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Analysis of relative dispersion of two particles by Lagrangian stochastic models and DNS methods

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Abstract

Comparisons of the Q1D against the known Lagrangian stochastic well-mixed quadratic form models and the moments approximation models are presented. In the case of modestly large Reynolds numbers turbulence ($Re_\lambda \simeq 240$) the comparison of the Q1D model with the DNS data is made. Being in a qualitatively agreemnet with the DNS data, the Q1D model predicts higher rate of separation. Realizability of Q1D model extracted from the transport equation with a quadratic form of the conditional acceleration is shown.

1 Introduction

A turbulent dispersion of a contaminant, for example pollutant dispersion in the atmosphere, is conveniently described in terms of Lagrangian statistics sampled along the paths of fluid particles. In practice, however, the Eulerian statistics sampled at fixed points in space are better known from experiments. Therefore, the basic problem of turbulent dispersion is to calculate the Lagrangian statistics from given Eulerian statistics. Lagrangian stochastic models of turbulent dispersion address the problem by statistically characterising the particle paths from an Eulerian input. The Lagrangian stochastic models simulate the time evolution of the particle coordinate and velocity in terms of stochastic differential equations. These models are best understood for the description of one-particle statistics, which contain only one-point statistical information. A modelled ensemble of single particles allows the calculation of mean concentrations, whereas an ensemble of particle pairs allows the calculation of concentration fluctuations. When we consider the motion of a pair particles, the modelling can be seen as the superposition of a relative motion and the motion of a single particle, or particle centeroid (e.g., see Durbin 1980, Thomson 1990, Sabelfeld & Kurbanmuradov 1997). The relative motion reflects more directly the internal turbulent structure because of the appearance of an internal lengths (particle distance), and its description permits the introduction of concepts developed within the statistical theory of turbulence (Monin & Yaglom 1975).

In this work we suggest comparisons of the Q1D model of relative dispersion of two particles against the known Lagrangian stochastic well-mixed quadratic form models and the moments approximation models. Recall that the Q1D model of relative dispersion is aimed to describe time evolution of the distanse and longitudional component of relative velocity between two particles. The main problem we deal with is the extraction of information needed for constructing Lagrangian stochastic models from DNS data in the case of modestly large Reynolds numbers ($Re_\lambda \simeq 240$) turbulence.

In section 2, basic assumptions used in the construction of the Lagrangian stochastic models are formulated. Models satisfying the well-mixed condition are given in section 3. Models based on the moments approximation method are presented in section 4.

Comparison of different models of relative dispersion for the inertial subrange is given in section 5. Section 6 presents the comparison of Q1D models with a bi-gaussian Eulerian pdf against the DNS data for modestly large Reynolds number turbulence.

Let us introduce the notations. The Eulerian velocity field is considered as a 3D random field denoted by $\mathbf{U}_E(\mathbf{x},t) = (U_{E1}(\mathbf{x},t), U_{E2}(\mathbf{x},t), U_{E3}(\mathbf{x},t))$, whose samples are incompressible: $\frac{\partial}{\partial x_i} U_{Ei}(\mathbf{x},t) = 0$. The concentration of a conservative passive scalar scattered by this field is governed by the transport equation

$$\frac{\partial c(\mathbf{x},t)}{\partial t} + U_{Ei}(\mathbf{x},t)\frac{\partial c}{\partial x_i} = 0, \quad t \geq 0; \quad c(\mathbf{x},0) = S(\mathbf{x})$$

where $S(\mathbf{x})$ is the initial distribution of the concentration. Here and in what follows, we use the summation convention over the repeated indices.

In practice, the following quantities are of special interest: $\langle c(\mathbf{x},t)\rangle$, the mean concentration, $\langle U_{Ei}(\mathbf{x},t)c(\mathbf{x},t)\rangle$, the mean fluxes of concentration, and $\langle c(\mathbf{x},t)c(\mathbf{x}',t)\rangle$, the concentration covariance. There are two main approaches to evaluate these quantities. The first one is based on the averaging of the transport equation to extract closed equations for the quantities in question. This is the so-called closure problem which faces well known difficulties (Monin and Yaglom, 1971).

Second approach is based on the Lagrangian description where the following representations are used

$$\langle c(\mathbf{x},t)\rangle = \int d\mathbf{x}_0\, S(\mathbf{x}_0)\, p_{1L}(\mathbf{x},t;\mathbf{x}_0),$$

$$\langle U_{Ei}(\mathbf{x},t)c(\mathbf{x},t)\rangle = \int d\mathbf{v} \int d\mathbf{x}_0\, v_i\, S(\mathbf{x}_0)\, p_{1L}(\mathbf{v},\mathbf{x},t;\mathbf{x}_0),$$

$$\langle c(\mathbf{x},t)c(\mathbf{x}',t)\rangle = \int d\mathbf{x}_0 \int d\mathbf{x}_0'\, S(\mathbf{x}_0)\, S(\mathbf{x}_0')\, p_{2L}(\mathbf{x},\mathbf{x}',t;\mathbf{x}_0,\mathbf{x}_0')\ .$$

Here p_{1L} and p_{2L} are the Lagrangian transition densities:

$$p_{1L}(\mathbf{x},t;\mathbf{x}_0) = \langle \delta(\mathbf{x} - \mathbf{X}(t;\mathbf{x}_0))\rangle,$$

$$p_{1L}(\mathbf{v},\mathbf{x},t;\mathbf{x}_0) = \langle \delta(\mathbf{x} - \mathbf{X}(t;\mathbf{x}_0))\delta(\mathbf{v} - \mathbf{V}(t;\mathbf{x}_0))\rangle,$$

$$p_{2L}(\mathbf{x},\mathbf{x}',t;\mathbf{x}_0) = \langle \delta(\mathbf{x} - \mathbf{X}(t;\mathbf{x}_0))\delta(\mathbf{x}' - \mathbf{X}(t;\mathbf{x}_0))\rangle,$$

In these formulae, the Lagrangian variables $\mathbf{X}, \mathbf{V}$ are defined through

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}(t;\mathbf{x}_0) = \mathbf{U}_E(t, \mathbf{X}(t;\mathbf{x}_0)), \qquad \mathbf{X}(0;\mathbf{x}_0) = \mathbf{x}_0.$$

In this paper we focus on the two-particle models which describe the motion of two fluid particles. It is governed by

$$\frac{d\mathbf{X}_1(t)}{dt} = \mathbf{U}_E(t,\mathbf{X}_1), \quad \frac{d\mathbf{X}_2(t)}{dt} = \mathbf{U}_E(t,\mathbf{X}_2),$$

where $\mathbf{X}_1(t) = \mathbf{X}(t;\mathbf{x}_0)$ and $\mathbf{X}_2(t) = \mathbf{X}(t;\mathbf{x}_0')$.

It is convenient to rewrite this system as follows

$$\begin{aligned} \frac{d\mathbf{R}}{dt} &= \frac{1}{2}\left\{\mathbf{U}_E(\mathbf{R}+\frac{\mathbf{r}}{2},t)+\mathbf{U}_E(\mathbf{R}-\frac{\mathbf{r}}{2},t)\right\}, \\ \frac{d\mathbf{r}}{dt} &= \mathbf{U}_E(\mathbf{R}+\frac{\mathbf{r}}{2},t)-\mathbf{U}_E(\mathbf{R}-\frac{\mathbf{r}}{2},t), \end{aligned} \tag{1.1 }$$

where

$$\mathbf{R}(t)=\frac{1}{2}\left(\mathbf{X}_1(t)+\mathbf{X}_2(t)\right), \quad \mathbf{r}(t)=\mathbf{X}_1(t)-\mathbf{X}_2(t).$$

This form clearly illustrates the division of large and small scales of turbulence. Indeed if the distance between the two particles is much less than the external scale ($r \ll L$), then the terms

$$\mathbf{U}(\mathbf{R}+\frac{\mathbf{r}}{2},t)+\mathbf{U}(\mathbf{R}-\frac{\mathbf{r}}{2},t), \quad and \quad \mathbf{U}(\mathbf{R}+\frac{\mathbf{r}}{2},t)-\mathbf{U}(\mathbf{R}-\frac{\mathbf{r}}{2},t).$$

are approximately statistically independent.

This property can be used to simulate the motion of two particles in small scales according to

$$\begin{aligned} \frac{d\mathbf{R}}{dt} &= \frac{1}{2}\left\{\tilde{\mathbf{U}}(\mathbf{R}+\frac{\mathbf{r}}{2},t)+\tilde{\mathbf{U}}(\mathbf{R}-\frac{\mathbf{r}}{2},t)\right\}, \\ dr_i(t) &= v_i(t)\,dt, \quad dv_i(t)=a_i(\mathbf{r},\mathbf{v},t)\,dt+b_{ij}(\mathbf{r},\mathbf{v},t)\,dW_j(t), \quad i=1,2,3. \end{aligned}$$

Here the large scale velocity field is approximated by a field marked by the tilde (e.g., extracted from DNS or LES methods), while the small scale motion is described as a diffusion process governed by a Langevin type equation.

Two alternative modelling approaches include Eulerian statistics in Lagrangian stochastic models: the **well-mixed** approach of Thomson (1987) and the **moments approximation** method of Novikov (1989). The importance of Thomson's approach is that when the material distribution is uniform, the model does not artifically an-mix material.

A one dimensional well-mixed Lagrangian stochastc model of relative dispersion of two particles has been proposed by Thomson (1986). Three dimensional models based on well mixed criterion have been considered in Thomson (1990) and Borgas & Sawford (1994). Gaussian Eulerian statistics are used in this articles. Well-mixed quasi-one-dimensional (Q1D) models of relative dispersion of two particles in the case of arbitrary Eulerian statistics was considerd in Kurbanmuradov (1995) and Kurbanmuradov & Sabelfeld (1995). A three dimensional well-mixed model of relative dispersion consistent with arbitrary Eulerian statistics was proposed in Kurbanmuradov (1997).

Lagrangian stochstic models of relative dispersion based on the moments approximation approach has been proposed by Novikov (1989), and has been developed by Pedrizzetti & Novikov (1984), Heppe (1998), and Pedrizzetti (1999).

2 Basic assumptions

We deal here with the process of relative dispersion of a pair of fluide particles in a stationary, spatially isotropic incompressible fully developed turbulent flow. We introduce the Eulerian velocity difference $\mathbf{u}_E(\mathbf{r}) = \mathbf{U}_E(\mathbf{x}+\mathbf{r},t) - \mathbf{U}_E(\mathbf{x},t)$ considered at two fixed points separated by vector $\mathbf{r}$.

Now we formulate the main assumptions underlying the models developed.

2.1 Markov assumption

Let $(\mathbf{r}(t), \mathbf{v}(t))$ be the Lagrangian variables for the separation vector and the relative velocity between two fluid particles at the time t. It is usually assumed (e.g., see Thomson, 1987) that $(\mathbf{r}(t), \mathbf{v}(t))$ is a $6D$ (continuous) Markov process (i.e., given the values of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ at time t, the values at time greater than t are independent of the values at times less than t). Under the Markov assumption for $(\mathbf{r}(t), \mathbf{v}(t))$ the most general equation used to describe the time evolution of $(\mathbf{r}(t), \mathbf{v}(t))$ is the Ito type stochastic differential equation:

$$dr_i = v_i\, dt, \quad dv_i = a_i(\mathbf{r},\mathbf{v})\, dt + b_{ij}(\mathbf{r},\mathbf{v})\, dW_j(t)\ , \quad i = 1,2,3 \tag{2.1 }$$

The main problem here is the following: how can we determine the functions $a_i(\mathbf{r},\mathbf{v})$ and $b_{ij}(\mathbf{r},\mathbf{v})$, (called *the drift* and *diffusion* terms, respectively) so that the model random process (2.1) is in a sense close to the true process (1.1). To this end, one uses two consistency principles: (i) consistency with Kolmogorov's similarity theory and (ii) consistency with Thomson's well-mixed condition.

2.2 Consistency with the second Kolmogorov similarity hypothesis

Given r, assume that there exists τ such that

$$\tau_\eta \ll \tau \ll \frac{r^{2/3}}{\bar{\varepsilon}^{1/3}}. \tag{2.2 }$$

The consistency with the second Kolmogorov similarity hypothesis requires that

$$b_{ij}(\mathbf{r},\mathbf{v}) = (2\, C_0\, \bar{\varepsilon})^{1/2}\, \delta_{ij}, \tag{2.3 }$$

where $\bar{\varepsilon}$ is the mean dissipation rate of the kinetic turbulence energy, C_0 is the Kolmogorov constant.

Let us comment this condition.

Denote $\mathbf{V}_1(t) = (V_{11}(t), V_{12}(t), V_{13}(t))$, $\mathbf{V}_2(t) = (V_{21}(t), V_{22}(t), V_{23}(t))$ the Lagrangian velocity of the first and the second particles, respectively. Then,

$$\Delta_\tau v_i(t) \equiv v_i(t+\tau) - v_i(t) =$$
$$V_{2i}(t+\tau) - V_{2i}(t) - (V_{1i}(t+\tau) - V_{1i}(t)) = \Delta_\tau V_{2i}(t) - \Delta_\tau V_{1i}(t),$$

In view of (2.2), the quantities

$$\mathbf{v}(t) = \mathbf{V}_2(t) - \mathbf{V}_1(t), \quad \Delta_\tau \mathbf{V}_2(t), \quad \Delta_\tau \mathbf{V}_1(t)$$

are approximately mutually independent. Therefore

$$\begin{aligned} &\langle \Delta_\tau v_i(t) \Delta_\tau v_j(t) \mid \mathbf{v}(t) = \mathbf{v},\ \mathbf{r}(t) = \mathbf{r} \rangle \\ &= \langle \{\Delta_\tau V_{2i}(t) - \Delta_\tau V_{1i}(t)\} \{\Delta_\tau V_{2j}(t) - \Delta_\tau V_{1j}(t)\} \mid \mathbf{v}(t) = \mathbf{v},\ \mathbf{r}(t) = \mathbf{r} \rangle \\ &\simeq \langle \Delta_\tau V_{2i}(t) \Delta_\tau V_{2j}(t) \rangle + \langle \Delta_\tau V_{1i}(t) \Delta_\tau V_{1j}(t) \rangle \simeq 2\, C_0\, \bar{\varepsilon} \delta_{ij}\, \tau, \end{aligned}$$

which implies (2.3).

2.3 Thomson's well mixed condition

The following relation between the true Eulerian and Lagrangian pdf's are known (Novikov, 1969):

$$p_E(\mathbf{v}; \mathbf{r}, t) = \int p_L(\mathbf{r}, \mathbf{v}; t, \mathbf{r}_0)\, d\mathbf{r}_0,$$

where

$$p_E(\mathbf{v}; \mathbf{r}, t) = \langle \delta(\mathbf{v} - \mathbf{u}_E(\mathbf{r})) \rangle$$
$$p_L(\mathbf{r}, \mathbf{v}; t, \mathbf{r}_0) = \langle \delta(\mathbf{r} - \mathbf{r}(t)) \delta(\mathbf{v} - \mathbf{v}(t)) \rangle$$

The model is considered consistent with Novikov's integral relation if its pdf also satisfies such a relation. It is well known that this leads to Thomson's well-mixed condition written in the form (Thomson, 1987):

$$v_i \frac{\partial p_E}{\partial r_i} + \frac{\partial}{\partial v_i}(a_i p_E) = C_0 \bar{\varepsilon} \frac{\partial^2 p_E}{\partial v_i \partial v_i} \; .$$

It should be noted that all this does not define the model uniquely (e.g., see Thomson, 1987; Borgas & Sawford, 1994).

3 Well-mixed Lagrangian stochastic models

Note that in the case of isotropic turbulence the structure of the drift term is defined by two scalar functions. Indeed,

$$a_i(\mathbf{r}, \mathbf{v}) = \varphi(r, v_\parallel, v_\perp) \frac{r_i}{r} + \psi(r, v_\parallel, v_\perp) \frac{v_i}{v}$$

where $r = (r_i r_i)^{1/2}, \quad v_\parallel = v_i r_i / r, \quad v_\perp = \left(v_i\, v_i - v_\parallel^2\right)^{1/2}$

It should be noted that if we could find an additional relation between the functions ϕ and ψ, then the well-mixed condition would provide a unique choice of the drift term. For instance, this is the case when $\phi \equiv 0$ (e.g., see the 1-particle model treated in Monti & Luezzi 1995), or $\psi \equiv 0$ (Kurbanmuradov & Sabelfeld 1995). But generally, since such relations are not known, different approaches can be used to extract the unique model. We present below two such approaches. In the first one, the drift term is assumed to be quadratic in velocity, and the Eulerian pdf p_E is Gaussian (Thomson, 1990; Borgas and Sawford, 1994). The second approach is based on a Markovian character of the evolution of the 2D process $r(t), u_r(t)$ where $r(t)$ is the distance between the two particles, and $u_r(t)$ is the longitudinal component of the relative velocity (Kurbanmuradov 1995; Kurbanmuradov & Sabelfeld, 1995; Kurbanmuradov, 1997).

3.1 Quadratic-form models

Following (Borgas & Sawford 1994) let us assume that pdf p_E is Gaussian:

$$p_E(\mathbf{u}, \mathbf{r}) = \frac{\lambda^{1/2}}{(2\pi)^{3/2}} \exp(-\frac{1}{2}\lambda_{ij} u_i u_j)$$

where

$$\langle u_{Ei}(\mathbf{r}) u_{Ej}(\mathbf{r}) \rangle = \lambda_{ij}^{-1}(\mathbf{r}), \quad \lambda = det(\lambda_{ij}), \quad \lambda_{ik}\lambda_{kj}^{-1} = \delta_{ij}.$$
$$u_{Ei}(\mathbf{r}) = U_{Ei}(\mathbf{x} + \mathbf{r}) - U_{Ei}(\mathbf{x})$$

Here λ_{ij}^{-1} are the entries of the relevant inverse matrix.

The drift term is seeked in the form:

$$a_i = \phi_i + C_0 \, \bar{\varepsilon} \, \frac{1}{p_E} \frac{\partial p_E}{\partial u_i},$$

with

$$\phi_i = \Gamma_i + \gamma_{ijk} \, u_j u_k.$$

As shown in (Borgas and Sawford, 1994), the well-mixed condition is satisfied for the following three cases:

$$\gamma_{ijk} = -\frac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{jk}}{\partial r_l}, \; \gamma_{ijk} = -\frac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{ik}}{\partial r_j} = \frac{1}{2}\lambda_{kl}\frac{\partial \lambda_{il}^{-1}}{\partial r_j}, \; \gamma_{ijk} = -\frac{1}{2}\lambda_{il}^{-1}\frac{\partial \lambda_{lj}}{\partial r_k} = \frac{1}{2}\lambda_{jl}\frac{\partial \lambda_{il}^{-1}}{\partial r_k},$$

where

$$\Gamma_i = \lambda_{ij}^{-1}\left(-\frac{1}{2\lambda}\frac{\partial \lambda}{\partial r_j} + \gamma_{kkj} + \gamma_{kjk} \right).$$

Thus this approach gives a specific structure of the drift term, but it also does not provide a unique solution. Let us consider now the second approach.

3.2 Quasi- one-dimensional models

As mentioned above, the motion of two particles is described here by the distance $r(t)$ and the longitudinal velocity component $u_r(t)$:

$$r(t) = (r_i(t) r_i(t))^{1/2}, \quad u_r(t) = v_i(t) r_i(t)/r(t)$$

Here $v_i(t)$ is the i-th component of the relative velocity $\mathbf{v}(t)$, and $\mathbf{r}(t)$ is the separation vector.

Now we fomulate the main assumption:
Assume that $(r(t), u_r(t))$ is a continuous 2D Markov process:

$$dr = u_r\, dt, \quad du_r = a(r, u_r)\, dt + (2C_0\bar{\varepsilon})^{1/2} dW(t) \tag{3.1 }$$

In the case of quasi-one dimensional model the well-mixed condition reads (Kurbanmuradov & Sabelfeld 1995):

$$v\frac{\partial r^2 p_E^{\|}(v,r)}{\partial r} + r^2\frac{\partial (X(r,v)p_E^{\|})}{\partial v} = C_0\bar{\varepsilon}r^2\frac{\partial^2 p_E^{\|}}{\partial v^2}, \tag{3.2 }$$

where $p_E^{\|}(v,r)$ is the pdf of the longitudinal component of the Eulerian velocity difference $\mathbf{u}_E(\mathbf{r}) = \mathbf{U}_E(\mathbf{x}+\mathbf{r}) - \mathbf{U}_E(\mathbf{x})$: $p_E^{\|}(v,r) = \langle \delta(v - u_{Ei}(\mathbf{r})r_i/r)\rangle$.
Assuming $Xp_E^{\|}\,|_{|v|\to\infty} = 0$ it is easy get

$$X(r,v) = C_0\bar{\varepsilon}\frac{\partial}{\partial v}\ln p_E^{\|}(v,r) - \frac{1}{p_E^{\|}(v,r)}\int_{-\infty}^{v}\frac{v'}{r^2}\frac{\partial}{\partial r}\left(r^2 p_E^{\|}(v',r)\right)\, dv'.$$

In the inertial subrange ($\eta \ll r \ll L$) this expression can be considerably simplified. In the inertial subrange the unique external parameter is $\bar{\varepsilon}$, so it is possible to turn to a dimensionalless density $f_E(\xi)$:

$$p_E^{\|}(v,r) = \frac{1}{(\bar{\varepsilon}r)^{1/3}} f_E\Big(\frac{v}{(\bar{\varepsilon}r)^{1/3}}\Big).$$

Consequently,

$$X(r,v) = \frac{\bar{\varepsilon}^{2/3}}{r^{1/3}}\tilde{X}\Big(\frac{v}{(\bar{\varepsilon}r)^{1/3}}\Big), \quad \tilde{X}(\xi) = C_0\frac{d\ln f_E}{d\xi} + \frac{\xi^2}{3} - \frac{\frac{7}{3}\int\limits_{-\infty}^{\xi}\xi' f_E(\xi')\, d\xi'}{f_E(\xi)}\ .$$

It is convenient to deal with the equation in dimensionalless form for $\xi_r(t) = u_r(t)/(\bar{\varepsilon}r)^{1/3}$:

$$dr = (\bar{\varepsilon}r)^{1/3}\xi_r\, dt, \quad d\xi_r = \Big(\frac{\bar{\varepsilon}}{r^2}\Big)^{1/3} a_0(\xi_r)dt + \Big(\frac{\bar{\varepsilon}}{r^2}\Big)^{1/6}\sqrt{2C_0}dW(t),$$

where

$$a_0(\xi) = \tilde{X}(\xi) - \frac{\xi^2}{3} = C_0\frac{d\ln f_E}{d\xi} - \frac{\frac{7}{3}\int\limits_{-\infty}^{\xi}\xi' f_E(\xi')\, d\xi'}{f_E(\xi)}\ .$$

In the case of Gaussian pdf f_E:

$$f_E(\xi) = \frac{1}{(2\pi C)^{1/2}}\exp\left(-\frac{\xi^2}{2C}\right)$$

the coefficient a_0 has the most simple form:

$$a_0(\xi) = -\frac{C_0}{C}\xi + \frac{7}{3}C\,,$$

where C is the Kolmogorov universal constant in the law of two-thirds.

3.3 Three dimensional extension of Q1D models

Assume that the turbulence is isotropic and stationary. Let us consider a 3D model of relative dispersion in the subrange $\eta \ll r$:

$$dr_i = v_i\,dt, \quad dv_i = a_i(\mathbf{r},\mathbf{v})\,dt + (C_0\bar{\varepsilon})^{1/2}\,dW_i(t)\ , \quad i = 1,2,3$$

where

$$a_i(\mathbf{r},\mathbf{v}) = \varphi(r, v_\parallel, v_\perp)\frac{r_i}{r} + \psi(r, v_\parallel, v_\perp)\frac{v_i}{v}$$

with unknown φ and ψ.

We derive from it the Q1D model:

$$dr = u_r\,dt, \quad du_r = \left(\frac{a_i r_i}{r} + \frac{u_\perp^2}{r}\right)dt + (C_0\bar{\varepsilon})^{1/2}\,dW(t),$$

where $u_\perp^2(t) = u^2(t) - u_r^2(t)$. *Assuming* that Q1D process $(r(t), u_r(t))$ is Markovian we can write

$$\frac{a_i r_i}{r} + \frac{v_\perp^2}{r} = \varphi(r, v_\parallel, v_\perp) + \frac{v_\parallel}{v}\psi(r, v_\parallel, v_\perp) + \frac{v_\perp^2}{r} = X(r, v_\parallel)$$

From the last relation and the 3D well-mixed condition it follows that (Kurbanmuradov, 1997)

$$\begin{aligned}\psi(r, v_\parallel, v_\perp) = \quad & C_0\bar{\varepsilon}\frac{v}{v_\perp}\frac{\partial}{\partial v_\perp}\ln p_E(v_\parallel, v_\perp; r) + \frac{v_\parallel v}{r} \\ -\frac{v}{v_\perp^2 p_E(v_\parallel, v_\perp; r)} & \int_0^{v_\perp}\Big\{\frac{\partial}{\partial v_\parallel}(Xp_E) + \frac{v_\parallel}{r^2}\frac{\partial}{\partial r}r^2 p_E(v_\parallel, v'_\perp; r)\Big\}v'_\perp\,dv'_\perp.\end{aligned}$$

$$\varphi(r, v_\parallel, v_\perp) = X(r, v_\parallel) - \frac{v_\perp^2}{r} - \frac{v_\parallel}{v}\psi(r, v_\parallel, v_\perp).$$

This yields in the case of Gaussian pdf p_E, for the inertial subrange (Sabelfeld & Kurbanmuradov, 1997) $\eta \ll r \ll L$:

$$\begin{aligned}a_i(\mathbf{r},\mathbf{v}) = \quad & \frac{\bar{\varepsilon}^{2/3}}{r^{1/3}}\Big[\frac{7}{3}C - \frac{C_0}{4C}\frac{v_\parallel}{(\bar{\varepsilon}r)^{1/3}} - \frac{v^2}{(\bar{\varepsilon}r)^{2/3}}\Big]\frac{r_i}{r} \\ + \quad & \frac{\bar{\varepsilon}^{2/3}}{r^{1/3}}\Big[\frac{4}{3}\frac{v_\parallel}{(\bar{\varepsilon}r)^{1/3}} - \frac{C_0}{C'}\Big]\frac{v_i}{(\bar{\varepsilon}r)^{1/3}},\end{aligned}$$

where $C' = \frac{4}{3}C$.

4 Stochastic Lagrangian models based on the moments approximation method

The evaluation of the pdf $p_E(v_\parallel, v_\perp; r)$ is very difficult problem because it needs to construct a family of solutions to Navier-Stokes equations. Therefore, in practice one uses the method of moments: one constructs an approximation to the Eulerian pdf under the condition that its first several moments coincide with those of the true velocity moments.

The true moments can be found via DNS method solving the Navier-Stokes equation, or extracted from experiments.

4.1 Moments approximation conditions

In a more general case when the intermittency is taken into account, the model of relative dispersion has the form:

$$dr_i = v_i\,dt, \quad dv_i = a_i(\mathbf{r},\mathbf{v})\,dt + b_{ij}(\mathbf{r},\mathbf{v})\,dW_j(t)\ , \quad i=1,2,3$$

The well-mixed condition reads in this case (e.g., see Thomson 1987, Novikov 1989):

$$v_i \frac{\partial p_E}{\partial r_i} + \frac{\partial}{\partial v_i}(a_i p_E) = \frac{1}{2}\frac{\partial^2 m_{ij} p_E}{\partial v_i \partial v_j}\ , \tag{4.1 }$$

where $m_{ij} = b_{ik} b_{jk}$. Multiplying this equation by v_j and integrating over $\mathbf{v}$, we get (Pedrizzetti & Novikov 1994):

$$\frac{\partial}{\partial r_i}\langle u_{Ei}(\mathbf{r}) u_{Ej}(\mathbf{r})\rangle = \langle a_j(\mathbf{r},\cdot)\rangle,$$

i.e. (due to incompressibility)

$$\langle a_j(\mathbf{r},\cdot)\rangle = 0, \qquad j=1,2,3.$$

Myltiplying (4.1) by $v_j v_k$ and integrating over $\mathbf{v}$ yields

$$\begin{aligned} &\frac{\partial}{\partial r_i}\langle u_{Ei}(\mathbf{r}) u_{Ej}(\mathbf{r}) u_{Ek}(\mathbf{r})\rangle \\ &= \langle a_j(\mathbf{r},\cdot)\, u_{Ek}(\mathbf{r}) + a_k(\mathbf{r},\cdot)\, u_{Ej}(\mathbf{r})\rangle + \langle m_{jk}(\mathbf{r},\cdot)\rangle, \qquad j,k=1,2,3. \end{aligned}$$

Kolmogorov's relation for third order moments in the inertial subrange (e.g., see Monin & Yaglom 1975, Novikov 1989):

$$\langle u_{Ei}(\mathbf{r}) u_{Ej}(\mathbf{r}) u_{Ek}(\mathbf{r})\rangle = -\frac{4}{15}\bar{\varepsilon}(r_i\delta_{jk} + r_j\delta_{ik} + r_k\delta_{ij}), \quad \eta \ll r \ll L.$$

Hence

$$\frac{\partial}{\partial r_i}\langle u_{Ei}(\mathbf{r}) u_{Ej}(\mathbf{r}) u_{Ek}(\mathbf{r})\rangle = -\frac{4}{3}\bar{\varepsilon}\delta_{jk}, \quad (\eta \ll r \ll L).$$

Therefore in the inertial subrange the moments approximation conditions have the form:

$$\begin{aligned} &\langle a_j(\mathbf{r},\cdot)\, u_{Ek}(\mathbf{r}) + a_k(\mathbf{r},\cdot)\, u_{Ej}(\mathbf{r})\rangle + \langle m_{jk}(\mathbf{r},\cdot)\rangle = -\frac{4}{3}\,\bar{\varepsilon}\delta_{jk}, \\ &\langle a_j(\mathbf{r},\cdot)\rangle = 0, \qquad j,k=1,2,3; \quad (\eta \ll r \ll L). \end{aligned} \tag{4.2 }$$

4.2 Realizability of Lagrangian stochastic models based on the moments approximation method

Here we have to analyse if the scheme presented in the previous subsection can be realized indeed.

The input parametrs of the moments approximation method are

$$\langle u_{Ei}(\mathbf{r})u_{Ej}(\mathbf{r})\rangle, \quad \langle u_{Ei}(\mathbf{r})u_{Ej}(\mathbf{r})u_{Ek}(\mathbf{r})\rangle, \quad i,j,k = 1,2,3.$$

Let us assume that the functions a_i and b_{ij} satisfying the moments approximation conditions (4.2) are found.

Now we have to check if there exists a *positive* solution $p_E(\mathbf{v},\mathbf{r})$ to the equation (4.1) which satisfies the conditions

$$\int_{\mathbb{R}^3} p_E(\mathbf{v},\mathbf{r})\,d\mathbf{v} = 1, \quad \int_{\mathbb{R}^3} v_i p_E(\mathbf{v},\mathbf{r})\,d\mathbf{v} = 0,$$

$$\int_{\mathbb{R}^3} v_i v_j p_E(\mathbf{v},\mathbf{r})\,d\mathbf{v} = \langle u_{Ei}(\mathbf{r})u_{Ej}(\mathbf{r})\rangle,$$

$$\int_{\mathbb{R}^3} v_i v_j v_k p_E(\mathbf{v},\mathbf{r})\,d\mathbf{v} = \langle u_{Ei}(\mathbf{r})u_{Ej}(\mathbf{r})u_{Ek}(\mathbf{r})\rangle, \quad i,j,k = 1,2,3.$$

Example (Pedrizzetti & Novikov 1994).

$$a_i = -k\frac{(\bar{\varepsilon}r)^{1/3}}{r}(v_i + \gamma v_j n_j\, n_i), \quad n_i = r_i/r, \tag{4.3 }$$

$$m_{ij} = \bar{\varepsilon}\Big\{\beta\delta_{ij} + (\alpha-\beta)n_i n_j\Big\}, \quad (\alpha \ge 0, \beta \ge 0), \tag{4.4 }$$

where α, β, γ and k are dimensionless coefficients of the model ($\alpha \ge 0, \beta \ge 0, k \ge 0$), and C is Kolmogorov's constant in the law of two-thirds ($C \simeq 2$). The moments appraximation conditions (4.2) imply the following relations between these coefficients (Pedrezotti & Novikov, 1994):

$$8Ck = 3\beta + 4, \quad (6\gamma - 2)Ck = (3\alpha - \beta).$$

From $\alpha \ge 0, \beta \ge 0$ it follows that

$$Ck \ge 0.5, \quad (9\gamma + 1)Ck \ge 2.$$

Thus for any γ and $k \ge 0$ satisfying these inequalities the model (4.3)–(4.4) satisfies the moments approximation conditions (4.2). However as shown in Pedrizzetti & Novikov (1994), the realizability conditions are satisfied only for a special subrange of the pairs (γ, k). Some examples of such pairs are: (1/3,3), (1,2.5), (3,1.3), (5,0.9).

A generalation of the model (4.3)–(4.4) is given in Pedrizzetti (1999) where the drift term is defined by

$$a_i = -\frac{k}{r}\left((\bar{\varepsilon}r)^{1/3}(v_i + \gamma v_j n_j\, n_i) + \nu\, v_j n_j \tilde{v}_i\right), \quad \tilde{v}_i = v_i - v_j n_j\, n_i. \tag{4.5 }$$

and with diffusion term (4.4). In absence of intermittency, the moments approximation conditions (4.2) imply (Pedrizzetti, 1999):

$$\alpha = -\frac{4}{3} + 2k(\gamma + 1), \quad \beta = -\frac{4}{3} + \frac{8}{15}k(5C - \nu).$$

Relizability conditions imply that the resting parameters k, γ and ν are not complitely free. For example, as shown in Pedrizzetti, in the case of isotropic forsing ($\alpha = \beta$) which implies $\gamma = 1/3 - 4\nu/15C$, and assuming $\nu = 0, 0.25, 1$; the corresponding values of parameter k, for which the relizability conditions are satisfied, are 3.8, 4.2, 6, respectively.

The examples cited will be used below in the next section when comparing different models of relative dispersion in the inertial subrange (see Table 1).

5 Comparison of different models of relative dispersion for the inertial subrange of a fully developed turbulence

5.1 Q1D quadratic-form model of Borgas & Yeung

The exact Eulerian transport equation for the Eulerian velocity pdf $p_E(\mathbf{v}, \mathbf{r})$ is (e.g., see Heppe 1998)

$$v_i \frac{\partial p_E}{\partial r_i} + \frac{\partial}{\partial v_i}(\langle a_i | \mathbf{v}, \mathbf{r} \rangle p_E) = 0,$$

where $a_i(t) = \frac{dv_i}{dt}$ is the relative acceleration between two fluid particles, and $\langle a_i|\mathbf{v}, \mathbf{r}\rangle$ is its conditionally averaged value under the condition that $\mathbf{v}(t) = \mathbf{v}, \mathbf{r}(t) = \mathbf{r}$. The Q1D analog of this equation is the following exact transport equation for $p_E^{\|}$ derived by Borgas (1998):

$$\frac{u_r}{r^2}\frac{\partial r^2 p_E^{\|}}{\partial r} + \frac{\partial}{\partial u_r}(\langle a_r | u_r, r\rangle p_E^{\|}) = 0, \quad (5.6\ )$$

where

$$a_r(t) = \frac{dv_{\|}(t)}{dt} = \frac{a_i(t) r_i(t)}{r(t)} + \frac{v_i(t)v_i(t) - u_r^2(t)}{r}$$

is the longitudinal component of the relative acceleration, and $\langle a_r|u_r, r\rangle$ is the conditional acceleration.

From (3.2) and (5.6) it follows that

$$X(r, u_r) = C_0 \bar{\varepsilon} \frac{1}{p_E^{\|}} \frac{\partial p_E^{\|}(u_r, r)}{\partial u_r} + \langle a_r | u_r, r\rangle. \quad (5.7\ )$$

The quadratic-form assumption for the conditional acceleration is

$$\langle a_r | u_r, r\rangle = \alpha(r) + \beta(r) u_r + \gamma(r) u_r^2 = \frac{\sigma^2}{r}(\alpha_0 + \beta_0 \xi + \gamma_0 \xi^2)\,, \quad (5.8\ )$$

where $\sigma^2 = \langle u_r^2 \rangle, \xi = u_r/\sigma$. Let $S(r)$ and $F(r)$ are the skewness and flattness of the Eulerian velocity u_r, respectively.

In order to satisfy the moment constraints

$$\langle u_r^2 \rangle = \sigma^2, \quad \langle u_r^3 \rangle = S\sigma^3, \quad \langle u_r^4 \rangle = F\sigma^4$$

the coefficients α, β and γ should be choosen from the following equations (see Borgas & Yeung 1998):

$$\frac{1}{r^2}\frac{d(r^2\sigma^2)}{dr} = \alpha + \gamma\sigma^2,$$
$$\frac{1}{r^2}\frac{d(r^2 S\sigma^3)}{dr} = 2\beta\sigma^2 + 2\gamma S\sigma^3,$$
$$\frac{1}{r^2}\frac{d(r^2 F\sigma^4)}{dr} = 3\alpha\sigma^2 + 3\beta S\sigma^3 + 3\gamma F\sigma^4,$$

which can be solved for parameters α, β and γ to give:

$$\alpha = \alpha_0\frac{\sigma^2}{r}, \quad \beta = \beta_0\frac{\sigma}{r}, \quad \gamma = \gamma_0\frac{1}{r} \tag{5.9 }$$

where the dimensionless parameters α_0, β_0 and γ_0 are

$$\begin{aligned} \alpha_0 &= \quad -\gamma_0 - 2\psi_0 + 2, \quad \beta_0 = -\gamma_0 - \frac{3}{2}S\psi_0 + S + \frac{r}{2}\frac{dS}{dr}, \quad \psi_0 = -r\frac{d\ln\sigma}{dr}, \\ \gamma_0 &= \quad \frac{\left(-6 - 3S^2 + 2F + \psi_0(6 + \frac{9}{2}S^2 - 4F) - \frac{3}{2}Sr\frac{dS}{dr} + r\frac{dF}{dr}\right)}{3(F - S^2 - 1)}. \end{aligned} \tag{5.10 }$$

In the inertial subrange ($\eta \ll r \ll L$) $\sigma^2(r) = C(\bar{\varepsilon}r)^{2/3}$, $S = const$, $F = const$ due to Kolmogorov's second similarity hypothesis (e.g., see Monin & Yaglom 1975) which yields $\psi_0 = -1/3$. Therefore (5.10) yields

$$\gamma_0 = \frac{\frac{10}{9}F - \frac{3}{2}S^2 - \frac{8}{3}}{F - S^2 - 1}, \quad \alpha_0 = \frac{8}{3} - \gamma_0, \quad \beta_0 = \frac{3}{2}S - \gamma_0.$$

In the inertial subrange there exists a dimensionless pdf $f(\xi)$ such that

$$p_E^{\|}(u_r, r) = \frac{1}{\sigma(r)} f\left(\frac{u_r}{\sigma(r)}\right).$$

Then the equation (5.6) in this case can be rewritten as the following ODE

$$(b_0 + b_1\xi + b_2\xi^2)\frac{df(\xi)}{d\xi} + (c_0 + c_1\xi)f(\xi) = 0, \tag{5.11 }$$

where b_0, b_1, b_2, c_0, c_1 are dimensionless constants

$$b_0 = \alpha_0, \quad b_1 = \beta_0, \quad b_2 = \gamma_0 - \frac{1}{3},$$
$$c_0 = b_1, \quad c_1 = 2 + 2\gamma_0 - \frac{1}{3}.$$

From (5.7), (5.8) and (5.11) we get

$$X(r, u_r) = \frac{\bar{\varepsilon}^{2/3}}{r^{1/3}}\tilde{X}(\frac{u_r}{\sigma(r)}), \quad \sigma(r) = \sqrt{C}(\bar{\varepsilon}r)^{1/3},$$

where

$$\tilde{X}(\xi) = -\frac{C_0}{\sqrt{C}}\frac{c_0 + c_1\xi}{b_0 + b_1\xi + b_2\xi^2} + C(\alpha_0 + \beta_0\xi + \gamma_0\xi^2).$$

5.2 Comparison of different models in the inertial subrange

In this subsection we study the process of relative dispersion of two particles in the inertial subrange $\eta \ll r \ll L$. We first consider Q1D models wich are determined by the Eulerian pdf $f_E(\xi)$ of the dimensionless velosity difference $\xi_r = u_r/(\bar{\varepsilon} r)^{1/3}$. Suppose

$$f_E(\xi) = \frac{p}{\sqrt{2\pi}\sigma_1} \exp\{-\frac{(\xi-\mu_1)^2}{2\sigma_1^2}\} + \frac{1-p}{\sqrt{2\pi}\sigma_2} \exp\{-\frac{(\xi-\mu_2)^2}{2\sigma_2^2}\},$$

where the unknown parameters $p,\ \mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$ should be chosen to fit the first four moments of ξ_r:

$$\langle \xi_r \rangle = p\mu_1 + (1-p)\mu_2 = 0,$$
$$\langle \xi_r^2 \rangle = \quad p(\mu_1^2+\sigma_1^2) + (1-p)(\mu_2^2+\sigma_2^2) = C,$$
$$\langle \xi_r^3 \rangle = p(\mu_1^3 + 3\mu_1\sigma_1^2) + (1-p)(\mu_2^3+3\mu_2\sigma_2^2) = -\frac{4}{5}$$
$$\langle \xi_r^4 \rangle = p(\mu_1^4 + 6\mu_1^2\sigma_1^2 + 3\sigma_1^4) + (1-p)(\mu_2^4 + 6\mu_2^2\sigma_2^2 + 3\sigma_2^4) = 3.4\, C^2.$$

Thus we have four equations for five unnown parameters $p,\ \mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$. To find these parameters we need an additional (cloasure) assumption. Further we will use two kind of cloasure assumptions:
(i) $\sigma_1 = \sigma_2$, and
(ii) $\mu_1/\sigma_1 = -\mu_2/\sigma_2$.

In the Table 1 we present the Richardson constant g in the cubic law $\langle r^2(t) = g\bar{\varepsilon}t^3$ obtained by different well-mixed models. The same constant obtained by the moments approximation models is given in Table 2, while Table 3 presents the results obtained by different theoretical approaches.

Table 1. The universal constant g in the Richardson law $\langle r^2(t)\rangle = g\varepsilon t^3$ calculated by different well-mixed Lagrangian stochastic models; the Kolmogorov constant C was taken equal to 2.

Well-mixed models	$C_0 = 4$	$C_0 = 5$	$C_0 = 6$	$C_0 = 7$	$C_0 = 10$
Borgas & Sawford (1994)	0.75	0.7	0.6	0.5	0.3
Thomson (1990)	1.8	1.4	1.1	0.85	0.45
Q1D, pdf of Borgas & Yeung (1998)	4.8	2.6	1.7	1.1	0.45
Q1D, bigaussian pdf $\sigma_1 = \sigma_2$.	4.4	2.55	1.67	1.15	0.47
Q1D, bigaussian pdf $\mu_1/\sigma_1 = -\mu_2/\sigma_2$.	8.25	4.	2.27	1.4	0.51
Q1D, gaussian pdf	7.	4.7	2.8	1.9	0.75

Table 2. The universal constant g evaluated by different moments approximation models.

Model	realizability	g
Pedrizzetti & Novikov (1994)	+	0.50 $(k = 3.8,\ C_0 = 9.47)$
Pedrizzetti (1999)	+	0.35 $(\nu = 0.25,\ k = 4.2,\ C_0 = 10.25)$
	+	0.20 $(\nu = 1,\ k = 6,\ C_0 = 13.73)$
Heppe (1998)	unknown	0.44 $(C_0 = 5.3)$
	unknown	0.31 $(C_0 = 9.47)$

In the models Pedrizzetti & Novikov (1994), and Pedrizzetti (1999), the isotropic forcing was considered ($\alpha = \beta$, see subsection 4.2)

Table 3. The universal constant g evaluated by other methods.

Method	g
LHDI, Kraichnan (1966)	2.42
Modified LHDI, Lundgren (1981)	3.
EDQNM, Larcheveque & Lesieur (1981)	3.5
Thomson's corrected EDQNM, Thomson (1996)	1.4
Effective Hamiltonian method, Nakao (1991)	3.5

6 Comparison of different Q1D models of relative dispersion for Modestely Large Reynolds Number Turbulence ($Re_\lambda \simeq 240$)

The models noted in the previous section were developed for the case of a turbulece with a reach inertial subrange. In this case the DNS methods cannot be used since nowdays, the DNS data are available around $Re_\lambda \simeq 240$ (Yeung, 2000). Therefore, to make the validation through comparisons with DNS method, we can do it only for models with modestely large Reynolds number turbulence.

6.1 Parametrisation of Eulerian statistics.

Here we turn to a not developed turbulent flow whose characteristic $L/\eta \simeq 500$. In this case the scales go from the viscous ones, pass through a transitional region and goes to the external one. But it should be noted that the transitional range can be considered as an analog of the inertial subrange only in terms of the Eulerian statistical characteristics. For the Lagrangian statistics however this interval does not show the inertial subrange behaviour (Yeung, 2000).

Thus we have to make the assumption that $\mathbf{r}(t), u_r(t)$ is a Markov random process governed by (3.1) with the value C_0 depending on the distance r such that C_0 tends to a constant value as r/η is getting large.

Dimensional arguments show that in the case of stationary and isotropic turbulence

$$p_E^{\|}(v,r) = \frac{1}{\sigma(r)} f_E(\frac{v}{\sigma(r)};r), \tag{6.12 }$$

where $\sigma(r) = \langle u_{Er}^2 \rangle^{1/2}$ is r.m.s of the longitudinal component of the Eulerian velocity difference $\mathbf{u}_E(\mathbf{r}) = \mathbf{U}_E(\mathbf{x}+\mathbf{r}) - \mathbf{U}_E(\mathbf{x})$:

$$u_{Er} = (u_{Ei}(\mathbf{x}+\mathbf{r}) - U_{Ei}(\mathbf{x}))\,\frac{r_i}{r},$$

and $f_E(\xi;r)$ is a dimensionaless pdf on the dimensionaless velocity $\xi_r = u_{Er}/\sigma(r)$. It follows from (5.7) that

$$X(r,v) = \frac{\sigma^2}{r}\tilde{X}(r,\frac{v}{\sigma(r)}),$$

$$\tilde{X}(r,\xi) = C_0(r)\frac{\bar{\varepsilon}r}{\sigma^3}\frac{\partial \ln f_E}{\partial \xi} - \psi_0(r)\xi^2 - (2-\psi_0)\,\frac{\int\limits_{-\infty}^{\xi} \xi' f_E(\xi';r)\,d\xi'}{f_E(\xi;r)},$$

where

$$\psi_0 = \psi_0(r) = -r\,\frac{d\ln\sigma}{dr} = -\frac{r}{2}\frac{d\ln\langle u_r^2\rangle}{dr}.$$

If we pass to the dimensionaless velocity $\xi_r(t) = u_r(t)/\sigma(r(t))$ then Q1D model reads:

$$dr = \sigma(r)\xi_r\,dt, \quad d\xi_r = \frac{\sigma(r)}{r}\left\{\tilde{X}(r,\xi_r) + \psi_0(r)\xi_r^2\right\} + \frac{(C_0(r)\bar{\varepsilon})^{1/2}}{\sigma(r)}dW(t). \tag{6.13 }$$

Now, we have to specify the Eulerian velocity $f_E(\xi;r)$ and the function $\sigma(r)$.

Concerning the density $f_E(\xi;r)$, we assume that the first four moments are given by (Borgas & Yeung, 1998):

$$\langle \xi_r \rangle = 0, \quad \langle \xi_r^2 \rangle = 1, \quad \langle \xi^3 \rangle = \frac{\langle u_{Er}^3\rangle}{\langle u_{Er}^2\rangle^{3/2}}, \quad \langle \xi^4 \rangle = \frac{\langle u_{Er}^4\rangle}{\langle u_{Er}^2\rangle^{2}}$$

where

$$u_{Er}^2(r) = 2\sigma_1^2\left(\frac{r^2}{\omega_2\eta^2 + r^2}\right)^{2/3}\left(\frac{r^2}{\Lambda_2 L^2 + r^2}\right)^{1/3},$$

$$\langle u_r^3 \rangle = \theta\sigma_1^3\left(\frac{r}{L}\right)^3\left(\frac{L^2}{\omega_3\eta^2 + r^2}\right)\left(\frac{L^2}{\Lambda_3 L^2 + r^2}\right)^4,$$

$$\langle u_r^4 \rangle = 4k\sigma_1^4\left(\frac{r^2}{\omega_4\eta^2 + r^2}\right)^{4/3}\left(\frac{r^2}{\Lambda_4 L^2 + r^2}\right)^{2/3}.$$

Here $\sigma_1 = [U_{Ei}U_{Ei}]^{1/2}$ is the one-point root-mean-square (rms) velocity fluctuations, $L = \sigma_1^3/\bar{\varepsilon}$, the dimensionaless constants $\theta, k, \omega_2, \omega_3, \omega_4, \Lambda_2, \Lambda_3, \Lambda_4$ in this parametrizations are

$$\Lambda_2 = \left(\frac{2}{C}\right)^3, \ \Lambda_3 = 1., \ \Lambda_4 = \Lambda_2\left(\frac{k}{K_i}\right)^{3/2}, \ k = 3., \ \theta = -\frac{4}{5}(\Lambda_3)^4,$$

$$\omega_2 = \frac{164.32}{\sqrt{\Lambda_2}}, \ \omega_3 = -\frac{4}{5}\frac{\omega_2\Lambda_2^{1/2}}{2^{3/2}S_0}, \ \omega_4 = \omega_2\left(\frac{K_i}{K_0}\right)^{3/4}, \ K_0 = 7.5, \ S_0 = -0.5, \ K_i = 3.4 \ .$$

6.2 Bi-gaussian pdf.

Here we present comparisons of the Lagrangian statistical characteristics obtained by the model (6.13) with the results obtained by DNS. The pdf $f_E(\xi; r)$ is choosen as the following bi-Gaussian density with unknown parameters $p, \mu_1, \mu 2, \sigma_1, \sigma_2$:

$$f_E(\xi, r) = \frac{p}{\sqrt{2\pi}\sigma_1} \exp\{-\frac{(\xi-\mu_1)^2}{2\sigma_1^2}\} + \frac{1-p}{\sqrt{2\pi}\sigma_2} \exp\{-\frac{(\xi-\mu_2)^2}{2\sigma_2^2}\}.$$

$$p\mu_1 + (1-p)\mu_2 = 0, \quad p(\mu_1^2+\sigma_1^2) + (1-p)(\mu_2^2+\sigma_2^2) = 1,$$

$$p(\mu_1^3 + 3\mu_1\sigma_1^2) + (1-p)(\mu_2^3 + 3\mu_2\sigma_2^2) = S$$

$$p(\mu_1^4 + 6\mu_1^2\sigma_1^2 + 3\sigma_1^4) + (1-p)(\mu_2^4 + 6\mu_2^2\sigma_2^2 + 3\sigma_2^4) = F,$$

where $S = S(r) = \frac{\langle u_{Er}^3 \rangle}{\langle u_{Er}^2 \rangle^{3/2}}$ and $F = F(r) = \frac{\langle u_{Er}^4 \rangle}{\langle u_{Er}^2 \rangle^2}$ are the skewness and the flatness of u_{Er}, respectively.

To determine uniquely the 5 unknown parameters, we use the closure assumption (Luhar et al. 1996) which has proven to be reasonable:

$$\frac{\mu_1}{\sigma_1} = -\frac{\mu_2}{\sigma_2} = m, \tag{6.14 }$$

where $m = m(r)$ is the solution to the equation

$$F = \left(1 + \frac{(1+m^2)^3 S^2}{(3+m^2)^2 m^2}\right) \frac{3 + 6m^2 + m^4}{(1+m^2)^2}.$$

The assumption (6.14) allows to obtain the unknown parameters σ_1 and σ_2 explicitly:

$$\sigma_1 = \left(\frac{1-p}{p(1+m^2)}\right)^{1/2}, \quad \sigma_2 = \left(\frac{p}{(1-p)(1+m^2)}\right)^{1/2},$$

where

$$p = \frac{1}{2}\left[1 - \left(\frac{a}{4+a}\right)^{1/2}\right], \quad a = \frac{(1+m^2)^3 S^2}{(3+m^2)^2 m^2}.$$

In Fig.1, the dimensionless rms of the relative separation as a function of dimensionless time is presented. Calculations by the model (6.13) were carried out for two initial separations: $r_0 = 16\eta$ (lower solid curve) and $r_0 = 64\eta$ (upper solid curve). The Kolmogorov constant was chosen as $C = 2.13$ while $C_0(r)$ is a piecewise linear function: $C_0(0) = 0, C_0(r) = 5$ for $r \geq 30\eta$. The relevant DNS data are shown as the dotted curves. In Fig.2 and Fig.3 the skewness and flatness factors, respectively are shown as functions of dimensionless time. In these curves, the upper solid lines correspond to the initial separation $r_0 = 16\eta$, and the lower solid lines - to $r_0 = 64\eta$. The DNS data are also shown as dotted curves, where $r_0 = 16\eta$ and $r_0 = 64\eta$ correspond to the upper and lower lines, respectively. As the results of Fig.1 show, the model describes the rms of the separation well for dimensionalless times from 0 to 10. There is some discrepancy between the model and DNS data for larger times. As to the skewness and flatness factors, we can only state a qualitative agreement with the DNS data: the model overestimates these factors, compared to the DNS results, in the time interval intermediate between the viscous and external ranges.

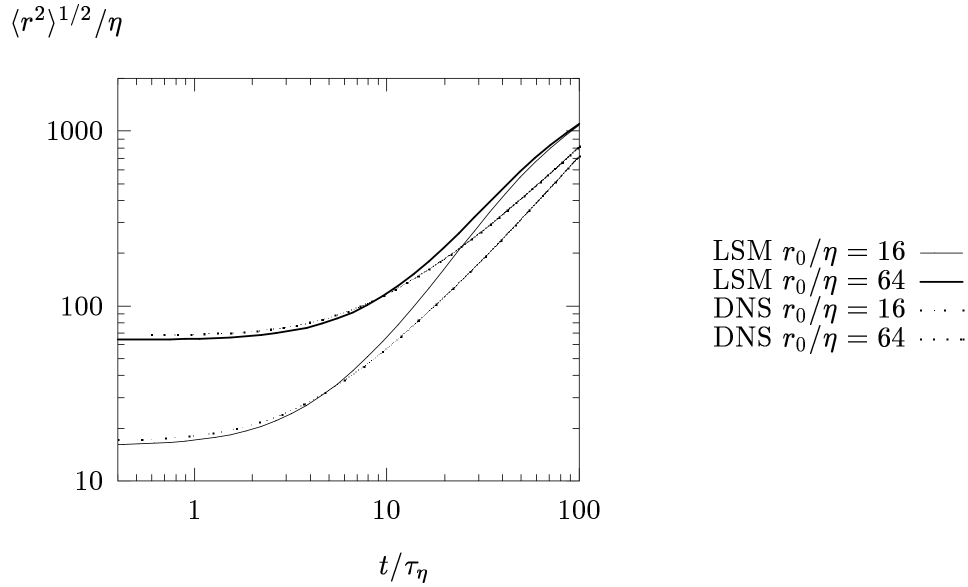

Fig.1: The dimensionless rms of the relative separation; LSM indicates the data obtained by the Lagrangian stochastic model.

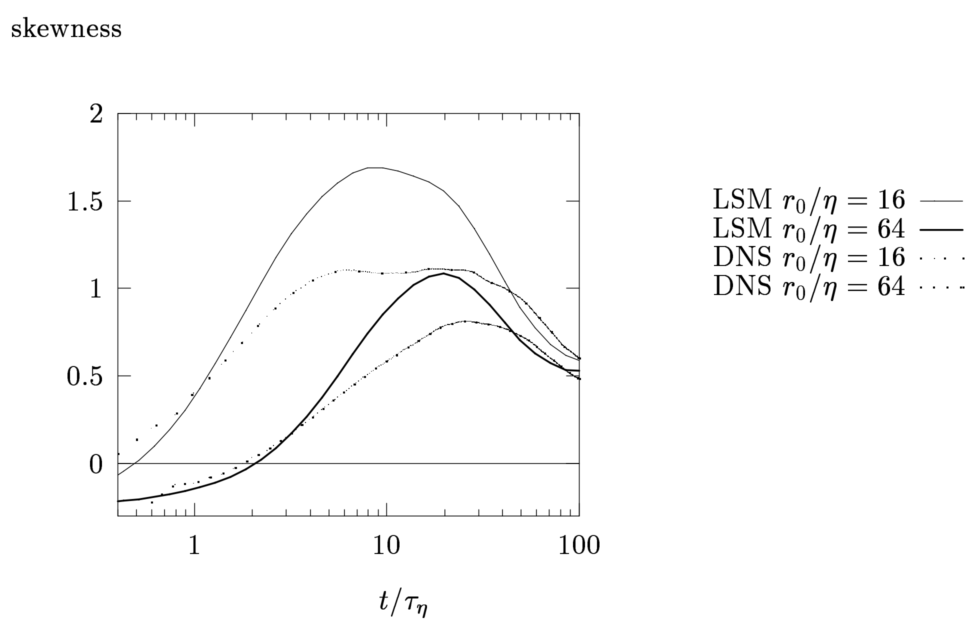

Fig.2: The skewness factor of the relative separation.

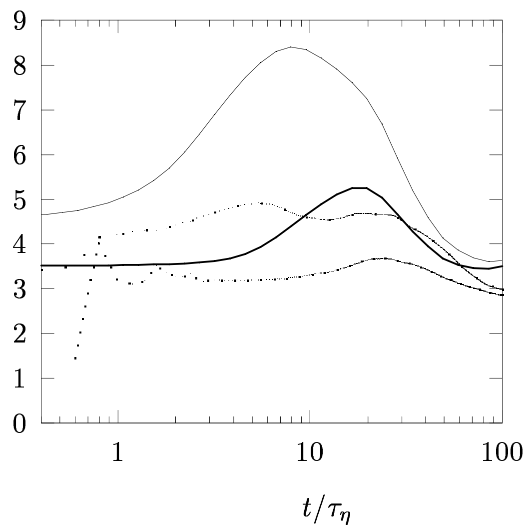

Fig.3: The flatness factor of the relative separation.

It should be noted that our calculations with the bi-gaussian pdf with other types of closure assumptions show qulitatively the same picture. We conclude thus that the bi-gaussian pdf is not the right choice. Therefore we try another class of densities, based on the quadratic-form approximation of the conditional acceleration. These densities have a different tail behaviour which might have a large impact on the skewness and flatness factors. Such indications were reported recently by M. Borgas (Borgas, 2000).

6.3 Q1D quadratic-form model

Let us show that realizability condition will be satisfied if there exists a positive solution $p_E^{\|}$ to the equation (5.6) with the quadratic-form of the conditional acceleration (5.8) which is a pdf of the variable u_r:

$$\int_{-\infty}^{\infty} p_E^{\|}(u_r; r)\, du_r = 1, \quad \forall r \geq 0. \tag{6.15 }$$

and there exists a point r_0 such that

$$\overline{u_r}(r_0) = 0, \quad \overline{u_r^2}(r_0) = \sigma^2(r_0), \quad \overline{u_r^3}(r_0) = \kappa(r_0), \quad \overline{u_r^4}(r_0) = \vartheta(r_0). \tag{6.16 }$$

Here and below we denote by overbar the average over this pdf. In order to proove this assertion let us integrate the equation (5.6) on u_r to get $r^2\overline{u_r} = const$ which yields $\overline{u_r}(r) = 0$ since $\overline{u_r}(r_0) = 0$.
Further multipying the equation (5.6) by u_r and then integrating over u_r we get

$$\frac{1}{r^2}\frac{dr^2\overline{u_r^2}}{dr} = \alpha + \gamma\overline{u_r^2}.$$

This yields $\overline{u_r^2}(r) = \sigma^2(r)$ since the functions $\overline{u_r^2}$ and $\sigma^2(r)$ satisfie the same equations and the common initial conditions at $r = r_0$ (see (6.16)).
Multiplying the equation (5.6) by u_r^2 and u_r^3, and integrating over u_r we find that the functions

$\overline{u_r^3}$ and $\overline{u_r^4}$ satisfy the same equations as the functions $\kappa(r)$ and $\vartheta(r)$, respectively. From these and the conditions (6.16) it follows that

$$\overline{u_r^3}(r) = \kappa(r), \quad \overline{u_r^4}(r) = \vartheta(r),$$

i.e., the realizability condition is established.

Let $f_E(\xi, r)$ be a dimensionless pdf defined by (6.12). Then the transport equation (5.6) can be rewritten in the form:

$$\xi r \frac{\partial f(\xi, r)}{\partial r} + (b_0 + b_1\xi + b_2\xi^2)\frac{\partial f(\xi, r)}{\partial \xi} + (c_0 + c_1\xi) f(\xi, r) = 0, \tag{6.17 }$$

where b_0, b_1, b_2, c_0, c_1 are dimensionless functions depending on r:

$$b_0(r) = \alpha_0(r), \quad b_1(r) = \beta_0(r), \quad b_2(r) = \gamma_0(r) + \psi_0(r),$$

$$c_0(r) = b_1(r), \quad c_1(r) = 2 + 2\gamma_0(r) + \psi_0(r),$$

and $\alpha_0(r)$, $\beta_0(r)$, $\gamma_0(r)$ and $\psi_0(r)$ are determined in (5.10).

Remark. If we assume that for some α, r_1 and r_2 the following conditions are valid

$$\frac{\langle u_r^2 \rangle}{r^\alpha} = const, \quad S(r) = const, \quad F(r) = const, \quad r_1 \ll r \ll r_2\,. \tag{6.18 }$$

then from (5.10) and (6.18) it follows that the coefficients b_0, b_1, b_2, c_0, c_1 are constants, say, $\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \tilde{c}_0, \tilde{c}_1$ in the subrange $r_1 \ll r \ll r_2$. For example in the far-viscous subrange $r \ll \eta$, in the inertial subrange $\eta \ll r \ll L$ and in the external subrange $r \gg L$ these conditions are fulfilled.

This property can be used to define the boundary conditions to the transport equation for the function $f(\xi, r)$. Indeed, let $f_0(\xi)$ be a solution to the following ODE:

$$(\tilde{b}_0 + \tilde{b}_1\xi + \tilde{b}_2\xi^2)\frac{df_0(\xi)}{d\xi} + (\tilde{c}_0 + \tilde{c}_1\xi) f_0(\xi) = 0$$

satisfying the condition

$$\int_{-\infty}^{\infty} f_0(\xi)\, d\xi = 1.$$

We can consider the pdf $f_0(\xi)$ as an approximation to the pdf $f(\xi, r)$ in the subrange $r_1 \ll r \ll r_2$.

Choosing r_0 in the interval $r_1 \ll r_0 \ll r_2$ we put the following boundary condition for the equation (6.17):

$$f(\xi, r = r_0) = f_0(\xi), \tag{6.19 }$$

and the condition (6.15) rewritten in the form:

$$\int_{-\infty}^{\infty} f(\xi, r)\, d\xi = 1, \quad \forall\, r \geq 0. \tag{6.20 }$$

It should be noted that the problem of existense of the solution to (6.17) with boundary condition (6.19) and normalization (6.20) requires a special attention and is currently studied by the authors.

7 Conclusion

Comparisons of the Q1D against the known Lagrangian stochastic well-mixed quadratic form models and the moments approximation models are presented. The comparison is made in the inertial sub-range of turbulence, where g, the Richardson constant in the cubic law of relative diffusion is calculated. The Q1D models give some overestimation of the Richardson constant, compared to the two mentioned models. The discrepance is larger for the Q1D model with the gaussian pdf p_E, while the Q1D models with nongaussian pdf lead to smaller difference. Agreemnet of Q1D model with other theoretical models (presented in Table 3) is more pronounced.

In the case of modestly large Reynolds numbers ($Re_\lambda \simeq 240$) turbulence the comparison of the Q1D model (with a bi-gaussian p_E) with the DNS data is made. Being in a qualitatively agreemnet with the DNS data, the Q1D model predicts higher rate of separation. The higher moments (e.g., skewnws and flatness of separation) show a larger discrepancy. This suggests a need to develope the Q1D model with a non bi-gaussian pdf, for instance, extracted from the transport equation with a quadratic form of the conditional acceleration. A realizable model of such type is mentioned in the last section.

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