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Geometric approach to vibrational control of singularly perturbed systems

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Abstract

We extend the theory of vibrational stabilizability to systems with fast and slow variables. The mathematical tools for establishing corresponding results are the persistence theory of normally hyperbolic invariant manifolds, the averaging theory and appropriate transformations. At the same time we introduce modified concepts of vibrational stabilizability compared with the 'classical' definitions.

1 Introduction

Vibrational control is an open-loop control strategy to modify the dynamical properties of a system by introducing fast oscillations with small amplitude into the system under consideration [10]. Compared with feedback or feedforward control, this method is in some sense unconventional since it does not need online-measurements of states, outputs and disturbances. A well-known example for vibrational control is the inverted pendulum that can be stabilized by vertically oscillating the pendulum pin at a sufficiently high frequency and small amplitude. The corresponding mathematical model reads

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= [c_1 - a\omega^2 c_3 \sin \omega t] \sin x_1 - c_2 x_2, \end{aligned} \tag{1.1}$$

where x_1 is the angular displacement measured from the inverted equilibrium position, x_2 is the angular velocity, c_1, c_2, c_3 are positive physical constants, a is the amplitude and ω the frequency of the applied vibration. >From the representation (1.1) it follows that the applied control can be viewed as a variation of the parameter c_1.

If we horizontally oscillate the pendulum pin of the inverted pendulum, then we get the system

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= c_1 \sin x_1 - a\omega^2 c_3 \sin \omega t \cos x_1 - c_2 x_2. \end{aligned} \tag{1.2}$$

Here, the applied control cannot be viewed as a parameter oscillation, and the origin is not more an equilibrium point.

If we introduce the notation

$$\varepsilon := 1/\omega, \quad a = \alpha\,\varepsilon, \tag{1.3}$$

then system (1.1) can be written in the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= c_1 \sin x_1 - c_2 x_2 + \frac{c_3 \alpha}{\varepsilon} \sin(\frac{t}{\varepsilon}) \sin x_1. \end{aligned} \tag{1.4}$$

It is well-known [1, 10, 14] that the coordinates x_1 and x_2 of (1.4) can be stabilized near $x_1 = x_2 = 0$ for sufficiently small ε and $\alpha^2 > 2c_1/c_3^2$ (that is the frequency ω and the amplitude a are sufficiently small).

Concerning system (1.2) we can prove that only the coordinate x_1 can be stabilized near $x_1 = 0$ (partial stabilization) under the same conditions.

Using (1.3), systems (1.1) and (1.2) can be represented in the form

$$\frac{dx}{dt} = f(x) + \frac{1}{\varepsilon} U(x, \alpha, \frac{t}{\varepsilon}), \tag{1.5}$$

where U is T-periodic in the last argument. By introducing the fast time τ by $t = \varepsilon\tau$ we get from (1.5)

$$\frac{dx}{d\tau} = \varepsilon f(x) + U(x, \alpha, \tau), \tag{1.6}$$

where U is T-periodic in τ.

First contributions towards a theory of vibrational control are due to S.M. Meerkov (see [10] for linear systems) and R.E. Bellmann, J. Bentsman and S.M. Meerkov (see [2, 3] for systems affine linear in the applied control).
Important applications of the method of vibrational control are the stabilization of plasmas [12], lasers [11], chemical reactors [2, 6].

In what follows we extend the theory of vibrational control to systems with slow and fast state variables where we apply the control to the slow components. In section 2 we describe the class of control systems under consideration and introduce modified definitions of vibrational stabilizability compared with the 'classical' definitions. Section 3 contains the reduction of our control problem to some normal form by means of normally hyperbolic invariant manifolds and appropriate transformations. In section 4 we derive conditions for strongly vibrational stabilizability and illustrate our result analytically by means of a linear singularly perturbed system. In the last section we treat the case of partial vibrational stabilizability and demonstrate it by considering the singularly perturbed van der Pol system.

2 Formulation of the problem.

We are given a process containing slow and fast variables and which can be described by the singularly perturbed differential system

$$\begin{aligned} \frac{dz}{dt} &= X(z,y), \\ \varepsilon\frac{dy}{dt} &= Y(z,y), \end{aligned} \tag{2.1}$$

where ε is a small positive parameter. Concerning the functions X and Y we suppose

(A_1). *$X : G \to R^n$, $Y : G \to R^m$ are twice continuously differentiable where G is a neighborhood of the origin in $R^n \times R^m$.*

(A_2). *$(x = 0, y = 0)$ is an equilibrium point of (2.1) that is possibly unstable.*

Our goal is to apply a vibrational control to (2.1) such that the controlled system has an attracting invariant manifold whose projection into the z, y-phase space is a compact set near the origin.

Let G^n be a neighborhood of the origin in R^n. We denote by $\mathcal{U}$ the set of all functions $U : G^n \times R \to R^n$ which are twice continuously differentiable with respect to all arguments and T-periodic in the second argument.

In the sequel we consider control systems of the type

$$
\begin{aligned}
\frac{dz}{dt} &= X(z,y) + \frac{1}{\varepsilon}U(z,\frac{t}{\varepsilon}), \\
\varepsilon\,\frac{dy}{dt} &= Y(z,y),
\end{aligned}
\tag{2.2}
$$

where ε is a small parameter and U belongs to the set $\mathcal{U}$. (The case that U is almost periodic in the second argument can be treated in the same way.) It is clear that $(z=0, y=0)$ is not necessarily a stationary solution of (2.2).

By means of the fast time τ we may rewrite (2.2) as

$$
\begin{aligned}
\frac{dz}{d\tau} &= \varepsilon X(z,y) + U(z,\tau), \\
\frac{dy}{d\tau} &= Y(z,y).
\end{aligned}
\tag{2.3}
$$

Definition 1. *We call the equilibrium point* $(z=0, y=0)$ *of system* (2.1) *strongly vibrationally stabilizable if to any* $\delta > 0$ *there are a sufficiently small positive number* ε_0 *and a function* $U \in \mathcal{U}$ *such that for* $0 < \varepsilon \le \varepsilon_0$ *system* (2.3) *has an exponentially attracting* T*-periodic solution* $(z_p(\tau,\varepsilon), y_p(\tau,\varepsilon))$ *satisfying* $|z_p(\tau,\varepsilon)| \le \delta$, $|y_p(\tau,\varepsilon)| \le \delta$ *for all* τ.

Remark. *This definition of vibrational stabilizability differs from the definition introduced by Meerkov and others* [3] *as follows: In* [3] *it is required that only the average of the periodic solution* $(z_p(\tau,\varepsilon), y_p(\tau,\varepsilon))$ *is located in a* δ*-neighborhood of the origin, and it is assumed that the time-average of the control is zero.*

Definition 2. *We call the equilibrium point* $(z=0, y=0)$ *of system* (2.1) *weakly vibrationally stabilizable if to any* $\delta > 0$ *there are sufficiently small positive numbers* ε_0, δ_0 *and a function* $U \in \mathcal{U}$ *such that for* $0 < \varepsilon \le \varepsilon_0$ *the solution of* (2.3) *starting for* $\tau = 0$ *at any point in a* δ_0*-neighborhood of the origin exists for all* $\tau \ge 0$ *and stays for all* τ *in a* δ*-neighborhood of the origin.*

In singularly perturbed systems the slow variables usually play a special role. Therefore, we introduce the concept of vibrational stabilizability with respect to the vector z of slow variables.

Definition 3. *We call the equilibrium point* $(z=0, y=0)$ *of system* (2.1) *strongly* *vibrationally stabilizable with respect to the slow variable z if to any $\delta > 0$ there are a sufficiently small positive number ε_0 and a function $U \in \mathcal{U}$ such that system (2.3) has for $0 < \varepsilon \leq \varepsilon_0$ an exponentially attracting T-periodic solution $(z_p(\tau, \varepsilon), y_p(\tau, \varepsilon))$ with the property $|z_p(\tau, \varepsilon)| \leq \delta$ for all τ.*

Definition 4. *We call the equilibrium point $(z = 0, y = 0)$ of system (2.1) weakly vibrationally stabilizable with respect to the slow variable z if to any $\delta > 0$ there are sufficiently small positive numbers ε_0, δ_0 and a function $U \in \mathcal{U}$ such that for $0 < \varepsilon \leq \varepsilon_0$ the following properties hold:*

(i) *any solution $(\bar{z}(\tau, z_0, y_0), \bar{y}(\tau, z_0, y_0))$ of (2.3) starting for $\tau = 0$ at a point (z_0, y_0) in a δ_0-neighborhood of the origin exists for all $\tau \geq 0$.*

(ii) *The inequality $|\bar{z}(\tau, z_0, y_0)| < \delta$ holds for all $\tau > 0$.*

Our aim is to find a vibrational control $U(z, \tau)$ stabilizing the equilibrium point $(z = 0, y = 0)$ of (2.1). To this purpose we first derive conditions on U and Y implying that we can reduce system (2.3) to a system in some normal form to which the method of averaging can be applied in order to prove the existence of an attracting periodic solution near the origin.

3 Reduction to some normal form

The first step in our reduction process consists in eliminating the term $U(z, \tau)$ in the first equation of (2.3) by means of an appropriate coordinate transformation. To this end we assume:

(A_3). *To any $\delta > 0$ there is a $\delta_1 > 0$ and a function $U \in \mathcal{U}$ such that the differential system*

$$\frac{d\zeta}{d\tau} = U(\zeta, \tau) \tag{3.1}$$

has the first integral $\zeta = h(\tau, c)$ where h is periodic in τ, and $|h(\tau, c)| \leq \delta$ for $|c| \leq \delta_1$.

The assumption that the image of h is in a small neighborhood of the origin is important for establishing the stabilizability property.

As examples for (3.1) we consider the simple cases $U(\zeta, \tau) \equiv a \cos \tau$ where we have

$|h(\tau,c)| := |a\sin\tau + c| \le |a| + |c|$, such that for $|a| \le \delta/2$ and $|c| \le \delta_1 = \delta/2$ it holds $|h(\tau,c)| \le \delta$; and $U(\zeta,\tau) \equiv \cos\tau\ z$ where $|h(\tau,c)| = |ce^{sin\tau}| \le e|c|$.

The solution $h(\tau,.)$ of (3.1) represents for all τ a diffeomorphism and can be used to introduce a new variable x by

$$z = h(\tau, x). \tag{3.2}$$

By hypothesis (A_3) we get from (3.2), (3.1) and (2.3)

$$\frac{dz}{d\tau} = U(z,\tau) + \frac{\partial h}{\partial x}\frac{dx}{d\tau} = \varepsilon\, X(h(\tau,x),y) + U(z,\tau).$$

Thus, we have

$$\begin{aligned} \frac{dx}{d\tau} &= \varepsilon\left(\frac{\partial h}{\partial x}\right)^{-1}(\tau,x)X(h(\tau,x),y), \\ \frac{dy}{d\tau} &= Y(h(\tau,x),y). \end{aligned} \tag{3.3}$$

The right hand side of (3.3) is periodic in τ, hence we consider system (3.3) in the extended phase space $R^n \times R^m \times S^1$.

For $\varepsilon = 0$ we get the system

$$\begin{aligned} \frac{dx}{d\tau} &= 0, \\ \frac{dy}{d\tau} &= Y(h(\tau,x),y). \end{aligned} \tag{3.4}$$

In the next step we will reduce system (3.3) to a system containing only slow variables by means of a compact exponentially attracting invariant manifold. To this purpose we assume

(A_4). *To any $\delta > 0$ there is a neighborhood G_x^n of the origin in R^n such that for $x \in G_x^n$ the differential system*

$$\frac{dy}{d\tau} = Y(h(\tau,x),y) \tag{3.5}$$

has an exponentially attracting T-periodic solution $y = p_0(\tau,x)$ with the properties

(i) *p_0 is differentiable with respect to x.*

(ii) p_0 *satisfies* $|p_0(\tau, x)| \le \delta/2$.

Remark 1. *Assumption* (A_4) *implies that* $\Gamma_0 := \{(x, y, \tau) \in G_x^n \times R^m \times S^1 : y = p_0(\tau, x)\}$ *is a compact normally hyperbolic invariant manifold of system* (3.4) [7, 15].

Remark 2. *Assumption* (A_1) *does not imply that* (3.4) *has only one exponentially attracting invariant manifold. But it is clear that exponentially attracting invariant manifolds cannot intersect each other.*

Remark 3. *Under the hypotheses* $(A_1) - (A_4)$ *the equilibrium point* $(z = 0, y = 0)$ *of system* (3.4) *is weakly vibrationally stabilizable. This follows immediately from the property that* $(x = c \in G_x^n, y = p_0(\tau, x))$ *is a solution of* (3.4).

According to the theory of normally hyperbolic invariant manifolds they persists under small perturbations [7, 15]. Thus we have

Theorem 1 *Under the assumptions* $(A_1) - (A_4)$ *there exists a sufficiently small positive* ε_0 *such that for* $0 \le \varepsilon \le \varepsilon_0$ *system* (3.3) *has a compact exponentially attracting invariant manifold* $\Gamma_\varepsilon := \{(x, y, \tau) \in G_x^n \times R^m \times S^1 : y = p(\tau, x, \varepsilon) = p_0(\tau, x) + O(\varepsilon)\}$ *where* p *is* T*-periodic in* τ *and has the same smoothness as* X.

Our aim is to prove the existence of an asymptotically stable T-periodic solution of system (3.3). Since Γ_ε is an exponentially attracting invariant manifold of (3.3) it is sufficient to consider system (3.3) on the manifold Γ_ε that is, we study the system

$$\frac{dx}{d\tau} = \varepsilon \left(\frac{\partial h}{\partial x}\right)^{-1}(\tau, x) X(h(\tau, x), p(\tau, x, \varepsilon)). \tag{3.6}$$

4 Existence of a small asymptotically stable T- periodic solution

Equation (3.6) can be written in the form

$$\frac{dx}{d\tau} = \varepsilon \left(\frac{\partial h}{\partial x}\right)^{-1}(\tau, x) X(h(\tau, x), p_0(\tau, x)) + O(\varepsilon^2). \tag{4.1}$$

Since the right hand side of (4.1) is T-periodic in τ we use the averaging theory to prove the existence of a T-periodic solution of (4.1). To this end we have to introduce the following assumption. Let

$$X^0(x) := \frac{1}{T}\int_0^T \left(\frac{\partial h}{\partial x}\right)^{-1}(\tau,x)X(h(\tau,x),p_0(\tau,x))d\tau.$$

(A_5). *$X^0(x)=0$ has a solution $x=x_0$ with $|x_0|\le \delta_1/2$. The spectrum of the Jacobian $A := X^0_x(x_0)$ is located in the left half plane.*

Then, applying the fundamental theorem of the theory of averaging [5, 13] we have the following result.

Theorem 2 *Assume the assumptions $(A_1)-(A_5)$ are valid. Then, there exists a sufficiently small positive ε_1 such that for $0<\varepsilon\le\varepsilon_1$ system (4.1) has an exponentially attracting periodic solution $x=q(\tau,\varepsilon)$ located in an δ_1-neighborhood of the origin.*

Under the assumptions of Theorem 2 it follows that for $0<\varepsilon\le\varepsilon_1$ $(z=h(\tau,q(\tau,\varepsilon)), y=p(\tau,q(\tau,\varepsilon),\varepsilon))$ is an exponentially attracting periodic solution of system (2.3) satisfying $|h(\tau,q(\tau,\varepsilon))|+|p(\tau,q(\tau,\varepsilon),\varepsilon))|<2\delta$. Thus, we have

Corollary 1 *Under the assumptions of Theorem 2 the equilibrium point $(z=0,y=0)$ of system (2.1) is strongly vibrationally stabilizable.*

We illustrate our result by considering the following singularly perturbed linear system

$$\begin{aligned}\frac{dz}{dt} &= az+by,\\ \varepsilon\frac{dy}{dt} &= z-y\end{aligned}\tag{4.2}$$

with $a+b>0,\ 0<\varepsilon\ll 1$. Hence, the equilibrium point $(z=0,y=0)$ is a saddle that is, an unstable equilibrium. We want to apply a high frequency control to the slow variable z in the first equation in order to stabilize the system near the origin. The corresponding control system has the form

$$\begin{aligned}\frac{dz}{dt} &= az+by+\frac{1}{\varepsilon}cos\left(\frac{t}{\varepsilon}\right)z,\\ \varepsilon\frac{dy}{dt} &= z-y.\end{aligned}\tag{4.3}$$

Introducing the fast time τ we get from (4.3)

$$\begin{aligned} \frac{dz}{d\tau} &= \varepsilon(az+by)+\cos\tau\, z, \\ \frac{dy}{d\tau} &= z-y. \end{aligned} \tag{4.4}$$

Using the coordinate transformation

$$z = e^{\sin\tau} x$$

we get from (4.4)

$$\begin{aligned} \frac{dx}{d\tau} &= \varepsilon(ax+bye^{-\sin\tau}), \\ \frac{dy}{d\tau} &= e^{\sin\tau}\, x-y. \end{aligned} \tag{4.5}$$

If we consider x as a parameter in the second equation in (4.5), then to given x this equation has a unique 2π-period solution $y_0(\tau,x) := p_0(\tau)\,x$ where $p_0(\tau)$ is defined by

$$p_0(\tau) := \frac{e^{-\tau}}{e^{2\pi}-1}\Big[\int_0^{2\pi} e^{\sigma+sin\sigma}\,d\sigma + (e^{2\pi}-1)\int_0^{\tau} e^{\sigma+sin\sigma}\,d\sigma\Big]. \tag{4.6}$$

It is easy to check that $y_0(\tau,x)$ is exponentially stable and that

$$\Gamma_0 := \{(x,y,\tau)\in G_x^n\times R^m\times S^1 : y = p_0(\tau)\,x\} \tag{4.7}$$

represents a compact exponentially attracting invariant manifold of system (4.5) for $\varepsilon=0$. Therefore, hypotheses $(A_1)-(A_4)$ are satisfied and for sufficiently small ε we get by Theorem 1 that (4.5) has a compact normally hyperbolic invariant manifold Γ_ε

$$\Gamma_\varepsilon := \{(x,y,\tau)\ \in G_x^n\times R^m\times S^1 : y = p(\tau,\varepsilon)\,x = (p_0(\tau)+p_1(\tau)\varepsilon+\ldots)\,x\},$$

where p is 2π-periodic in τ. On Γ_ε (4.5) reads

$$\frac{dx}{d\tau} = \varepsilon(a+bp(\tau,\varepsilon)e^{-\sin\tau})x = \varepsilon(a+bp_0(\tau)e^{-\sin\tau})x + O(\varepsilon^2). \tag{4.8}$$

Using

$$m := \frac{1}{2\pi}\int_0^{2\pi} p_0(\tau)e^{-\sin\tau}d\tau \approx 1.29 \neq 0$$

the averaged equation to (4.8) has the form

$$\frac{dx}{d\tau} = \varepsilon(a + bm)x.$$

For $a + bm < 0$ hypothesis (A_5) is satisfied. Consequently, by Theorem 2 system (4.2) is vibrationally stabilizable.

5 Partial vibrational stabilizability

In the sequel we replace hypothesis (A_4) by the following assumption.

(A_4^*). *To any* $\delta > 0$ *there is a neighborhood* G_x^n *of the origin in* R^n *such that for* $x \in G_x^n$ *system* (3.5) *has an exponentially attracting* T*-periodic solution* $y = p_0(\tau, x)$ *that is differentiable with respect to* x.

Compared with assumption (A_4) we do not assume that the periodic solution p_0 is located in a small neighborhood of the origin. A consequence of this hypothesis is that we are not able to guarantee that the y-component of system (2.1) can be vibrationally stabilized near $y = 0$.

The following observation is obvious.

Lemma 1 *Assume the hypotheses* $(A_1) - (A_3)$ *and* (A_4^*) *are valid. Then system* (2.1) *is weakly vibrationally stabilizable with respect to the slow component* z.

The following theorem can be proved in the same way as Theorem 2.

Theorem 3 *Assume the assumptions* $(A_1) - (A_3), (A_4^*), (A_5)$ *are valid. Then there exists a sufficiently small positive* ε_1 *such that for* $0 < \varepsilon \leq \varepsilon_1$ *system* (3.3) *has an exponentially attracting* T*-periodic solution* $(z = h(\tau, q(\tau, \varepsilon)), y = p(\tau, q(\tau, \varepsilon), \varepsilon))$ *satisfying* $|h(\tau, q(\tau, \varepsilon))| < \delta$.

>From Theorem 3 we get immediately

Corollary 2 *Under the assumptions of Theorem* 3 *the equilibrium point* $(z = 0, y = 0)$ *of system* (2.1) *is strongly vibrationally stabilizable with respect to the slow component* z.

We illustrate Theorem 3 by considering the van der Pol equation with large damping [8]. In that case, it can be represented by the singularly perturbed system

$$\begin{aligned} \frac{dz}{dt} &= -y, \\ \varepsilon\,\frac{dy}{dt} &= z + y - y^3. \end{aligned} \tag{5.1}$$

It is well-known that system (5.1) has for $0 < \varepsilon \ll 1$ a unique exponentially stable relaxation oscillation [8]. The corresponding closed curve in the (z, y)-phase plane contains the origin as unique equilibrium point which is unstable. Our goal is a strong vibrational stabilization of the z-component near the origin by applying an additive high-frequency control. We consider the control system

$$\begin{aligned} \frac{dz}{dt} &= -y + \frac{a}{\varepsilon} cos\Big(\frac{t}{\varepsilon}\Big), \\ \varepsilon\,\frac{dy}{dt} &= z + y - y^3. \end{aligned} \tag{5.2}$$

Introducing the fast time τ and the new coordinate x by $z = x + \sin\tau$ we get from (5.2)

$$\begin{aligned} \frac{dx}{d\tau} &= -\varepsilon\, y, \\ \frac{dy}{d\tau} &= a\sin\tau + x + y - y^3. \end{aligned} \tag{5.3}$$

For $|x|$ sufficiently small, the differential equation

$$\frac{dy}{d\tau} = x + y - y^3$$

has three equilibria $y^x_{-1} < y^x_0 < y^x$ where y^x_{-1} and y^x_1 are hyperbolic stable equilibria which are located near -1 and 1 respectively. Consequently, for sufficiently small $|a|$ and $|x|$, the second differential equation in (5.3) has two T-periodic solutions $p^x_{-1}(\tau)$ and $p^x_1(\tau)$ which are exponentially attracting and satisfy $|p^x_{-1}(\tau) - y^x_{-1}| < \varrho$ and $|p^x_1(\tau) - y^x_1| < \varrho$ respectively, where ϱ is a small number [9]. Therefore, according to Theorem 3, there exists a sufficiently small ε_1 such that for $0 < \varepsilon \le \varepsilon_1$ system (5.3) has two exponentially attracting T-periodic solutions $(x_{-1}(\tau,\varepsilon), y_{-1}(\tau,\varepsilon)), (x_1(\tau,\varepsilon), y_1(\tau,\varepsilon))$. Therefore, system (5.1) can be strongly vibrationally stabilized with respect to the slow component.

Note, if we want to stabilize (5.1) by the linear multiplicative control

$$
\begin{aligned}
\frac{dz}{dt} &= -y + \frac{a}{\varepsilon}\cos\left(\frac{t}{\varepsilon}\right)z, \\
\varepsilon\frac{dy}{dt} &= z + y - y^3,
\end{aligned}
\tag{5.4}
$$

then computer experiments indicate that the equilibrium point $(z = 0, y = 0)$ of (5.1) cannot be vibrationally stabilized that way.

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