mathcal{E}_{x_i} \{ L^{x_j}(t) \} \big) \Big] \Big) \Big\} \qquad (173) \\
&\quad \le \theta \, \mathsf{E} \, \mathcal{E}_0 \Big\{ \exp\Big(-2\theta \sum_{|x_i| \le 1} a_i \, L^{x_i}(t) \Big[\exp\Big(-2\theta c_0 \sum_{|x_j| \le 1} a_j \Big) \\
&\qquad\qquad \exp\Big(-2\theta \sum_{|x_j| > 1} a_j f\big(t, |x_j| - 1\big) \Big) \Big] \Big) \Big\}.
\end{aligned}
$$

Now denote

$$
L_1 := \sum_{|x_j| \le 1} a_j \qquad \text{and} \qquad L_2 := \sum_{|x_j| > 1} a_j \, f(t, |x_j| - 1) = \int_{\mathbb{R}} \tilde{f} \, d\Gamma, \qquad (174)
$$

for $\tilde{f}(x) = 1_{\{|x|>1\}} f\big(t, |x| - 1\big)$. Under P the random variables L_1 and L_2 are independent, almost surely finite, and stable of index γ. We infer that, for arbitrary fixed $c > 0$, the event

$$
A := \Big\{ \inf_{|x| \le 1} L^x(t) \ge c \Big\} \cap \Big\{ L_2 \le \frac{1}{\theta} \Big\} \cap \Big\{ \frac{1}{\theta} \le L_1 \le \frac{2}{\theta} \Big\} \qquad (175)
$$

has (annealed) probability

$$\begin{aligned}\mathsf{E}\,\mathcal{P}_0(A) &= \mathcal{E}_0\Big\{\inf_{|x|\le 1} L^x(t)\ge c\Big\}\,\mathsf{P}\Big\{L_2\le\frac{1}{\theta}\Big\}\,\mathsf{P}\Big\{\frac{1}{\theta}\le L_1\le\frac{2}{\theta}\Big\} \\ &\ge \mathcal{E}_0\Big\{\inf_{|x|\le 1} L^x(t)\ge c\Big\}\,\big(1-c_1\theta^\gamma\big)\big(c_2\theta^\gamma\big)\ge c_3\theta^\gamma\,,\end{aligned}\tag{176}$$

for a suitable choice of the constants $c_1, c_2, c_3 > 0$. On A we have

$$\begin{aligned}&\exp\Big(-2\,\theta c L_1\exp\big(-2\,\theta c_0 L_1\big)\exp\big(-2\,\theta L_2\big)\Big) \\ &\le \exp\Big(-2\,\theta \mathrm{e}^{-2} c L_1\exp\big(-2\,\theta c_0 L_1\big)\Big)\le \exp\Big(-2\mathrm{e}^{-2}c\,\mathrm{e}^{-4c_0}\Big)<1\,.\end{aligned}\tag{177}$$

We can thus continue the estimate (173) with

$$\begin{aligned}&\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{|x_i|\le 1} a_i L^{x_i}(t)\Big[\exp\Big(-2\,\theta c_0\sum_{|x_j|\,\le\,1} a_j\Big)\exp\Big(-2\,\theta\sum_{|x_j|\,>\,1} a_j f(t,|x_j|-1)\Big)\Big]\Big)\Big\} \\ &\qquad\le \theta-\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{1_A\Big(1-\exp\Big[-2\,\theta c L_1\exp\big(-2\,\theta c_0 L_1\big)\exp\big(-2\,\theta L_2\big)\Big]\Big)\Big\} \\ &\qquad\qquad\le \theta-\theta\,\mathsf{E}\,\mathcal{P}_0(A)\Big(1-\exp\big(-2\mathrm{e}^{-2}c\,\mathrm{e}^{-4c_0}\big)\Big)\;=\;\theta-C_2\theta^{\gamma+1},\end{aligned}\tag{178}$$

by (176) for a suitable choice of $C_2 = C_2(t) > 0$. This finishes the proof. □

The bounds for $\mathsf{E}U_t\theta$ in Lemma 32 translate easily into bounds for the Laplace transform

$$\Lambda^t(\theta)=\mathbf{E}_\ell\big\{\exp\big(-\theta\,\beth_t(t)\big)\big\}\,,\ \text{ for }\ \theta\ge 0,\tag{179}$$

of the mass of a clump alive at a macroscopic time t.

Lemma 33 (Asymptotic behaviour of Λ^t)**.** *For all $t > 0$, as $\theta \downarrow 0$,*

$$1-\frac{1}{\lambda(t)}\,\theta+\frac{C_2(t)}{\lambda(t)}\,\theta^{\gamma+1}\;\le\;\Lambda^t(\theta)\;=\;1-\frac{1}{\lambda(t)}\,\theta+\frac{C_1(t)}{\lambda(t)}\,\theta^{\gamma+1}.\tag{180}$$

Proof. Using the Laplace transform of a general compound Poisson point field,

$$\mathsf{E}U_t\theta\,(0)=-\log\mathbf{E}_\ell\big\{-\theta X_t^\infty[0,1]\big\}=\lambda(t)\big(1-\Lambda^t(\theta)\big),\tag{181}$$

hence the statement follows by applying Lemma 32 and the scaling relation of $\lambda(t)$. □

Finally, to get the tail behaviour we observe that Theorem 10 (iii) follows directly from the previous lemma together with the following version of the Tauberian Theorem of Bingham and Doney, see [2, Theorem 8.1.6] for the original statement. Here we can apply the relation $\approx$ occurring in Theorem 10 (iii) in a t–independent situation.

Lemma 34 (Tauberian Theorem)**.** *Suppose ξ is a nonnegative random variable defined on a probability space $(\Omega, \mathcal{A}, P)$ with positive and finite mean m and Laplace transform Λ. Then*

$$\Lambda(\theta)-\big(1-m\theta\big)\approx\theta^{\gamma+1},\ \text{ as }\theta\downarrow 0,\tag{182}$$

implies

$$P\{\xi>x\}\approx\frac{1}{x^{\gamma+1}}\,,\ \text{ as }x\uparrow\infty.\tag{183}$$

Proof. We denote $F(x) := P\{\xi \le x\}$ and

$$h(x) := \frac{1}{m}\int_x^\infty \big(1 - F(y)\big)\, dy\,. \tag{184}$$

Then, using integration by parts twice and then plugging in the assumption,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta x} h(x)\, dx \;&=\; \frac{1}{\theta} - \frac{1}{\theta m}\int_0^\infty \mathrm{e}^{-\theta x}\,\big(1 - F(x)\big)\, dx \\ &= \frac{1}{\theta} - \frac{1}{\theta m}\,\frac{1-\Lambda(\theta)}{\theta} \;\approx\; \theta^{\gamma - 1}.\end{aligned} \tag{185}$$

We now apply the Tauberian Theorem of de Haan and Stadtmüller [2, Theorem 2.10.2] to infer

$$\int_0^x h(y)\, dy \approx x^{1-\gamma}\,,\ \text{ as } \ x \uparrow \infty. \tag{186}$$

Next we use the O-version of the Monotone Density Theorem [2, Proposition 2.10.3] twice to conclude

$$\text{first that } h(x) \approx x^{-\gamma} \text{ and then } 1 - F(x) \approx \frac{1}{x^{\gamma+1}}\,,\ \text{ as } \ x \uparrow \infty, \tag{187}$$

as claimed. □

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