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The Brockett problem in the theory of nonstationary stabilization of linear differential equations

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Abstract

In the present work a review of algorithms for nonstationary linear stabilization is given. In many cases these algorithms, together with the criterion of nonstabilizing, allow us to obtain a solution of the Brockett problem.

In the book [1], R. Brockett has stated the following problem.

Let be given a triple of constant matrices (A, B, C). Under what circumstances does there exist a time-dependent matrix $K(t)$ such that the system

$$\frac{dx}{dt} = Ax + BK(t)Cx, \qquad x \in R^n \tag{1}$$

is asymptotically stable ?

Note that for system (1) the problem of stabilization by means of a constant matrix K is the classic one in the automatic control theory [2, 3]. From this point of view the Brockett problem can be restated in the following way.

How much does the introduction of matrices $K(t)$, depending on time t, enlarge the possibilities of a classic stabilization ?

In studying the stabilization problems of mechanical systems it is sometimes necessary to consider a more narrow class of stabilizing matrices $K(t)$. These matrices must be periodic and have a zero mean value on the period $[0, T]$:

$$\int_0^T K(t)\, dt = 0. \tag{2}$$

Consider, for example, a linear approximation in a neighborhood of the equilibrium position of a pendulum with vertically oscillating pendulum pin

$$\ddot{\theta} + \alpha\dot{\theta} + (K(t) - \omega_o^2)\theta = 0, \tag{3}$$

where α and ω_o are positive numbers. Here the most frequently considered functions $K(t)$ are of the form [4] $\beta \sin \omega t$ and those of the following form [5,6]

$$K(t) = \begin{cases} \beta \text{ for } t \in [0, T/2) \\ \\ -\beta \text{ for } t \in [T/2, T) \end{cases} \tag{4}$$

For such functions $K(t)$ the effect of stabilization of the upper equilibrium position for large ω and small T is well known.

In the present paper, algorithms for construction of periodic piecewise constant functions $K(t)$ are given, which solve the Brockett problem in some cases.

In order to show in the most simple manner the main peculiarities and the advantages of these algorithms we consider first equation (3) and prove the following

Proposition 1. *Suppose $\alpha^2 < 4(\beta - \omega_o^2)$. Then for any number $\tau > 0$ there exists a number $T > \tau$ such that equation (3) with the function $K(t)$ of the form (4) is asymptotically stable.*

This fact, in particular, implies the possibility to stabilize the upper position of a pendulum under low-frequency vertical oscillations of a pendulum pin. In this case the amplitude a of oscillations is sufficiently large:

$$a = \frac{lT^2\beta}{8}$$

where l is a pendulum length, β is an acceleration divided by l.

To prove Proposition 1, note that we may assume without loss of generality that $\beta - \omega_o^2 - \alpha^2/4 = 1$. (For this purpose it is sufficient to made a change of time).

Together with equation (3), consider the system equivalent to it

$$\begin{aligned} \dot{\theta} &= \eta \\ \dot{\eta} &= -\alpha\eta - (K(t) - \omega_o^2)\theta. \end{aligned} \tag{5}$$

At first, consider some properties of this system for $K(t) \equiv -\beta$ and $K(t) \equiv \beta$, which are needed for the sequel.

A system

$$\begin{aligned} \dot{\theta} &= \eta \\ \dot{\eta} &= -\alpha\eta + (\beta + \omega_o^2)\theta \end{aligned} \tag{6}$$

has a saddle singular point with a stable manifold $\eta = L_1\theta$ and an unstable manifold $\eta = L_2\theta$. Here

$$L_1 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + (\beta + \omega_o^2)},$$

$$L_2 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + (\beta + \omega_o^2)}.$$

We now consider a fundamental matrix $X(t)$ of system

$$\begin{aligned} \dot{\theta} &= \eta \\ \dot{\eta} &= -\alpha\eta - (\beta - \omega_o^2)\theta \end{aligned} \tag{7}$$

with the initial data $X(0) = I$.

From the condition $\alpha^2 < 4(\beta - \omega_o^2)$ it follows that the characteristic polynomial of system (7) has complex roots and therefore there exists a number $T_1 > 0$ such that a linear operator $X(T_1)$ transforms the straight line $\eta = L_2\theta$ into the line $\eta = L_1\theta$.

From the equality $\beta - \omega_o^2 - \alpha^2/4 = 1$ it follows that the straight line $\eta = L_2\theta$ is also transformed into the straight line $\eta = L_1\theta$ by operators $X(T_1 + 2\pi j)$. Here j are integer numbers.

Show that as a number T we can choose a value $2(T_1 + 2\pi j)$ with sufficiently large j. Consider a ball

$$\Omega = \{\eta^2 + \theta^2 \leq 1\}$$

and system (5) such that

$$K(t) = \begin{cases} -\beta \text{ for } t \in [0, T/4) \\ \beta \text{ for } t \in [T/4, 3T/4) \\ -\beta \text{ for } t \in [3T/4, T] \end{cases}$$

Now we prove that points from the ball Ω, moving along trajectories of system (5), get into a ball of radius $1/2$ for $t = T$. Note first that solutions of system (6) for $t = 0$ with the initial data from Ω, for $t = T/4$ get into an ε — neighborhood of the straight line $\eta = L_2\sigma$, where

$$\varepsilon = \nu_1 e^{L_1 T/4}.$$

Here ν_1 is some number. In addition for these solutions the following inequality holds

$$\eta\left(\frac{T}{4}\right)^2 + \theta\left(\frac{T}{4}\right)^2 \leq \nu_2 e^{L_2 T/2} \tag{8}$$

Here ν_2 is some number.

On the interval $(T/4, 3T/4)$ the motion occurs along trajectories of system (7). Under the action of the operator $X(T/2)$ an ε — neighborhood of the straight line $\eta = L_2\theta$ is transformed into a $\nu_3\varepsilon$ — neighborhood of the straight line $\eta = L_1\theta$. Here ν_3 is a number. In this case by inequality (8) and the relation $\beta - \omega_o^2 - \alpha^2/4 = 1$ we have the following estimate

$$\eta\left(\frac{3T}{4}\right)^2 + \theta\left(\frac{3T}{4}\right)^2 \leq \nu_2 e^{L_2 T/2} \tag{9}$$

On the interval $(3T/4, T)$ the motion occurs along trajectories of system (6). Here from the fact that the points $\theta(3T/4)$, $\eta(3T/4)$ are in a $\nu_3\varepsilon$ — neighborhood of the straight line $\eta = L_1\theta$ and from inequality (9) it follows that the points $\theta(T)$, $\eta(T)$ belong to an ε_1 — neighborhood of the straight line $\eta = L_1\theta$ and

$$\eta(T)^2 + \theta(T)^2 \leq \nu_4 e^{(L_1+L_2)T/2}. \tag{10}$$

Here

$$\varepsilon_1 = \nu_5\nu_3\nu_1 e^{(L_1+L_2)T/4}, \tag{11}$$

ν_4 and ν_5 are some numbers.

Relations (10), (11), and the inequality $L_1 + L_2 < 0$ imply that, choosing sufficiently large T, the inequality

$$\eta(T)^2 + \theta(T)^2 \leq 1/4 \tag{12}$$

is satisfied. It is well known [5] that inequalities (12) are the sufficient condition for an asymptotic stability of linear systems with the periodic coefficients.

Thus, the algorithm for stabilization turns out to be very simple and is based on two properties of linear systems (6) and (7). Firstly, solutions of system (6) approach to an unstable manifold faster than they stretch along this manifold. Secondly, this unstable manifold may be turned after a switching along trajectories of system (7) in such a way that to the next switching it coincides with the stable manifold. Acting on the interval $(3T/4, T)$, we use the prevailing value of compression in comparison with the stretching and, in large, by the time $t = T$ we can completely eliminate the stretching, embedding solutions into a ball of arbitrary small radius.

We now describe a similar algorithm for system (1).

Suppose, there exists a matrix K_1 such that the system

$$\frac{dx}{dt} = (A + \mu B K_1 C)x \tag{13}$$

with the scalar parameter μ has a stable linear invariant manifold $L(\mu)$ for $\mu \geq \mu_0$. Here μ_0 is some number. We also assume that

$$\lim_{\mu \to +\infty} L(\mu) = L_o \tag{14}$$

and for any number $\delta > 0$ there exists a number $\mu_1 \geq \mu_0$ such that

$$|x(1, x_0)| \leq \delta, \quad \forall\, x_0 \in \{|x| = 1\} \bigcap L(\mu), \quad \mu \geq \mu_1 \tag{15}$$

Here $x(0, x_0) = x_0$ and limit (14) is understood in the sense that the set $L(\mu) \cap \{|x| \leq 1\}$ is in an ε — neighborhood of $L_o \cap \{|x| \leq 1\}$, where $\varepsilon \to 0$ as $\mu \to +\infty$.

This assumption means the fast convergence of trajectories on the manifold $L(\mu)$ for sufficiently large parameter μ. Denote by $M(\mu)$ a linear invariant manifold of system (13) such that

$$\lim_{\mu \to +\infty} M(\mu) = M_o,$$

$$\dim M(\mu) + \dim L(\mu) = n,$$

$$M(\mu) \cap L(\mu) = \{0\}.$$

We assume that $M(\mu)$ is a manifold of slow motions, i.e., there exists a number R such that for all $\mu \geq \mu_0$ the following inequality

$$|x(1, x_0)| \leq R, \quad \forall x_0 \in \{|x| = 1\} \cap M(\mu) \tag{16}$$

is valid. Suppose now that there exists a matrix K_2 such that for a system

$$\frac{dy}{dt} = (A + BK_2C)y \tag{17}$$

there exists in turn a number τ such that

$$Y(\tau)M_o \subset L_o. \tag{18}$$

Here $Y(t)$ is a fundamental matrix of system (17), $Y(0) = I$.

Define a $(2+\tau)$ — periodic matrix $K(t)$ in the following way

$$K(t) = \begin{cases} \mu K_1 \text{ for } t \in [0,1), \\ K_2 \text{ for } t \in [1, 1+\tau), \\ \mu K_1 \text{ for } t \in [1+\tau, 2+\tau) \end{cases} \tag{19}$$

Theorem 1. *System* (1) *with the matrix* $K(t)$ *of the form* (19) *is asymptotically stable for sufficiently large* μ.

Proof. From the construction of the sets L_o and M_o we can see that for any number $\varepsilon > 0$ there exists $\mu_2 \geq \mu_0$ such that it has the following property. For $|x_0| = 1$ and $\mu \geq \mu_2$ a solution of system (1) $x(1, x_0)$ is in an ε — neighborhood of the set

$$M_o \cap \{|x| \leq R\}.$$

From this and condition (18) it follows that there exists a number R_1 such that the following statement is valid. For $|x_0| = 1$, $\mu \geq \mu_2$ a solution $x(1+\tau, x_0)$ of system (1) is in a $R_1\varepsilon$ — neighborhood of the set

$$L_o \cap \{|x| \leq R_1\}.$$

Hence it follows that for the solutions considered there exists a number R_2 such that

$$|x(2+\tau, x_0)| \leq R_2\varepsilon.$$

Choosing ε sufficiently small (and, consequently, μ sufficiently large) we find that for all x_0 from the sphere $\{|x| = 1\}$ an estimate holds

$$|x(2+\tau, x_0)| < 1/2.$$

This means the asymptotic stability of system (1) with the periodic matrix $K(t)$ of the form (19).

To check condition (18), for a periodic solution $z(t)$ of the system

$$\dot{z} = Qz, \quad z \in R^n, \tag{20}$$

where Q is a constant nonsingular $n \times n$ — matrix, it is sometimes useful to use the following

Lemma 1. *For any vector $h \in R^n$ there exists a number τ such that $h^* z(\tau) = 0$.*

Proof. Suppose the contrary. We obtain the inequality $h^* z(t) \neq 0$, $\forall t \in R^1$. Without loss of generality it can be assumed that $h^* z(t) > 0$, $\forall t \in R^1$. The above and the periodicity of $z(t)$ result in the following relation

$$\lim_{t \to +\infty} \int_0^t h^* z(t)\, dt = +\infty, \tag{21}$$

On the other hand we have

$$\int_0^t h^* z(t)\, dt = h^* Q^{-1}(z(t) - z(0)).$$

From the above and the periodicity of $z(t)$ the periodicity of the function

$$\int_0^t h^* z(t)\, dt$$

follows, what contradicts relation (21). The lemma is proved.

Theorem 2. *Let there exist matrices K_1 and K_2 such that they satisfy the following conditions:*

1) *The matrix BK_1C has $n-1$ eigenvalues with negative real parts and* $\det BK_1C = 0$,

2) *for a vector $u \neq 0$, which satisfies the equality $BK_1Cu = 0$, and for some number λ a vector function*

$$\exp[(A + BK_2C + \lambda I)t]u$$

is periodic.

Then there exists a periodic matrix $K(t)$ such that system (1) is asymptotically stable.

Proof. Condition 1) of Theorem 2 results in that relations (14)–(16) are satisfied. Here L_o is a stable manifold of the system

$$\frac{dz}{dt} = BK_1Cz,$$

$M_o = \{\gamma u |\ \gamma \in R^1\}$.

By virtue of the lemma condition 2) of Theorem 2 implies the existence of a number τ such that

$$\exp[(A + BK_2C)\tau]u \in L_o.$$

Thus we arrive at condition (18). Hence by Theorem 1 system (1) with the matrix of the form (19) is asymptotically stable.

In the two-dimensional case Theorem 2 has the following simple form.

Theorem 3. *Let $n = 2$ and there exist matrices K_1, K_2, satisfying the following conditions:*

1) $\det BK_1C = 0$ *and* $TrBK_1C \neq 0$,

2) *The matrix* $A + BK_2C$ *has complex eigenvalues.*

Then there exists a periodic matrix $K(t)$ *of the form* (19) *such that system* (1) *is asymptotically stable.*

Proof. Condition 1) of Theorem 3 implies the existence of nonzero eigenvalue of the matrix BK_1C. If this eigenvalue is negative, then condition 1) of Theorem 2 is satisfied; if it is positive, then condition 1) of Theorem 2 is satisfied with $-K_1$.

Condition 2) of Theorem 2 follows directly from condition 2) of Theorem 3. The proof of Theorem 3 is completed.

We proceed now to the necessary conditions of stabilization.

Consider the case being substantial for the control theory, namely, B is a column vector, C is a row sector, $K(t)$ is a sectionally continuous function: $\mathbf{R}^1 \to \mathbf{R}^1$.

In the case that the transfer function $W(p)$ of system (1) is nondegenerate we have

$$W(p) = c^*(A - pI)^{-1}B = \frac{c_n p^{n-1} + \ldots + c_1}{p^n + a_n p^{n-1} + \ldots + a_1},$$

where c_j and a_j are real numbers, and system (1) may be written in the following scalar form [7]:

$$\begin{aligned} &\dot{x}_1 = x_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\dot{x}_{n-1} = x_n, \\ &\dot{x}_n = -(a_n x_n + \ldots + a_1 x_1) - K(t)(c_n x_n + \ldots + c_1 x_1). \end{aligned} \tag{22}$$

Then

$$C = (c_1, \ldots, c_n), \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & 1 \\ -a_1 & \ldots & \ldots & -a_n \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}.$$

Recall that the nondegeneracy of the transfer function $W(p)$ indicates that the following polynomials

$$\begin{gathered} c_n p^{n-1} + \ldots + c_1, \\ p^n + a_n p^{n-1} + \ldots + a_1 \end{gathered}$$

have not the common zeros.

Let $c_n \neq 0$. We can assume without loss of generality that $c_n = 1$.

Theorem 4. *Suppose, the following conditions hold:*

1) *for* $n > 2 \quad c_1 \leq 0, \ldots, c_{n-2} \leq 0,$

2)

$$\begin{array}{l} c_1(a_n - c_{n-1}) > a_1, \\ c_1 + (a_n - c_{n-1})c_2 > a_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n-2} + (a_n - c_{n-1})c_{n-1} > a_{n-1}. \end{array}$$

Then there exists no function $K(t)$ *such that system* (1) *is asymptotically stable.*

Proof. Consider a set

$$\Omega = \{x_1 \geq 0, \ldots, x_{n-1} \geq 0, \;\; x_n + c_{n-1}x_{n-1} + \ldots + c_1 x_1 \geq 0\}.$$

Let us prove that Ω is positively invariant, i.e., if $x(t_0) \in \Omega$, then $x(t) \in \Omega$, $\forall\, t \geq t_0$. Note that for $j = 1, \ldots, n-1$

$$\begin{array}{c} x_j(\tau) = 0, \quad x_i(\tau) > 0, \;\; \forall\, i \neq j, \; i \leq n-1, \\ x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \ldots + c_1 x_1(\tau) > 0 \end{array}$$

the following inequality holds

$$\dot{x}_j(\tau) > 0. \tag{23}$$

Indeed, for $j = 1, \ldots, n-2$ we have

$$\dot{x}_j(\tau) = x_{j+1}(\tau) > 0.$$

For $n = 2$

$$\dot{x}_1(\tau) = x_2(\tau) > -c_1 x_1(\tau) = 0$$

and for $n > 2$

$$\dot{x}_{n-1}(\tau) = x_n(\tau) > -c_{n-2}x_{n-2}(\tau) - c_1 x_1(\tau) \geq 0.$$

Note also that

$$(x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \ldots + c_1 x_1(\tau))^{\bullet} > 0 \tag{24}$$

for $x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \ldots + c_1 x_1(\tau) = 0$ and $x_j(\tau) > 0$, $j = 1, \ldots, n-1$.

Really,

$$\begin{array}{l} (x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \ldots + c_1 x_1(\tau))^{\bullet} = \\ = (-a_{n-1} + c_{n-2} + (a_n - c_{n-1})c_{n-1})x_{n-1}(\tau) + \ldots \\ \ldots + (-a_2 + c_1 + (a_n - c_{n-1})c_2)x_2(\tau) + \\ +(-a_1 + (a_n - c_{n-1})c_1)x_1(\tau). \end{array}$$

From here and from condition 2) of the theorem inequality (24) follows.

Relations (23) and (24) imply that almost everywhere the boundary of set Ω is non-contact with respect to the vector field of system (22) and the solutions of system (22) are "sewing" this boundary almost everywhere into the set Ω. From here and from the continuous dependence of solutions of system (22) on the initial data it follows that the set Ω is positively invariant. The positive invariance of Ω implies the lack of asymptotic stability of system (22). The proof of the theorem is completed.

We apply the results obtained to the case that $n = 2$, B is a column vector, C is a row vector, and $K(t)$ is a scalar function.

Introduce a transfer function of system (1)

$$W(p) = C(A - pI)^{-1}B = \frac{\rho p + \gamma}{p^2 + \alpha p + \beta},$$

where p is a complex variable.

With this objection in mind we put $\rho \neq 0$. Then it can be assumed without loss of generality that $\rho = 1$. Suppose also that the function $W(p)$ is nondegenerate, i.e., the inequality

$$\gamma^2 - \alpha\gamma + \beta \neq 0$$

is true. In this case [7] system (1) may be written as follows

$$\begin{aligned} \dot{\sigma} &= \eta \\ \dot{\eta} &= -\alpha\eta - \beta\sigma - K(t)(\eta + \gamma\sigma). \end{aligned} \tag{25}$$

Stabilization of system (25) by means of the constant $K(t) \equiv K_0$ is possible iff

$$\alpha + K_0 > 0, \quad \beta + \gamma K_0 > 0.$$

For the existence of a number K_0, satisfying these two inequality, it is necessary and sufficient that either the condition $\gamma > 0$ or the inequalities $\gamma \leq 0$, $\alpha\gamma < \beta$ should be satisfied.

Consider the case that by means of the constant $K(t) \equiv K_0$ the stabilization is impossible:

$$\gamma \leq 0, \quad \alpha\gamma > \beta.$$

Let us make the use of Theorem 3. Condition 1) of Theorem 3 is satisfied since $\det BK_1C = K_1 \det BC = 0$ and $TrBK_1C = K_1CB = -K_1 \neq 0$.

Condition 2) of Theorem 3 is satisfied if for some K_2 the polynomial

$$p^2 + \alpha p + \beta + K_2(p + \gamma)$$

has complex roots. We can see that for existence of such K_2 it is necessary and sufficient that the inequality

$$\gamma^2 - \alpha\gamma + \beta > 0 \tag{26}$$

should be satisfied.

Thus if inequality (26) holds, then there exists a periodic function $K(t)$ such that system (26) is asymptotically stable.

If the inequality

$$\gamma^2 - \alpha\gamma + \beta < 0 \tag{27}$$

is valid, then the conditions of Theorem 4 are obviously satisfied. Thus we have the following result.

Theorem 5 [8]. *If inequality* (26) *holds, then there exists a periodic function* $K(t)$ *such that system* (25) *is asymptotically stable.*

If inequality (27) *holds, then there exists no function* $K(t)$ *such that system* (25) *is asymptotically stable.*

In another class of the stabilizing functions $K(t)$ of the form

$$K(t) = (k_0 + k_1\omega \cos \omega t), \quad \omega \gg 1,$$

this result was also obtained in [9] by means of the averaging method.

Now we show that the method, suggested here for construction piecewise constant stabilizing functions, makes it possible to consider the Brocket problem also in the case when $n = 3$, B is a column vector, C is a row vector, $K(t)$ is a scalar function.

Assume that the following conditions hold:

1) the inequality $CB < 0$ holds,

2) a matrix A has two complex eigenvalues with the positive real parts and one negative eigenvalue,

3) for some number k the function $G(t) = C \exp[(A + kBC)t]B$ has at least one root in the interval $(-\infty, 0)$,

4) the inequality $\det(B, AB, A^2B) \neq 0$ holds.

Theorem 6. *If conditions* 1)–4) *are satisfied, then there exists a periodic function* $K(t)$ *such that system* (1) *is asymptotically stable.*

Proof. Let us describe an algorithm for construction the desirable function $K(t)$. The specific character of the problem is that we can obtain the intense pressure in the phase space $R^3 = \{x\}$ by the choice of $K(t) = \mu$, $\mu \gg 1$ in one direction parallel to the vector B only. Therefore the algorithm for constructing the stabilizing function $K(t)$ involves a greater number of steps than that in proving the previous statements. Consider sequentially each of these steps, observing the transformations of a ball Ω of radius 1

$$\Omega = \{|x| = 1\}$$

along trajectories in R^3.

1) Define $K(t)$ on the set $[0, 1)$ in the following way: $K(t) = \mu$, where μ is a large parameter. The ball Ω is collapsed into an ellipsoid Ω_1, placed in an ε –

neighborhood of the plane $\{Cx = 0\}$. Here $\varepsilon = \varepsilon(\mu)$ is a small number. The ellipsoid $\Omega_1 = x(1, \Omega)$, obtained in such a way, has one principal semiaxis of order $O(\varepsilon)$ and two other principal semiaxes depending on A, B, and C.

2) Consider the segment $[1, 1-\tau]$, where τ is some zero of the function $G(t) = C\exp[(A+kBC)t]B$ in the interval $(-\infty, 0)$. Define $K(t)$ on this segment as follows: $K(t) = k$. In this case for a solution $z(t, B)$ of the system

$$\frac{dz}{dt} = (A + kBC)z \tag{28}$$

with the initial data $z(0, B) = B$, the following equality holds

$$Cz(\tau, B) = 0.$$

Whence it follows that the ellipse

$$\Omega_1 \cap \{Cx = 0\},$$

transformed along trajectories of system (1) on $[1, 1-\tau]$, at time $t = 1-\tau$ intersects the straight line $\{\lambda B | \lambda \in R^1\}$.

3) Define $K(t)$ on the interval $(1-\tau, 2-\tau)$ in the same way as at the first step, namely, $K(t) = \mu$, $\mu \gg 1$.

Here the ellipse, intersected by the straight line $\{\lambda B | \lambda \in R^1\}$ that has been transformed along trajectories of system (1), at time $t = 2-\tau$ becomes a prolate ellipsoid, placed in an ε — neighborhood of a certain segment of the straight line $\{\nu d, \nu \in [-1, 1]\}$, placed on the plane $\{Cx = 0\}$: $Cd = 0$.

Thus at time $t = 2-\tau$ the ball Ω of radius unit that has been transformed along the trajectories of system (1) on the interval $(0, 2-\tau)$ has become transformed into the ellipsoid, placed in an $O(\varepsilon)$ – neighborhood of the segment $\{\nu d, \nu \in [-1, 1]\}$.

4) Put $K(t) = 0$ on the segment $[2-\tau, 2-\tau+T_1]$. The number T_1 is defined in the following way o. Denote by $\{\lambda e | \lambda \in R^1\}$ a stable linear manifold of system (28). Here $e \in R^3$. Consider further a plane Ψ, spanned by vectors e and B. Such a plane exists by virtue of condition 4) of Theorem 5. A number T_1 is the first intersect time of the plane Ψ by the solution $z(t, d)$ of system (28) with the initial data $z(0, d) = d$ on the set $[0, +\infty)$. The existence of such a number T_1 follows from condition 1) of Theorem 6.

5) On the set $(2-\tau+T_1, 2-\tau+T_1+T_2]$, put $K(t) = \mu$ or $K(t) = -\mu$, $\mu \gg 1$. In this case we choose the number T_2 and sign of $K(t)$ in such a way that the vector $z(T_1, d)$ is transformed into a vector $x(T_1 + T_2, z(T_1, d))$, placed in an ε — neighborhood of the stable manifold $\{\lambda e | \lambda \in R^1\}$.

6) Let $K(t) = 0$ on the set $(2-\tau+T_1+T_2, 2-\tau+T_1+T_2+T_3]$. In this case we choose a number T_3 so large that

$$|x(T_3, x(T_1 + T_2, z(T_1, d)))| < \frac{1}{4}. \tag{29}$$

Such a number exists if the number $\varepsilon = \varepsilon(\mu)$, mentioned at the previous step, is sufficiently small.

Since at time $t = 2 - \tau + T_1 + T_2 + T_3$ the image of a unit ball Ω, shifted along solutions of system (1), is placed in a small neighborhood of the vector $x(T_3, x(T_1 + T_2, z(T_1, d)))$, by (29) we can state that this image belongs to a ball of radius 1/2. The latter is equivalent to the asymptotic stability of system (1) with $(2 - \tau + T_1 + T_2 + T_3)$ being the periodic function $K(t)$, constructed above. This completes the proof of Theorem 6.

Note that conditions 1)–3) of Theorem 6 may be replaced by those formally less limitative:

1) $CB \neq 0$,

and there exist numbers k_1 and k_2 such that:

2) a matrix $A + k_1 BC$ has two complex eigenvalues with nonzero real parts and one negative eigenvalue,

3) a function

$$C \exp[(A + k_2 BC)t]B$$

has at least one root in the interval $(-\infty, 0)$.

Note also that Theorem 6 may be considered as an extension of Theorem 5 to the three-dimensional case.

Further, an example of application of Theorem 6 will be given.

If $CB \neq 0$, without loss of generality we can assume that $CB = -1$.

Condition 3) is satisfied if the matrix $A + k_2 BC$ has one negative eigenvalue λ, two eigenvalues with positive real parts, and the following inequality holds

$$\lim_{p \to \lambda} (\lambda - p) G(p) > 0,$$

where $G(p) = C(A + k_2 BC - pI)^{-1} B$.

We can easily see that for some number ρ the following relation

$$\lim_{t \to -\infty} [C e^{(A + k_2 BC)t} B - \rho e^{\lambda t}] = 0$$

is satisfied. From here and from the equality $CB = -1$ it follows that for condition 3) to be satisfied it is sufficient that $\rho > 0$. On the other hand we have

$$\rho = \lim_{p \to \lambda} (\lambda - p) G(p).$$

Consider now system (1) with a nondegenerate transfer function

$$W(p) = \frac{p^2 + c_2 p + c_1}{p^3 + a_3 p + a_2 p}.$$

Since for $a_3 > 0$, $a_2 > 0$ there exists $K(t) \equiv K$ such that system (1) is asymptotically stable, we assume that $a_2 > 0$, $a_3 < 0$.

The nondegeneracy implies that $c_1 \neq 0$. Therefore it is possible to choose a small number k_2 such that $k_2 c_1 > 0$. From here and from the inequality $a_3 < 0$ it follows that one zero of the polynomial

$$p^3 + (a_3 + k_2)p^2 + (a_2 + c_2 k_2)p + c_1 k_2$$

is negative and two other zeros have positive real parts.

Then

$$\lambda_1 = -\frac{c_1}{a_2}\, k_2 + O(k_2^2),$$

$$\lambda_{2,3} = -\frac{a_3}{2} \pm \sqrt{\frac{a_3^2}{4} - a_2} + O(k_2).$$

Whence it follows that

$$\lim_{p \to \lambda_1} (\lambda_1 - p)G(p) = -\frac{c_1}{a_2} + O(k_2),$$

where

$$G(p) = \frac{p^2 + c_2 p + c_1}{p^3 + (a_3 + k_2)p^2 + (a_2 + k_2 c_2)p + k_2 c_1}.$$

Thus condition 3) of Theorem 6 is satisfied if $c_1 < 0$.

To check condition 2), for a polynomial

$$p^3 + \alpha p^2 + \beta p + \gamma, \tag{30}$$

where $\gamma > 0$, the following lemma will be useful.

Lemma 2. *In order that polynomial* (30) *has no positive real zeros it is necessary and sufficient that one of the following inequalities*

1) $\alpha^2 < 3\beta$,

2) $\alpha^2 \geq 3\beta$, $\ -\alpha + \sqrt{\alpha^2 - 3\beta} < 0$,

3) $\alpha^2 \geq 3\beta$, $\gamma - \frac{1}{3}\,\alpha\beta + \frac{2}{27}\,\alpha^3 - \frac{2}{27}\,(\alpha^2 - 3\beta)^{3/2} > 0$

should be satisfied.

In the case considered we have $a_2 > 0$, $a_3 < 0$, $c_1 < 0$. Therefore we choose $k_1 < 0$ and by the lemma have the following inequalities

$$\begin{array}{ll} 1) & (a_3 + k_1)^2 < 3(a_2 + k_1 c_2), \\ 2) & (a_3 + k_1)^2 \geq 3(a_2 + k_1 c_2), \end{array}$$

$$k_1 c_1 - \frac{1}{3}\,(a_3 + k_1)(a_2 + k_1 c_2) + \frac{2}{27}\,(a_3 + k_1)^3 -$$

$$-\frac{2}{27}\,((a_3 + k_1)^2 - 3(a_2 + k_1 c_2))^{3/2} > 0,$$

which ensure that the condition 2) of Theorem 6 is satisfied: Whence it follows that if $a_3^2 < 4a_2$, then for sufficiently small negative k_1 condition 2) of Theorem 6 holds.

Thus for $c_1 < 0$, $a_3 < 0$, $4a_2 > a_3^2$ all the conditions of Theorem 6 are satisfied and system (1) is stabilizable.

Theorem 4 implies the following conditions of nonstabilization

$$c_1 < 0, \quad a_3 < c_2, \quad (a_3 - c_2)c_2 > a_2 - c_1.$$

From the above for $c_2 = a_3/2$, $a_3 < 0$, $c_1 < 0$, $a_2 > 0$ we obtain the following condition of stabilization

$$4a_2 > a_3^2$$

and the condition of non-stabilization

$$4(a_2 - c_1) < a_3^2.$$

As $c_1 \to 0$ Theorems 6 and 4 give the asymptotically sharp estimate of the domain of stabilization.

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