MUTUALLY CATALYTIC BRANCHING IN THE PLANE: FINITE MEASURE STATES

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ABSTRACT. We study a pair of populations in $\mathbb{R}^2$ which undergo diffusion and branching. The system is interactive in that the branching rate of each type is proportional to the local density of the other type. For a diffusion rate sufficiently large compared with the branching rate, the model is constructed as the unique pair of finite measure-valued processes which satisfy a martingale problem involving the collision local time of the solutions. The processes are shown to have densities at fixed times which live on disjoint sets and explode as they approach the interface of the two populations. In the long-term limit, global extinction of one type is shown. The process constructed is a rescaled limit of the corresponding $\mathbb{Z}^2$–lattice model studied by Dawson and Perkins (1998) and resolves the large scale mass-time-space behavior of that model.

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Date: September 11, 2000 WIAS Berlin, Preprint No. 615 ISSN 0946-8633.

1991 *Mathematics Subject Classification.* Primary 60 K 35; Secondary 60 G57, 60 J 80.

Key words and phrases. Catalytic super-Brownian motion, catalytic super-random walk, collision local time, duality, martingale problem, segregation of types, stochastic pde.

Dawson: Supported in part by an NSERC Research Grant and a Max Planck Award.

Etheridge: Supported in part by EPSRC Advanced Fellowship.

Fleischmann: Supported in part by the DFG.

Perkins: Supported in part by an NSERC Research Grant.

1. Introduction and statement of results

1.1. **Background and motivation.** In [DP98] solutions to the following system of stochastic partial differential equations were studied:

$$\frac{\partial}{\partial t} X_t^i(x) \;=\; \frac{\sigma^2}{2}\Delta X_t^i(x) + \sqrt{\gamma\, X_t^1(x)\, X_t^2(x)}\, \dot W_t^i(x), \tag{1}$$

$(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $i = 1,2$. Here Δ is the one-dimensional Laplacian, σ, γ are (strictly) positive constants (the migration and collision rate, respectively), and $\dot W^1, \dot W^2$ are independent standard time-space white noises on $\mathbb{R}_+ \times \mathbb{R}$. Our goal is to study the same system of equations for $x \in \mathbb{R}^2$. As we explain below, from one point of view, existence in two dimensions appears to be counter-intuitive. This is the reason why six different people were attracted to this question and finally combined their efforts.

Recall that

$$\frac{\partial}{\partial t} X_t(x) \;=\; \frac{\sigma^2}{2}\Delta X_t(x) + \sqrt{\varrho_t(x)\, X_t(x)}\, \dot W_t(x) \quad \text{on } \mathbb{R}_+ \times \mathbb{R} \tag{2}$$

is the stochastic partial differential equation for the density of a one-dimensional *super-Brownian motion* (SBM) ([KS88, R89]) with branching rate at time t at x equal to $\varrho_t(x)$ (bounded in t and x). As a measure-valued process it arises as the large population (N particles), small mass (N^{-1}) per particle limit of a system of critical binary branching Brownian motions with diffusion rate σ^2 which branch at rate $N\varrho_t(x)$ at site x at time t. Equivalently each Brownian particle with path $s \mapsto \xi_s$ branches according to the additive functional $t \mapsto N\int_0^t ds\; \varrho_s(\xi_s)$. Although the limit exists in higher dimensions as the unique solution of an appropriate martingale problem, the resulting process takes values in the space of singular measures and it is easy to use this fact to see that (2) has no solutions in higher dimensions (see Remark 1.4 of [DP99]). The problem is that in higher dimensions the critical branching (which tends to cluster the population on a small set) overpowers the diffusion. This situation is typical of parabolic spde's driven by white noise: Solutions seem to only exist in one spatial dimension (see [Wal86]).

One way to rectify this situation in the branching context is to replace $(\varrho_t(x)dx, t \geq 0)$ by a collection of singular measures, i.e., have the branching only take place on singular sets. Delmas [Del96] showed if the branching takes place on a Lebesgue null set (the catalyst) independent of time and satisfying a mild regularity condition guaranteeing that the null set is not polar for Brownian motion (more precisely, particles branch according to an additive functional with Revuz measure supported by this null set) then the associated super-Brownian motion (reactant) has a density at all times with probability one.

A particular time-dependent case was introduced by Dawson and Fleischmann [DF97a] and different aspects of this model were investigated in [DF97b],[EF98], [FK98] and [DF98]. In this model the catalyst itself is a super-Brownian motion

ϱ and the resulting reactant model X^{ϱ} exists and has a nice density in 3 dimensions and less. In higher dimensions an intrinsic Brownian reactant particle's path will not hit the support of an independent super-Brownian catalyst and hence the reactant process degenerates into heat flow as there can be no branching. The construction of such a model poses no difficulties in principle as one first constructs the super-Brownian catalyst and then builds a super-Brownian motion (reactant) whose branching rate is governed by this catalyst.

The situation in (1) is quite different as one has a truly interacting system consisting of two types in which the branching rate of one type is given by the local density of mass of the other, that is, each type catalyzes the branching of the other. Let $S(\mu)$ denote the closed support of a measure μ. Assume for the moment that $\mathbf{X} = (X^1, X^2)$ is a solution to (1) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$, where the $\dot{W}^1, \dot{W}^2$ are independent white noises on $\mathbb{R}_+ \times \mathbb{R}^2$. Then the singularity of ordinary (2-dimensional) SBM (or of SBM with a strictly positive branching rate) suggests that $S(X_t^1) \cap S(X_t^2)$ is Lebesgue null, and the requirement in (1) that X^i solves the heat equation away from this null set shows that X_t^i should have a density away from this null set. In fact this would suggest that $X_t^1(x) X_t^2(x) = 0$ for almost all x and so (1) degenerates into a pair of heat flows which of course do not solve (1).

To circumvent this non-existence argument we will work with the following martingale problem formulation of (1) in two dimensions. We write $\langle \mu, \varphi \rangle$ to denote the integral of a function φ with respect to a measure μ. For fixed constants $\sigma, \gamma > 0$, let $\mathbf{X} = \bigl(X^1, X^2\bigr)$ be a pair of continuous measure-valued processes such that for an appropriate class of test functions φ_i,

$$M_t^i(\varphi_i) \ := \ \bigl\langle X_t^i, \varphi_i \bigr\rangle - \bigl\langle \mu^i, \varphi_i \bigr\rangle - \int_0^t ds \, \Bigl\langle X_s^i, \, \frac{\sigma^2}{2} \Delta \varphi_i \Bigr\rangle, \tag{3}$$

$t \geq 0$, $i = 1, 2$, are orthogonal continuous square integrable martingales starting from 0 at time $t = 0$ and with continuous square function

$$\bigl\langle\!\bigl\langle M^i(\varphi_i) \bigr\rangle\!\bigr\rangle_t \ = \ \gamma \int_{[0,t] \times \mathbb{R}^2} L_{\mathbf{X}}\bigl(d[s,x]\bigr) \, \varphi_i^2(x). \tag{4}$$

Here $L_{\mathbf{X}}$ is the collision local time of X^1 and X^2, loosely described by

$$L_{\mathbf{X}}\bigl(d[s,x]\bigr) \ = \ ds \, X_s^1(dx) \int_{\mathbb{R}^2} X_s^2(dy) \, \delta_x(y) \tag{5}$$

(a precise description is given in Definition 1 below via a smoothing procedure). It is not hard to see that *if* a solution to (1) (for 2 dimensions) is locally bounded (in both space and time) and has the appropriate square integrability properties then the associated measure-valued processes will satisfy (3) and (4), and so the above martingale problem is a natural generalization of (1). We will show (see Theorem 11 and 17 below) that under appropriate conditions on the finite initial measures and for γ/σ^2 sufficiently small, solutions to this martingale problem exist and satisfy the intuitive description given in the paragraph prior to (3): Each population X_t^i has a density denoted by the same symbol X_t^i, and $X_t^1(x) X_t^2(x) = 0$ for Lebesgue-a.a. x. Indeed we will give an explicit expression for the joint law of these densities for fixed values of t and x (see Theorem 17). Evidently these densities cannot be locally bounded since in that case we can easily show that

$$L_{\mathbf{X}}\left([0,\infty) \times \mathbb{R}^2\right) \ = \ \int_0^{\infty} ds \int_{\mathbb{R}^2} dx \, X_s^1(x) \, X_s^2(x) \ = \ 0 \ \text{a.s.}, \tag{6}$$

and again our solutions become a pair of solutions to the heat equation, hence $L_{\mathbf{X}}\left([0,\infty)\times\mathbb{R}^2\right) > 0$ contradicting (6). In fact we will show that each of these densities becomes unbounded near any point in the interface of the two types given by the support of the collision local time (Corollary 19). This bad behavior of the densities near the interface is borne out by simulations of Achim Klenke which you can find on his webpage `http://www.mi.uni-erlangen.de/~klenke`.

The question of uniqueness of solutions to the above martingale problem is also of interest. Although there has been some progress recently in establishing uniqueness for a variety of interactive measure-valued branching processes (e.g. Dawson and March [DM95], Perkins [Per95], Donnelly and Kurtz [DK99], Athreya and Tribe [AT00]) this question for interactive branching diffusions in which the branching rate depends on the present state of the system remains unresolved in general. For the one-dimensional case (1), Mytnik [Myt98] obtained uniqueness by an exponential self-duality. It will be more difficult to implement this approach here due to the bad behavior of the densities. Nevertheless, the problem of uniqueness will be resolved in a companion paper [DFMPX00a] under an additional integrability condition (**IntC**) involving the trajectories of $\mathbf{X}$, introduced in Definition 7 below. In the latter paper this condition will be verified for the solutions constructed in Theorem 11 by means of the moment calculations in Section 3 which are carried out in terms of a function-valued dual. We state the uniqueness result and associated Markov property as Theorem 11 (b) as it will play an important role in our study of the longtime behavior of the solutions (Theorem 20) and the proof of segregation of the two populations (Theorem 17 (b)).

The existence of our solutions will be established by means of rescaling the lattice versions of (1), constructed in [DP98] (in any number of dimensions). We will use the moment bounds in Sections 3 and 4 (for finite initial conditions satisfying a suitable energy condition) to establish tightness of these rescaled processes providing γ/σ^2 is small enough. This restriction on the parameters is needed to ensure that the higher (specifically fourth) moments used in the tightness arguments are finite. It is not hard to show that the approximating fourth moments blow up for γ/σ^2 large enough, but we have not tried to find the best value of this ratio here. We conjecture that solutions to (3) and (4) should exist for any positive values of γ and σ. This is because $2+\delta$ moments should suffice and as $\delta\to 0$, this should allow any values of these parameters. The situation in higher dimensions is intriguing and unresolved.

Many of the results of this paper had been obtained independently and at the same time by two subgroups of the present authors and others were obtained after we coalesced.

The present paper is completely restricted to the finite measure-valued case. For the infinite measure case, we refer to our forthcoming paper [DEFMPX00b].

1.2. **A Martingale Problem for Mutually Catalytic Branching.** We start by formulating our martingale problem for finite measures. Let $\mathcal{M}_{\mathrm{f}} = \mathcal{M}_{\mathrm{f}}(\mathbb{R}^2)$ denote the space of finite measures on the Borel subsets $\mathcal{B}(\mathbb{R}^2)$ of $\mathbb{R}^2$, with the topology of weak convergence. $C_{\mathrm{b}}(\mathbb{R}^2)$ is the space of bounded continuous functions on $\mathbb{R}^2$ with the supnorm $\|\cdot\|_\infty$ topology, and $C^n_{\mathrm{b}}(\mathbb{R}^2)$ is the subspace consisting of those functions whose partial derivatives of order n or less are also in C_{b} (n could be a natural number or ∞). We let $\mathcal{C}_{\mathrm{com}} = \mathcal{C}_{\mathrm{com}}(\mathbb{R}^2)$ denote the space of continuous function on $\mathbb{R}^2$ with compact support. γ and σ are fixed positive constants. Write

$\left(\xi, \Pi_x\, , x \in \mathbb{R}^2\right)$ for the Brownian motion in $\mathbb{R}^2$ with variance parameter σ^2,

$$\mathrm{p}_t(x,y) \;:=\; \frac{1}{2\pi\sigma^2 t}\,\exp\Big[-\frac{|y-x|^2}{2\sigma^2 t}\Big], \qquad t>0, \quad x,y\in\mathbb{R}^2, \tag{7}$$

for its transition density ($|\cdot|$ denotes the Euclidean norm), and $\{S_t : t \geq 0\}$ for the corresponding semigroup. If μ is a measure on $\mathbb{R}^2$, set $S_t\mu\,(x) := \int d\mu(y)\;\mathrm{p}_t(x,y)$.

Definition 1 (Collision Local Time)**.** Let $\mathbf{X} = \left(X^1, X^2\right)$ denote an $\mathcal{M}_{\mathrm{f}}^2$ –valued continuous process where $\mathcal{M}_{\mathrm{f}}^2 = \mathcal{M}_{\mathrm{f}} \times \mathcal{M}_{\mathrm{f}}$. The collision local time of $\mathbf{X}$ (if it exists) is a continuous non-decreasing $\mathcal{M}_{\mathrm{f}}$–valued stochastic process $t \mapsto L_{\mathbf{X}}(t) = L_{\mathbf{X}}(t,\,\cdot\,)$ such that

$$\left\langle L_{\mathbf{X}}^{*,\delta}(t), \varphi\right\rangle \;\longrightarrow\; \left\langle L_{\mathbf{X}}(t), \varphi\right\rangle \quad \text{as } \delta\downarrow 0 \text{ in probability,} \tag{8}$$

for all $t > 0$ and $\varphi \in \mathcal{C}_{\mathrm{com}}(\mathbb{R}^2)$, where

$$L_{\mathbf{X}}^{*,\delta}(t,dx) \;:=\; \frac{1}{\delta}\int_0^\delta dr \int_0^t ds\;\, S_rX_s^1(x)\,S_rX_s^2(x)\;dx, \qquad t\geq 0, \quad \delta>0. \tag{9}$$

The collision local time $L_{\mathbf{X}}$ will also be considered as a (locally finite) measure $L_{\mathbf{X}}\,(ds,dx)$ on $\mathbb{R}_+ \times \mathbb{R}^2$. ◇

Note that we used an additional smoothing in time in defining the collision local time, compared with other sources, as e.g. [BEP91]. Clearly if it exists as in [BEP91], it will exist in the above sense and the processes will coincide.

All filtrations will be assumed to be right-continuous and contain the null sets at time 0.

Definition 2 (Martingale Problem $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$)**.** A continuous $\mathcal{F}_\cdot$–adapted and $\mathcal{M}_{\mathrm{f}}^2(\mathbb{R}^2)$–valued process $\mathbf{X} = (X^1, X^2)$ on some probability space $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ is said to satisfy the martingale problem $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$, if for all $\varphi_i \in C_{\mathrm{b}}^2(\mathbb{R}^2)$, $i=1,2$,

$$M_t^i(\varphi_i) \;=\; \langle X_t^i, \varphi_i\rangle - \langle X_0^i, \varphi_i\rangle - \int_0^t ds\,\Big\langle X_s^i, \frac{\sigma^2}{2}\Delta\varphi_i\Big\rangle, \quad t\geq 0, \;\; i=1,2, \tag{10}$$

are orthogonal continuous L^2 $\mathcal{F}_\cdot$–martingales such that $M_0^i(\varphi_i) = 0$ and

$$\big\langle\!\big\langle M^i(\varphi_i)\big\rangle\!\big\rangle_t \;=\; \gamma\left\langle L_{\mathbf{X}}(t), \varphi_i^2\right\rangle, \quad t\geq 0, \;\; i=1,2. \tag{11}$$

◇

Note that in this definition the initial state $\mathbf{X}_0$ may be random. To construct solutions to this martingale problem we will need to impose a bivariate regularity condition on the initial state.

Notation 3 (Energy Function)**.** Introduce the energy function

$$g(x_1\,, x_2) \;:=\; 1 + \log^+ \frac{1}{|x_2 - x_1|}\,, \qquad x_1, x_2 \in \mathbb{R}^2, \tag{12}$$

(recall that $|\cdot|$ denotes the Euclidean norm). ◇

Definition 4 (State Space Versions)**.**

(a) (Energy Condition): Write $\mu = (\mu^1, \mu^2) \in \mathcal{M}_{\mathrm{f,e}}$ and say μ satisfies the energy condition, iff $\mu \in \mathcal{M}_{\mathrm{f}}^2(\mathbb{R}^2)$ and

$$\|\mu\|_g \;:=\; \left\langle \mu^1 \times \mu^2, g\right\rangle \;<\; \infty. \tag{13}$$

(b) (Strong Energy Condition): Write $\mu = (\mu^1, \mu^2) \in \mathcal{M}_{\rm f,se}$ and say μ satisfies the strong energy condition, iff $\mu \in \mathcal{M}_{\rm f}^2(\mathbb{R}^2)$ and for any $p \in (0,1)$ there is a constant $c = c(p,\mu)$ such that

$$\max_{1 \le i,j \le 2} \left\langle \mu^i \times \mu^j, \mathrm{p}_r \right\rangle \ \le\ c\, r^{-p}, \qquad r > 0. \tag{14}$$ ◇

Remark 5. **(a)** Inequality (14) is trivially fulfilled for $r \ge 1$, and so we only need to consider $0 < r < 1$. By an elementary interpolation argument it actually suffices to consider only $r = 2^{-n}$ and so $\mathcal{M}_{\rm f,e}$ is clearly a Borel subset of $\mathcal{M}_{\rm f}^2$.
(b) An elementary calculation shows that for all $T > 0$ there are constants c_T and C_T such that

$$c_T\, g \ \le\ 1 + \int_0^T dr\ \mathrm{p}_r \ \le\ C_T\, g. \tag{15}$$

In particular, by (14),

$$\mathcal{M}_{\rm f,se} \subseteq \mathcal{M}_{\rm f,e}\,. \tag{16}$$ ◇

Next we introduce a *lattice system* of *approximating processes* we will use to construct solutions to $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$.

Fix a deterministic $\mathbf{X}_0 \in \mathcal{M}_{\rm f,e}$ and $\varepsilon \in (0,1]$. Set

$$X_0^{i,\varepsilon}(x) \ =\ \varepsilon^{-2} X_0^i\left(\varepsilon x + [0,\varepsilon)^2\right), \qquad x = (x_1,x_2) \in \mathbb{Z}^2, \quad i = 1,2. \tag{17}$$

Let $\{W^i(x) : x \in \mathbb{Z}^2, i = 1,2\}$ be a collection of independent standard one-dimensional Brownian motions on $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$, and consider the unique (in law) solution of

(18)
$$X_t^{i,\varepsilon}(x) \ =\ X_0^{i,\varepsilon}(x) + \int_0^t ds\ \frac{\sigma^2}{2}{}^1\Delta X_s^{i,\varepsilon}(x) \ +\ \int_0^t dW_s^i(x)\, \sqrt{\gamma\, X_s^{1,\varepsilon}(x)\, X_s^{2,\varepsilon}(x)},$$

$i = 1,2,\ t \ge 0,\ x \in \mathbb{Z}^2$. See [DP98, Theorems 2.2 and 2.4] for the existence and uniqueness of these solutions.

Via scaling we pass to processes indexed by $\varepsilon\mathbb{Z}^2$ (instead of $\mathbb{Z}^2$) :

$${}^\varepsilon X_t^i(x) \ :=\ X_{t\varepsilon^{-2}}^{i,\varepsilon}(x\varepsilon^{-1}), \qquad i = 1,2,\ t \ge 0,\ x \in \varepsilon\mathbb{Z}^2. \tag{19}$$

Write $x \sim^\varepsilon y$ if x and y are neighbors in $\varepsilon\mathbb{Z}^2$, and introduce the discrete Laplacian on $\varepsilon\mathbb{Z}^2$:

$${}^\varepsilon\Delta\varphi(x) \ :=\ \sum_{y\ \sim^\varepsilon x} \frac{\varphi(y) - \varphi(x)}{\varepsilon^2}, \qquad x \in \varepsilon\mathbb{Z}^2. \tag{20}$$

If $\ell^\varepsilon := \sum_{y \in \varepsilon\mathbb{Z}^2} \varepsilon^2 \delta_y$ and $d^\varepsilon x$ denotes integration with respect to ℓ^ε, let ${}^\varepsilon\mathcal{M}_f(\mathbb{R}^2)$ denote the subspace of $\mathcal{M}_{\rm f}(\mathbb{R}^2)$ of measures with densities with respect to ℓ^ε. Also denote by $t \mapsto {}^\varepsilon X_t^i$ the ${}^\varepsilon\mathcal{M}_f(\mathbb{R}^2)$-valued process with densities ${}^\varepsilon X_t^i(x)$, i.e.,

$$\langle {}^\varepsilon X_t^i, \varphi\rangle \ =\ \int_{\varepsilon\mathbb{Z}^2} d^\varepsilon x\ {}^\varepsilon X_t^i(x)\,\varphi(x) \ =\ \sum_{x \in \varepsilon\mathbb{Z}^2} {}^\varepsilon X_t^i(x)\,\varphi(x)\,\varepsilon^2. \tag{21}$$

Then ${}^\varepsilon X_0^i\left(\{x\}\right) = X_0^i\left(x + [0,\varepsilon)^2\right)$ for $x \in \varepsilon\mathbb{Z}^2$ and so clearly these initial states satisfy ${}^\varepsilon X_0^i \to X_0^i$ in $\mathcal{M}_{\rm f}(\mathbb{R}^2)$ as $\varepsilon \downarrow 0$. The following lemma can easily be derived.

Lemma 6 (Martingale Problem $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mathbf{X}_0}$). *The process* ${}^\varepsilon\mathbf{X}$ *on* $(\Omega,\mathcal{F},\mathcal{F}_\cdot,P)$ *defined via* (21), (19), (18), *and* (17), *based on* $\mathbf{X}_0\in\mathcal{M}_{\mathrm{f,e}}$, *satisfies the following approximate martingale problem* $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mathbf{X}_0}$:

For each pair of bounded functions $\varphi_i:\varepsilon\mathbb{Z}^2\to\mathbb{R},\ i=1,2,$

$$\langle {}^\varepsilon X^i_t,\varphi_i\rangle = \langle {}^\varepsilon X^i_0,\varphi_i\rangle + \int_0^t ds\,\Big\langle {}^\varepsilon X^i_s,\,\frac{\sigma^2}{2}\,{}^\varepsilon\Delta\varphi_i\Big\rangle + {}^\varepsilon M^i_t(\varphi_i), \tag{22}$$

where

$${}^\varepsilon M^i_t(\varphi_i) = \int_{\varepsilon\mathbb{Z}^2} d^\varepsilon x\ \varphi_i(x)\int_0^{t\varepsilon^{-2}} dW^i_s(x\varepsilon^{-1})\sqrt{\gamma X^{1,\varepsilon}_s(x\varepsilon^{-1})X^{2,\varepsilon}_s(x\varepsilon^{-1})} \tag{23}$$

$(i=1,2)$ *are orthogonal continuous* L^2 $(\mathcal{F}_\cdot)$*–martingales such that*

$$\langle\!\langle {}^\varepsilon M^i(\varphi_i)\rangle\!\rangle_t = \gamma\int_0^t ds\int_{\varepsilon\mathbb{Z}^2} d^\varepsilon x\ \varphi_i^2(x)\ {}^\varepsilon X^1_s(x)\ {}^\varepsilon X^2_s(x) =: \gamma\big\langle {}^\varepsilon L_{{}^\varepsilon\mathbf{X}}(t),\varphi_i^2\big\rangle, \tag{24}$$

$i=1,2.$

Existence of solutions to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X_0}}$ will later follow by taking a weak limit point of ${}^\varepsilon\mathbf{X}$ as $\varepsilon\downarrow 0$. Our proof of uniqueness will require an additional integrability condition:

Definition 7 (Integrability Conditions on Path Space)**.** For $\varepsilon>0$ and a pair $\mu=(\mu^1,\mu^2)$ of measures in $\mathcal{M}_{\mathrm f}(\mathbb{R}^2)$ we write

$$H_\varepsilon(\mu) := \int_{\mathbb{R}^2} dx\int_{\mathbb{R}^2} dy\,\Big[1+\frac{1}{|x-y|}\Big] S_\varepsilon\mu^1(x)\,S_\varepsilon\mu^2(x)\,S_\varepsilon\mu^1(y)\,S_\varepsilon\mu^2_\varepsilon(y). \tag{25}$$

(Integrability Condition (IntC)): A continuous $\mathcal{M}^2_{\mathrm f}$–valued process $\mathbf{X}=(X^1,X^2)$ on a probability space $(\Omega,\mathcal{F},\mathcal{F}_\cdot,P)$ is said to satisfy the integrability condition (**IntC**), if for all $0<\delta<T<\infty$,

$$E\Big\{\int_\delta^T ds\ H_\varepsilon(\mathbf{X}_s)\ \Big|\ \mathcal{F}_\delta\Big\}\quad\text{is bounded in probability as } \varepsilon\downarrow 0.$$

that is, for all $\eta>0$ there is an M such that

$$\overline{\lim_{\varepsilon\downarrow 0}}\,P\bigg(E\Big\{\int_\delta^T ds\ H_\varepsilon(X_s)\ \Big|\ \mathcal{F}_\delta\Big\}>M\bigg) < \eta. \tag{26}$$

(Strong Integrability Condition (SIntC): X is said to satisfy the stronger (and simpler) integrability condition (**SIntC**) if

$$\overline{\lim_{\varepsilon\downarrow 0}}\,E\int_0^T ds\ H_\varepsilon(\mathbf{X}_s) < \infty,\qquad T>0. \tag{27}$$

$\diamond$

To describe the restriction on γ/σ^2, let $({}^\varepsilon\xi,\Pi^\varepsilon_x, x\in\varepsilon\mathbb{Z}^2)$ denote the continuous time simple symmetric random walk on $\varepsilon\mathbb{Z}^2$ with generator $\frac{\sigma^2}{2}\,{}^\varepsilon\Delta$. That is, ${}^\varepsilon\xi$ jumps to a nearest neighbor site at rate $2\varepsilon^{-2}\sigma^2$. Introduce the corresponding transition density ${}^\varepsilon\mathrm{p}_t(x,y)=\varepsilon^{-2}\Pi_x({}^\varepsilon\xi_t=y),\ x,y\in\varepsilon\mathbb{Z}^2$ with respect to ℓ^ε, and $\{{}^\varepsilon S_t: t\ge 0\}$ the related semigroup.

The following elementary result is proved in Appendix A for the sake of completeness.

Lemma 8 (Random Walk Kernel Estimates)**.**

(a) (Local Central Limit Theorem): *For all* $s > 0$, *with the heat kernel* p *from* (7),

$$\lim_{\varepsilon \to 0} \sup_{x,y \in \varepsilon \mathbb{Z}^2} \left| {}^{\varepsilon}p_s(x,y) - \mathrm{p}_s(x,y) \right| \;=\; 0. \tag{28}$$

(b) (Uniform Bound): *There is a universal constant* c_8 (*independent of* σ^2) *such that*

$$\sup_{s \geq 0,\, x,y \in \varepsilon \mathbb{Z}^2} {}^{\varepsilon}\mathrm{p}_s(x,y) s\,\sigma^2 \;=\; \sup_{s \geq 0} {}^{\varepsilon}\mathrm{p}_s(0,0) s\,\sigma^2 \;=\; c_8\,, \tag{29}$$

for all $\varepsilon > 0$.

Remark 9 (Size of c_8)**.** Statement (a) is of course a standard local central limit theorem. The value of the constant c_8 of (b) enters in Theorem 11 below. To estimate its value, write $\tilde{p}$ instead of ${}^{\varepsilon}\mathrm{p}$ in the case $\varepsilon = \sigma = 1$. Then,

$$c_8 \;=\; \sup_{t \geq 0} t\,\tilde{p}_t(0,0). \tag{30}$$

Now a direct calculation and exploiting Stirling's Formula (see [Fel68, p.52]) gives $c_8 \leq \mathrm{e}^{1/12}/2 < 0.55$. On the other hand, $c_8 \geq {}^{\varepsilon}\mathrm{p}_t(0,0) t\,\sigma^2$, and it follows from (a) that

$$c_8 \;\geq\; t\,\mathrm{p}_t(0,0) \;=\; (2\pi)^{-1} \;>\; 0.15. \tag{31}$$

Consequently, $c_8 \in (0.15,\, 0.55)$. ◇

Notation 10 (Path Space)**.** Let $\Omega_\circ := C\big(\mathbb{R}_+, \mathcal{M}_{\mathrm{f}}^2(\mathbb{R}^2)\big)$ with the usual topology of uniform convergence on compact subsets of $\mathbb{R}_+$. ◇

Recall the spaces $\mathcal{M}_{\mathrm{f,e}}$ and $\mathcal{M}_{\mathrm{f,se}}$ introduced in Definition 4.

Theorem 11 (Mutually Catalytic SBM in $\mathbb{R}^2$)**.** *Assume*

$$\gamma/\sigma^2 \;<\; (3\sqrt{6}\pi c_8)^{-1} \tag{32}$$

and $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,e}}$.

(a) (Existence): *There is a process* $\mathbf{X}$ *on some* $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ *satisfying the martingale problem* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *and the integrability condition* (**IntC**), *and such that* $\mathbf{X}_t \in \mathcal{M}_{\mathrm{f,e}}$ *for all* $t \geq 0$ *a.s. If moreover* $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,se}}$, *then* $\mathbf{X}$ *will satisfy* (**SIntC**).

(b) (Strong Markov and Uniqueness): *There is a (time-homogeneous) Borel Markov transition kernel* $\mathsf{P} = \big\{\mathsf{P}_t(\mu, d\nu) :\; t > 0,\; \mu \in \mathcal{M}_{\mathrm{f,e}}\big\}$ *on* $\mathcal{M}_{\mathrm{f,e}}$ *such that any process satisfying* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *and* (**IntC**) *on* $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ *is* $(\mathcal{F}_\cdot)$-*strong Markov with transition kernel* P. *In particular, the law* $P_{\mathbf{X}_0}$ *on* $\Omega_\circ$ *of the solution in* (a) *is unique.*

(c) (Lattice Approximation): *Let* ${}^{\varepsilon}\mathbf{X}$ *denote the lattice system of approximating processes given by (18), (19), with initial conditions (17) and let* ${}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}$ *be as defined in Lemma 6. As* $\varepsilon \downarrow 0$,

$$P\big(({}^{\varepsilon}\mathbf{X}, {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}) \in \cdot\,\big) \;\Longrightarrow\; P\big((\mathbf{X}, L_{\mathbf{X}}) \in \cdot\,\big) \tag{33}$$

weakly on $C\big(\mathbb{R}_+, \mathcal{M}_{\mathrm{f}}^3(\mathbb{R}^2)\big)$, *where* $\mathbf{X}$ *satisfies* (**IntC**) *and is a solution to the martingale problem* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *with* $L_{\mathbf{X}}$ *as its collision local time.*

(d) (Scaling Property): *Assume that* $\mathbf{X}$ *satisfies* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *and* $(\mathbf{IntC})$, $\varepsilon,\theta > 0$, $z \in \mathbb{R}^2$ *and* $\widehat{X}^i_t(A) := \theta X^i_{\varepsilon^2 t}(z+\varepsilon A)$, $t \geq 0$, $A \in \mathcal{B}(\mathbb{R}^2)$, $i = 1,2$. *Then* $(\widehat{X}^1, \widehat{X}^2)$ *satisfies* $(\mathbf{MP})^{\sigma,\gamma}_{\widehat{\mathbf{X}}_0}$ *and* $(\mathbf{IntC})$ *and so has law* $P_{\widehat{\mathbf{X}}_0}$.

The proof of (b) will be completed in a companion paper [DFMPX00a], but much of the groundwork is laid in Section 3 below. The verification of the integrability conditions $(\mathbf{IntC})$ and $(\mathbf{SIntC})$ is also deferred to [DFMPX00a] as its main use is the proof of (b) (although $(\mathbf{SIntC})$ is also used in our description of the long term behavior (Theorem 20)). The main ingredient in the proof of $(\mathbf{IntC})$ is a bound on its conditional 4th moments in terms of a function-valued dual (Theorem 53 below).

Remark 12. (i) Part (c) remains true for a wider class of lattice approximations of the initial measure. It suffices that ${}^{\varepsilon}\mathbf{X}_0$ approaches $\mathbf{X}_0$ weakly and satisfies the conclusions of Lemmas 35 and 45(a) below.
(ii) Part (a) of Theorem 11 is valid if we only assume $\gamma/\sigma^2 < 2/\sqrt{6}$. To allow for this weaker condition, solutions may be constructed as limits as $\varepsilon \downarrow 0$ of smoothed models in $\mathbb{R}^2$ in which the branching rate of type i at time t at site x is $dx \int_{\mathbb{R}^2} X^j_t(dy)\, \mathrm{p}_\varepsilon(x,y)$ (where $j \neq i$), instead of $X^j_t(dx)$. The proof in fact is simpler than that for our lattice approximation but the latter is in many ways more natural and is used in [DEFMPX00b] to shed some light on the large mass-time-space behavior of the lattice systems studied in [DP98]. Part (b) remains valid for $\gamma/\sigma^2 < 1/\sqrt{6}$.

(iii) The space $\mathcal{M}_{\mathrm{f,se}}$ seems to be needed to get unconditional fourth moment bounds (see, e.g. Theorem 54) and a simple second moment argument shows that $X_t \in \mathcal{M}_{\mathrm{f,se}}$ a.s. $\forall t > 0$ (see Proposition 24 (a) below). We have not, however, been able to show $X_t \in \mathcal{M}_{\mathrm{f,se}}$ $\forall t > 0$ a.s. and this leads to an additional conditioning argument in our construction and the use of the larger $\mathcal{M}_{\mathrm{f,e}}$ as our state space. $\diamond$

We now state the key self-duality result, Proposition 2.13 from [DFMPX00a] both because it is used below and because its proof uses our existence results Theorem 11(a).

Proposition 13. *Assume (32),* $\mathbf{X}_0 \in \mathcal{M}_{f,e}$ *and* $\tilde{\mathbf{X}}_0 = (\tilde{x}^1_0(x), \tilde{x}^2_0(x))$ *where* $\tilde{x}^i_0$ *is bounded, non-negative and continuous. Then*

$$P_{\mathbf{X}_0}\left(\exp\left\{-\left\langle X^1_t + X^2_t, \tilde{x}^1_0 + \tilde{x}^2_0\right\rangle + i\left\langle X^1_t - X^2_t, \tilde{x}^1_0 - \tilde{x}^2_0\right\rangle\right\}\right)$$
$$= \lim_{\varepsilon\downarrow 0} P_{\tilde{\mathbf{X}}_0}\left(\exp\left\{-\left\langle X^1_0 + X^2_0, S_\varepsilon \tilde{X}^1_t + S_\varepsilon \tilde{X}^1_t\right\rangle + i\left\langle X^1_0 - X^2_0, S_\varepsilon \tilde{X}^1_t - S_\varepsilon \tilde{X}^2_t\right\rangle\right\}\right).$$

In [DFMPX00a] this proposition plays a major role in the proof of uniqueness in Theorem 11(b) which is assumed implicitly in our notation. The result is therefore stated there for any solution $\mathbf{X}$ of $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ and for a particular limit point, $\tilde{\mathbf{X}}$ from Theorem 11(a).

We now introduce an integrability hypothesis on a possibly random initial state. Recall the norm $\|\cdot\|_g$ introduced in (13).

Definition 14 (Random Energy Condition **(EnC)**)**.** We say a possibly random initial state $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,e}}$ satisfies the random energy condition $(\mathbf{EnC})$ if

$$\sum_{i=1,2} E\left\langle X^i_0, 1\right\rangle^2 + E\|\mathbf{X}_0\|_g < \infty. \tag{34}$$

(If $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,e}}$ is deterministic, then (**EnC**) clearly holds.) ◇

Although we will need either a dual process calculation or some explicit differential equation calculations to handle some higher moments, the covariance structure of the solutions to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ only requires some integrability conditions and (**IntC**) is more than enough.

Proposition 15 (First Two Moments). *Let* $\mathbf{X}$ *satisfy* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *on some filtered space* $(\Omega,\mathcal{F},\mathcal{F}_\cdot,P)$ *for a possibly random* $\mathbf{X}_0$ *satisfying* (**EnC**).

(a) (Expectation): *Let* $\varphi:\mathbb{R}^2\to R_+$ *be a bounded Borel map. Then*

$$E\langle X^i_t,\varphi\rangle \;=\; E\langle X^i_0,S_t\varphi\rangle \;<\;\infty,\qquad t\ge 0,\;\; i=1,2. \tag{35}$$

(b) (Correlation): *For bounded measurable* $\psi:(\mathbb{R}^2)^2\to\mathbb{R}_+$, $t\ge 0$, *and* $i,j=1,2,$

$$\begin{aligned} E\big\langle X^i_t\times X^j_t,\psi\big\rangle \;\le\; & E\int_{\mathbb{R}^2}dx_1\; S_tX^i_0(x_1)\int_{\mathbb{R}^2}dx_2\; S_tX^j_0(x_2)\,\psi(x_1,x_2)\\ &+\delta_{ij}\,\gamma\,E\int_0^t ds\int_{\mathbb{R}^2}dx\; S_sX^1_0(x)\,S_sX^2_0(x)\\ &\times\int_{\mathbb{R}^2}dx_1\;\mathrm{p}_{t-s}(x,x_1)\int_{\mathbb{R}^2}dx_2\;\mathrm{p}_{t-s}(x,x_2)\,\psi(x_1,x_2)\end{aligned}$$

where all expressions are finite. Moreover, equality holds if $i\ne j$.

(c) (Expected Collision Local Time): *For measurable* $\psi:\mathbb{R}_+\times\mathbb{R}^2\to\mathbb{R}_+$, *bounded on each* $[0,T]\times\mathbb{R}^2$, $T>0$,

$$E\int_{[0,T]\times\mathbb{R}^2}dL_{\mathbf{X}}\,\psi\;\le\;\int_0^T ds\int_{\mathbb{R}^2}dx\;\psi(s,x)ES_sX^1_0(x)\,S_sX^2_0(x)\;<\;\infty. \tag{36}$$

(d) (Identities under (IntC)): *If, in addition,* $\mathbf{X}$ *satisfies the integrability condition* (**IntC**), *then equality holds in both* (b) *and* (c).

Note that it follows from (a) that the solution to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ constructed in Theorem 11 is not deterministic since $\langle X^i_t,\varphi\rangle\equiv\langle X^i_0,S_t\varphi\rangle$ will not satisfy $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$. Alternatively we can see from (d) that the covariance structure of this solution is not trivial.

We will now be able to state some more interesting properties of the solutions to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$. We begin by stating the absolute continuity and segregation of types results mentioned in the introduction.

1.3. **Segregated densities.**

Notation 16 (Brownian Exit Time). Consider the (planar) Brownian motion $\xi=(\xi^1,\xi^2)$ with law Π_x, $x\in\mathbb{R}^2_+$, and introduce its exit time

$$\tau_{\mathrm{ex}}\;:=\;\inf\big\{t:\;\xi^1_t\,\xi^2_t=0\big\}, \tag{37}$$

from the first quadrant. ◇

Let $\ell(dx)=dx$ denote Lebesgue measure. Here and elsewhere we will identify integrable functions $X(x)$ in C^+_{b} with the finite absolutely continuous measure $X(x)dx$.

Here is our segregation result.

Theorem 17 (Segregated Densities). *Fix $t > 0$.*

(a) (Absolute Continuity): *If $\mathbf{X}$ is a solution to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X_0}}$ on $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ with a possibly random initial condition $\mathbf{X}_0 \in \mathcal{M}^2_f(\mathbb{R}^2)$, then $X^i_t \ll \ell$ a.s. and so $X^i_t(dx) = X^i_t(x)dx$ a.s. where*

$$X^i_t(x) = \begin{cases} \lim_{n\to\infty} S_{2^{-n}} X^i_t(x) & \text{if it exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

(b) (Local Segregation): *Let $\mathbf{X}_0 \in M_{\mathrm{f,e}}$ be fixed and $\mathbf{X}_t = \left(X^1_t, X^2_t\right)$ the functions from (38), and set $\mathbf{S}_t\mathbf{X}_0(x) := \left(S_t X^1_0(x), S_t X^2_0(x)\right)$. Then the following two statements hold:*

(b1): *For ℓ–a.a. x,*

$$P_{\mathbf{X_0}}\left(\mathbf{X}_t(x) \in \cdot\,\right) \;=\; \Pi_{\mathbf{S}_t\mathbf{X}_0(x)}\left(\xi_{\tau_{\mathrm{ex}}} \in \cdot\,\right). \tag{39}$$

(b2): *With $P_{\mathbf{X_0}}$–probability one, $X^1_t(x)\, X^2_t(x) = 0$ for ℓ–a.a. x, and so*

$$\int_{\mathbb{R}^2} dx\; X^1_t(x)\, X^2_t(x) \;=\; 0 \quad P_{\mathbf{X_0}}\text{–a.s.} \tag{40}$$

Remark 18 (Infinite Variance). (i) Note that (b1) implies

$$E_{\mathbf{X_0}}\left(X^i_t(x)\right)^2 = \infty \quad \text{for } \ell\text{–a.a. } x \in \mathbb{R}^2_+ \text{ and } i = 1,2,$$

for any $\mathbf{X}_0 \in M_{\mathrm{f,e}}$ with $X^i_0 \neq 0,\ i = 1,2$.

(ii) It follows from (b) that the two populations segregate at each fixed time. The "interface" between the two types, although Lebesgue null must be rather active to generate a non-trivial collision local time and we show below (Corollary 19) that the densities typically explode near it. The particular distribution arising in (b1) also gave the large time limit for the lattice system (18) starting in constant initial states. In fact, the counterpart of this latter result for solutions to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X_0}}$ (Theorem 20 below) plays a central role in the proof. Basically a scaling argument shows that locally the joint densities $x \mapsto \mathbf{X}_t(x)$ relax to an equilibrium state instantaneously. In fact, when both types are present, the infinitely large branching rate will immediately drive one type to local extinction. The type to die is determined by the exit distribution of planar Brownian motion from the first quadrant. ◇

Let (38) define our canonical and jointly measurable densities

$$X^i : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega_\circ \to [0, \infty), \qquad i = 1,2. \tag{41}$$

Let $\|X^i\|_U$ denote the essential supremum of X^i (with respect to Lebesgue measure) on the open set $U \subseteq \mathbb{R}_+ \times \mathbb{R}^2$.

Corollary 19 (Explosion at the Interface). *If $\mathbf{X}_0 \in M_{\mathrm{f,e}}$, then $P_{\mathbf{X_0}}$–a.s. for any open set $U \subseteq \mathbb{R}_+ \times \mathbb{R}^2$,*

$$L_{\mathbf{X}}(U) > 0 \quad \text{implies} \quad \|X^1\|_U = \infty = \|X^2\|_U\,. \tag{42}$$

Example. Here is a simple time-independent example on the line which shows how (unbounded) densities with disjoint supports may nonetheless have a non-zero collision local time. Let $1 > \alpha_1, \alpha_2 > 0$ and set

$$X^1(dx) = u^1(x)dx = x^{-\alpha_1} 1(x > 0)dx, \;\; X^2(dx) = u^2(x)dx = |x|^{-\alpha_2} 1(x < 0)dx.$$

Then clearly $u^1(x)u^2(x) \equiv 0$ but if $\alpha_1 + \alpha_2 = 1$, the analogue of collision local time is ($\varphi \in \mathcal{C}_{\rm com}(\mathbb{R}^2)$)

$$\begin{aligned}\langle L_{\mathbf{X}}, \varphi\rangle &= \lim_{\varepsilon\downarrow 0} \int S_\varepsilon X^1(x) S_\varepsilon X^2(x)\varphi(x)dx \\ &= \lim_{\varepsilon\downarrow 0} \int\int\int \varphi(\sqrt{\varepsilon}w)p_1(w-z_1)p_1(w-z_2)z_1^{-\alpha_1}|z_2|^{-\alpha_2}1(z_2<0<z_1)dz_1dz_2dw \\ &= \varphi(0)\iint p_2(z_1-z_2)z_1^{-\alpha_1}|z_2|^{-\alpha_2}1(z_2<0<z_1)dz_1dz_2,\end{aligned}$$

where we have used Dominated Convergence in the last line. Therefore the collision local time of $\mathbf{X}$ is a (non-zero) constant multiple of the δ_0.

1.4. **Global Extinction of One Type.** The one-dimensional version of the following theorem is proved in [DP98, Theorem 6.6].

Theorem 20 (Global Extinction of One Type)**.** *Let* $\mathbf{X}_0 \in M_{\rm f,e}$. *Then*

$$\big(\langle X_t^1, 1\rangle, \langle X_t^2, 1\rangle\big) \underset{t\uparrow\infty}{\longrightarrow} \big(X_\infty^1, X_\infty^2\big) \;\; P_{\mathbf{X}_0}\text{–}a.s., \tag{43}$$

where

$$P\big((X_\infty^1, X_\infty^2) \in \cdot\,\big) \;=\; \Pi_{(\langle X_0^1,1\rangle,\, \langle X_0^2,1\rangle)}(\xi_{\tau_{\rm ex}} \in \cdot\,). \tag{44}$$

The a.s. convergence is immediate from the martingale convergence theorem, as $t \mapsto \langle X_t^i, 1\rangle$ are non-negative martingales by $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$. The fact that $X_\infty^1 X_\infty^2 = 0$ a.s. will require a refinement of the proof for the lattice case given in [DP98, Theorem 1.2 (b)]. In particular, we need to consider the rate of convergence in that result.

2. Preliminaries

In this section we prove Proposition 15 and identify the natural state space for $\mathbf{X}$.

2.1. **Green Function Representation.** Assume $\mathbf{X}$ is a solution of $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ on $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ where $\mathbf{X}_0$ is an $\mathcal{F}_0$–measurable $\mathcal{M}_{\rm f}^2(\mathbb{R}^2)$–valued initial state. Let $\mathbf{M}_{\rm loc}$ denote the space of continuous $(\mathcal{F}_\cdot)$-local martingales such that $M_0 = 0$ and, for $T > 0$ fixed, $\mathbf{M}^2[0,T]$ the space of continuous square integrable $(\mathcal{F}_\cdot)$–martingales on $[0,T]$, where processes which agree off an evanescent set are identified. Let $\mathbf{M}^2$ be the space of continuous square integrable $(\mathcal{F}_t)$-martingales on $\mathbb{R}_+$.

Let $\mathcal{P}$ denote the σ-field of $(\mathcal{F}_\cdot)$-predictable sets in $\mathbb{R}_+ \times \Omega$ and define

$$\begin{aligned}\mathcal{L}^2_{\rm loc} \;:=\; \Big\{&\psi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \to \mathbb{R} :\; \psi \text{ is } \mathcal{P}\times\mathcal{B}(\mathbb{R}^2)\text{–measurable} \\ &\text{and } \int_{[0,t]\times\mathbb{R}^2} L_{\mathbf{X}(\omega)}\big(d(s,x)\big)\,\psi^2(s,\omega,x) \;<\; \infty \;\; \forall t>0, \;\text{a.s.}\Big\}\end{aligned} \tag{45}$$

By starting with functions ψ of the form

$$\psi(s,\omega,x) \;=\; \sum_{m=1}^{k} \psi_{m-1}(\omega)\,\varphi_m(x)\,\mathbf{1}_{(t_{m-1},t_m]}(s) \tag{46}$$

for some $\varphi_m \in C_{\mathrm{b}}^2(\mathbb{R}^2)$, $\psi_{m-1} \in b\mathcal{F}_{t_{m-1}}$ (the space of bounded $\mathcal{F}_{t_{m-1}}$–measurable maps), and $0 = t_0 < \cdots < t_k \leq \infty$, and defining [with M^i from the martingale problem $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$], for $i = 1, 2$,

(47)

$$M_t^i(\psi) = \int_{[0,t]\times\mathbb{R}^2} dM^i(s,x)\,\psi(s,x) \;:=\; \sum_{m=1}^{k} \psi_{m-1}\big(M_{t\wedge t_m}^i(\varphi_j) - M_{t\wedge t_{j-1}}^i(\varphi_j)\big),$$

we may uniquely extend M^i to linear maps $M^i : \mathcal{L}_{\mathrm{loc}}^2 \to \mathbf{M}_{\mathrm{loc}}$, such that

$$\big\langle\!\big\langle M^i(\psi_i), M^j(\psi_j)\big\rangle\!\big\rangle_t \;=\; \gamma\,\delta_{i\,j}\int_{[0,t]\times\mathbb{R}^2} L_{\mathbf{X}}\left(d(s,x)\right)\psi_i(s,x)\,\psi_j(s,x) \tag{48}$$

$t \geq 0$ a.s. for all $\psi_i \in \mathcal{L}_{\mathrm{loc}}^2$. This may be done as in [Per00, Proposition II.5.4] or [Wal86, Chapter 2]. The M^i are orthogonal martingale measures. If in addition,

$$\psi \in \mathcal{L}^2 \;:=\; \left\{\psi \in \mathcal{L}_{\mathrm{loc}}^2 :\; E\int_{[0,t]\times\mathbb{R}^2} dL_{\mathbf{X}}\;\psi^2 < \infty,\;\; \forall t > 0\right\}, \tag{49}$$

then $M^i(\psi) \in \mathbf{M}^2$. The martingale problem $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ shows that $M^i(1)$ belongs to $\mathbf{M}^2$, hence the constant function $\mathbf{1}$ is in $\mathcal{L}^2$ and so

(50) every bounded and $\mathcal{P} \times \mathcal{B}(\mathbb{R}^2)$–measurable ψ is in $\mathcal{L}^2$ and $M^i(\psi) \in \mathbf{M}^2$.

We need to extend $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ to time-dependent test functions.

Notation 21 (Time-space Test Functions)**.** If $T > 0$, let $\mathcal{D}_T$ denote the set of all bounded Borel maps $\psi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfying:

- **(a):** For any $x \in \mathbb{R}^2$, the map $t \mapsto \psi(t,x)$ is absolutely continuous and $\dot{\psi}(t,x) = \frac{\partial\psi}{\partial t}(t,x)$ is uniformly bounded in (t,x) and continuous in x for each $t \in [0,T]$.
- **(b):** For each t in $[0,T]$, the mapping $x \mapsto \psi(t,x)$ belongs to $C_{\mathrm{b}}^2(\mathbb{R}^2)$, and $\Delta\psi(t,\,\cdot\,)(x)$ is uniformly bounded in (t,x). $\diamond$

Lemma 22 (Extension of the Martingale Problem $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$)**.** *If $\psi_i \in \mathcal{D}_T$, $i = 1, 2$, then*

$$\langle X_t^i, \psi_i(t)\rangle \;=\; \big\langle X_0^i, \psi_i(0)\big\rangle + \int_0^t ds\,\left\langle X_s^i,\, \dot{\psi}_i(s) + \frac{\sigma^2}{2}\Delta\psi_i(s)\right\rangle + M_t^i(\psi_i), \tag{51}$$

$t \in [0,T]$, *where $M^i(\psi_i)$ belongs to $\mathbf{M}^2$, and*

$$\big\langle\!\big\langle M^i(\psi_i), M^j(\psi_j)\big\rangle\!\big\rangle_t \;=\; \delta_{i\,j}\,\gamma\int_{[0,t]\times\mathbb{R}^2} L_{\mathbf{X}}\left(d(s,x)\right)\;\psi_i(s,x)\,\psi_j(s,x). \tag{52}$$

Proof. This may be done just as for ordinary superprocesses; see, e.g., [Per00, Proposition II.5.7]. The argument proceeds by approximating $\psi(s,x)$ by an appropriate sequence of step functions in t. ∎

Corollary 23 (Green Function Representation)**.** *Let $i = 1, 2$. If $\varphi_i : \mathbb{R}^2 \to \mathbb{R}$ is bounded and measurable, then for any fixed $T > 0$,*

$$\langle X_t^i, S_{T-t}\varphi_i\rangle \;=\; \langle X_0^i, S_T\varphi_i\rangle + N_t^{i,T}(\varphi_i), \quad 0 \leq t \leq T,\ a.s., \tag{53}$$

where

$$t \mapsto N_t^{i,T}(\varphi_i) \;=\; \int_{[0,t]\times\mathbb{R}^2} dM^i(r,x)\,S_{T-r}\varphi_i(x) \quad \textit{belongs to } \mathbf{M}^2[0,T], \tag{54}$$

and

$$\big\langle\!\big\langle N^{i,T}(\varphi_i), N^{j,T}(\varphi_j)\big\rangle\!\big\rangle_t \;=\; \delta_{ij}\,\gamma \int_{[0,t]\times\mathbb{R}^2} L_{\mathbf{X}}\big(d(s,x)\big)\, S_{T-s}\varphi_i(x)\, S_{T-s}\varphi_j(x). \tag{55}$$

In particular,

$$\langle X^i_T, \varphi_i\rangle \;=\; \langle X^i_0, S_T\varphi_i\rangle + N^{i,T}_T(\varphi_i) \quad a.s.\ \forall T>0. \tag{56}$$

Proof. Let $\varphi \in C^2_{\mathrm{b}}(\mathbb{R}^2)$ and $\psi(s,x) = S_{T-s}\varphi(x)$ for $(s,x)\in[0,T]\times\mathbb{R}^2$. Then $\psi \in \mathcal{D}_T$ because $\dot\psi(s,x) = (-\sigma^2/2)\Delta S_{T-s}\varphi(x) = (-\sigma^2/2)S_{T-s}\Delta\varphi(x)$. The result follows for such $\varphi \in C^2_{\mathrm{b}}(\mathbb{R}^2)$ by Lemma 22. Now pass to the bounded pointwise closure to get the result for all bounded measurable ϕ. ∎

2.2. First and Second Moments: Proof of Proposition 15.

We proceed in several steps.

Step 1° (Proof of (a)) The equality in (a) is immediate upon taking expectations in Corollary 23 and using **(EnC)** (14) for the finiteness of the mean.

Step 2° Assume that $\psi = \varphi_1 \otimes \varphi_2$ with $\varphi_1, \varphi_2 \in b\mathcal{B}(\mathbb{R}^2)$. Corollary 23 shows that

$$\begin{aligned} E\langle X^i_t,\varphi_i\rangle\langle X^j_t,\varphi_j\rangle \;&=\; E\langle X^i_0, S_t\varphi_i\rangle\langle X^j_0, S_t\varphi_j\rangle \\ &\quad +\; \delta_{ij}\,\gamma\, E\int_{[0,t]\times\mathbb{R}^2} L_{\mathbf{X}}\big(d(s,x)\big)\, S_{t-s}\varphi_i(x)\, S_{t-s}\varphi_j(x), \end{aligned} \tag{57}$$

since by conditioning on $\mathbf{X}_0$ the cross terms vanish.

Step 3° (Proof of (c)) Before completing the proof of (b) we will consider (c). Assume $\psi(s,x) = \varphi(x)$ with $\varphi \in \mathcal{C}^+_{\mathrm{com}}(\mathbb{R}^2)$. By Definition 1 and Fatou's lemma,

$$E\big\langle L_{\mathbf{X}}(T), \varphi\big\rangle \;\le\; \liminf_{\delta\downarrow 0}\, E\big\langle L^{*,\delta}_{\mathbf{X}}(T),\varphi\big\rangle \tag{58}$$

$$= \liminf_{\delta\downarrow 0} E\frac{1}{\delta}\int_0^\delta dr \int_0^T ds \int_{\mathbb{R}^2} dx\, S_r X^1_s(x)\, S_r X^2_s(x)\,\varphi(x) \tag{59}$$

$$= \liminf_{\delta\downarrow 0} E\int_{\mathbb{R}^2} X^1_0(dy_1)\int_{\mathbb{R}^2} X^2_0(dy_2) \tag{60}$$

$$\times\; \frac{1}{\delta}\int_0^\delta dr\int_0^T ds\int_{\mathbb{R}^2} dx\, \mathrm{p}_{r+s}(x,y_1)\,\mathrm{p}_{r+s}(x,y_2)\,\varphi(x), \tag{61}$$

where we used (57) to continue after (59). The term in (61) is bounded by

$$\|\varphi\|_\infty \frac{1}{\delta}\int_0^\delta dr\int_0^T ds\, \mathrm{p}_{2(r+s)}(y_1,y_2) \;\le\; c\,\|\varphi\|_\infty\, g(y_1,y_2), \tag{62}$$

where in the last step we used (15). But by (**EnC**) the bound in (62) is integrable with respect to $EX^1_0\times X^2_0$. Hence, the limit inferior can be taken through the three integrals in (60). It is then easy to let $\delta\to 0$ in the resulting integrand as we only need to consider $y_1 \neq y_2$ by (**EnC**). This gives

$$E\big\langle L_{\mathbf{X}}(T),\varphi\big\rangle \;\le\; E\int_0^T ds\int_{\mathbb{R}^2} dx\; S_sX^1_0(x)\, S_sX^2_0(x)\,\varphi(x) \tag{63}$$

$$\le\; c_T\,\|\varphi\|_\infty\, E\|\mathbf{X}_0\|_g \;<\; \infty. \tag{64}$$

By an obvious monotone class argument, claim (c) follows for bounded measurable ψ on $[0,t]\times\mathbb{R}^2$.

Step 4° (Proof of (b)) We may apply (c) to (57) to get the claim (b) for the special functions ψ used in step 2°. A monotone class argument then gives the desired extension.

Step 5° (Proof of (d)) Assume (**IntC**). First consider again the case $\psi(s,x) = \varphi(x)$ with a function $\varphi\in\mathcal{C}^+_{\mathrm{com}}(\mathbb{R}^2)$. Fix $0<\varepsilon<T$. Suppose we can show

$$(65)\qquad E\Big\{\big\langle L_{\mathbf{X}}(T)-L_{\mathbf{X}}(\varepsilon),\,\varphi\big\rangle\;\Big|\;\mathcal{F}_\varepsilon\Big\} \;=\; \int_0^{T-\varepsilon} ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,S_sX^1_\varepsilon(x)\,S_sX^2_\varepsilon(x).$$

Then by (57),

$$(66)\qquad E\big\langle L_{\mathbf{X}}(T)-L_{\mathbf{X}}(\varepsilon),\,\varphi\big\rangle \;=\; \int_\varepsilon^{T} ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,ES_sX^1_0(x)\,S_sX^2_0(x).$$

Now let $\varepsilon\downarrow 0$. By (c), the left hand side of (66) converges to $E\big\langle L_{\mathbf{X}}(T),\varphi\big\rangle$, whereas by monotone convergence on the right hand side we obtain the required expression. Provided we have (65), this proves equality in (c) under (**IntC**) for the considered special ψ, hence for all required ψ by Dominated Convergence and (c).

By (57), we then also get the equality in (b) under (**IntC**) for functions ψ of the form $\varphi_1\otimes\varphi_2$ with $\varphi_1,\varphi_2\in b\mathcal{B}(\mathbb{R}^2)$, thus for all required ψ.

Step 6° To finish the proof, it remains to show (65). First of all, (56) and (53) in Corollary 23 give

$$(67)\qquad \big\langle X^i_s,\varphi\big\rangle-\big\langle X^i_\varepsilon,S_{s-\varepsilon}\varphi\big\rangle \;=\; N^{i,s}_s(\varphi)-N^{i,s}_\varepsilon(\varphi),\quad\text{a.s. } s\geq\varepsilon,\ i=1,2.$$

Therefore,

$$(68)\qquad E\Big\{\big\langle X^1_s,\varphi\big\rangle\big\langle X^2_s,\varphi\big\rangle\;\Big|\;\mathcal{F}_\varepsilon\Big\} \;=\; \big\langle X^1_\varepsilon,S_{s-\varepsilon}\varphi\big\rangle\big\langle X^2_\varepsilon,S_{s-\varepsilon}\varphi\big\rangle,\quad\text{a.s. } s\geq\varepsilon.$$

On the other hand, for $\delta>0$, by the Definition (1) of $L^{*,\delta}_{\mathbf{X}}$,

$$(69)\qquad \Big\langle L^{*,\delta}_{\mathbf{X}}(T)-L^{*,\delta}_{\mathbf{X}}(\varepsilon),\,\varphi\Big\rangle \;=\; \frac{1}{\delta}\int_0^\delta dr\int_\varepsilon^T ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,S_rX^1_s(x)\,S_rX^2_s(x).$$

Thus, by (68),

$$\begin{aligned}
&E\Big\{\Big\langle L^{*,\delta}_{\mathbf{X}}(T)-L^{*,\delta}_{\mathbf{X}}(\varepsilon),\,\varphi\Big\rangle\;\Big|\;\mathcal{F}_\varepsilon\Big\}\\
&=\frac{1}{\delta}\int_0^\delta dr\left[\int_\varepsilon^T ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,S_{r+s-\varepsilon}X^1_\varepsilon(x)\,S_{r+s-\varepsilon}X^2_\varepsilon(x)\right]\\
(70)\qquad&=\frac{1}{\delta}\int_0^\delta dr\left[\int_r^{r+T-\varepsilon} ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,S_sX^1_\varepsilon(x)\,S_sX^2_\varepsilon(x)\right].
\end{aligned}$$

Since $r\in[0,\varepsilon]$, the term in square brackets in (70) can be bounded above by

$$(71)\qquad \int_0^T ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,S_sX^1_\varepsilon(x)\,S_sX^2_\varepsilon(x).$$

But by (57), the expectation of this can be computed and equals

$$(72)\qquad \int_\varepsilon^{T+\varepsilon} ds\int_{\mathsf{R}^2} dx\;\varphi(x)\,S_sX^1_0(x)\,S_sX^2_0(x)$$

which is finite by (36). Hence, (71) is finite a.s. Therefore we may let $\delta \downarrow 0$ in (70) and conclude that for any sequence $\delta_n \downarrow 0$,

$$\lim_{n\uparrow\infty} E\left\{\left\langle L_{\mathbf{X}}^{*,\delta_n}(T) - L_{\mathbf{X}}^{*,\delta_n}(\varepsilon),\, \varphi\right\rangle \,\Big|\, \mathcal{F}_\varepsilon\right\} = \int_0^{T-\varepsilon} ds \int_{\mathsf{R}^2} dx\ \varphi(x)\, S_s X_\varepsilon^1(x)\, S_s X_\varepsilon^2(x), \text{ a.s.} \tag{73}$$

Thus, to prove (65) it suffices to show that in probability

$$E\left\{\left\langle L_{\mathbf{X}}^{*,\delta_n}(T) - L_{\mathbf{X}}^{*,\delta_n}(\varepsilon),\, \varphi\right\rangle \,\Big|\, \mathcal{F}_\varepsilon\right\} \underset{n\uparrow\infty}{\longrightarrow} E\left\{\left\langle L_{\mathbf{X}}(T) - L_{\mathbf{X}}(\varepsilon),\, \varphi\right\rangle \,\Big|\, \mathcal{F}_\varepsilon\right\}. \tag{74}$$

Note that by the Definition 1 of the collision local time there is convergence in probability of the corresponding expressions inside the conditional expectations. On the other hand, by (69) and Jensen's inequality, we have

$$\begin{aligned}
&\left\langle L_{\mathbf{X}}^{*,\delta_n}(T) - L_{\mathbf{X}}^{*,\delta_n}(\varepsilon),\, \varphi\right\rangle^2 \\
&\le\ \|\varphi\|_\infty^2\, \frac{T}{\delta_n} \int_0^{\delta_n} dr \int_\varepsilon^T ds \int_{\mathsf{R}^2} dx \int_{\mathsf{R}^2} dy\ S_r X_s^1(x)\, S_r X_s^2(x)\ S_r X_s^1(y)\, S_r X_s^2(y) \\
&\le\ \|\varphi\|_\infty^2\, \frac{T}{\delta_n} \int_0^{\delta_n} dr \int_\varepsilon^T ds\ H_r(\mathbf{X}_s)
\end{aligned}$$

[recall notation (25)]. Therefore,

$$\begin{aligned}
&E\left\{\left\langle L_{\mathbf{X}}^{*,\delta_n}(T) - L_{\mathbf{X}}^{*,\delta_n}(\varepsilon),\, \varphi\right\rangle^2 \,\Big|\, \mathcal{F}_\varepsilon\right\} \\
&\quad\le\ \|\varphi\|_\infty^2\, \frac{T}{\delta_n} \int_0^{\delta_n} dr \int_\varepsilon^T ds\ E\left\{H_r(\mathbf{X}_s) \,\Big|\, \mathcal{F}_\varepsilon\right\}
\end{aligned} \tag{75}$$

which is bounded in probability as $\delta_n \downarrow 0$ by our assumption (**IntC**) (recall Definition 7). A standard uniform integrability argument for conditional expectations (Lemma 63 in Appendix B) now gives (74), and completes the proof of (d). ■

2.3. **State spaces for X.** Recall the state space versions $\mathcal{M}_{\mathrm{f,s}}$ and $\mathcal{M}_{\mathrm{f,se}}$ from Definition 4.

Proposition 24 (State Spaces)**.** *Assume* $\mathbf{X}_0$ *is a random initial state in* $\mathcal{M}_{\mathrm{f,e}}$ *satisfying the random energy condition* (**EnC**) *from Definition 14, and* $\mathbf{X}$ *satisfies* $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$*. Then:*

- **(a):** $X_t \in \mathcal{M}_{\mathrm{f,se}}$ *a.s. for each* $t > 0$.
- **(b):** $X_t \in \mathcal{M}_{\mathrm{f,e}}$ *for all* $t \ge 0$ *a.s.*

Proof. **(a)** Fix $t > 0$. By Remark 5, for the verification of (14) it suffices to consider $0 < r < 1$. By Proposition 15 (b),

$$\begin{aligned}
E\left\langle X_t^i \times X_t^j, \mathrm{p}_r\right\rangle\ \le\ & E\int_{\mathbb{R}^2} dx_1\ S_t X_0^i(x_1) \int_{\mathbb{R}^2} dx_2\ S_t X_0^j(x_2)\, \mathrm{p}_r(x_1,x_2) \\
&+\ \delta_{ij}\,\gamma\, E\int_0^t ds \int_{\mathbb{R}^2} dx\ S_s X_0^1(x)\, S_s X_0^2(x) \\
&\times \int_{\mathbb{R}^2} dx_1\ \mathrm{p}_{t-s}(x,x_1) \int_{\mathbb{R}^2} dx_2\ \mathrm{p}_{t-s}(x,x_2)\, \mathrm{p}_r(x_1,x_2).
\end{aligned}$$

The right hand side of this inequality can be written as

$$E\left\langle X_0^i \times X_0^j, \mathrm{p}_{2t+r}\right\rangle + \delta_{ij}\,\gamma\, E\int_0^t ds\; \mathrm{p}_{r+2(t-s)}(0,0) E\left\langle X_0^1 \times X_0^2, \mathrm{p}_{2s}\right\rangle . \tag{76}$$

For the first term in (76), use $\mathrm{p}_{2t+r}(y_1,y_2) \le \mathrm{p}_{2t+r}(0,0) \le c(t)$, to get the bound $c(t)\, E\langle X_0^i,1\rangle\langle X_0^j,1\rangle$. In the second term of (76), break the integral at $t/2$. For the lower part, apply $\mathrm{p}_{r+2(t-s)}(0,0) \le c(t)$, whereas for the second part, use $\mathrm{p}_{2s}(y_1,y_2) \le c(t)$. This gives the bound

$$c(t)\int_0^{t/2} ds\; E\left\langle X_0^1 \times X_0^2, \mathrm{p}_{2s}\right\rangle \tag{77}$$

$$+\; c(t)\int_{t/2}^t ds\; \mathrm{p}_{r+2(t-s)}(0,0)\, E\langle X_0^1,1\rangle\langle X_0^2,1\rangle \tag{78}$$

for the second term in (76). For (77) use (15) to bound it by $c(t)\,\|\mathbf{X}_0\|_g$, whereas in (78) the ds–integral can be bounded by $c(t)\;\bigl[1+\log(1/r)\bigr]$. Altogether,

$$\begin{aligned} &E\left\langle X_t^i \times X_t^j, \mathrm{p}_r\right\rangle \\ &\quad\le\; c(t)\;\bigl[1+\log(1/r)\bigr]\; E\Bigl[\sum_{i=1,2}\langle X_0^i,1\rangle^2 + \|\mathbf{X}_0\|_g\Bigr] \;=\; c\;\bigl[1+\log(1/r)\bigr], \end{aligned}\tag{79}$$

where in the last step we used our assumption (**EnC**), and the constant c is independent of r.

Next we want to apply this estimate for special values of r. In fact, if r belongs to $[2^{-n-1},2^{-n})$, $n \ge 0$, then $\mathrm{p}_r \le 2\,\mathrm{p}_{2^{-n}}$, and if $p \in (0,1)$, then from (79),

$$E\sup_{0<r<1} r^p\left\langle X_t^i \times X_t^j, \mathrm{p}_r\right\rangle \;\le\; 2\,c\sum_{n=0}^{\infty} 2^{-np}\,[1+\log 2^n] \;<\; \infty.$$

This proves $\mathbf{X}_t \in M_{\mathrm{f,se}}$ a.s.

(b) We will use a Tanaka formula approach from [BEP91]. To prepare for this, for $\alpha,\varepsilon \ge 0$, set

$$g_{\alpha,\varepsilon}(x_1,x_2) \;:=\; \frac{1}{2}\,\mathrm{e}^{\varepsilon\,\alpha/2}\int_\varepsilon^\infty du\; \mathrm{e}^{-\alpha u/2}\,\mathrm{p}_u(x_1,x_2), \qquad x_1,x_2 \in \mathbb{R}^2.$$

Note that

$$g_{\alpha,\varepsilon} \;\le\; \mathrm{e}^{\alpha}\, g_{\alpha,0}, \qquad 0 \le \varepsilon \le 1, \quad \alpha \ge 0, \tag{80}$$

and we have pointwise convergence

$$\lim_{\varepsilon\downarrow 0} g_{\alpha,\varepsilon} \;=\; g_{\alpha,0}, \qquad \alpha \ge 0. \tag{81}$$

It is easy to see ([BEP91, (5.6)]) that to each $\alpha > 0$ there are positive constants c_α and C_α such that

$$c_\alpha\, g \;\le\; 1 + g_{\alpha,0} \;\le\; C_\alpha\, g \tag{82}$$

[with the energy function g from (12)].

Let $\mathbf{X}_t = X_t^1 \times X_t^2$. It follows from $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ and a bit of stochastic calculus, just as in the derivation of (T_ε) in Section 5 of [BEP91], that

(83)

$$\begin{aligned}\langle \mathbf{X}_t, g_{\alpha,\varepsilon}\rangle &= \langle \mathbf{X}_0, g_{\alpha,\varepsilon}\rangle \\ &\quad + \int_0^t \iint g_{\alpha,\varepsilon}(x_1,x_2)\left[X_s^1(dx_1)M^2(ds,dx_2) + X_s^2(dx_2)M^1(ds,dx_1)\right] \\ &\quad + \alpha \int_0^t \iint g_{\alpha,\varepsilon}(x_1,x_2)X_s^1(dx_1)X_s^2(dx_2)ds - \hat{L}_t^\varepsilon(\mathbf{X})\end{aligned}$$

where $\hat{L}_t^\varepsilon(\mathbf{X}) = \int_0^t \iint p_\varepsilon(x_1-x_2)X_s^1(dx_1)X_s^2(dx_2)ds$. As $g_{\alpha,\varepsilon}$ is bounded the above stochastic integral in (83), $I^\varepsilon(t)$, is a continuous local martingale and we may choose stopping times $T_n \uparrow \infty$ a.s. such that $\sup_{t\le T_n} I^\varepsilon(t) \le n$. Then (83) implies

$$\begin{aligned}E(\langle \mathbf{X}_{t\wedge T_n}, g_{\alpha,\varepsilon}\rangle) &\le E(\langle \mathbf{X}_0, g_{\alpha,\varepsilon}\rangle) + \alpha \int_0^t E(\langle \mathbf{X}_{s\wedge T_n}, g_{\alpha,\varepsilon}\rangle)ds \\ &\le C_\alpha E(\langle \mathbf{X}_0, g\rangle) + \alpha \int_0^t E(\langle \mathbf{X}_{s\wedge T_n}, g_{\alpha,\varepsilon}\rangle)ds \quad \text{(by (80) and (82))}.\end{aligned} \tag{84}$$

Note also that $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ implies that $\langle \mathbf{X}_t, 1\rangle = \langle X_t^1, 1\rangle \langle X_t^2, 1\rangle$ is a martingale (we also use **EnC** here) and so

$$E(\langle \mathbf{X}_{t\wedge T_n}, g_{\alpha,\varepsilon}\rangle) \le ||g_{\alpha,\varepsilon}||_\infty E(\langle \mathbf{X}_{t\wedge T_n}, 1\rangle) = ||g_{\alpha,\varepsilon}||_\infty E(\langle \mathbf{X}_0, 1\rangle) < \infty.$$

It therefore follows from (84) that

$$E(\langle \mathbf{X}_{t\wedge T_n}, g_{\alpha,\varepsilon}\rangle) \le c(\alpha)E(\langle X_0, g\rangle)e^{\alpha t}, \quad \forall \quad t \ge 0,\ n \in \mathbb{N}. \tag{85}$$

Note also by Proposition 15(b),

$$\begin{aligned}E(\hat{L}_t^\varepsilon(\mathbf{X})) &= E\left(\int_0^t \iint p_\varepsilon(y_1-y_2)S_sX_0^1(y_1)S_sX_0^2(y_2)dy_1dy_2ds\right) \\ &= E\left(\int_0^t \iint p_{\varepsilon+2s}(y_1-y_2)X_0^1(dy_1)X_0^2(dy_1)ds\right) \\ &\le c'(t)E(\langle \mathbf{X}_0, g\rangle).\end{aligned} \tag{86}$$

It follows from (83) and the integrability implied by (85) and (86) that
$Y_t^n \equiv \langle \mathbf{X}_{t\wedge T_n}, g_{\alpha,\varepsilon}\rangle + \hat{L}_{t\wedge T_n}^\varepsilon(\mathbf{X})$ is a non-negative submartingale. Therefore by the weak maximal inequality for any $t, K > 0$ fixed

$$\begin{aligned}P\left(\sup_{s\le t\wedge T_n} \langle \mathbf{X}_s, g_{\alpha,\varepsilon}\rangle > K\right) &\le P\left(\sup_{s\le t} Y_s^n > K\right) \\ &\le K^{-1}E(Y_t^n) \\ &\le K^{-1}\left[c(\alpha)e^{\alpha t} + c'(t)\right]E(\langle \mathbf{X}_0, g\rangle).\end{aligned}$$

First let $n \to \infty$ and then $\varepsilon \to 0$ in the above and use Fatou's Lemma and (81) to see that

$$P\left(\sup_{s\le t} \langle \mathbf{X}_s, g_{\alpha,0}\rangle > K\right) \le K^{-1}\left[c(\alpha)e^{\alpha t} + c'(t)\right]E(\langle \mathbf{X}_0, g\rangle).$$

In view of the lower bound in (82), the required result is immediate. ■

3. A function-valued dual for higher moments

In this section function-valued duals which are used to compute higher moments are presented.

3.1. **Lattice approximation moment dual $\mathbf{V}^\varepsilon$ and self-duality.** Since it has not been explicitly mentioned in [DP98], we start by pointing out that our lattice approximations have finite moments of all orders:

Lemma 25 (Moments of all Orders)**.** *Let $\varepsilon > 0$. Assume ${}^\varepsilon\mathbf{X}$ satisfies the martingale problem $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mathbf{X}_0}$ of Lemma 6 with deterministic initial condition, $\mathbf{X}_0 \in \mathcal{M}^2_f(\mathbb{R}^2)$. Then for any integer $m \geq 1$, and $T > 0$ there is a constant $C = C\big(\varepsilon, T, m, \langle \mathbf{X}_0, 1\rangle\big)$ such that*

$$\sum_{i=1,2} E\,\big(\sup_{t\leq T} \langle {}^\varepsilon X^i_t, 1\rangle^m\big) \;\leq\; C. \tag{87}$$

Proof. Clearly we may assume $m \geq 2$ and $\varepsilon = 1$, and we will suppress the index $\varepsilon = 1$ in our notation. Then, for $i \in \{1,2\}$ fixed, $t \mapsto \langle X^i_t, 1\rangle - \langle X^i_0, 1\rangle = M^i_t(1)$ is a continuous L^2–martingale such that for $T > 0$ fixed and $t \leq T$,

$$\begin{aligned} \big\langle\!\big\langle M^i(1)\big\rangle\!\big\rangle^{m/2}_t \;&=\; \left(\gamma \int_0^t ds \sum_{x\in\mathbb{Z}^2} X^1_s(x)\, X^2_s(x)\right)^{m/2} \\ &\leq\; c\int_0^t ds \left(\sum_{x\in\mathbb{Z}^2} X^1_s(x)\, X^2_s(x)\right)^{m/2} \;\leq\; c\int_0^t ds \sum_{i=1,2} \langle X^i_s, 1\rangle^m \end{aligned}$$

(where $c = c_{m,\gamma,T}$). Fix for the moment $K \geq 1$, and consider the stopping time $\tau_K := T \wedge \inf\Big\{t : \sum_{i=1}^2 \langle X^i_t, 1\rangle \geq K\Big\}$ Burkholder's inequality then shows that for any $r \in [0,T]$,

$$\begin{aligned} &E\left(\sum_{i=1,2} \sup_{t\leq r\wedge\tau_K} \langle X^i_t, 1\rangle^m\right) \\ &\quad\leq\; c\sum_{i=1,2} \langle X^i_0, 1\rangle^m + c\int_0^r ds\; E\left(\sum_{i=1,2} \langle X^i_{s\wedge\tau_K}, 1\rangle^m\right), \end{aligned} \tag{88}$$

with the constant c independent of r (and K). Since the expectation in the integrand on the right hand side of this inequality can further be bounded from above by $E(\sum_{i=1,2} \sup_{t\leq s\wedge\tau_K} \langle X^i_t, 1\rangle^m)$, Gronwall's Lemma implies

$$E\left(\sum_{i=1,2} \sup_{t\leq \tau_K} \langle X^i_t, 1\rangle^m\right) \;\leq\; C \tag{89}$$

where $C = C(T, m, \langle \mathbf{X}_0, 1\rangle)$ is independent of K. Letting $K \uparrow \infty$ completes the proof since $\tau_K \uparrow T$. ∎

Although in this paper we only use fourth order moments, we now introduce a *function-valued dual process* $\mathbf{V}^\varepsilon = \mathbf{V}^{\varepsilon,m}$ which will describe moments of arbitrary but fixed order $m \geq 1$ for solutions ${}^\varepsilon\mathbf{X}$ of $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mathbf{X_0}}$, with a fixed $\varepsilon \in (0,1]$. The state space of the dual is $\mathcal{S}^\varepsilon = \mathcal{S}^{\varepsilon,m} := C^+_{\mathrm{b}}\big((\varepsilon\mathbb{Z}^2)^m\big) \times 2^{\{1,\ldots,m\}}$ (with $2^{\{1,\ldots,m\}}$ denoting the power set of $\{1,\ldots,m\}$), and elements in $\mathcal{S}^\varepsilon$ are denoted by (ϕ, I).

It is convenient to think of the argument of ϕ as the spatial positions of a system of m particles. Particles take two types: those corresponding to a coordinate whose index is in I are of type 1, those corresponding to indices in I^c are of type 2. These m particles have positions described by $x \in (\varepsilon\mathbb{Z}^2)^m$. We give $C_{\rm b}^+\big((\varepsilon\mathbb{Z}^2)^m\big)$ the topology of pointwise convergence, to make $\mathcal{S}^\varepsilon$ a separable metric space.

Let ${}^\varepsilon S_t^{(m)}$ denote the semigroup on $C_{\rm b}((\varepsilon\mathbb{Z}^2)^m)$ obtained by running m independent copies of our simple random walk ${}^\varepsilon\xi$ (each with generator $\frac{\sigma^2}{2}\,{}^\varepsilon\Delta$), and let $\frac{\sigma^2}{2}\,{}^\varepsilon\Delta^{(m)}$ denote the associated generator.

For $1 \le j, j' \le m$ with $j \ne j'$, define maps $\pi_{j,j'} : (\mathbb{R}^2)^m \to (\mathbb{R}^2)^m$ and $f_{j,j'} : C_{\rm b}^+\big((\varepsilon\mathbb{Z}^2)^m\big) \to C_{\rm b}^+\big((\varepsilon\mathbb{Z}^2)^m\big)$ by

$$(\pi_{j,j'}x)_i \;:=\; \begin{cases} x_i & \text{if } i \ne j', \\ x_j & \text{if } i = j', \end{cases} \qquad x = (x_1, ..., x_m) \in (\mathbb{R}^2)^m, \tag{90}$$

and

$$\begin{aligned} f_{j,j'}(\phi)\,(x) \;&:=\; \phi(\pi_{j,j'}x)\,\varepsilon^{-2}\,\mathbf{1}(x_j = x_{j'}) \\ &= \phi(\pi_{j,j'}x)p_0^\varepsilon(x_j, x_{j'}). \end{aligned} \tag{91}$$

Definition 26 (Dual Process $\mathbf{V}^\varepsilon$)**.** For fixed $m \ge 1$, denote by $\mathbf{V}^\varepsilon = \mathbf{V}^{\varepsilon,m} = \big\{\mathbf{V}_t^\varepsilon : t \ge 0\big\}$ the Markov process which has sample paths in the Skorohod space $D(\mathbb{R}_+, \mathcal{S}^\varepsilon)$, and evolves as follows:

- **(a) (jumps):** If $\mathbf{V}^\varepsilon$ is in the state (ϕ, I), for each (ordered) pair (j, j') in I^2 satisfying $j \ne j'$, the process $\mathbf{V}^\varepsilon$ jumps to $\big(f_{j,j'}(\phi),\, I\backslash\{j'\}\big)$ with rate $\gamma/2$, and for each $(j, j') \in (I^{\rm c})^2$ with $j \ne j'$, it jumps to $\big(f_{j,j'}(\phi),\, I \cup \{j'\}\big)$, also with rate $\gamma/2$. (In particular, a jumping particle changes its type.) In these cases we say j' *switches via* j.

  Let $\{T_j : j \ge 1\}$ denote the successive jump times, and set $T_0 = 0$.
- **(b) (between jumps):** Between jump times, the component ϕ of $\mathbf{V}^\varepsilon$ evolves according to the semigroup ${}^\varepsilon S^{(m)}$, whereas the component I is frozen. That is,

$$\text{if } T_n \le t < T_{n+1}\,, \text{ then } \;\phi_t(x) = {}^\varepsilon S_{t-T_n}^{(m)}\phi_{T_n}(x), \text{ and } I_t = I_{T_n}\,. \tag{92}$$

Let $\mathcal{A}^\varepsilon = \mathcal{A}^{\varepsilon,m}$ denote the (weak) infinitesimal generator of $\mathbf{V}^\varepsilon$, and $\hat{P}_{\mathbf{V}_0^\varepsilon}^\varepsilon$ the law of $\mathbf{V}^\varepsilon$ if $\mathbf{V}^\varepsilon$ starts in $\mathbf{V}_0^\varepsilon$ (deterministic). ◇

Define a *duality function* $F : \mathcal{S}^\varepsilon \times \mathcal{M}_{\rm f}^2(\varepsilon\mathbb{Z}^2) \to \mathbb{R}_+$ by

$$F(\phi, I, \mu^1, \mu^2) \;:=\; \prod_{i\in I}\int_{\varepsilon\mathbb{Z}^2}\mu^1(dx_i)\,\prod_{j\in I^{\rm c}}\int_{\varepsilon\mathbb{Z}^2}\mu^2(dx_j)\,\phi(x). \tag{93}$$

Then, for $(\phi, I, \mu) \in \mathcal{S}^\varepsilon \times \mathcal{M}_{\rm f}^2(\varepsilon\mathbb{Z}^2)$,

$$\begin{aligned} \mathcal{A}^\varepsilon F(\cdot,\cdot,\mu)(\phi, I) \;&=\; F\Big(\frac{\sigma^2}{2}\,{}^\varepsilon\Delta^{(m)}\phi,\, I, \mu\Big) \\ &\quad + \frac{\gamma}{2}\sum_{\substack{(j,j')\,\in\, I^2 \\ j\ne j'}}\Big(F\big(f_{j,j'}^\varepsilon(\phi),\, I\backslash\{j'\},\, \mu\big) - F(\phi, I, \mu)\Big) \\ &\quad + \frac{\gamma}{2}\sum_{\substack{(j,j')\,\in\,(I^{\rm c})^2 \\ j\ne j'}}\Big(F\big(f_{j,j'}^\varepsilon(\phi), I \cup \{j'\}, \mu\big) - F(\phi, I, \mu)\Big). \end{aligned} \tag{94}$$

Hence, for $\mu \in \mathcal{M}_{\rm f}^2(\varepsilon\mathbb{Z}^2)$,

$$F(\mathbf{V}_t^\varepsilon,\mu) - F(\mathbf{V}_0^\varepsilon,\mu) - \int_0^t ds\; \mathcal{A}^\varepsilon F(\mathbf{V}_s^\varepsilon,\mu) \tag{95}$$

is a $\hat{P}^\varepsilon_{\mathbf{V}_0^\varepsilon}$–martingale. [See (97) below for the integrability of $F(\mathbf{V}_t^\varepsilon,\mu)$ with respect to $\hat{P}^\varepsilon_{\mathbf{V}_0^\varepsilon}$.]

Let ${}^\varepsilon\mathbf{X}$ be our solution to $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mathbf{X}_0}$ from Lemma 6 and denote the underlying probability by $P^\varepsilon_{\mathbf{X}_0}$. As usual $\mathbf{X}_0$ is a fixed element in $\mathcal{M}_{\rm f,e}$. If $(\phi,I)\in\mathcal{S}^\varepsilon$, then Itô's Lemma and the system of stochastic equations (18) defining the process ${}^\varepsilon\mathbf{X}$ show that

$$\begin{aligned} F(\phi,I,{}^\varepsilon\mathbf{X}_t) \;&=\; F(\phi,I,{}^\varepsilon\mathbf{X}_0)\\ &\quad+\int_0^t ds\,\left[\mathcal{A}^\varepsilon F(\phi,I,{}^\varepsilon\mathbf{X}_s)+\gamma\left\{\binom{|I|}{2}+\binom{|I^{\rm c}|}{2}\right\}F(\phi,I,{}^\varepsilon\mathbf{X}_s)\right] \;+\; M_t^{\phi,I}, \end{aligned}$$

where $M^{\phi,I}$ is a continuous L^2–martingale which can be explicitly written in terms of the Brownian motions arising in (18). (Note that the integrals in the duality function (93) are actually sums.)

On the other hand, if A^ε is the weak generator of ${}^\varepsilon\mathbf{X}$, then we have

$$A^\varepsilon F(\phi,I,\cdot)(\mu) \;=\; \mathcal{A}^\varepsilon F(\cdot,\cdot,\mu)(\phi,I)+\gamma\left\{\binom{|I|}{2}+\binom{|I^{\rm c}|}{2}\right\}F(\phi,I,\mu), \tag{96}$$

$(\phi,I,\mu)\in\mathcal{S}^\varepsilon\times\mathcal{M}_f^2(\mathbb{R}^2)$.

Proposition 27 (Moment Duality for $\mathbf{X}^\varepsilon$). *For any* $\mathbf{V}_0^\varepsilon\in\mathcal{S}^\varepsilon$, $\mathbf{X}_0\in\mathcal{M}_{\rm f,e}(\mathbb{R}^2)$, $\varepsilon\in(0,1]$, *and* $t>0$,

$$E^\varepsilon_{\mathbf{X}_0}F(\mathbf{V}_0^\varepsilon,{}^\varepsilon\mathbf{X}_t) \;=\; \hat{E}^\varepsilon_{\mathbf{V}_0^\varepsilon}\left(F(\mathbf{V}_t^\varepsilon,{}^\varepsilon\mathbf{X}_0)\exp\left[\gamma\int_0^t ds\,\left\{\binom{|I_s|}{2}+\binom{|I_s^{\rm c}|}{2}\right\}\right]\right)<\infty.$$

Proof. In view of (96) we only need to check the hypotheses (4.50) and (4.51) of [EK86, Theorem 4.11] with $\alpha=0$ and $\beta(\phi,I)=\binom{|I|}{2}+\binom{|I^{\rm c}|}{2}$. Note that $\beta(\phi,I)\le 2\binom{m}{2}$, so that (4.51) is obvious. Let N_s be the number of jumps of $\mathbf{V}^\varepsilon$ up to time s. Note that

$$\begin{aligned} &\hat{E}^\varepsilon_{\mathbf{V}_0^\varepsilon}\times E^\varepsilon_{{}^\varepsilon\mathbf{X}_0}\left(\sup_{0\le s,t\le T}F(\phi_s,I_s,{}^\varepsilon\mathbf{X}_t)\right)\\ &\le\; c\hat{E}^\varepsilon_{\mathbf{V}_0^\varepsilon}(\varepsilon^{-2N_T}\|\phi_0\|_\infty)E^\varepsilon_{{}^\varepsilon\mathbf{X}_0}\left(\sup_{t\le T}\langle{}^\varepsilon X_t^1,1\rangle^m+\sup_{t\le T}\langle{}^\varepsilon X_t^2,1\rangle^m\right)<\infty, \end{aligned} \tag{97}$$

by Lemma 25. Then (4.50) in Theorem 4.11 of [EK86] is a simple consequence of this. ∎

It is not hard to see that the above moments grow too quickly for the moment problem to be well-posed and hence do not characterize the law of ${}^\varepsilon\mathbf{X}$. Mytnik's exponential self-duality [Myt98] is still required for this. At times we will write ${}^\varepsilon\phi$ for ϕ in $\mathbf{V}^\varepsilon$, but note that we may define ${}^\varepsilon I_t = I_t$ to be independent of ε.

A slight modification of the proof of Theorem 2.4 (b) in [DP98] gives the following self-duality relation for the discrete space processes:

Lemma 28 (self-duality). *Fix* $0 < \varepsilon \leq 1$. *Let* ${}^{\varepsilon}\mathbf{X} = ({}^{\varepsilon}X^1, {}^{\varepsilon}X^2)$ *and* ${}^{\varepsilon}\widetilde{\mathbf{X}} = ({}^{\varepsilon}\widetilde{X}^1, {}^{\varepsilon}\widetilde{X}^2)$ *denote independent mutually catalytic symmetric simple super-random walks in* $\varepsilon\mathsf{Z}^2$ *with initial states* ${}^{\varepsilon}\mathbf{X}_0 = {}^{\varepsilon}\mu = ({}^{\varepsilon}\mu^1, {}^{\varepsilon}\mu^2) \in {}^{\varepsilon}\mathcal{M}_{\mathrm{f}}^2$ *and* ${}^{\varepsilon}\widetilde{\mathbf{X}}_0 = {}^{\varepsilon}\varphi = ({}^{\varepsilon}\varphi^1, {}^{\varepsilon}\varphi^2) \in (L^1(d\ell^{\varepsilon}))^2$, *respectively. Then the following duality relation holds for these* ${}^{\varepsilon}\mathcal{M}_{\mathrm{f}}^2$*–valued processes:*

$$\begin{aligned}
&P_{{}^{\varepsilon}\mu} \exp\left[-\left\langle {}^{\varepsilon}X_t^1 + {}^{\varepsilon}X_t^2, {}^{\varepsilon}\varphi^1 + {}^{\varepsilon}\varphi^2 \right\rangle + i\left\langle {}^{\varepsilon}X_t^1 - {}^{\varepsilon}X_t^2, {}^{\varepsilon}\varphi^1 - {}^{\varepsilon}\varphi^2 \right\rangle\right] \\
&= P_{{}^{\varepsilon}\varphi} \exp\left[-\left\langle {}^{\varepsilon}\mu^1 + {}^{\varepsilon}\mu^2, {}^{\varepsilon}\widetilde{X}_t^1 + {}^{\varepsilon}\widetilde{X}_t^2 \right\rangle + i\left\langle {}^{\varepsilon}\mu^1 - {}^{\varepsilon}\mu^2, {}^{\varepsilon}\widetilde{X}_t^1 - {}^{\varepsilon}\widetilde{X}_t^2 \right\rangle\right], \quad t \geq 0,
\end{aligned}$$

(with $i = \sqrt{-1}$*), where the terms* $\left\langle {}^{\varepsilon}\mu^j, {}^{\varepsilon}\widetilde{X}_t^k, \right\rangle$, $j,k = 1,2$, *occurring in the exponent at the right hand side are finite* $P_{{}^{\varepsilon}\varphi}$*–a.s.*

3.2. **Limiting moment dual V.** In order to let $\varepsilon \downarrow 0$ in Proposition 27 we specialize to $m = 4$ and introduce the natural candidate for a limiting dual process $\mathbf{V}$. In order to define the state space we introduce some notation.

Notation 29. *For* $\mathbf{x} = (x^1, \ldots, x^n) \in (\mathbb{R}^d)^n$, $n \geq 1$, *we set*

$$|\mathbf{x}| := |\mathbf{x}^1| + \cdots + |x^n|.$$

We introduce the mollifier

$$\rho(x) := c1_{\{|\mathbf{x}|<1\}} \exp\left[-1/(1-x^2)\right], \ x \in \mathbb{R},$$

with c the normalizing constant so that $\int dx \rho(x) = 1$. *For* $\lambda \in \mathbb{R}$, *set*

$$\phi_\lambda^1(x) := \int dy \ e^{-\lambda|y|} \rho(y-x), \quad x \in \mathbb{R},$$

and introduce the reference function

$$\phi_\lambda^1(x) := \phi_\lambda^1(x_1) \cdots \phi_\lambda^1(x_d), \ x = (x_1, \ldots x_d) \in \mathbb{R}^d.$$

If φ *is a (real-valued) function on* $\mathbb{R}^d$, *put*

$$|\varphi|_\lambda := \sup_{x \in \mathbb{R}^d} |\varphi(x)|/\phi_\lambda(x), \quad \lambda \in \mathbb{R}.$$

For $\lambda \in \mathbb{R}$, *let* $\mathcal{C}_\lambda$ *denote the set of all continuous functions such that* $|\varphi|_\lambda$ *is finite. Introduce the space*

$$\mathcal{C}_{\mathrm{rap}} = \mathcal{C}_{\mathrm{rap}}(\mathbb{R}^d) := \cup_{\lambda>0} \mathcal{C}_\lambda$$

of rapidly decreasing continuous functions. Let $\mathcal{M}_{\mathrm{tem}} = \mathcal{M}_{\mathrm{tem}}(\mathbb{R}^d)$ *denote the subset of all measures* μ *on* $\mathbb{R}^d$ *such that* $\langle \mu, \phi_\lambda \rangle < \infty$ *for all* $\lambda > 0$. *We topologize the set of tempered measures* $\mathcal{M}_{\mathrm{tem}}$ *by the metric*

$$d_{\mathrm{tem}}(\mu, \nu) := d_0(\mu, \nu) + \sum_{n=1}^{\infty} 2^{-n} (|\mu - \nu|_{1/n} \wedge 1), \ \mu, \nu \in \mathcal{M}_{\mathrm{tem}}.$$

Here d_0 *is a complete metric on the space of Radon measures on* $\mathbb{R}^d$ *inducing the vague topology, and* $|\mu - \nu|_\lambda$ *is an abbreviation for* $|\langle \mu, \phi_\lambda \rangle - \langle \nu, \phi_\lambda \rangle|$. *Note that* $(\mathcal{M}_{\mathrm{tem}}, d_{\mathrm{tem}})$ *is a Polish space and that* $\mu_n \to \mu$ *in* $\mathcal{M}_{\mathrm{tem}}$ *if and only if* $\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle$ *for all* $\varphi \in \mathcal{C}_{\mathrm{rap}}$.

The state space for this dual $\mathbf{V}$ will be $\mathcal{S} = \mathcal{M}_{\text{tem}}((\mathbb{R}^2)^4) \times 2^{\{1,\dots,4\}}$, although our starting point $\mathbf{V}_0$ will be in

$$C_{\mathrm{b}}^+((\mathbb{R}^2)^4) \times 2^{\{1,\dots,4\}} \;=:\; \mathcal{S}_0\,. \tag{98}$$

As before, we will identify functions ϕ_0 in C_{b}^+ with the finite measure $\phi_0(x)dx$ in $\mathcal{M}_{\text{tem}}$. We abuse our earlier notation slightly and define $F : \mathcal{S} \times \mathcal{M}_{\mathrm{f}}^2(\mathbb{R}^2) \mapsto \mathbb{R}_+$ by

$$F(\phi, I, \mu) = \begin{cases} \int \phi(x_1,\dots,x_4) \prod_{i\in I} \mu^1(dx_i) \prod_{j\notin I} \mu^2(dx_j), & \text{if } (\phi, I) \in \mathcal{S}_0 \\ 0, \quad \text{otherwise} \end{cases}$$

and define $\pi_{j,j'} : (\mathbb{R}^2)^4 \mapsto (\mathbb{R}^2)^4$ for $1 \le j, j' \le 4$ as before. If $1 \le j, j' \le 4$, then $f_{j,j'} : C_{\mathrm{b}}^+((\mathbb{R}^2)^4) \mapsto \mathcal{M}_{\text{tem}}((\mathbb{R}^2)^4)$ is given by $f_{j,j'}(\phi) = \phi \circ \pi_{j,j'} \delta_{x_j - x_{j'}}$, that is

$$f_{j,j'}(\phi)(A) = \int_A dx_1 \dots dx_4 \; \phi(\pi_{j,j'} x)\, \delta_{x_j - x_{j'}}(x). \tag{99}$$

It is easy to check this measure is in $\mathcal{M}_{\text{tem}}$.

Definition 30 (Dual Process $\mathbf{V}$)**.**

Let $\mathbf{S}_t$ be the 8-dimensional Brownian semigroup with variance parameter σ^2, let $\frac{\sigma^2\Delta}{2}$ denote its generator and $\mathbf{p}_t(x,y)$ the associated transition function. The dynamics of the dual process $\mathbf{V} = \mathbf{V}^m = (\phi_{\cdot}, I_{\cdot}) \in D(\mathbb{R}_+, \mathcal{S})$ are as follows:

(a) For each $(j,j') \in I_t^2$, $j \neq j'$, with rate $\gamma/2$, (ϕ_{t-}, I_{t-}) jumps to $(f_{j,j'}(\phi_{t-}), I_{t-}\backslash\{j'\})$, and for each $(j,j') \in (I_t^c)^2$, $j \neq j'$, with rate $\gamma/2$, (ϕ_{t-}, I_{t-}) jumps to $(f_{j,j'}(\phi_{t-}), I_{t-} \cup \{j'\})$.

Let $0 = T_0 < T_1 < T_2 < \dots$ be the successive jump times.

(b) For $T_n \le t < T_{n+1}$, $\mathbf{V}_t = (\mathbf{S}_{t-T_n}\phi_{T_n}, I_{T_n})$. $\diamond$

Remark 31. *To ensure that this does define a process V_t we need to check that $\phi_{T_n-} \in C_b^+((\mathbb{R}^2)^4)$ for all $n \ge 1$ a.s. so that $f_{j,j'}(\phi_{T_n-})$ is well-defined. For this we will use induction to show if $T_n < T_{n+1}$ for all $n \ge 0$, then*

(100) *On $[T_n, T_{n+1})$, $\phi_{\cdot}$ is a continuous $\mathcal{M}_{\text{tem}}$-valued process taking values in $C_b^+(\mathbb{R}^8)$ for $t \in (T_n, T_{n+1})$, and $\phi_{T_{n+1}-} = \mathbf{S}_{T_{n+1}-T_n}\phi_{T_n} \in C_b^+(\mathbb{R}^8)$.*

For $n = 0$ this is clear as $\phi_0 \in C_b^+$. Assume (100) for $n-1$ and consider n. Then $\phi_{T_n} = f_{j,j'}(\phi_{T_n-}) \in \mathcal{M}_{\text{tem}}$ and for $t \in [T_n, T_{n+1})$

$$\phi_t(x) = \mathbf{S}_{t-T_n}\phi_{T_n}(x) = \int \mathbf{p}_{t-T_n}(x,y)\phi_{T_n}(dy).$$

It is easy to see that if $f \in \mathcal{C}_{\text{rap}}$, then $\langle \mathbf{S}_{t-T_n}\phi_{T_n}, f\rangle = \langle \phi_{T_n}, \mathbf{S}_{t-T_n} f\rangle$ is continuous in t (e.g., use Dominated Convergence and Lemma 6.2(ii) of [Shig94]*) and so ϕ_t is continuous on $[T_n, T_{n+1})$ and $\phi_{T_{n+1}-} = \mathbf{S}_{T_{n+1}-T_n}\phi_{T_n}$. For $t > T_n$ use the bound $\mathbf{p}_{t-T_n}(x,y) \le ce^{\lambda|x|}e^{-\lambda|y|}$ (c, λ may depend on (t, T_n)) and Dominated Convergence to conclude that $\phi_t(\cdot)$ is continuous for all $t \in (T_n, T_{n+1})$ and the same is true for $\phi_{T_{n+1}-}(\cdot)$. For boundedness use the induction hypothesis to see that*

$\phi_{T_n} \leq \|\phi_{T_n-}\|_\infty \delta_{x_j - x_{j'}} dx$ *and so (take* $j = 1$ *and* $j' = 2$ *for definiteness)*

$$\begin{aligned}\phi_t(x) &\leq \|\phi_{T_n-}\|_\infty \int \mathbf{p}_{t-T_n}(x, y_1, y_1, y_3, y_4) dy_1 dy_3 dy_4 \\ &\leq \|\phi_{T_n-}\|_\infty p_{2(t-T_n)}(x_1, x_2) \\ &\leq c(t - T_n)^{-1} \|\phi_{T_n-}\|_\infty < \infty.\end{aligned}$$

The same reasoning shows that $\phi_{T_{n+1}-}$ *is bounded. This completes the inductive proof of (100).*

It is clear from (100) that $V_\cdot$ has sample paths in $D(\mathbb{R}_+, \mathcal{S})$ a.s. Let $\hat{P}_{\mathbf{V}_0}$ denote the law of $\mathbf{V}$ on $D(\mathbb{R}_+, \mathcal{S})$.

Theorem 32 (Limiting Moment Dual $\mathbf{V}$)**.** *Assume* $\gamma/\sigma^2 < (c_8 \pi \sqrt{6})^{-1}$, $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,e}}$ *where* c_8 *is given by (30) and* ${}^\varepsilon\mathbf{X}$ *is the solution to* $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mathbf{X}_0}$ *of Lemma 6. Let* $\psi : \mathcal{M}_{\mathrm{f}}^2(\mathbb{R}^2) \to \mathbb{R}_+$ *be a bounded continuous map and let* $\{\varepsilon_m\}_{m \geq 1}$ *be a sequence of positive numbers with* $\varepsilon_m \downarrow 0$*. Assume either*
(**a**) $0 = \delta < t$ *and* $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,se}}$*, or*
(**b**) $0 < \delta < t$, $\{\psi \neq 0\} \subseteq \{(\mu^1, \mu^2) : \mu^1(\mathbb{R}^2) + \mu^2(\mathbb{R}^2) \leq K\}$ *for some* K *and the law of* ${}^{\varepsilon_m}\mathbf{X}_\delta$ *converges weakly in* $\mathcal{M}_{\mathrm{f}}^2(\mathbb{R}^2)$ *as* $m \to \infty$ *to a law* $P_{\mathbf{X}_0}(\mathbf{X}_\delta \in \cdot\,)$.
Then for any $\phi_0 \in C_{\mathrm{b}}^+(\mathbb{R}^8)$, $I_0 \subseteq \{1, \ldots, 4\}$,

$$\begin{aligned}&\lim_{m\to\infty} E_{{}^{\varepsilon_m}\mathbf{X}_0} F(\mathbf{V}_0, {}^{\varepsilon_m}\mathbf{X}_t)\, \psi({}^{\varepsilon_m}\mathbf{X}_\delta) \\ &= \hat{E}_{\mathbf{V}_0} \times E_{\mathbf{X}_0} \left(F(\mathbf{V}_{t-\delta}, \mathbf{X}_\delta)\, \psi(\mathbf{X}_\delta) \exp\left[\gamma \int_0^{t-\delta} ds \left\{ \binom{|I_s|}{2} + \binom{|I_s^{\mathrm{c}}|}{2} \right\} \right] \right) < \infty.\end{aligned}$$

Remark 33. *The proof (given below) is independent of the uniqueness results in Theorem 11and will in fact be used in the derivation of uniqueness in* [DFMPX00a]*. By (100),* $\phi_{t-\delta} \in C_{\mathrm{b}}^+(R^8)$ *a.s. and so on the right-hand side of the above,*

$$F(\phi_{t-\delta}, I_{t-\delta}, \mathbf{X}_\delta) \;=\; \int \phi_{t-\delta}(x_1, \ldots, x_4) \prod_{i \in I_{t-\delta}} X_\delta^1(dx_i) \prod_{j \notin I_{t-\delta}} X_\delta^2(dx_j) \ a.s. \tag{101}$$

The proof requires the following bound on ${}^\varepsilon\mathrm{p}$ which is proved in Appendix A.

Lemma 34. *If* $r \in (0, 1)$*, then*

$$\sup\left\{ {}^\varepsilon\mathrm{p}_s(x, y) :\ 0 < s,\ 0 < \varepsilon,\ |y - x| > s^{r/2} + \varepsilon^r \right\} \;=:\; c_{34} \;<\; \infty. \tag{102}$$

If $p, \varepsilon > 0$, define

$$\mathcal{E}_{\varepsilon,p}({}^\varepsilon\mathbf{X}_t) \;:=\; \sup_{s>0} s^p \left\langle {}^\varepsilon X_t^1 \times {}^\varepsilon X_t^2,\, {}^\varepsilon\mathrm{p}_s \right\rangle \tag{103}$$

and

$$\mathcal{E}_p(\mathbf{X}_t) \;:=\; \sup_{s>0} s^p \left\langle X_t^1 \times X_t^2, \mathrm{p}_s \right\rangle \;+\; \left\langle X_t^1, 1 \right\rangle \left\langle X_t^2, 1 \right\rangle. \tag{104}$$

The proof of case (a) also uses the following result which is a simple consequence of the previous Lemma.

Lemma 35. *If* $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,se}}$*, then for any* $0 < p' < p < 1$ *there is a* $c_{35} = c_{35}(p, p', \sigma)$ *so that*

$$\sup_{0<\varepsilon} \mathcal{E}_{\varepsilon,p}({}^\varepsilon\mathbf{X}_0) \leq c_{35} \mathcal{E}_{p'}(\mathbf{X}_0) < \infty. \tag{105}$$

Proof. By Lemma 8 and the definition of ${}^{\varepsilon}\mathrm{p}_s(x)$ we have

$$ {}^{\varepsilon}\mathrm{p}_s(x) \le c_0(\sigma)(s^{-1} \wedge \varepsilon^{-2}). \tag{106} $$

If $\varepsilon > 0$ and $p' \in (0,1)$, then

$$ \begin{aligned} &\int_{\mathbb{R}^2} X_0^1(dx_1) \int_{\mathbb{R}^2} X_0^2(dx_2)\, \mathbf{1}(|x_1 - x_2| \le \varepsilon) \\ &\le\ c_1(\sigma)\varepsilon^2 \left\langle X_0^1 \times X_0^2, \mathrm{p}_{\varepsilon^2} \right\rangle \ \le\ c_1 \varepsilon^{2(1-p')} \mathcal{E}_{p'}(\mathbf{X}_0). \end{aligned} \tag{107} $$

If $p, r \in (0,1)$, then (106) and (107) show that

$$ \begin{aligned} &s^p \int\int 1(|x_1 - x_2| \le s^{r/2} + \varepsilon^r)\, {}^{\varepsilon}\mathrm{p}_s(x_1,x_2)\, {}^{\varepsilon}X_0^1(dx_1)\, {}^{\varepsilon}X_0^2(dx_2) \\ &\le s^p c_0(\sigma)(s^{-1} \wedge \varepsilon^{-2}) \int\int 1(|x_1 - x_2| \le 4(s^{r/2} + \varepsilon^r)) X_0^1(dx_1) X_0^2(dx_2) \\ &\le s^p c_0(\sigma)(s^{-1} \wedge \varepsilon^{-2}) \int\int 1(|x_1 - x_2| \le 8(s^{r/2} \vee \varepsilon^r)) X_0^1(dx_1) X_0^2(dx_2) \\ &\le c_2 \mathcal{E}_{p'}(\mathbf{X}_0) s^p (s \vee \varepsilon^2)^{r(1-p')-1} \\ &\le 2c_2 \mathcal{E}_{p'}(\mathbf{X}_0)(s \vee \varepsilon^2)^{p-1+r(1-p')}. \end{aligned} $$

Let $0 < p' < p < 1$ and choose $r = r(p',p)$ sufficiently close to 1 so that the exponent of s in the above is positive. Use the above to bound $s \le 1$ and Lemma 8(b) to handle $s > 1$ and conclude that

$$ \begin{aligned} &\sup_{0<\varepsilon,s} s^p \int\int 1(|x_1 - x_2| \le s^{r/2} + \varepsilon^r)\, {}^{\varepsilon}p_s(x_1,x_2) {}^{\varepsilon}X_0^1(dx_1)\, {}^{\varepsilon}X_0^2(dx_2) \\ &\le 2c_2 \mathcal{E}_{p'}(\mathbf{X}_0) + c_8 \sigma^{-2} \langle X_0^1, 1\rangle \langle X_0^2, 1\rangle. \end{aligned} \tag{108} $$

Combine this with Lemma 34 and (107) to see that

$$ \sup_{0<\varepsilon} \mathcal{E}_{\varepsilon,p}({}^{\varepsilon}\mathbf{X}_0) \le 2c_2 \mathcal{E}_{p'}(\mathbf{X}_0) + (c_{34} + c_8\sigma^{-2}) \langle X_0^1, 1\rangle \langle X_0^2, 1\rangle. \tag{109} $$

The result follows. ■

The proof of case (b) of Theorem 32 will use

Lemma 36. *Let $0 < p < 1$ and $\delta > 0$.*
(a) There is a $c_{36} = c_{36}(\sigma, p)$ so that for any $\varepsilon > 0, \eta \in (0,1]$ there is a random variable $Z(\varepsilon, \eta, p, \delta)$ satisfying

$$ \mathcal{E}_{\varepsilon,p}({}^{\varepsilon}\mathbf{X}_\delta) \le c_{36} \eta^{p-1} \langle {}^{\varepsilon}X_\delta^1, 1\rangle \langle {}^{\varepsilon}X_\delta^2, 1\rangle + Z(\varepsilon, \eta, p, \delta), $$

and $E(Z(\varepsilon,\eta,p,\delta)) \le c_{36}\delta^{-1}\eta^{p/2} \langle X_0^1, 1\rangle \langle X_0^2, 1\rangle$.

(b) $\sup_{0<\varepsilon} E(\mathcal{E}_{\varepsilon,p}({}^{\varepsilon}\mathbf{X}_\delta)) \le c_{36}(1 + \delta^{-1}) \langle X_0^1, 1\rangle \langle X_0^2, 1\rangle$.

Proof. (a) Lemma 8 implies that ${}^{\varepsilon}\mathrm{p}_s \le c_1(s^{-1} \wedge \varepsilon^{-2})$. This, together with Lemma 34, implies for $\varepsilon > 0$, and $r = 1 - \frac{p}{2}$,

$$ \begin{aligned} \mathcal{E}_{\varepsilon,p}({}^{\varepsilon}\mathbf{X}_\delta) &\le c_1 \langle {}^{\varepsilon}X_\delta^1, 1\rangle \langle {}^{\varepsilon}X_\delta^2, 1\rangle + \sup_{s \le 1} s^p \int {}^{\varepsilon}\mathrm{p}_s(x_1,x_2)\, {}^{\varepsilon}X_\delta^1(dx_1)\, {}^{\varepsilon}X_\delta^2(dx_2) \\ &\le (c_1 + c_{34}) \langle {}^{\varepsilon}X_\delta^1, 1\rangle \langle {}^{\varepsilon}X_\delta^2, 1\rangle \\ &\quad + \sup_{s\le 1} c_1 s^p (s^{-1} \wedge \varepsilon^{-2}) \int 1(|x_1 - x_2| \le (s^{r/2} + \varepsilon^r))\, {}^{\varepsilon}X_\delta^1(dx_1)\, {}^{\varepsilon}X_\delta^2(dx_2). \end{aligned} \tag{110} $$

The second term in (110) is bounded by

$$\sup_{s\le 1} c_1(s\vee\varepsilon^2)^p(s\vee\varepsilon^2)^{-1}\int 1(|x_1-x_2|\le 2(s\vee\varepsilon^2)^{r/2})\,{}^\varepsilon X^1_\delta(dx_1)\,{}^\varepsilon X^2_\delta(dx_2)$$
$$\le c_1\eta^{p-1}\,\langle{}^\varepsilon X^1_\delta,1\rangle\,\langle{}^\varepsilon X^2_\delta,1\rangle$$
$$+c_1\sup_{\varepsilon^2\le s\le\eta} s^{p-1}\int 1(|x_1-x_2|\le 2s^{r/2})\,{}^\varepsilon X^1_\delta(dx_1)\,{}^\varepsilon X^2_\delta(dx_2), \tag{111}$$

where the second term is defined to be 0 if $\varepsilon^2>\eta$. If $s\in[2^{-k-1},2^{-k}]$, then

$$s^{p-1}\int 1(|x_1-x_2|\le 2s^{r/2})\,{}^\varepsilon X^1_\delta(dx_1)\,{}^\varepsilon X^2_\delta(dx_2)$$
$$\le 2^{1-p}2^{-k(p-1)}\int 1(|x_1-x_2|\le 2^{1-rk/2}){}^\varepsilon X^1_\delta(dx_1)\,{}^\varepsilon X^2_\delta(dx_2).$$

Use this in (111) and then (110) to see that

$$\mathcal{E}_{\varepsilon,p}({}^\varepsilon\mathbf{X}_\delta)\le c_2(\sigma,p)\eta^{p-1}\,\langle{}^\varepsilon X^1_\delta,1\rangle\,\langle{}^\varepsilon X^2_\delta,1\rangle+Z(\varepsilon,\eta,p,\delta), \tag{112}$$

where

$$Z(\varepsilon,\eta,p,\delta)=c_2\sum_{2^{-k}\le 2\eta}2^{k(1-p)}\int 1(|x_1-x_2|\le 2^{1-rk/2})\,{}^\varepsilon X^1_\delta(dx_1)\,{}^\varepsilon X^2_\delta(dx_2).$$

Proposition 15 (b) shows that

$$E(Z(\varepsilon,\eta,p,\delta))=c_2\sum_{2^{-k}\le 2\eta}2^{k(1-p)}\int\int\int\int 1(|x_1-x_2|\le 2^{1-rk/2})$$
$${}^\varepsilon\mathrm{p}_\delta(x_1,y_1)\,{}^\varepsilon\mathrm{p}_\delta(x_2,y_2)d^\varepsilon x_1\,d^\varepsilon x_2{}^\varepsilon X^1_0(dy_1)\,{}^\varepsilon X^2_0(dy_2)$$
$$\le c_2c_8\sigma^{-2}\delta^{-1}\sum_{2^{-k}\le 2\eta}2^{k(1-p)}\pi 2^{2-rk}\langle X^1_0,1\rangle\langle X^2_0,1\rangle$$
$$\le c_3(p,\sigma)\delta^{-1}\langle X^1_0,1\rangle\langle X^2_0,1\rangle\sum_{2^{-k}\le 2\eta}2^{-kp/2}$$
$$\le c_4(p,\sigma)\delta^{-1}\langle X^1_0,1\rangle\langle X^2_0,1\rangle\eta^{p/2}.$$

(112) therefore implies (a). To derive (b), take $\eta=1$ in (a) and note that $E(\langle{}^\varepsilon X^1_\delta,1\rangle\,\langle{}^\varepsilon X^2_\delta,1\rangle)=\langle X^1_0,1\rangle\langle X^2_0,1\rangle$ by Proposition 15 (b) . ∎

Notation 37. *Let $c_{37}(\sigma^2)=c_8\sigma^{-2}$. Then Lemma 8 implies*

$${}^\varepsilon p_t(x)\le c_{37}t^{-1}\quad\forall\,\varepsilon>0,t>0\ \ x\in\varepsilon\mathbb{Z}^2. \tag{113}$$

Let $U_n=T_n-T_{n-1}$ $(n\ge 1)$ be the inter-jump times for the dual process $(\mathbf{V}^\varepsilon_t,I_t)$.

Lemma 38. *Let $\phi_0\in C^+_b(\mathbb{R}^8)$, $I_0\subset\{1,2,3,4\}$ and $n_0\in\mathbb{Z}_+$. Assume there are distinct random indices $\{i_1,i_2\}\subset\{1,2,3,4\}$ and a measurable map $f:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ such that $t\to f(t,\omega)$ is continuous $\hat P^\varepsilon_{\phi_0,I_0}-$a.s. and*

$$\phi^\varepsilon_t(y_1,y_2,y_3,y_4)\le f(t,\omega)^\varepsilon p_{2(t-T_{n_0})}(y_{i_1}-y_{i_2}),\ \ i_1\in I_t,i_2\notin I_t$$
$$\text{for } T_{n_0}\le t<T_{n_0+1},\ \ \hat P^\varepsilon_{\phi_0,I_0}\ \text{a.s.} \tag{114}$$

Let

$$\rho^f_{n_0}(s)=\begin{cases} f(T_{n_0+1})\left(\prod_{k=n_0+2}^n\left(\frac{c_{37}}{U_{k-1}+U_k}\right)\right)\frac{c_{37}}{U_n+s-T_n} & \text{if } T_n\le s<T_{n+1},\ n>n_0\\ f(s) & \text{if } T_{n_0}\le s<T_{n_0+1}.\end{cases}$$

Then there are random indices $\{i_1^n,i_2^n:n\ge n_0\}\subset\{1,2,3,4\}$ *such that*

$$\phi^\varepsilon_s(y)\le\rho^f_{n_0}(s)^\varepsilon p_{2(s-T_n)}(y_{i_1^n}-y_{i_2^n}),\quad i_1^n\in I_s,\ i_2^n\in I_s^c,\ T_n\le s<T_{n+1},\ \forall\, n\ge n_0,$$
$$\hat P^\varepsilon_{\phi_0,I_0}-a.s.$$

Proof. We proceed by induction on $n\ge n_0$. If $n=n_0$, the required result is our hypothesis (114). Assume the result holds for $n-1$ ($n-1\ge n_0$) and consider n. Then

$$\phi^\varepsilon_{T_n-}(y)\le\rho^f_{n_0}(T_n-)\,^\varepsilon p_{2U_n}(y_{i_1^{n-1}}-y_{i_2^{n-1}}),\ i_1^{n-1}\in I_{T_n-},\ i_2^{n-1}\notin I_{T_n-}.$$

We consider several cases in analyzing the jump at T_n. We will write (i_1,i_2) for (i_1^{n-1},i_2^{n-1}) and use i_3,i_4 to denote the distinct indices in $\{1,2,3,4\}-\{i_1,i_2\}$.

Case 1. i_1 switches via $i_3\in I_{T_n-}$.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)^\varepsilon p_{2U_n}(y_{i_3}-y_{i_2})^\varepsilon p_0(y_{i_1}-y_{i_3}),\ I_{T_n}\supset\{i_3\},\ I^c_{T_n}\supset\{i_1,i_2\}.$$

Case 2. i_2 switches via $i_3\in I^c_{T_n-}$.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)^\varepsilon p_{2U_n}(y_{i_1}-y_{i_3})^\varepsilon p_0(y_{i_2}-y_{i_3}),\ I_{T_n}\supset\{i_1,i_2\},\ I^c_{T_n}\supset\{i_3\}.$$

Case 3. $i_3\in I_{T_n-}$ switches via i_1.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)^\varepsilon p_{2U_n}(y_{i_1}-y_{i_2})^\varepsilon p_0(y_{i_3}-y_{i_1}),\ I_{T_n}\supset\{i_1\},\ I^c_{T_n}\supset\{i_2,i_3\}.$$

Case 4. $i_3\in I^c_{T_n-}$ switches via i_2.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)^\varepsilon p_{2U_n}(y_{i_1}-y_{i_2})^\varepsilon p_0(y_{i_3}-y_{i_2}),\ I_{T_n}\supset\{i_1,i_3\},\ I^c_{T_n}\supset\{i_2\}.$$

Case 5. $i_3\in I_{T_n-}$ switches via $i_4\in I_{T_n-}$.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)^\varepsilon p_{2U_n}(y_{i_1}-y_{i_2})^\varepsilon p_0(y_{i_3}-y_{i_4}),\ I_{T_n}=\{i_1,i_4\},\ I^c_{T_n}=\{i_2,i_3\}.$$

Case 6. $i_3\in I^c_{T_n-}$ switches via $i_4\in I^c_{T_n-}$.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)^\varepsilon p_{2U_n}(y_{i_1}-y_{i_2})^\varepsilon p_0(y_{i_3}-y_{i_4}),\ I_{T_n}=\{i_1,i_3\},\ I^c_{T_n}=\{i_2,i_4\}.$$

We can now introduce new random indices $\{\hat\imath_j:j\le 4\}=\{1,2,3,4\}$ and reduce these 6 cases to essentially 2 cases.

Case A.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)\,^\varepsilon p_{2U_n}(y_{\hat\imath_1}-y_{\hat\imath_3})\,^\varepsilon p_0(y_{\hat\imath_1}-y_{\hat\imath_2}),\ \hat\imath_1\in I_{T_n},\ \{\hat\imath_2,\hat\imath_3\}\subset I^c_{T_n}$$
$$\text{or } \hat\imath_1\in I^c_{T_n},\ \{\hat\imath_2,\hat\imath_3\}\subset I_{T_n}.$$

Case B.

$$\phi^\varepsilon_{T_n}(y)\le\rho^f_{n_0}(T_n-)\,^\varepsilon p_{2U_n}(y_{\hat\imath_3}-y_{\hat\imath_4})\,^\varepsilon p_0(y_{\hat\imath_1}-y_{\hat\imath_2}),\ I_{T_n}=\{\hat\imath_1,\hat\imath_3\},\quad I^c_{T_n}=\{\hat\imath_2,\hat\imath_4\}$$

For Case A use (113) to see that for $T_n\le t<T_{n+1}$,

$$\begin{aligned}\phi^\varepsilon_t&\le\rho^f_{n_0}(T_n-)\int{}^\varepsilon p_{2U_n+t-T_n}(z_{\hat\imath_1}-y_{\hat\imath_3})\,^\varepsilon p_{t-T_n}(z_{\hat\imath_1}-y_{\hat\imath_2})\,^\varepsilon p_{t-T_n}(z_{\hat\imath_1}-y_{\hat\imath_1})d^\varepsilon z_{\hat\imath_1}\\&\le\rho^f_{n_0}(T_n-)c_{37}(2U_n+t-T_n)^{-1}\,^\varepsilon p_{2(t-T_n)}(y_{\hat\imath_2}-y_{\hat\imath_1})\\&\le\rho^f_{n_0}(t)\,^\varepsilon p_{2(t-T_n)}(y_{\hat\imath_2}-y_{\hat\imath_1}),\end{aligned}$$

where $\hat\imath_1\in I_t,\ \hat\imath_2\in I^c_t$ or conversely.

For Case B we use (113) to see that for $T_n \le t < T_{n+1}$,

$$\begin{aligned}\phi_t^\varepsilon &\le \rho_{n_0}^f(T_n-)\ {}^\varepsilon p_{2U_n+t-T_n}(y_{\hat{\imath}_3} - y_{\hat{\imath}_4})^\varepsilon p_{2(t-T_n)}(y_{\hat{\imath}_2} - y_{\hat{\imath}_1}) \\ &\le \rho_{n_0}^f(T_n-)c_{37}(U_n + t - T_n)^{-1\varepsilon} p_{2(t-T_n)}(y_{\hat{\imath}_2} - y_{\hat{\imath}_1}) \\ &\le \rho_{n_0}^f(t)\ {}^\varepsilon p_{2(t-T_n)}(y_{\hat{\imath}_2} - y_{\hat{\imath}_1}),\end{aligned}$$

where $\hat{\imath}_1 \in I_t$ and $\hat{\imath}_2 \in I_t^c$.

In either case it is clear how to define i_j^n so that the required result holds on $T_n \le t < T_{n+1}$, $\hat{P}^\varepsilon_{\phi_0,T_0}-$a.s. This completes the inductive proof. ■

Notation 39. *Write* $\rho_{n_0}(s)$ *for* $\rho_{n_0}^f(s)$ *when* $f \equiv 1$.

Corollary 40. *Let* $\phi_0 \in C_b^+((\mathbb{R}^2)^4)$ *and* $I_0 \subset \{1,2,3,4\}$. *There are random indices* $\{i_1^n, i_2^n : n \ge 1\} \subset \{1,2,3,4\}$ *such that* $\hat{P}^\varepsilon_{\phi_0,I_0}-$*a.s.* $\forall\ n \ge 1$

$$i_1^n \in I_s,\ \ i_2^n \in I_s^c \quad \text{if}\ \ T_n \le s < T_{n+1}$$

and

(115)
$$\phi_s^\varepsilon(y) \le ||\phi_0||_\infty \left[1(s < T_1) + \sum_{n=1}^\infty 1(T_n \le s < T_{n+1})\rho_1(s)\ {}^\varepsilon p_{2(s-T_n)}(y_{i_1^n} - y_{i_2^n})\right]$$

(116)
$$\le ||\phi_0||_\infty \left[1(s < T_1) + \sum_{n=1}^\infty 1(T_n \le s < T_{n+1})\rho_1(s)\ c_{37}(s - T_n)^{-1}\right]$$
$$\equiv \bar{\phi}(s).$$

Proof. We check (114) of the previous lemma for $n_0 = 1$ and $f \equiv ||\phi_0||_\infty$. Clearly $\phi^\varepsilon_{T_1} = \lim_{t\uparrow T_1}\ {}^\varepsilon S^(4)_t\phi_0 \le ||\phi_0||_\infty$. Therefore the definition of ϕ_t^ε shows that $\phi^\varepsilon_{T_1}(y) \le ||\phi_0||_\infty\ {}^\varepsilon p_0(y_{i_1} - y_{i_2})$ for some $i_1 \in I_{T_1}$, $i_2 \notin I_{T_1}$. It follows that for $T_1 \le t < T_2$, $i_1 \in I_t$, $i_2 \notin I_t$, and

$$\phi_t^\varepsilon(y) \le ||\phi_0||_\infty \int {}^\varepsilon p_{s-T_1}(z - y_{i_1})\ {}^\varepsilon p_{s-T_1}(z - y_{i_2}) d^\varepsilon z = ||\phi_0||_\infty\ {}^\varepsilon p_{2(s-T_1)}(y_{i_1} - y_{i_2}).$$

This verifies (114) and (115) follows from Lemma 38, as this inequality is trivial for $s < T_1$. The second inequality is then clear by (113). ■

Lemma 41. *Let* ${}^\varepsilon S_t$ *denote the semigroup of the nearest neighbor continuous time random walk* ξ_t^ε *on* $\varepsilon\mathbb{Z}^d$ *which jumps to a nearest neighbor at rate* $d\varepsilon^{-2}\sigma^2$ *and let* S_t *denote the semigroup of the d-dimensional Brownian motion with variance parameter* σ^2. *Let* $f^\varepsilon : \varepsilon\mathbb{Z}^d \to \mathbb{R}$, $f : \mathbb{R}^d \to \mathbb{R}$ *satisfy* $\sup_{\varepsilon>0}||f^\varepsilon||_\infty < \infty$ *and* $\lim_{\varepsilon\downarrow 0} f^\varepsilon(x_\varepsilon) = f(x)$ *whenever* $\lim_{\varepsilon\downarrow 0} x_\varepsilon = x$ ($x_\varepsilon \in \varepsilon\mathbb{Z}^d$). *Then* $||f||_\infty < \infty$ *and*

$$\lim_{\varepsilon\downarrow 0} {}^\varepsilon S_t f^\varepsilon(x_\varepsilon) = S_t f(x).$$

Proof. The first assertion is obvious. Let $\varepsilon_n \downarrow 0$. By Skorohod's theorem we may assume $\xi_t^{\varepsilon_n} \to B_t$ a.s. where $\xi_0^{\varepsilon_n} = B_0 = 0$. Then $x_{\varepsilon_n} + \xi_t^{\varepsilon_n} \to x + B_t$ a.s. and the result follows by Dominated Convergence. ■

Notation 42. *If* $x \in \mathbb{R}$, *let* $[x]_\varepsilon = [\varepsilon^{-1}x]\varepsilon$ *for each* $\varepsilon > 0$. *If* $x = (x_1, \ldots, x_d)$, *let* $[x]_\varepsilon = ([x_1]_\varepsilon, \ldots, [x_d]_\varepsilon) \in (\varepsilon\mathbb{Z})^d$.

Lemma 43. *If $\phi_0 \in C_b^+(\mathbb{R}^8)$ and $I_0 \subset \{1,2,3,4\}$, then for each $t > 0$,*
(a) $\sup_{\varepsilon>0} \sup_{x\in\varepsilon\mathbb{Z}^8} \phi_t^\varepsilon(x) < \infty \;\; \hat{P}_{\phi_0,I_0}-a.s.$
(b) $\lim_{\varepsilon\downarrow 0} \sup_{x\in\varepsilon\mathbb{Z}^8,|x|\le K} \;|\phi_t^\varepsilon(x) - \phi_t(x)| = 0 \;\forall\; K > 0 \;\; \hat{P}_{\phi_0,I_0}-a.s.$

Proof. (a) follows from Corollary 40 since $\bar{\phi}(t) < \infty$ for $t \notin \{T_n : n \ge 1\}$ which holds $\hat{P}_{\phi_0,I_0}$–a.s.

For (b), it suffices to show that for a fixed sequence $\varepsilon_k \downarrow 0$,

$$\lim_{k\to\infty} \phi_t^{\varepsilon_k}(x_k) = \phi_t(x) \;\; \hat{P}_{\phi_0,I_0} - \text{a.s. whenever } x_k \in \varepsilon_k\mathbb{Z}^8,\; x\in\mathbb{R}^8 \quad (117)$$

$$\text{are random points satisfying } \lim_{k\to\infty} x_k = x \;\; \hat{P}_{\phi_0,I_0} - \text{a.s.}$$

This in turn will follow by establishing

(118)

(118_n) $\quad (a)(117)$ holds a.s. on $\{\omega : T_n < t < T_{n+1}\}$ for $\{x_k\}, x$ as above, and

(118_n) $\quad (b) \lim_{k\to\infty} \phi_{T_{n+1}-}^{\varepsilon_k}(x_k) = \phi_{T_{n+1}-}(x)$ for $\{x_k\}, x$ as above, for all $n \in \mathbb{Z}_+$.

Clearly (118_n)(a) $\forall\; n \ge 0$ suffices but (b) helps in our inductive proof. On $\{t < T_1\}$, $\phi_t^{\varepsilon_k}(x_k) = \mathbf{S}_t^{\varepsilon_k}\phi_0(x_k)$, $\phi_t(x) = \mathbf{S}_t\phi_0(x)$ and so Lemma 41 implies (118_0)(a). Since $\phi_{T_1-}^{\varepsilon_k}(x_k) = {}^{\varepsilon_k}\mathbf{S}_{T_1}\phi_0(x_k)$ and $\phi_{T_1-}(x_k) = \mathbf{S}_{T_1}(x_k)$, the same result also gives (118_0)(b). Assume (118_m) for $m < n$. Consider $\{T_n < t < T_{n+1}\} \cap \{j$ switches via j' at $T_n\}$. On this event

(119)

$$\phi_t^{\varepsilon_k}(x_k) = \int \phi_{T_n-}^{\varepsilon_k}([\pi_{j,j'}y]_{\varepsilon_k})\, {}^{\varepsilon_k}p_{t-T_n}([y_j]_{\varepsilon_k} - (x_k)_{j'}) \prod_{i\ne j'} {}^{\varepsilon_k}p_{t-T_n}([y_i]_{\varepsilon_k} - (x_k)_i)\, d\hat{y}^{j'}$$

and

$$\phi_t(x) = \int \phi_{T_n-}(\pi_{j,j'}y) p_{t-T_n}(y_j - x_{j'}) \prod_{i\ne j'} p_{t-T_n}(y_i - x_i)\, d\hat{y}^{j'}, \quad (120)$$

where $d\hat{y}^{j'}$ is the 3-dimensional Lebesgue integral with the $y_{j'}$ variable omitted and we write $(x_k)_j$ for the jth component of $x_k \in \varepsilon_k\mathbb{Z}^8$. By (118_{n-1}) (b) and Lemma 8 if $(y_i^k)_{i\ne j'} \to (y_i)_{i\ne j'}$ as $k\to\infty$ when $(y_i^k)_{i\ne j'} \in (\varepsilon_k\mathbb{Z}^2)^3$ and $(y_i)_{i\ne j'} \in (\mathbb{R}^2)^3$, then

$$\lim_{k\to\infty} \phi_{T_n-}^{\varepsilon_k}([\pi_{j,j'}y^k]_{\varepsilon_k})\, {}^{\varepsilon_k}p_{t-T_n}([y_j^k]_{\varepsilon_k} - (x_k)_{j'}) = \phi_{T_n-}(\pi_{j,j'}y)p_{t-T_n}(y_j - x_{j'}).$$

Moreover (a) and Lemma 8(a) show that $\sup_{\varepsilon_k,y} \phi_{T_n-}^{\varepsilon_k}([\pi_{j,j'}y]_{\varepsilon_k})\, {}^{\varepsilon_k}p_{t-T_n}([y_j] - (x_k)_{j'}) < \infty \;\; \hat{P}_{\phi_0,I_0}$–a.s. Now apply Lemma 41 to the 6-dimensional random walks with transition function $\prod_{i\ne j'} {}^{\varepsilon_k}p_{t-T_n}([y_i]_{\varepsilon_k} - (x_k)_i)$ to see that $\lim_{k\to\infty}\phi_t^{\varepsilon_k}(x_k) = \phi_t(x)$ on $\{T_n < t < T_{n+1}\} \cap \{j$ switches via j' at $T_n\}$. The same reasoning also proves (118_n)(b). This completes the induction and hence the proof of (b). ■

Proof of Theorem 32. Use the Markov property of ${}^{\varepsilon_m}\mathbf{X}$ at $t = \delta \ge 0$ and Proposition 27 to see that it suffices to prove

$$\begin{aligned}\lim_{m\to\infty} \hat{E}_{\phi_0,I_0}^{\varepsilon_m} \times E_{{}^{\varepsilon_m}\mathbf{X}_0}^{\varepsilon_m}(F(\phi_{t-\delta}^{\varepsilon_m}, I_{t-\delta}, {}^{\varepsilon_m}\mathbf{X}_\delta)\psi({}^{\varepsilon_m}\mathbf{X}_\delta)\mathcal{E}_{t-\delta}) \\ = \hat{E}_{\phi_0,I_0} \times E_{\mathbf{X}_0}(F(\phi_{t-\delta}, I_{t-\delta}, \mathbf{X}_\delta)\psi(\mathbf{X}_\delta)\mathcal{E}_{t-\delta}),\end{aligned} \quad (121)$$

where

$$\mathcal{E}_{t-\delta} = \exp\left\{\gamma \int_0^{t-\delta} \binom{|I_s|}{2} + \binom{|I_s^c|}{2} ds\right\}.$$

By Skorohod's theorem we may assume $\{^{\varepsilon_m}\mathbf{X}_\delta\}$ and $\mathbf{X}_\delta$ are defined on a common $(\Omega, \mathcal{F}, \mathcal{P})$ such that $^{\varepsilon_m}\mathbf{X}_\delta \to \mathbf{X}_\delta$ $P-$a.s. and replace the expectations $E_{^{\varepsilon_m}\mathbf{X}_0}$ and $E_{\mathbf{X}_0}$ in (121) with E. We now claim that

(122)
$$\lim_{m\to\infty} F(\phi_{t-\delta}^{\varepsilon_m}, I_{t-\delta}, {}^{\varepsilon_m}\mathbf{X}_\delta)\psi({}^{\varepsilon_m}\mathbf{X}_\delta) = F(\phi_{t-\delta}, I_{t-\delta}, \mathbf{X}_\delta)\psi(\mathbf{X}_\delta) \;\; \hat{P}_{\phi_0,I_0} \times P - \text{a.s.}$$

As ψ is continuous we only need focus on the "F terms". Since $^{\varepsilon_m}\mathbf{X}_\delta \to \mathbf{X}_\delta$ in $\mathcal{M}_f^2(\mathbb{R})^2$ a.s., $\{^{\varepsilon_m}\mathbf{X}_\delta : m \in \mathbb{N}\}$ are a.s. tight. This together with Lemma 43(b), the fact that $\phi_{t-\delta} \in C_b^+(\mathbb{R}^8)$ $\hat{P}_{\phi_0,I_0}-$a.s. (recall (100)) and $^{\varepsilon_m}\mathbf{X}_\delta \to \mathbf{X}_\delta$ a.s. allow one to prove (122) by an elementary weak convergence argument.

To prove (121) it now suffices to show

$$\{F(\phi_{t-\delta}^{\varepsilon_m}, I_{t-\delta}, {}^{\varepsilon_m}\mathbf{X}_\delta)\psi({}^{\varepsilon_m}\mathbf{X}_\delta) : m \in \mathbb{N}\} \text{ is uniformly integrable with respect to } \hat{E}_{\phi_0,I_0} \times E. \tag{123}$$

Bound ϕ_0 by $\|\phi\|_\infty$ and hence verify (114) with $n_0 = 1$ and $f = \|\phi_0\|_\infty$ through a short calculation. Lemma 38 shows (recall $\rho_{n_0} = \rho^1_{n_0}$) that if $M = M^{m,\delta}(\omega) = {}^{\varepsilon_m}X^1_\delta(\mathbb{R}^2) + {}^{\varepsilon_m}X^2_\delta(\mathbb{R}^2)$ and $p \in (0, \frac{1}{2})$, then $\hat{P}_{\phi_0,I_0} \times P-$a.s.

(124)
$$\begin{aligned}
&F(\phi_{t-\delta}^{\varepsilon_m}, I_{t-\delta}, {}^{\varepsilon_m}\mathbf{X}_\delta) \\
&\le ||\phi_0||_\infty \Big[1(t-\delta < T_1)(({}^{\varepsilon_m}X^1_\delta(\mathbb{R}^2))^4 + ({}^{\varepsilon_m}X^2_\delta(\mathbb{R}^2))^4) \\
&+ \sum_{n=1}^\infty 1(T_n \le t-\delta < T_{n+1})\rho_1(t-\delta)(2(t-\delta-T_n))^{-p}\mathcal{E}_{\varepsilon_m,p}({}^{\varepsilon_m}\mathbf{X}_\delta) \\
&\qquad\qquad \times (({}^{\varepsilon_m}X^1_\delta(\mathbb{R}^2))^2 + ({}^{\varepsilon_m}X^2_\delta(\mathbb{R}^2))^2)\Big] \\
&\le ||\phi_0||_\infty (M^4 + M^2)\Big[1(t-\delta < T_1) + \Big(1(T_1 \le t-\delta < T_2)(t-\delta-T_1)^{-p} \\
&+ \sum_{n=2}^\infty 1(T_n \le t-\delta < T_{n+1}) c_{37}^{n-1} \left(\prod_{k=3}^n (U_{k-1} + U_k)^{-1}\right)(U_n + t-\delta-T_n)^{-1}(t-\delta-T_n)^{-p}\Big) \\
&\times \; \mathcal{E}_{\varepsilon_m,p}({}^{\varepsilon_m}\mathbf{X}_\delta)\Big].
\end{aligned}$$

Now for $n \geq 2$, either U_n is exponential with rate 2γ and U_{n+1} is exponential with rate 3γ or conversely. Therefore if α_n is the rate of U_n we have

$$\begin{aligned}
&\hat{E}_{\phi_0,I_0}(1(T_n \leq s < T_{n+1}) \left(\prod_{k=3}^{n}(U_{k-1}+U_k)^{-1}\right)(U_n+s-T_n)^{-1}(s-T_n)^{-p}) \\
&\quad = \int_0^\infty du_1 \cdots \int_0^\infty du_n \; 1(\sum_1^n u_i \leq s) e^{-\alpha_{n+1}(s-\sum_1^n u_j)} e^{-\sum_1^n \alpha_j u_j} \\
&\quad \times \prod_1^n \alpha_j \left(\prod_{k=3}^{n}(u_{k-1}+u_k)^{-1}\right)(s-\sum_1^{n-1} u_i)^{-1}(s-\sum_1^n u_i)^{-p} \\
&\quad \leq \alpha_1 6^{\frac{n}{2}} \gamma^{n-1} \int_0^\infty du_1 \cdots \int_0^\infty du_n \; 1(\sum_1^n u_i \leq s) \\
&\quad\quad \times \left(\prod_{k=2}^{n-1}(u_{k+1}+u_k)^{-1}\right)(s-\sum_1^{n-1} u_i)^{-1}(s-\sum_1^n u_i)^{-p}.
\end{aligned} \tag{125}$$

Now change variables and set $v_k = u_{k+1},\ 1 \leq k \leq n-1,\ v_n = \sum_1^n u_i$. Note also $\alpha_1 \leq 6\gamma$ as the largest jump rate occurs when $I_0 = \emptyset$ or $I_0^c = \emptyset$. If $J_n(s,T)$ is as in Corollary 61 in Appendix B, then (125) is at most

$$\begin{aligned}
&\gamma^n 6^{1+\frac{n}{2}} \int_0^s (s-v_n)^{-p} \\
&\quad \cdot \left[\int_{\mathbb{R}_+^{n-1}} 1(\sum_1^{n-1} v_i \leq v_n) \prod_{i=1}^{n-2}(v_i+v_{i+1})^{-1}(s-v_n+v_{n-1})^{-1} dv_1 \ldots dv_{n-1}\right] dv_n \\
&\leq \gamma^n 6^{1+\frac{n}{2}} \int_0^s (s-v_n)^{-p} J_{n-1}(s-v_n, v_n) dv_n \\
&\leq c_{61}\gamma^n 6^{1+\frac{n}{2}} \pi^{n-2} \int_0^s (s-v_n)^{-p}(v_n)^{\frac{1}{2}}(s-v_n)^{-1/2} dv_n,
\end{aligned}$$

where we have used Corollary 61 with $p = \frac{1}{2}$ in the last line. A simple change of variables shows that if we use the above to bound (125) we arrive at

$$\begin{aligned}
&\hat{E}_{\phi_0,I_0}(1(T_n \leq s < T_{n+1}) \left(\prod_{k=3}^{n}(U_{k-1}+U_k)^{-1}\right)(U_n+s-T_n)^{-1}(s-T_n)^{-p}) \\
&\quad \leq c_{61}(1/2)\gamma^n 6^{1+\frac{n}{2}} \pi^{n-2} \int_0^1 w^{\frac{1}{2}}(1-w)^{-\frac{1}{2}-p} dw \; s^{1-p} \\
&\quad \leq c_{126}(p)(\gamma\sqrt{6}\pi)^n s^{1-p}.
\end{aligned} \tag{126}$$

We first establish (123) in case (a). As $\delta = 0$, $M = X_0^1(\mathbb{R}^2) + X_0^2(\mathbb{R}^2)$ is a constant. Lemma 35 and (124) imply that if $p' \in (0,p)$, and

$$\begin{aligned}
W(s) = {} & 1 + 1(T_1 \leq s < T_2)(s-T_1)^{-p} \\
&+ \sum_{n=2}^\infty 1(T_n \leq s < T_{n+1}) c_{37}^{n-1} \prod_{k=3}^{n}(U_{k-1}+U_k)^{-1} \times (U_n+s-T_n)^{-1}(s-T_n)^{-p},
\end{aligned}$$

then

$$F(\phi_t^{\varepsilon_m}, I_t, {}^{\varepsilon_m}\mathbf{X}_0) \le ||\phi_0||_\infty (M^4 + M^2)(1 + c_{35}\mathcal{E}_{p'}(\mathbf{X}_0))W(t). \tag{127}$$

Our assumption on $\gamma\sigma^{-2}$ implies $c_{37}\gamma\sqrt{6}\pi < 1$ and (126) easily implies

$$\hat{E}_{\phi_0 I_0}(W(s)) < \infty \ \ \forall\, s > 0. \tag{128}$$

As the upper bound in (127) is $\hat{P}_{\phi_0, I_0}$–integrable and independent of m, and ψ is bounded, the required uniform integrability in (123) follows and the proof is complete in case (a).

Consider the case (b) and write $(\hat{\omega}, \omega)$ for our sample points under $\hat{P}_{\phi_0,I_0} \times P$. Note that $W(t-\delta) \equiv W(t-\delta, \hat{\omega})$. Our hypothesis on ψ and (124) imply for some $0 < c(\psi) < \infty$,

$$\psi({}^{\varepsilon_m}\mathbf{X}_\delta)F(\phi_{t-\delta}^{\varepsilon_m} I_{t-\delta}, {}^{\varepsilon_m}\mathbf{X}_\delta) \le c(\psi)W(t-\delta, \hat{\omega})(1 + \mathcal{E}_{\varepsilon_m, p}({}^{\varepsilon_m}\mathbf{X}_\delta(\omega)). \tag{129}$$

Fix $\eta > 0$. By Lemma 36 there are random variables $Z(\varepsilon_m, \eta, p, \delta) \equiv Z_m(\omega)$ such that

$$\begin{aligned} \mathcal{E}_{\varepsilon_m,p}({}^{\varepsilon_m}\mathbf{X}_\delta(\omega)) &\le c_{36}\eta^{p-1} X_\delta^1(\mathbb{R}^2) X_\delta^2(\mathbb{R}^2)(\omega) + Z_m(\omega) \\ E(Z_m) &\le c_{36}\delta^{-1}\eta^{p/2} X_0^1(\mathbb{R}^2) X_0^2(\mathbb{R}^2). \end{aligned} \tag{130}$$

By (128) and Proposition 15(b) we may choose $\varepsilon > 0$ so that $\hat{P}_{\phi_0,I_0} \times P(A) < \varepsilon$ implies
$\hat{E}_{\phi_0,I_0} \times E(1_A W(t-\delta)(1 + X_\delta^1(\mathbb{R}^2)X_\delta^2(\mathbb{R}^2))) < \eta(1 + c_{36}\eta^{p-1})^{-1}$. Then (129) and (130) imply for A as above,

$$\begin{aligned} &\hat{E}_{\phi_0,I_0} \times E(1_A F(\phi_{t-\delta}^{\varepsilon_m}, I_{t-\delta}, {}^{\varepsilon_m}\mathbf{X}_\delta)\psi({}^{\varepsilon_m}\mathbf{X}_\delta)) \\ &\le c(\psi)(1 + c_{36}\eta^{p-1})\hat{E}_{\phi_0,I_0} \times E(1_A W(t-\delta)(1 + X_\delta^1(\mathbb{R}^2)X_\delta^2(\mathbb{R}^2)) \\ &\qquad + c(\psi)\hat{E}_{\phi_0,I_0}(W(t-\delta))E(Z_m) \\ &\le c(\psi)\eta + c(\psi)\hat{E}_{\phi_0,I_0}(W(t-\delta))c_{36}\delta^{-1}X_0^1(\mathbb{R}^2)X_0^2(\mathbb{R}^2)\eta^{p/2}. \end{aligned}$$

This goes to zero as $\eta \downarrow 0$, independently of m and so (123) holds and the proof is complete in case (b). ∎

4. Construction of a Solution

In this section we prove Theorem 11(a),(c). Recall that $[x]_\varepsilon = (y_1, y_2) \in \varepsilon\,\mathbb{Z}^2$ iff $x \in \prod_{i=1}^{2}[y_i, y_i + \varepsilon) \equiv C_\varepsilon((y_1, y_2))$. In Section 1.2 we fixed $\mathbf{X}_0 \in \mathcal{M}_{f,e}$ and constructed a solution ${}^\varepsilon\mathbf{X}$ to the approximate martingale problem $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma,\varepsilon}$ starting at

$${}^\varepsilon X_0^i(\{x\}) = X_0^i\left(C_\varepsilon(x)\right), \quad x \in \varepsilon\mathbb{Z}^2.$$

We assume (32) throughout this Section. We use this stronger condition in the proof of a ket L^2 estimate in Proposition 46. The following elementary bound is proved in Appendix A.

Lemma 44. *There is a constant $c_{44} = c_{44}(\sigma^2)$ such that*

$$\int_0^\delta {}^\varepsilon p_s(x)ds \le c_{44}\left[\left(\frac{\sqrt{\delta}}{||x||}\right) \wedge 1 + \log^+\left(\frac{\sqrt{\delta}}{||x||}\right)\right] \quad \forall x \in \varepsilon\mathbb{Z}^2 \quad \forall \delta, \varepsilon > 0.$$

Lemma 45. *(a)* $\lim_{\delta\downarrow 0}\sup_{\varepsilon>0}\int\left[\int_0^\delta {}^\varepsilon p_s(x_1-x_2)ds\right]{}^\varepsilon X_0^1(dx_1){}^\varepsilon X_0^2(dx_2)=0$

(b) $\sup_{\varepsilon>0}\int\left[\int_0^T {}^\varepsilon p_s(x_1-x_2)ds\right]{}^\varepsilon X_0^1(dx_1){}^\varepsilon X_0^2(dx_2)=c_{45}(T)<\infty\quad\forall T>0.$

Proof. Define

$$G_\varepsilon(\mathbf{X}_0)=\int 1(\|x_1-x_2\|<\sqrt{2}\varepsilon)g(x_1-x_2)X_0^1(dx_1)X_0^2(dx_2).$$

Note that if $[x_1]_\varepsilon\neq[x_2]_\varepsilon$, then $\|[x_1]_\varepsilon-[x_2]_\varepsilon\|\geq\varepsilon$ and so

$$\frac{\|[x_1]_\varepsilon-[x_2]_\varepsilon\|}{\|x_1-x_2\|}\geq\frac{\|[x_1]_\varepsilon-[x_2]_\varepsilon\|}{\|[x_1]_\varepsilon-[x_2]_\varepsilon\|+2\sqrt{2}\varepsilon}\geq(1+2\sqrt{2})^{-1}\equiv c_0. \tag{131}$$

We have ${}^\varepsilon p_s(0)\leq c_1(s^{-1}\wedge\varepsilon^{-2})$ by Lemma 8, and so by Lemma 44,

$$\begin{aligned}
&\iint\left[\int_0^\delta {}^\varepsilon p_s(x_1-x_2)ds\right]{}^\varepsilon X_0^1(dx_1){}^\varepsilon X_0^2(dx_2)\\
&=\iint {}^\varepsilon p_s([x_1]_\varepsilon-[x_2]_\varepsilon)X_0^1(dx_1)X_0^2(dx_2)\\
&\leq\iint\left[\int_0^\delta c_1(s^{-1}\wedge\varepsilon^{-2})ds\right]1([x_1]_\varepsilon=[x_2]_\varepsilon)X_0^1(dx_1)X_0^2(dx_2)\\
&+\iint c_{44}\left[1\wedge\left(\frac{\sqrt{\delta}}{||[x_1]_\varepsilon-[x_2]_\varepsilon||}\right)+\log^+\left(\frac{\sqrt{\delta}}{||[x_1]_\varepsilon-[x_2]_\varepsilon||}\right)\right]\\
&\times 1([x_1]_\varepsilon\neq[x_2]_\varepsilon)X_0^1(dx_1)X_0^2(dx_2)\\
&\leq c_1\left(\frac{\varepsilon^2\wedge\delta}{\varepsilon^2}+\log^+\frac{\delta}{\varepsilon^2}\right)\left(1+\log^+\frac{1}{\sqrt{2}\varepsilon}\right)^{-1}G_\varepsilon(\mathbf{X}_0)\\
&+c_{44}\iint\left[1\wedge\left(\frac{\sqrt{\delta}}{c_0||x_1-x_2||}\right)+\log^+\left(\frac{\sqrt{\delta}}{c_0||x_1-x_2||}\right)\right]dX_0^1(x_1)dX_0^2(x_2).
\end{aligned}\tag{132}$$

We have used (131) in the last line. The second term approaches 0 as $\delta\downarrow 0$ by Dominated Convergence since $\mathbf{X}_0\in\mathcal{M}_{f,e}$. This also implies $\lim_{\varepsilon\downarrow 0}G_\varepsilon(\mathbf{X}_0)=0$ and so the first term in (132) clearly approaches 0 uniformly in $\delta\in(0,1]$ as $\varepsilon\downarrow 0$. As $G_\varepsilon(\mathbf{X}_0)$ is uniformly bounded in ε, it then follows easily that the first term in (132) approaches 0 uniformly in $\varepsilon>0$ as $\delta\downarrow 0$. This proves (a). (b) is immediate from (a). ■

Tightness of ${}^\varepsilon\mathbf{X}$ will be proved using bounds on its moments. First and second moments for $\varepsilon=1$ are easy to derive from (18) and were given in Theorem 2.2 (b) (iii) of [DP98]. Using our definition of ${}^\varepsilon X_t^i$, we then easily get for $\phi_i:\varepsilon\mathbb{Z}^2\to\mathbb{R}_+$, $i=1,2$,

$$\begin{aligned}
&(i)\ E(\langle{}^\varepsilon X_t^i,\phi_i\rangle)=\langle{}^\varepsilon X_0^i,{}^\varepsilon S_t\phi_i\rangle\\
&(ii)\ E(\langle{}^\varepsilon X_t^1,\phi_1\rangle\langle{}^\varepsilon X_t^2,\phi_2\rangle)=\langle{}^\varepsilon X_0^1,{}^\varepsilon S_t\phi_1\rangle\langle{}^\varepsilon X_0^2,{}^\varepsilon S_t\phi_2\rangle
\end{aligned}\tag{133}$$

Our key L^2-bound is on the increments of

$$\langle L^\varepsilon(t),\phi\rangle\equiv\langle L^\varepsilon_{{}^\varepsilon\mathbf{X}}(t),\phi\rangle=\int_0^t\int\phi(x){}^\varepsilon X_s^1(x){}^\varepsilon X_s^2(x)d^\varepsilon x ds.$$

Recall that $d^\varepsilon x$ denotes integration with respect to $\ell^\varepsilon = \sum_{y \in \varepsilon\mathbb{Z}^2} \varepsilon^2 \delta_y$. Recall the notation $\mathcal{E}_{\varepsilon,p}\left({}^\varepsilon\mathbf{X}_0\right)$ from Lemma 34 and let

$$\overline{\mathcal{E}}_{\varepsilon,p}\left({}^\varepsilon\mathbf{X}_0\right) = \mathcal{E}_{\varepsilon,p}\left({}^\varepsilon\mathbf{X}_0\right)\left[\left({}^\varepsilon X_0^1(\mathbb{R}^2)\right)^2 + \left({}^\varepsilon X_0^2(\mathbb{R}^2)\right)^2\right].$$

Proposition 46. *There is an $\varepsilon_0 = \varepsilon_0(\gamma, \sigma^2) > 0$, and for any $T > 0$ there is a $c_{46} = c_{46}(T, \gamma, \sigma^2) > 0$ such that for any bounded Borel $\phi : \mathbb{R}^2 \to \mathbb{R}_+$ and any $0 < \varepsilon < \varepsilon_0$,*

$$E\left(\left(\langle L^\varepsilon(t_2), \phi\rangle - \langle L^\varepsilon(t_1), \phi\rangle\right)^2\right) \le c_{46}\overline{\mathcal{E}}_{\varepsilon,1/2}\left({}^\varepsilon\mathbf{X}_0\right) \quad |t_2 - t_1|^{3/2} t_2^{-1} \|\phi\|_\infty^2 \quad \text{for all } 0 \le t_1 < t_2 \le T.$$

Remark 47. *The power $3/2$ is by no means sharp and can easily be improved to $2 - \delta$ for any $\delta > 0$, at the cost of a stronger assumption on $\gamma\sigma^{-2}$. The factor t_2^{-1} will not pose any problems as t_2 is the greater of the two times.*

The proof will be given at the end of this Section and uses the following bound on a family of iterated integrals for $p = \frac{1}{2}$. We include the more general case here because it will be used in [DFMPX00a] to verify (IntC).

Notation 48. *If $n \in \mathbb{N}^{\ge 2}$, $p \in (0,1)$ and $s_0 > s_1 > 0$, let*

$$K_n^{(p)}(s_0, s_1) = \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \prod_{k=2}^n (s_{k-2} - s_k)^{-1} s_{n-1}^{-1} s_n^{-p}\left(1 + [(s_{n-1} - s_n)/s_n]^{-p}\right).$$

Lemma 49. *Let $p \in (0,1)$ and $c_{49}(p) = 3\pi/\sin(\pi(1-p))$.*
(a) If $\phi_p(x) = x\left(1 + (x-1)^{-p}\right)^{-1} \int_0^1 (x-w)^{-1}\left(w^{-p} + (1-w)^{-p}\right) dw$, $x > 1$, then $\sup_{x>1} \phi_p(x) \le c_{49}(p)$.
(b) $K_n^{(p)}(s_0, s_1) \le c_{49}(p)^{n-1} s_1^{-p} s_0^{-1}\left(1 + (\frac{s_0}{s_1} - 1)^{-p}\right)$ $\forall n \in \mathbb{N}^{\ge 2}$, $s_0 > s_1 > 0$.

Proof. See Appendix B.

Lemma 50. *If $\phi \in C_b(\mathbb{R}^2)$, then $\forall T > 0$*

$$\lim_{\varepsilon \downarrow 0} E\left(\langle L^\varepsilon(T), \phi\rangle\right) = \int_0^T \int S_s X_0^1(x) S_s X_0^2(x) \phi(x) dx ds \in \mathbb{R} \tag{134}$$

Proof. By (133)

$$E\left({}^\varepsilon X_s^1(x) {}^\varepsilon X_s^2(x)\right) = {}^\varepsilon S_s {}^\varepsilon X_0^1(x) {}^\varepsilon S_s {}^\varepsilon X_0^2(x), \tag{135}$$

and therefore

$$E\left(\langle L^\varepsilon(T), \phi\rangle\right) = \int_0^T \int {}^\varepsilon S_s {}^\varepsilon X_0^1(x) {}^\varepsilon S_s {}^\varepsilon X_0^2(x) \phi(x) d^\varepsilon x ds \tag{136}$$

Lemma 45(a) shows that

$$\begin{aligned} &\limsup_{\delta \downarrow 0} \sup_{\varepsilon > 0} \int_0^\delta \int {}^\varepsilon S_s {}^\varepsilon X_0^1(x) {}^\varepsilon S_s {}^\varepsilon X_0^2(x) |\phi(x)| d^\varepsilon x ds \\ &\le \|\phi\|_\infty \limsup_{\delta \downarrow 0} \sup_{\varepsilon > 0} \int \left[\int_0^\delta {}^\varepsilon p_{2s}(y_1 - y_2) ds\right] {}^\varepsilon X_0^1(dy_1) {}^\varepsilon X_0^2(dy_2) \;= 0. \end{aligned} \tag{137}$$

If $\delta > 0$, then

$$\lim_{\varepsilon\downarrow 0}\int_\delta^T \int {}^\varepsilon S_s {}^\varepsilon X_0^1(x) {}^\varepsilon S_s {}^\varepsilon X_0^2(x)\phi(x) d^\varepsilon x ds$$
$$= \lim_{\varepsilon\downarrow 0}\int_\delta^T \int \left[\int {}^\varepsilon p_s([y_1]_\varepsilon - [x]_\varepsilon) {}^\varepsilon p_s([y_2]_\varepsilon - [x]_\varepsilon)\phi([x]_\varepsilon)dx\right] X_0^1(dy_1)X_0^2(dy_2)ds$$
$$= \int_\delta^T \int \left[\int p_s(y_1 - x)p_s(y_2 - x)\phi(x)dx\right] X_0^1(dy_1)X_0^2(dy_2)ds$$
$$= \int_\delta^T \int S_s X_0^1(x) S_s X_0^2(x)\phi(x)dxds, \tag{138}$$

where in the next to last line we used Lemma 8 and Dominated Convergence. Note that the finiteness of the right-hand side of (134) is clear since $\mathbf{X}_0 \in \mathcal{M}_{f,e}$. (136), (137) and (138) now easily give (134). ∎

Proposition 51. *If $\varepsilon_n \downarrow 0$, then $\{({}^{\varepsilon_n}X^1, {}^{\varepsilon_n}X^2, L^{\varepsilon_n}_{{}^{\varepsilon_n}\mathbf{X}}) : n \in \mathbb{N}\}$ is a tight sequence in $C\left(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}^2)^3\right)$.*

Proof. Write $({}^n\mathbf{X}, L^n)$ for $({}^{\varepsilon_n}\mathbf{X}, L^{\varepsilon_n}_{{}^{\varepsilon_n}\mathbf{X}})$. It suffices to show tightness of each of the three coordinates separately ([JS87], p. 317) and to this end we specialize a result of Jakubowski [J86] (see Theorem II.4.1 of [Per00]). To show a sequence of processes, $\{Y^n\}$, with sample paths in $C\left(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}^2)\right)$ is tight it suffices to show:

(i) $\forall \varepsilon, T > 0$ there is a compact set, $K_{T,\varepsilon}$, in $\mathbb{R}^2$ such that
$$\sup_n P\left(\sup_{t\le T} Y_t^n(K^c_{T,s}) > \varepsilon\right) < \varepsilon.$$

(ii) $\forall \phi \in C_b^2(\mathbb{R}^2)$, $\{\langle Y_\cdot^n, \phi\rangle : n \in \mathbb{N}\}$ is tight in $C(\mathbb{R}_+, \mathbb{R})$.

We start by proving (i) for $Y^n = L^n$. Fix $\psi : \mathbb{R}^2 \to [0,1]$ in $C_b^2(\mathbb{R}^2)$ such that $[-1,1]^2 \subset \{\psi = 0\} \subset \{\psi < 1\} \subset [-2,2]^2$ and define $\psi_k(x) = \psi(xk^{-1})$. Lemma 50 implies

$$\lim_{n\to\infty} E\left(\langle L^n(T), \psi_k\rangle\right) = \int_0^T \int S_s X_0^1(x) S_s X_0^2(x)\psi_k(x)dxds < \infty. \tag{139}$$

The right-hand side of (139) approaches 0 as $k \to \infty$ and so it follows from (139) that for any $\eta > 0$ there is a k_0 such that

$$\sup_n E\left(\langle L^n(T), \psi_{k_0}\rangle\right) < \eta. \tag{140}$$

This proves (i) for $Y^n = L^n$ by the monotonicity of $L^n(t)$ in t.

(ii) would be a simple consequence of Proposition 46 and Lemma 35 if $\mathbf{X}_0 \in \mathcal{M}_{f,se}$. To handle $\mathbf{X}_0 \in \mathcal{M}_{f,e}$ we will condition on $\mathcal{F}^\varepsilon_\delta \equiv \sigma({}^\varepsilon\mathbf{X}_s : s \le \delta)$ and use the elementary equivalence between (ii) and

(ii)$_\mathrm{a}$ $\forall\delta > 0\quad \forall\phi \in C_b^2(\mathbb{R}^2), \{\langle Y_\cdot^n, \phi\rangle : n \in \mathbb{N}\}$ is tight in $C\left([\delta,\infty], \mathbb{R}\right)$, and

(ii)$_\mathrm{b}$ $\forall\phi \in C_b^2(\mathbb{R}^2), \{\langle Y_0^n, \phi\rangle : n \in \mathbb{N}\}$ is tight in $\mathbb{R}$ and $\forall\eta > 0$
$$\lim_{\delta\downarrow 0}\sup_n P\left(\sup_{t\le\delta} |\langle Y_t^n, \phi\rangle - \langle Y_0^n, \phi\rangle| > \eta\right) = 0.$$

To verify $(\mathrm{ii})_{\mathrm{a}}$ we may choose $\delta = k_0 2^{-m_0}$ for some $k_0, m_0 \in \mathbb{N}$. For $\phi \in C_b^2(\mathbb{R}^2)$ and $m \geq m_0$, use the Markov property of ${}^n\mathbf{X}$ and Proposition 46 to see that

$$\begin{aligned}
&P\left(\max_{k_0 2^{m-m_0} < k \leq N 2^m} \left\langle L^n(k2^{-m}) - L^n\left((k-1)2^{-m}\right), \phi\right\rangle > 2^{-m/8} \mid \mathcal{F}_\delta^{\varepsilon_n}\right) \\
&\leq \sum_{k=k_0 2^{m-m_0}+1}^{N2^m} 2^{m/4} c_{46}(N) \overline{\mathcal{E}}_{\varepsilon_n,1/2}\left({}^n\mathbf{X}_\delta\right) 2^{-m3/2}\left(k2^{-m} - k_0 2^{-m_0}\right)^{-1} \|\phi\|_\infty^2 \\
&\leq c(N, \|\phi\|_\infty) \overline{\mathcal{E}}_{\varepsilon_n,1/2}\left({}^n\mathbf{X}_\delta\right) 2^{-m/4} \sum_{k=1}^{N2^m} k^{-1} \leq c(N, \|\phi\|_\infty) \overline{\mathcal{E}}_{\varepsilon_n,1/2}\left({}^n\mathbf{X}_\delta\right) 2^{-m/4} m,
\end{aligned}$$

which is summable over m. The standard binary expansion argument of Lévy shows that for some $c_1 > 0$ and any $\eta, M > 0$,

$$\begin{aligned}
&P\left(\sup_{\substack{\delta \leq s < t \leq N \\ t-s<\eta}} \left|\langle L^n(t), \phi\rangle - \langle L^n(s), \phi\rangle\right| |t-s|^{-1/8} > c_1\right) \\
&\leq P\left(\mathcal{E}_{\varepsilon_n,1/2}({}^n\mathbf{X}_\delta) > M\right) + P\left({}^nX_\delta^1(\mathbb{R}^2) + {}^nX_\delta^2(\mathbb{R}^2) > M\right) + c'(N, \|\phi\|_\infty) M^3 \delta_1(\eta),
\end{aligned}$$

where $\lim_{\eta \downarrow 0} \delta_1(\eta) = 0$. Lemma 36(b) and (133)(i) allow us to choose M so that the first two terms are small, uniformly in n. Then choose η small enough to make the last term small. This proves $(\mathrm{ii})_{\mathrm{a}}$. $(\mathrm{ii})_{\mathrm{b}}$ is immediate from (136) and (137) which imply

$$\lim_{\delta \downarrow 0} \sup_n E\left(\langle L^n(\delta), |\phi|\rangle\right) = 0. \tag{141}$$

This proves the tightness of $\{L^n(\cdot) : n \in \mathbb{N}\}$ in $C\left(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}^2)\right)$.

Next consider (i) for $Y^n = {}^nX^i$. If ψ_k is as above, then a second order Taylor expansion shows that

$$|{}^\varepsilon\Delta\psi_k(x)| \leq c_\psi k^{-2}. \tag{142}$$

Let $\eta > 0$. The definition of ${}^nX_0^i$ shows we may choose k_0 so that

$$\sup_n \left\langle {}^nX_0^i, \psi_k\right\rangle < \eta \quad \forall k \geq k_0, \tag{143}$$

and (140) holds but with η^3 in place of η. Let $k \geq k_0$. Then $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma,\varepsilon_n}$ implies

$$\sup_{t \leq T} {}^nX_t^i(\psi_k) \leq \eta + (c_\psi \sigma^2/2) k^{-2} \int_0^T {}^nX_s^i(\mathbb{R}^2) ds + \sup_{t \leq T} \left|{}^{\varepsilon_n}M_t^i(\psi_k)\right|.$$

Therefore for $k \geq k_0$,

$$\begin{aligned}
P\left(\sup_{t \leq T} {}^nX_t^i(\psi_k) > 3\eta\right) &\leq \frac{c_\psi \sigma^2}{2\eta k^2} \int_0^T E\left({}^nX_s^i(\mathbb{R}^2)\right) ds + \eta^{-2} E\left(\sup_{t \leq T} \left|{}^{\varepsilon_n}M_t^i(\psi_k)\right|^2\right) \\
&\leq \frac{c_\psi \sigma^2 T X_0^i(\mathbb{R}^2)}{2\eta k^2} + c\gamma\eta,
\end{aligned}$$

where in the last line we have used (133)(i), Burkholder's inequality, and (140) (with η^3 in place of η). Take k larger still to ensure the above bound is at most $c'\eta$, thus verifying (i) for $Y^n = {}^nX^i$.

Let $\phi \in C_b^2(\mathbb{R}^2)$ and consider (ii)$_{\rm a}$ for $Y^n = {}^nX^i$. A second order Taylor approximation shows that

$$\left|\frac{\sigma^{2\ \varepsilon_n}\Delta\phi(x)}{2}\right| \le c_\phi \quad \text{all} \quad x, n. \tag{144}$$

Use the Markov property of ${}^{\varepsilon_n}\mathbf{X}$ together with $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma,\varepsilon_n}$ and Burkholder's inequality to see that for

$$\begin{aligned}
E\left(\left({}^nX_s^i(\mathbb{R}^2)^4\right) \mid \mathcal{F}_\delta^{\varepsilon_n}\right)(\omega) &= E\left(\left({}^nX_s^i(\mathbb{R}^2)^4\right) \mid {}^nX_0^i = {}^nX_\delta^i(\omega)\right) \\
&\le c\left[{}^nX_\delta^i(\mathbb{R}^2)(\omega)^4 + \gamma^2 E\left(\langle L^n(s-\delta), 1\rangle^2 \mid {}^nX_0^i = {}^nX_\delta^i(\omega)\right)\right] \\
&\le c\left[{}^nX_\delta^i(\mathbb{R}^2)(\omega)^4 + \gamma^2 c_{46}(T)\overline{\mathcal{E}}_{\varepsilon_n,1/2}({}^n\mathbf{X}_\delta(\omega))T^{1/2}\right],
\end{aligned} \tag{145}$$

where Proposition 46 is used in the last line. Now use (144) and (145) in $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma,\varepsilon_n}$ to conclude that for $0 < \delta \le t_1 < t_2 \le T$,

$$\begin{aligned}
&E\left(\left(\langle {}^nX_{t_2}^i, \phi\rangle - \langle {}^nX_{t_1}^i, \phi\rangle\right)^4 \mid \mathcal{F}_\delta^{\varepsilon_n}\right) \\
&\le c\left[E\left(\left[\int_{t_1}^{t_2} \langle {}^nX_s^i, \frac{\sigma^{2\ \varepsilon_n}\Delta\phi}{2}\rangle ds\right]^4 \mid \mathcal{F}_\delta^{\varepsilon_n}\right) + E\left(\left[{}^{\varepsilon_n}M_{t_2}^i(\phi) - {}^{\varepsilon_n}M_{t_1}^i(\phi)\right]^4 \mid \mathcal{F}_\delta^{\varepsilon_n}\right)\right] \\
&\le c\left[c_\phi^4(t_2-t_1)^3\int_{t_1}^{t_2} E\left({}^nX_s^i(\mathbb{R}^2)^4 \mid \mathcal{F}_\delta^{\varepsilon_n}\right) ds + c\gamma^2 E\left((\langle L^n(t_2), \phi\rangle - \langle L^n(t_1), \phi\rangle)^2 \mid \mathcal{F}_\delta^{\varepsilon_n}\right)\right] \\
&\le c(T, \phi, \gamma, \sigma^2)\left[{}^nX_\delta^i(\mathbb{R}^2)^4(t_2-t_1)^3 + \overline{\mathcal{E}}_{\varepsilon_n,1/2}({}^n\mathbf{X}_\delta)\,(t_2-t_1)^{3/2}(t_2-\delta)^{-1}\right],
\end{aligned}$$

by Proposition 46. Lemma 36(b) and the fact that $E\left({}^nX_\delta^i(\mathbb{R}^2)\right) = X_0^i(\mathbb{R}^2)$ (from (133)(i)) show that ${}^nX_\delta^i(\mathbb{R}^2)^4 + \overline{\mathcal{E}}_{\varepsilon_n,1/2}({}^nX_\delta^i)$ remains bounded in probability as $n \to \infty$. We can therefore argue just as for L^n, using the above conditional L^4 bound, to see that (ii)$_{\rm a}$ holds for $Y^n = {}^nX^i$.

To check (ii)$_{\rm b}$, let $\delta \in (0,1]$ and use $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma,\varepsilon_n}$ to see

$$\begin{aligned}
&E\left(\sup_{t\le\delta}\left(\langle {}^nX_t^i, \phi\rangle - \langle {}^nX_0^i, \phi\rangle\right)^2\right) \\
&\le 2E\left(\left(\int_0^\delta \langle {}^nX_s^i, \frac{\sigma^{2\ \varepsilon_n}\Delta\phi}{2}\rangle ds\right)^2\right) + 2E\left(\sup_{s\le\delta}\left({}^{\varepsilon_n}M_s^i(\phi)\right)^2\right) \\
&\le 2c_\phi^2\delta\int_0^\delta E\left(\langle {}^nX_s^i, 1\rangle^2\right) ds + c\gamma\|\phi\|_\infty^2 E\left(\langle L^n(\delta), 1\rangle\right) \\
&\le 2c_\phi^2\delta\left[2\langle X_0^i, 1\rangle^2 + 2\delta E\left(\langle L^n(\delta), 1\rangle\right)\right] + c\gamma\|\phi\|_\infty^2 E\left(\langle L^n(\delta), 1\rangle\right).
\end{aligned}$$

The above bound converges to zero uniformly in n as $\delta \downarrow 0$ by (141). Since $\langle {}^nX_0^i, \phi\rangle \to \langle X_0^i, \phi\rangle$ as $n \to \infty$, (ii)$_{\rm b}$ follows for $Y^n = {}^nX^i$ and we are done. ■

To show the limit points obtained from Proposition 51 solve $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ we first introduce some notation:

$$\begin{aligned}
\Omega_{X,L} = &C\left(\mathbb{R}_+, \mathcal{M}_f(\mathbb{R}^2)^3\right) \text{ with its Borel sigma-field } \mathcal{F}_{X,L} \\
&\text{and canonical right-continuous filtration } (\mathcal{F}_t^{X,L}).
\end{aligned}$$

Let $(\mathbf{X}, L) = (X^1, X^2, L)$ denote the coordinate maps on $\Omega_{X,L}$.

Proposition 52. *Let P be a weak limit point of the laws of $\{({}^{\varepsilon_n}\mathbf{X}, L^{\varepsilon_n}_{\varepsilon_n\mathbf{X}}) : n \in \mathbb{N}\}$ on $(\Omega_{X,L}, \mathcal{F}_{X,L})$, as $\varepsilon_n \downarrow 0$. Let $\mathcal{F}$ and $\mathcal{F}_t$ be the P-completions of $\mathcal{F}_{X,L}$ and $\mathcal{F}^{X,L}_t$, respectively. Then $\mathbf{X}$ solves $(\mathbf{MP})^{\sigma^2,\gamma}_{\mathbf{X}_0}$ on $(\Omega_{X,L}, \mathcal{F}, \mathcal{F}_t, P)$ and $L = L_{\mathbf{X}}$ is the collision local time of $\mathbf{X}$ P-a.s. Moreover*

$$(146) \qquad \langle L^{*,\delta}_{\mathbf{X}}(t), \phi\rangle \to \langle L_{\mathbf{X}}(t), \phi\rangle \quad in \quad L^1(P) \quad as \quad \delta \downarrow 0 \quad \forall \phi \in C_b(\mathbb{R}^2)$$

where $L^{*,\delta}_{\mathbf{X}}$ was defined in (1).

Proof. By Skorohod's theorem we may assume that on some $(\Omega', \mathcal{F}', P')$

$$(147) \qquad ({}^n\mathbf{X}, L^n) \equiv ({}^{\varepsilon_n}\mathbf{X}, L^{\varepsilon_n}_{\varepsilon_n\mathbf{X}}) \xrightarrow{\text{a.s.}} (\mathbf{X}, L) \quad \text{in} \quad \Omega_{X,L}, \quad \varepsilon_n \downarrow 0.$$

Let $\mathcal{F}'_t$ (respectively, $\mathcal{F}^n_t$) be the right-continuous P'-complete filtration generated by $(\mathbf{X}, L)$ (respectively, ${}^n\mathbf{X}$) and let $\phi \in C^2_b(\mathbb{R}^2)$. An elementary argument shows that

$$(148) \qquad {}^{\varepsilon_n}\Delta\phi([x]_{\varepsilon_n}) \to \Delta\phi(x) \quad \text{boundedly and uniformly on compacts.}$$

¿From $(MP)^{\sigma,\gamma,\varepsilon_n}_{\mathbf{X}_0}$ we have

$$(149) \qquad \begin{aligned} &\langle {}^nX^i_t, \phi\rangle = \langle {}^nX^i_0, \phi\rangle + \int_0^t \langle {}^nX^i_s, \frac{\sigma^2\, {}^{\varepsilon_n}\Delta\phi}{2}\rangle ds + {}^{\varepsilon_n}M^i_t(\phi), \quad t \geq 0 \\ &{}^{\varepsilon_n}M^i_t(\phi) \quad \text{is a continuous } L^2\ (\mathcal{F}^n_t) - \text{martingale}, \ i = 1, 2, \text{ and} \\ &\langle {}^{\varepsilon_n}M^i(\phi), {}^{\varepsilon_n}M^j(\phi)\rangle_t = \delta_{ij}\gamma\langle L^n(t), \phi^2\rangle. \end{aligned}$$

As each of the first three terms in (149) converges a.s. in $C(\mathbb{R}_+, \mathbb{R})$, we see that ${}^{\varepsilon_n}M^i_t(\phi) \to M^i_t(\phi)$ a.s. in $C(\mathbb{R}_+, \mathbb{R})$ for some $\mathcal{F}'_t$-adapted continuous process $M^i_t(\phi)$. Lemma 50, (149) and Burkholder's inequality imply that $\{\sup_{t\leq T} |{}^{\varepsilon_n}M^i_t(\phi)| : n \in \mathbb{N}\}$ is L^2-bounded for each $T > 0$. It follows easily that $M^i_t(\phi)$ is a continuous L^2 $(\mathcal{F}'_t)$-martingale. Theorem VI.6.1(b) of [JS87] implies that $\langle M^i(\phi), M^j(\phi)\rangle_t = \gamma\delta_{ij}\langle L(t), \phi^2\rangle$ $\forall t \geq 0$ a.s. We may now let $n \to \infty$ in (149) to see that for $\phi \in C^2_b(\mathbb{R}^2)$,

$$(150) \qquad \begin{aligned} &\langle X^i_t, \phi\rangle = \langle X^i_0, \phi\rangle + \int_0^t \langle X^i_s, \frac{\sigma^2\Delta\phi}{2}\rangle ds + M^i_t(\phi), \quad i = 1, 2, \\ &M^i_t(\phi) \text{ is a continuous } L^2(\mathcal{F}'_t)\text{-martingale such that} \\ &\langle M^i(\phi), M^j(\phi)\rangle_t = \delta_{ij}\gamma\langle L(t), \phi^2\rangle \quad \forall t \geq 0 \quad \text{a.s.} \end{aligned}$$

By polarization the last equality implies $\langle M^1(\phi_1), M^2(\phi_2)\rangle = 0$ a.s. for all $\phi_1, \phi_2 \in C^2_b(\mathbb{R}^2)$.

To show $\mathbf{X}$ satisfies $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ it remains to prove $L = L_{\mathbf{X}}$, P'-a.s. The Markov property of ${}^n\mathbf{X}$ and (136) imply

$$(151) \qquad \begin{gathered} \text{For } 0 \leq s < t \text{ and } \phi \in C_b(\mathbb{R}^2), \\ E\left(\langle L^n(t) - L^n(s), \phi\rangle \mid \mathcal{F}^n_s\right) = \int_0^{t-s} \left[\int {}^{\varepsilon_n}S_r\, {}^nX^1_s(x)\, {}^{\varepsilon_n}S_r\, {}^nX^2_s(x)\phi(x) d^{\varepsilon_n}x\right] dr \text{ a.s.} \end{gathered}$$

For each $r > 0$, it is straightforward to use Lemma 8, (147) and a Dominated Convergence argument to see that

$$\lim_{n\to\infty} \int {}^{\varepsilon_n} S_r{}^n X_s^1(x) {}^{\varepsilon_n} S_r{}^n X_s^2(x)\phi(x) d^{\varepsilon_n} x = \int S_r X_s^1(x) S_r X_s^2(x)\phi(x)dx \quad (152)$$
$$< \infty \quad \text{a.s.} \quad \forall r > 0.$$

To take an L^1 limit on the right side of (151) we first show

$$f_n(r) = \int {}^{\varepsilon_n} S_r{}^n X_s^1(x) {}^{\varepsilon_n} S_r{}^n X_s^2(x) d^{\varepsilon_n} x \quad \text{is a uniformly integrable}$$
$$\text{sequence on } ([0, t-s] \times \Omega', dr \times P')\,. \quad (153)$$

If $s = 0$ this is an easy consequence of Lemma 45(a), so assume $s > 0$. We see from (152) that

$$\lim_{n\to\infty} f_n(r) = \int S_r X_s^1(x) S_r X_s^2(x) dx \equiv f(r) \quad \text{a.s.} \quad \forall r > 0. \quad (154)$$

Now use (150) just as in the proof of Proposition 15(b) (more specifically, (57)) to conclude that

$$E'\left(S_r X_s^1(x) S_r X_s^2(x)\right) = S_{r+s} X_0^1(x) S_{r+s} X_0^2(x) \quad \forall r > 0, \quad x \in \mathbb{R}^2. \quad (155)$$

From (133) we have

$$\begin{aligned} \lim_{n\to\infty} E'\left(\int_0^{t-s} f_n(r)dr\right) &= \lim_{n\to\infty} \int_0^{t-s} \int {}^{\varepsilon_n} S_{r+s}{}^n X_0^1(x) {}^{\varepsilon_n} S_{r+s}{}^n X_0^2(x) d^{\varepsilon_n} x dr \\ &= \int_0^{t-s} \int S_{r+s} X_0^1(x) S_{r+s} X_0^2(x) dx dr \end{aligned}$$

by Dominated Convergence and $s > 0$, as for (138). This with (154) and (155) shows that $\lim_{n\to\infty} E'\left(\int_0^{t-s} f_n(r)dr\right) = E'\left(\int_0^{t-s} f(r)dr\right)$, which, together with (154), gives (153). Therefore the same uniform integrability holds for

$$\left\{\int {}^{\varepsilon_n} S_r{}^n X_s^1(x) {}^{\varepsilon_n} S_r X_s^2(x)\phi(x) d^{\varepsilon_n} x : n \in \mathbb{N}\right\}.$$

Use this and (152) to let $n \to \infty$ in the right-hand side of (151) and conclude

$$\begin{aligned} E'\left(\langle L^n(t) - L^n(s), \phi\rangle \mid \mathcal{F}_s^n\right) &\xrightarrow{L^1} \int_0^{t-s} \int S_r X_s^1(x) S_r X_s^2(x)\phi(x) dx dr \quad \text{as} \quad n \to \infty \\ &\equiv \Phi(\mathbf{X}_s). \end{aligned} \quad (156)$$

Now let $\psi : \Omega_{X,L} \to \mathbb{R}$ be bounded continuous and $\mathcal{F}_s^{X,L}$-measurable and satisfy $\psi(\mathbf{X}, L) = 0$ if $X_s^1(\mathbb{R}^2) + X_s^2(\mathbb{R}^2) > K$ for some $K > 0$. If $J > 0$, then (156) implies

$$\begin{aligned}
&E'\left(\Phi(\mathbf{X}_s)\psi(\mathbf{X}, L)\right) \\
&= \lim_{n\to\infty} E'\left(E'\left(\langle L^n(t) - L^n(s), \phi\rangle 1\left(|\langle L^n(t) - L^n(s), \phi\rangle| > J\right) |\mathcal{F}_s^n\right) \psi\left({}^n\mathbf{X}, L^n\right)\right) \\
&\quad + E'\left(E'\left(\langle L^n(t) - L^n(s), \phi\rangle 1\left(|\langle L^n(t) - L^n(s), \phi\rangle| \le J\right) |\mathcal{F}_s^n\right) \psi\left({}^n\mathbf{X}, L^n\right)\right) \\
&= \lim_{n\to\infty} T_n^{(1)} + T_n^{(2)}.
\end{aligned} \tag{157}$$

Choose J so that $P'\left(|\langle L^n(t) - L^n(s), \phi\rangle| = J\right) = 0$. By Proposition 46, the Markov property of ${}^n\mathbf{X}$, and our assumption on the support of ψ, if $s > 0$, then

$$\begin{aligned}
|T_n^{(1)}| &\le J^{-1} E'\left(E'\left(\langle L^n(t) - L^n(s), \phi\rangle^2 \mid \mathcal{F}_s^n\right) \psi\left({}^n\mathbf{X}, L^n\right)\right) \\
&\le J^{-1} c_{46}(t) t^{1/2} \|\phi\|_\infty^2 K^2 E'\left(\mathcal{E}_{\varepsilon_n,1/2}\left({}^n\mathbf{X}_s\right)\right) \|\psi\|_\infty \\
&\le c(t,\phi,\psi) c_{36}(1 + s^{-1}) K^4 J^{-1},
\end{aligned} \tag{158}$$

the last by Lemma 36(b). Next use (147), our choice of J and Dominated Convergence to see that

$$\begin{aligned}
\lim_{n\to\infty} T_n^{(2)} &= E'\left(\langle L(t) - L(s), \phi\rangle 1\left(|\langle L(t) - L(s), \phi\rangle| \le J\right) \psi(\mathbf{X}, L)\right) \\
&\to E'\left(\langle L(t) - L(s), \phi\rangle \psi(\mathbf{X}, L)\right) \quad \text{as} \quad J \to \infty.
\end{aligned} \tag{159}$$

The last line is clear from $E'\left(\langle L(t), 1\rangle\right) < \infty$ (by (150)). Use (158) and (159) in (157) and then let $J \to \infty$ to conclude

$$E'\left(\Phi(\mathbf{X}_s)\psi(\mathbf{X}, L)\right) = E'\left(\langle L(t) - L(s), \phi\rangle \psi(\mathbf{X}, L)\right),\ t > s > 0.$$

It follows that for $\phi \in C_b(\mathbb{R}^2)$

$$E'\left(\langle L(t) - L(s), \phi\rangle \mid \mathcal{F}'_s\right) = \int_0^{t-s} \int S_r X_s^1(x) S_r X_s^2(x) \phi(x)\, dx\, dr \quad \text{a.s.} \quad \forall s > 0,$$

and therefore, by the definition of $L_{\mathbf{X}}^{*,\delta}(t)$, that

$$\langle L_{\mathbf{X}}^{*,\delta}(t), \phi\rangle = \int_0^t \frac{1}{\delta} E'\left(\langle L(s+\delta) - L(s), \phi\rangle \mid \mathcal{F}'_s\right) ds \ P' \text{ a.s.} \tag{160}$$

Theorem 37 on p. 126 of [Mey66], and the continuity and integrability of $\langle L(t), \phi\rangle$ yield that the right-hand side of (160) converges in $L^1(P')$ to $\langle L(t), \phi\rangle$ as $\delta \downarrow 0$ for each $t \ge 0$ and $\phi \in C_b(\mathbb{R}^2)$. Therefore $L_{\mathbf{X}}$ exists and equals L a.s., and (146) holds on $(\Omega', \mathcal{F}', P')$. It is now trivial to transfer these results over to the canonical space in Proposition 52. ■

Proof of Theorem 11. (a) and (c) are immediate from Propositions 51, 52 and 24(b), except for the verification of (IntC) and (SIntC), the latter for $\mathbf{X}_0 \in \mathcal{M}_{f,se}$. These are derived in [DFMPX00a] using the dual process from Section 3 and, more specifically, Theorems 53, 54 and Remark 55 below.

(b) is proved in [DFMPX00a].

(d) The first assertion follows by a direct change of variables calculation in $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ and the second assertion then follows from (b). ■

Let $P_{\mathbf{X}_0}$ denote the law on $\Omega_o = C\left(\mathbb{R}_+, \mathcal{M}_f^2(\mathbb{R}^2)\right)$ of the process $\mathbf{X}$ constructed in Proposition 52.

Theorem 53. *Assume (32) and $\mathbf{X}_0 \in \mathcal{M}_{f,e}$. For any $0 < \delta < t$, any bounded continuous $\phi_0 : (\mathbb{R}^2)^4 \to \mathbb{R}_+$, $I_0 \subset \{1, \ldots, 4\}$, and any Borel map $\psi : \mathcal{M}_f^2(\mathbb{R}^2) \to \mathbb{R}_+$,*

$$E_{\mathbf{X}_0}\left(\int \phi_0(x_1,\ldots,x_4)\prod_{i\in I_0} X_t^1(dx_i)\prod_{j\notin I_0} X_t^2(dx_j)\psi(\mathbf{X}_\delta)\right)$$

$$\le \hat{E}_{\phi_0,I_0}\times E_{\mathbf{X}_0}\left(\int \phi_{t-\delta}(x_1,\ldots,x_4)\prod_{i\in I_{t-\delta}} X_\delta^1(dx_i)\prod_{j\notin I_{t-\delta}} X_\delta^2(dx_j)\psi(\mathbf{X}_\delta)\right.$$

$$\left.\times\exp\left\{\gamma\int_0^{t-\delta}\binom{|I_s|}{2}+\binom{|I_s^c|}{2}ds\right\}\right). \tag{161}$$

If, in addition, ψ is bounded and

$$\{\psi \neq 0\} \subset \left\{(\mu^1,\mu^2) : \mu^1(\mathbb{R}^2)+\mu^2(\mathbb{R}^2) \le K\right\} \quad \text{for some} \quad K > 0, \tag{162}$$

then the above expressions are both finite.

Proof. If $\delta > 0$, ψ is bounded, continuous, and satisfies (162), then both the above results are immediate from Theorems 32(b) and 11(c), and Fatou's Lemma. By taking bounded pointwise limits in ψ, these results extend to bounded non-negative Borel ψ satisfying (162). Next, use Monotone Convergence to get the first inequality for all non-negative Borel ψ and $\delta > 0$. ■

Theorem 54. *Assume (32) and $\mathbf{X}_0 \in \mathcal{M}_{f,se}$. For any $t > 0$, any bounded continuous $\phi_0 : (\mathbb{R}^2)^4 \to \mathbb{R}_+$, any $I_0 \subset \{1, \ldots, 4\}$, and any Borel map $\psi : \mathcal{M}_f(\mathbb{R}^2)^2 \to \mathbb{R}_+$,*

$$E_{\mathbf{X}_0}\left(\int \phi_0(x_1,\ldots,x_4)\prod_{i\in I_0} X_t^1(dx_i)\prod_{j\notin I_0} X_t^2(dx_j)\psi(\mathbf{X}_0)\right)$$

$$\le \hat{E}_{\phi_0,I_0}\left(\int \phi_0(x_1,\ldots,x_4)\prod_{i\in I_t} X_\delta^1(dx_i)\prod_{j\notin I_t} X_\delta^2(dx_j)\psi(\mathbf{X}_0)\right.$$

$$\left.\times\exp\left\{\gamma\int_0^{t-\delta}\binom{|I_s|}{2}+\binom{|I_s^c|}{2}ds\right\}\right) < \infty.$$

In particular,

$$E_{\mathbf{X}_0}\left(\sup_{t\le T} X_t^1(\mathbb{R}^2)^4 + X_t^2(\mathbb{R}^2)^4\right) < \infty \quad \textit{for all} \quad T > 0.$$

Proof. The first two inequalities are proved as in Theorem 53 but using Theorem 32 (a) instead of Theorem 32(b) (the proof is simpler as the $\psi(\mathbf{X}_0)$ term is deterministic and hence trivial to include). The last result is obtained by taking $\phi_0 = 1$, $I_0 = \emptyset$ or $I_0^c = \emptyset$, and using the L^4 maximal inequality for martingales. ■

Remark 55. *We will use Theorem 53 in* [DFMPX00a] *to show the solution constructed in Theorem 11(a),(c) satisfies (IntC). Note that without any uniqueness result, the above proof and Propositions 51 and 52 show that any weak limit point of* $\{{}^{\varepsilon_n}\mathbf{X}\}$ *satisfies* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *and the conclusions of Theorems 53 and 54.*

We complete this section with the proof of our key L^2 estimate.

Proof of Proposition 46. Clearly it suffices to consider the case $\phi \equiv 1$. Let $(\mathcal{F}^\varepsilon_t)$ denote the right continuous filtration generated by ${}^\varepsilon\mathbf{X}$. Use the Markov property of ${}^\varepsilon\mathbf{X}$ and (133)(ii) to see that for T fixed and $0 \le t_1 < t_2 \le T$,

$$
\begin{aligned}
&E\left(\langle L^\varepsilon(t_2) - L^\varepsilon(t_1), 1\rangle^2\right)\\
&= 2\int_{t_1}^{t_2} dr \int_{t_1}^{r} dt \int d^\varepsilon y \int d^\varepsilon x\, E\big(E\left({}^\varepsilon X^1_r(x)\,{}^\varepsilon X^2_r(x)|\mathcal{F}^\varepsilon_t\right)\,{}^\varepsilon X^1_t(y)\,{}^\varepsilon X^2_t(y)\big)\\
&= 2\int_{t_1}^{t_2} dr \int_{t_1}^{r} dt \int d^\varepsilon y \int d^\varepsilon x\, E\big({}^\varepsilon S_{r-t}\,{}^\varepsilon X^1_t(x)\,{}^\varepsilon S_{r-t}\,{}^\varepsilon X^2_t(x)\,{}^\varepsilon X^1_t(y)\,{}^\varepsilon X^2_t(y)\big)\\
&= 2\int_{t_1}^{t_2} dr \int_{t_1}^{r} dt\, E\left(\int {}^\varepsilon p_{2(r-t)}(y_1 - y_2)\,{}^\varepsilon p_0(y_3 - y_4)\,{}^\varepsilon X^1_t(dy_1)\,{}^\varepsilon X^2_t(dy_2)\,{}^\varepsilon X^1_t(dy_3)\,{}^\varepsilon X^2_t(dy_4)\right).
\end{aligned}
$$

Let $\phi^{\varepsilon,v}_0(y) = {}^\varepsilon p_v(y_1 - y_2)\,{}^\varepsilon p_0(y_3 - y_4)$ $(v \ge 0)$ and let $(\phi^{\varepsilon,v}_t, I_t)$ denote the dual process in Proposition 27 starting at $(\phi^{\varepsilon,v}_0, I_0 = \{1,3\})$. Then a simple change of variables in the above, together with Proposition 27, implies

$$
\begin{aligned}
E\left(\langle L^\varepsilon(t_2) - L^\varepsilon(t_1), 1\rangle^2\right) &= \int_{t_1}^{t_2} dt \int_0^{2(t_2-t)} dv\, E\left(F(\phi^{\varepsilon,v}_0, I_0, {}^\varepsilon\mathbf{X}_t)\right)\\
&\le e^{3\gamma T}\int_{t_1}^{t_2} dt \int_0^{2(t_2-t)} dv\, \hat{E}^\varepsilon_{\phi^{\varepsilon,v}_0, I_0}\left(F(\phi^{\varepsilon,v}_t, I_t, {}^\varepsilon\mathbf{X}_0)\right). \qquad (163)
\end{aligned}
$$

To bound the expectation in(163) we will use Lemma 38. Note first that

$$
\begin{aligned}
\phi^{\varepsilon,v}_t(y) &= {}^\varepsilon p_{v+2t}(y_1 - y_2)\,{}^\varepsilon p_{2t}(y_3 - y_4) \text{ for } T_0 \le t < T_1 \qquad (164)\\
\phi^{\varepsilon,v}_{T_1-}(y) &= {}^\varepsilon p_{v+2T_1}(y_1 - y_2)\,{}^\varepsilon p_{2T_1}(y_3 - y_4).
\end{aligned}
$$

We now will verify (114) for $n_0 = 1$. Suppose i_1 switches via i_2, where $\{i_1, i_2\} = \{2,4\}$ are distinct random indices. Then $I_{T_1} = \{1,3,i_1\}$ and

$$
\phi^{\varepsilon,v}_{T_1}(y) = {}^\varepsilon p_{v+2T_1}(y_1 - y_{i_2})\,{}^\varepsilon p_{2T_1}(y_3 - y_{i_2})\,{}^\varepsilon p_0(y_{i_1} - y_{i_2}).
$$

It follows that for $T_1 \le t < T_2$

$$
\begin{aligned}
\phi^{\varepsilon,v}_t(y) &= \int {}^\varepsilon p_{v+2T_1+t-T_1}(y_1 - z_{i_2})\,{}^\varepsilon p_{2T_1+t-T_1}(y_3 - z_{i_2})\,{}^\varepsilon p_{t-T_1}(z_{i_2} - y_{i_1})\,{}^\varepsilon p_{t-T_1}(z_{i_2} - y_{i_2})\,d^\varepsilon z_{i_2}\\
&\le \left(\frac{c_{37}}{v + T_1 + t}\right)\left(\frac{c_{37}}{T_1 + t}\right)\,{}^\varepsilon p_{2(t-T_1)}(y_{i_1} - y_{i_2})
\end{aligned}
$$

and $i_1 \in I_t$, $i_2 \notin I_t$. A similar result holds if 1 switches via 3, or conversely, at T_{n_0}. This establishes (114) with

$$
f(t,\omega) = c_{37}^2 (v+t)^{-1} t^{-1}. \qquad (165)
$$

Then according to the definition in Lemma 38, after some algebra

$$\rho(s) \equiv \rho_1^f(s) = \begin{cases} c_{37}^{n+1} \prod_{k=1}^{n-1} (U_k + U_{k+1})^{-1}(v+T_2)^{-1}(s-T_{n-1})^{-1}, & T_n \le s < T_{n+1},\ n \ge 2 \\ c_{37}^2 (v+s)^{-1} s^{-1}, \quad T_1 \le s < T_2 & \end{cases}$$

$$= c_{37}^{n+1} \prod_{k=1}^{n-1} (U_k + U_{k+1})^{-1}(v + T_2 \wedge s)^{-1}(s - T_{n-1})^{-1} \quad T_n \le s < T_{n+1},\ n \ge 1.$$

Extend $\rho(s)$ to $[0, T_1]$ by defining

$$\rho(s) = c_{37}(v+s)^{-1} \quad \text{if} \ \ 0 \le s < T_1.$$

Then Lemma 38, (113) and (164) imply there are random indices$\{i_1^n, i_2^n\} \subset \{1,2,3,4\}$, such that

$$\phi_s^{\varepsilon,v}(y) \le \rho(s)\, {}^{\varepsilon}p_{2(s-T_n)}(y_{i_1^n} - y_{i_2^n}),\ \ i_1^n \in I_s, i_2^n \in I_s^c,\ T_n \le s < T_{n+1},\ n \ge 0,\ \hat{P}_{\phi_0^{\varepsilon,v}, I_0} - \text{a.s.}$$

Therefore, if $\hat{E}^{\varepsilon}$ denotes $\hat{E}^{\varepsilon}_{\phi_0^{\varepsilon,v}, I_0}$ and $N(t) = n$ iff $T_n \le t < T_{n+1}$, then

$$\hat{E}^{\varepsilon}(F(\phi_t^{\varepsilon,v}, I_t, {}^{\varepsilon}\mathbf{X}_0))$$
$$\le \hat{E}^{\varepsilon}\left(\rho(t) \int {}^{\varepsilon}p_{2(t-T_{N(t)})}(y_1 - y_2)\, {}^{\varepsilon}X_0^1(dy_1)\, {}^{\varepsilon}X_0^2(dy_2)\, {}^{\varepsilon}X_0^1(\mathbb{R}^2)\, {}^{\varepsilon}X_0^2(\mathbb{R}^2)\right)$$
$$\le \bar{\mathcal{E}}_{\varepsilon,1/2}({}^{\varepsilon}\mathbf{X}_0)\hat{E}^{\varepsilon}(\rho(t)(t - T_{N(t)})^{-1/2}). \tag{166}$$

Let α_n be the rate of the exponential time U_n. Then $\alpha_{2n} = 3\gamma,\ \alpha_{2n+1} = 2\gamma$. The definition of $\rho(t)$ gives

$$\hat{E}^{\varepsilon}(\rho(t)(t - T_{N(t)})^{-1/2})$$
$$= \hat{E}^{\varepsilon}(1(t \in [0, T_1))c_{37}(v+t)^{-1}t^{-1/2})$$
$$+ \sum_{n=1}^{\infty} \hat{E}^{\varepsilon}(1(T_n \le t < T_{n+1})c_{37}^{n+1} \prod_{k=1}^{n-1}(U_k + U_{k+1})^{-1}(v + T_2 \wedge t)^{-1}(t - T_{n-1})^{-1}(t-T_n)^{-1/2})$$
$$\le c_{37}(v+t)^{-1}t^{-1/2} + \sum_{n=1}^{\infty} c_{37}^{n+1} \prod_{i=1}^{n} \alpha_i \int_{\mathbb{R}_+^n} 1(\sum_1^n u_i \le t) e^{-\alpha_{n+1}(t - \sum_1^n u_i)} e^{-\sum_1^n \alpha_i u_i}$$
$$\times \prod_{i=1}^{n-1}(u_i + u_{i+1})^{-1}(v + (u_1+u_2)\wedge t)^{-1}(t - \sum_1^{n-1} u_i)^{-1}(t - \sum_1^n u_i)^{-1/2} du_1 \dots du_n$$
(167)
$$\le c_{37}(v+t)^{-1}t^{-1/2} + c_{37}e^{3\gamma t}\sum_{n=1}^{\infty}(c_{37}\sqrt{6}\gamma)^n I_n(t,v),$$

where

$$I_n(t,v)$$
$$= \int_{\mathbb{R}_+^n} 1(\sum_1^n u_i \le t) \prod_{i=1}^{n-1}(u_i + u_{i+1})^{-1}(v + (u_1+u_2)\wedge t)^{-1}(t - \sum_1^{n-1} u_i)^{-1}(t - \sum u_i)^{-1/2} du_1 \dots du_n.$$

Let $s_j = t - \sum_{i=1}^j u_i$, $j = 0, 1, \ldots, n$, and recall the notation $K_n^{(p)}(s_0, s)$ in Lemma 49. Then for $n \geq 1$,

$$
\begin{aligned}
&I_n(t,v)\\
&= \int_{\mathbb{R}_+^n} 1(s_n \leq s_{n-1} \leq \cdots \leq s_1 \leq s_0 = t) \prod_{i=1}^n (s_{i-2} - s_i)^{-1} (v + (s_0 - s_1))^{-1} s_{n-1}^{-1} s_n^{-1/2} ds_1 \ldots ds_n\\
&\leq K_{n+1}^{(1/2)}(v + s_0, s_0)\\
&\leq (3\pi)^n t^{-1/2} (v+t)^{-1} (1 + (v/t)^{-1/2}),
\end{aligned}
$$

the last by Lemma 49(b). Our hypothesis (32) on $\gamma\sigma^{-2}$ implies $c_{37} 3\pi\sqrt{6}\gamma < 1$ so we may use the above bound in (167) to conclude that for $t \leq T$,

$$
\begin{aligned}
&\hat{E}^\varepsilon(\rho(t)(t - T_{N(t)})^{-1/2})\\
&\leq c_{37}(v+t)^{-1} t^{-1/2} + c_{37} e^{3\gamma t}(1 - c_{37} 3\pi\sqrt{6}\gamma)^{-1} (v+t)^{-1} t^{-1/2} (1 + (v/t)^{-1/2})\\
&\leq c_1(\gamma, \sigma^2, T)(v+t)^{-1} t^{-1/2} (1 + (v/t)^{-1/2}).
\end{aligned}
$$

Employing this bound in (166) and (163), we get (for $0 \leq t_1 < t_2 \leq T$)

$$
\begin{aligned}
&E\big(\langle L^\varepsilon(t_2) - L^\varepsilon(t_1), 1\rangle^2\big)\\
&\leq c_2(\gamma, \sigma^2, T) \bar{\mathcal{E}}_{\varepsilon,1/2}({}^\varepsilon\mathbf{X}_0) \int_{t_1}^{t_2} dt \int_0^{2(t_2 - t)} dv (v+t)^{-1} (t^{-1/2} + v^{-1/2})\\
&\equiv c_2 \bar{\mathcal{E}}_{\varepsilon,1/2}({}^\varepsilon\mathbf{X}_0) I(t_1, t_2).
\end{aligned}
\tag{168}
$$

Substitute $u = v/t$ for v to see that

$$
\begin{aligned}
I(t_1, t_2) &= \int_{t_1}^{t_2} dt\, t^{-1/2} \int_0^{2(\frac{t_2}{t} - 1)} du \left(\frac{1 + u^{-1/2}}{1 + u} \right)\\
&\leq c_3 \int_{t_1}^{t_2} dt\, t^{-1/2} \left(\left(\frac{t_2}{t} - 1 \right) \wedge 1 \right)^{1/2} dt\\
&= c_3 \int_{t_1 \vee (t_2/2)}^{t_2} t^{-1} (t_2 - t)^{1/2} dt + c_3 \int_{t_1}^{t_1 \vee (t_2/2)} t^{-1/2} dt\\
&\leq \frac{2c_3}{t_2} \int_{t_1}^{t_2} (t_2 - t)^{1/2} dt + c_3 1(t_2 > 2t_1) 2(t_2/2)^{1/2}\\
&\leq \frac{c_4}{t_2} \left((t_2 - t_1)^{3/2} + 1(t_2 > 2t_1) t_2^{3/2} \right)\\
&\leq \frac{c_5}{t_2} (t_2 - t_1)^{3/2},
\end{aligned}
$$

where we use $t_2 - t_1 > t_2/2$ if $t_2 > 2t_1$ in the last line. Use this in (168) to complete the proof. ■

5. Long-term behavior

In this Section we prove Theorem 20. Recall this gives the limiting law of $(X_t^1(\mathbb{R}^2), X_t^2(\mathbb{R}^2))$ as $t \to \infty$. We will adapt the proof of the corresponding result for the lattice case (Theorem 1.2(b) from [DP98]). Assume X_0 is a fixed initial state in $\mathcal{M}_{\mathrm{f,e}}$ and (32) holds throughout this Section. The following third moment bound is simpler than the fourth moment bounds in Section 4 but we include a proof for completeness.

Recall the notation $\mathcal{E}_p(\mathbf{X}_0)$ introduced prior to Lemma 35. We set

$$\bar{\mathcal{E}}_p(\mathbf{X}_0) = \mathcal{E}_p(\mathbf{X}_0)\left[\langle X_0^1, 1\rangle + \langle X_0^2, 1\rangle\right].$$

For those keeping track, in this particular argument (32) could be weakened to $\gamma/\sigma^2 < (c_8\pi)^{-1}$.

Lemma 56. *Assume $X_0 \in \mathcal{M}_{\mathrm{f,se}}$. For any $p' \in (0,1/2)$ there is a $c_{56} = c_{56}(\gamma,\sigma,p')$ so that the law $P_{\mathbf{X}_0}$ in Theorem 11 satisfies*

$$E_{X_0}\int_0^T \int \mathrm{p}_s(x_1,x_2) X_r^1(dx_1) L_{\mathbf{X}}(d[r,x_2]) \ \le\ c_{56}\,\bar{\mathcal{E}}_{p'}(\mathbf{X}_0) s^{-1/2} < \infty \quad \forall T > 0.$$

Proof. Fix $s > 0$. Let ${}^{\varepsilon}X^i$ and $L^{\varepsilon} = {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}$ denote our usual rescalings of the process and its collision local time on $\varepsilon\mathbb{Z}^2$. An application of Fatou's Lemma, Theorem 11 (c), Skorohod's a.s. representation and Lemma 8 show that it suffices to prove that for all $\varepsilon > 0$ sufficiently small,

$$E\int_0^T \int {}^{\varepsilon}\mathrm{p}_s(x_1,x_2)\, {}^{\varepsilon}X_r^1(dx_1) L^{\varepsilon}\left[d(r,x_2)\right] \ \le\ c_{56}\,\bar{\mathcal{E}}_{p'}(\mathbf{X}_0) s^{-1/2}.$$

We calculate the left side using the moment dual process in Proposition 27 with $p = 3$.

Let $T_n = U_1 + \cdots + U_n$ ($T_0 = 0$) be the jump times of the dual process $\big(\phi_t(x_1,x_2,x_3), I_t\big)$ for third order moments with $\phi_0(x_1,x_2,x_3) = {}^{\varepsilon}\mathrm{p}_s(x_1,x_2)\,{}^{\varepsilon}\mathrm{p}_0(x_2,x_3)$ and $I_0 = \{1,2\}$. Then $\{U_i\}$ are i.i.d. exponential with rate γ and Proposition 27 gives

$$\begin{aligned} &E\left(\textstyle\int \phi_0(x_1,x_2,x_3)\,{}^{\varepsilon}X_r^1(dx_1)\,{}^{\varepsilon}X_r^1(dx_2)\,{}^{\varepsilon}X_r^2(dx_3)\right) \\ &= \hat{E}^{\varepsilon}_{\mathbf{V}_0^{\varepsilon}}\big(e^{\gamma r}\textstyle\int \phi_r(x_1,x_2,x_3) \displaystyle\prod_{i\in I_r} {}^{\varepsilon}X_0^1(dx_i) \prod_{j\in I_r^c} {}^{\varepsilon}X_0^2(dx_j)\big). \end{aligned} \tag{169}$$

Recall from Lemma 8 that ${}^{\varepsilon}\mathrm{p}_r \le c_8\sigma^{-2} r^{-1} \equiv c_1 r^{-1}$. We claim that setting $U_0 \equiv s$ for all $n \in \mathbb{Z}_+$,

(170)

$$T_n \le r < T_{n+1} \quad \text{implies}$$

(170_n)

$$\phi_r(x_1,x_2,x_3) \le (c_1)^{n+1} \prod_{\ell=0}^{n-1} (U_\ell + U_{\ell+1})^{-1} (U_n + r - T_n)^{-1}\ {}^{\varepsilon}\mathrm{p}_{2(r-T_n)}(x_i,x_j), \text{ and}$$

$$I_r = \{i,k\} \quad \text{or} \quad I_r = \{i\} \qquad (i,j,k \quad \text{distinct random indices}).$$

Assume (170_n) with, say, $I_{T_{n+1}-} = \{i,k\}$ (a similar argument goes through if $I_{T_{n+1}-} = \{i\}$). Then, if k changes type at T_{n+1},

$$\phi_{T_{n+1}}(x_1,x_2,x_3) \le c_1{}^{n+1} \prod_{\ell=0}^{n} (U_\ell + U_{\ell+1})^{-1}\ {}^{\varepsilon}\mathrm{p}_{2U_{n+1}}(x_i,x_j)\,{}^{\varepsilon}\mathrm{p}_0(x_i,x_k), \quad I_{T_{n+1}} = \{i\}.$$

Therefore if $T_{n+1} \le r < T_{n+2}$,

$$\begin{aligned}
&\phi_r(x_1,x_2,x_3)\\
&\le {c_1}^{n+1}\prod_{\ell=0}^{n}(U_\ell+U_{\ell+1})^{-1}\int {}^{\varepsilon}\mathrm{p}_{2U_{n+1}+(r-T_{n+1})}(x_j,\,y_i)^{\varepsilon}\mathrm{p}_{r-T_{n+1}}(x_k,y_i)^{\varepsilon}\mathrm{p}_{r-T_{n+1}}(x_i,y_i)\,d^{\varepsilon}y_i\\
&\le c_1^{n+2}\prod_{\ell=0}^{n}(U_\ell+U_{\ell+1})^{-1}\big(U_{n+1}+(r-T_{n+1})\big)^{-1}\,{}^{\varepsilon}\mathrm{p}_{2(r-T_{n+1})}(x_i,x_k).
\end{aligned}$$

If i changes type at T_{n+1}, then

$$\phi_{T_{n+1}}(x_1,x_2,x_3) \le c_1^{n+1}\prod_{\ell=0}^{n}\big(U_\ell+U_{\ell+1}\big)^{-1}\,{}^{\varepsilon}\mathrm{p}_{2U_{n+1}}(x_k,x_j)\,{}^{\varepsilon}\mathrm{p}_0(x_i,x_k),\quad I_{T_{n+1}}=\{k\},$$

and so if $T_{n+1} \le r < T_{n+2}$

$$\begin{aligned}
&\phi_r(x_1,x_2,x_3)\\
&\le c_1^{n+1}\prod_{\ell=0}^{n}\big(U_\ell+U_{\ell+1}\big)^{-1}\int {}^{\varepsilon}\mathrm{p}_{2U_{n+1}+(r-T_{n+1})}(x_j,y_k)\,{}^{\varepsilon}\mathrm{p}_{r-T_{n+1}}(x_i,y_k)\,{}^{\varepsilon}\mathrm{p}_{r-T_{n+1}}(x_k,y_k)d^{\varepsilon}y_k\\
&\le c_1^{n+2}\prod_{\ell=0}^{n}\big(U_\ell+U_{\ell+1}\big)^{-1}\big(U_{n+1}+(r-T_{n+1})\big)^{-1}\,{}^{\varepsilon}\mathrm{p}_{2(r-T_{n+1})}(x_i,x_k),
\end{aligned}$$

which gives (170_{n+1}) . Finally if $T_0 \le r < T_1$, $I_r=\{1,2\}$ and

$$\begin{aligned}
\phi_r(x_1,x_2,x_3) &= \int {}^{\varepsilon}\mathrm{p}_{s+r}(x_1,y_2)\,{}^{\varepsilon}\mathrm{p}_r(y_2,x_3)\,{}^{\varepsilon}\mathrm{p}_r(x_2,y_2)d^{\varepsilon}y_2\\
&\le c_1(s+r)^{-1}\,{}^{\varepsilon}\mathrm{p}_{2r}(x_2,x_3)
\end{aligned}$$

and so (170_0) holds. This completes the inductive proof of (170_n) for $n\in\mathbb{Z}_+$.
It follows from (169) and (170_n) that

$$\begin{aligned}
&E_{{}^{\varepsilon}\mathbf{X}_0}\int \phi_0\,{}^{\varepsilon}X_r^1(dx_1)\,{}^{\varepsilon}X_r^1(dx_2)\,{}^{\varepsilon}X_r^2(dx_3)\\
&\le \sum_{n=0}^{\infty}E(1(T_n\le r<T_{n+1})e^{\gamma r}c_1^{n+1}\prod_{\ell=0}^{n-1}(U_\ell+U_{\ell+1})^{-1}(U_n+r-T_n)^{-1}\\
&\qquad\int {}^{\varepsilon}\mathrm{p}_{2(r-T_n)}(x_1,x_2)\,{}^{\varepsilon}X_0^1(dx_1)\,{}^{\varepsilon}X_0^2(dx_2))(\langle X_0^1,1\rangle+\langle X_0^2,1\rangle)\\
&\le \sum_{n=0}^{\infty}(c_1\gamma)^{n+1}\int_{\mathbb{R}_+^n}1\left(\sum_1^n u_i\le r\right)e^{\gamma r}\exp\left(-\gamma\left(r-\sum_1^n u_i\right)\right)\exp\left(-\gamma\sum_1^n u_i\right)\\
&\times(s+u_1)^{-1}\prod_{\ell=1}^{n-1}(u_\ell+u_{\ell+1})^{-1}\left(u_n+r-\sum_1^n u_i\right)^{-1}c_{35}\bar{\mathcal{E}}_{p'}(\mathbf{X}_0)\left(r-\sum_1^n u_i\right)^{-1/2}\,d\vec{u},
\end{aligned}$$

where $0<p'<1/2$ and we have used Lemma 35 in the last line.

Therefore

$$
\begin{aligned}
&E\int_0^T \int {}^\varepsilon\mathrm{p}_s(x_1,x_2)\,{}^\varepsilon X_r^1(dx_1)L^\varepsilon(d[r,x_2])\\
=&E\int_0^T dr\int {}^\varepsilon\mathrm{p}_s(x_1,x_2)\,{}^\varepsilon\mathrm{p}_0(x_2,x_3)\,{}^\varepsilon X_r^1(dx_1)\,{}^\varepsilon X_r^1(dx_2)\,{}^\varepsilon X_r^2(dx_3)\\
\le&\sum_{n=0}^\infty (c_1\gamma)^{n+1}\int_0^T dr\int_{\mathbb{R}_+^n} 1\left(\sum_1^n u_i\le r\right)(s+u_1)^{-1}\prod_{\ell=1}^{n-1}(u_\ell+u_{\ell+1})^{-1}\left(u_n+r-\sum_1^n u_i\right)^{-1}\\
&\quad\times (r-\sum_1^n u_i)^{-1/2}\,d\vec{u}\cdot c_{35}\bar{\mathcal{E}}_{p'}(\mathbf{X}_0)\\
\le& c_{35}\bar{\mathcal{E}}_{p'}(\mathbf{X}_0)\sum_{n=0}^\infty (c_1\gamma)^{n+1}\int_{\mathbb{R}_+^{n+1}} 1\left(\sum_1^{n+1} u_i\le T\right)(s+u_1)^{-1}\prod_{\ell=1}^{n}(u_\ell+u_{\ell+1})^{-1}u_{n+1}^{-1/2}\,d\vec{u}\\
&\le c_{35}\bar{\mathcal{E}}_{p'}(\mathbf{X}_0)\sum_{n=0}^\infty (c_1\gamma\pi)^{n+1}s^{-1/2},
\end{aligned}
$$

the last by Lemma 60 in Appendix B below. Our choice of γ/σ^2 in (32) ensures the series is summable and so the above expected valued is bounded by the required quantity. ■

Proof of Theorem 20. The a.s. convergence of $\mathbf{X}_t(\mathbb{R}^2)$ is immediate from the martingale convergence theorem as $X_t^i(\mathbb{R}^2)$ is a non-negative (hence L^1-bounded) martingale. Since $\mathbf{X}_t(\mathbb{R}^2)$ is a conformal martingale ($X_t^i(\mathbb{R}^2)$ are orthogonal martingales with the same square function) $\mathbf{X}_t(\mathbb{R}^2)=B(A_t)$ for some planar Brownian motion B starting at $\mathbf{X}_0(\mathbb{R}^2)$, where $A_t=L_X(t)(\mathbb{R}^2)$. Clearly $\mathbf{X}_\infty(\mathbb{R}^2)=B(A_\infty)$ where $A_\infty\le\tau_{\mathrm{ex}}$ because $\mathbf{X}_\cdot(\mathbb{R}^2)$ stays in the first quadrant. To complete the proof we need only prove

$$X_\infty^1(1)X_\infty^2(1)=0\quad\text{a.s.,}\tag{171}$$

as this clearly implies $A_\infty=\tau_{\mathrm{ex}}$ a.s.

To prove (171) we may assume $\mathbf{X}_0\in\mathcal{M}_{\mathrm{f,se}}$ by applying the Markov property at a fixed time $\delta>0$ and using Proposition 24(a). Let $\vec{S}_t$ denote the 4-dimensional Brownian semigroup, M^i denote the martingale measures associated with $X^i(i=1,2)$, and $\Pi_t=X_t^1\times X_t^2$ denote the product measure on $\mathbb{R}^4$. We claim that if ϕ is bounded and Borel measurable on $\mathbb{R}^4$, then for each $s>0$, with probability 1,

(172)

$$\Pi_s(\phi)=\langle\Pi_0,\vec{S}_s\phi\rangle+\int_0^s\int \vec{S}_{s-r}\phi(x_1,x_2)\left[X_r^1(dx_1)M^2(dr,dx_2)+X_r^2(dx_2)M^1(dr,dx_1)\right].$$

If $\phi(x_1,x_2)=\phi_1(x_1)\phi_2(x_2)$ for bounded measurable ϕ_i, then this is immediate from Corollary 23 and an integration by parts. The general result follows by passing to the bounded pointwise closure of the linear span of this class. Let $\varepsilon > 0$ and define

$$M_s = \int_0^s \int \mathrm{p}_{\varepsilon+2(s-r)}(x_1, x_2)\big[X_r^1(dx_1)M^2(dr, dx_2) + X_r^2(dx_2)M^1(dr, dx_1)\big].$$

Now let $\phi(x_1, x_2) = \mathrm{p}_\varepsilon(x_1, x_2)$ in (172) to get

$$\int \mathrm{p}_\varepsilon(x_1, x_2)\Pi_s(dx_1, dx_2) = \int \mathrm{p}_{\varepsilon+2s}(x_1, x_2)\Pi_0(dx_1, dx_2) + M_s. \tag{173}$$

Integrate s over $[0, T]$ and use a stochastic Fubini theorem [[Wal86], Theorem 2.6] to conclude

$$\begin{aligned} &\int_0^T \int \mathrm{p}_\varepsilon(x_1, x_2)\Pi_s(dx_1, dx_2)\, ds \\ &= \int_0^T \int \mathrm{p}_{\varepsilon+2s}(x_1, x_2)\Pi_0(dx_1, dx_2) ds + \int_0^T M_s ds \\ &= \int \left[\frac{1}{2}\int_\varepsilon^{\varepsilon+2T} \mathrm{p}_r(x_1, x_2)\, dr\right] \Pi_0(dx_1, dx_2) + \int_0^T \int \left[\frac{1}{2}\int_\varepsilon^{\varepsilon+2(T-r)} \mathrm{p}_u(x_1, x_2)\, du\right] \\ &\qquad \big[X_r^1(dx_1)M^2(dr, dx_2) + X_r^2(dx_2)M^1(dr, dx_1)\big]. \end{aligned} \tag{174}$$

To check the integrability condition required for the stochastic Fubini Theorem, note first that the expression on the left-hand side of (173) is L^2-bounded in s (by Theorem 54 and our assumption that the initial measure is in $\mathcal{M}_{\mathrm{f,se}}$) and the first term on the right-hand side of (173) is bounded. This shows that M_s is also L^2 bounded in s and so $E(\int_0^T \langle M\rangle_s\, ds) < \infty$, which is the required condition in [Wal86].

Let $h_{\delta,T} : \mathbb{R}_+ \to [0, 1]$ be the piecewise linear function satisfying $h_{\delta,T}(0) = h_{\delta,T}(x) = 0$ for all $x \geq 2T + \delta$ and $h_{\delta,T}(r) = 1$ for $r \in [\delta, 2T]$. Let $q_\delta(x, y) = \delta^{-1}\int_0^\delta \mathrm{p}_r(x, y)\, dr$. The left side of (174) equals

$$\int_0^T \int S_{\varepsilon/2}X_s^1(x)S_{\varepsilon/2}X_s^2(x) dx ds \tag{175}$$

by Chapman-Kolmogorov. By Theorem 11(a) (**SIntC**) holds, and this (we do not require the factor $|x - y|^{-1}$ in the definition of H_ε in this application of (**SIntC**)), together with the Cauchy-Schwarz inequality, shows that (175), and so the left side of (174), remains L^2-bounded as $\varepsilon \downarrow 0$. The first term on the right hand side of (174) approaches

$$\frac{1}{2}\int G_{2T}(x_1, x_2)\Pi_0(dx_1, dx_2) < \infty,$$

where $G_{2T}(x, y) = \int_0^{2T} \mathrm{p}_r(x, y)\, dr$ and the above is finite since $\mathbf{X}_0 \in \mathcal{M}_{\mathrm{f,se}}$. This means the stochastic integral on the far right side of (174) is also L^2-bounded as $\varepsilon \downarrow 0$. This allows us to integrate (174) with respect to $\varepsilon \in (0, \delta]$ and again use the stochastic Fubini theorem to see that

$$\begin{aligned}
&\int_0^T \int q_\delta(x_1,x_2)\Pi_s(dx_1,dx_2)\,ds \\
&= \int \frac{1}{2} \int_0^{\delta+2T} \mathrm{p}_r(x_1,x_2)h_{\delta,T}(r)\,dr\Pi_0(dx_1,dx_2) \\
&+ \int_0^T \int \left[\frac{1}{2}\int_0^{\delta+2(T-r)} \mathrm{p}_u(x_1,x_2)h_{\delta,T-r}(u)du\right] \\
&\qquad \left[X_r^1(dx_1)M^2(dr,dx_2)+X_r^2(dx_2)M^1(dr,dx_1)\right]. \qquad (176)
\end{aligned}$$

As $\delta\downarrow 0$, the first term on the right approaches $\frac{1}{2}\int G_{2T}(x_1,y_2))\Pi_0(dx_1,dx_2)<\infty$ by Dominated Convergence (recall that $\mathbf{X}_0\in M_{\mathrm{f,se}}$). The left side converges in L^1 to $L_{\mathbf{X}}(T)(\mathbb{R}^2)$ by (146), and is L^2 bounded as $\delta\downarrow 0$, by (**SIntC**) and the Cauchy-Schwarz inequality (as above), respectively. It follows that the square function of the stochastic integral remains L^1 bounded as $\delta\downarrow 0$ and so by Fatou's lemma

$$\begin{aligned}
E\Bigg(&\int_0^T \int \left[\int G_{2(T-r)}(x_1,x_2)X_r^1(dx_1)\right]^2 L_X(dr,dx_2) \\
&+\int_0^T \int \left[\int G_{2(T-r)}(x_1,x_2)X_r^2(dx_2)\right]^2 L_X(dr,dx_1)\Bigg) \;<\;\infty.
\end{aligned}$$

This and the above L^2-boundedness readily allow us to see that the above integrals are still finite if $G_{2(T-r)}(x_1,x_2)$ is replaced with $\int_0^{\delta+2(T-r)} \mathrm{p}_u(x_1,x_2)\tilde h_{\delta,T-r}(u)\,du$ where $\tilde h_{\delta,T-r}(u)=1$ for $0\le u\le 2(T-r)$ and agrees with $h_{\delta,T-r}$ elsewhere. Therefore we may apply Dominated Convergence to see that the stochastic integral in (176) converges in L^2 and conclude that

$$\begin{aligned}
A_T \equiv{}& L_X(T)(\mathbb{R}^2) = \frac{1}{2}\int G_{2T}(x_1,x_2)\Pi_0(dx_1,dx_2) \\
&+\int_0^T \int \left[\frac{1}{2}G_{2(T-r)}(x_1,x_2)\right]\ [X_r^1(dx_1)M^2(dr,dx_2) \\
&\qquad\qquad +X_r^2(dx_2)M^1(dr,dx_1)] \\
\equiv{}& A_T^1+N_T, \qquad (177)
\end{aligned}$$

where N_T is in L^2.

Choose $M=M(X_0)\in\mathbb{N}$ so that $X_0^i(B(0,M/2))\ge\frac{1}{2}X_0^i(\mathbb{R}^2)$. If $T\ge M^4$ and $G_{2T}(M)=\inf_{|x_1-x|2|<M}G_{2T}(x_1,x_2)$, then

$$A_T^1 \ge \frac{1}{8}G_{2T}(M)X_0^1(1)X_0^2(1)\ge c_1(\log T)X_0^1(1)X_0^2(1) \qquad (178)$$

for some universal constant $c_1(M)$ by an elementary calculation. Let $p > 1$ and set

$$\eta_T^i = \int_0^T \int \left\{ \int \left[\int_0^{T^{-p}} \mathrm{p}_s(x_1, x_2)\, ds \right] X_r^i(dx_1) \right\}^2 L_{\mathbf{X}}(dr, dx_2), \text{ and}$$

$$\delta_T^i = \int_0^{T^{-p}} \log\left(T^{-p}/s\right) \left[\int_0^T \int \mathrm{p}_s(x_1, x_2) X_r^i(dx_1) L_{\mathbf{X}}(dr, dx_2) \right] ds.$$

If $X_T^{i^*}(1) = \sup_{r \le T} X_r^i(1)$, then (for $T \ge 2$),

$$\begin{aligned}
\langle N \rangle_T &= \frac{\gamma}{4} \sum_{i=1,2} \int_0^T \int \left[\int G_{2(T-r)}(x_1, x_2) X_r^i(dx_1) \right]^2 L_{\mathbf{X}}(dr, dx_2) \\
&\le \frac{\gamma}{2} \sum_{i=1,2} \left[\int_0^T \int \left[\int \int_{T^{-p}}^{2T} \mathrm{p}_s(x_1, x_2)\, ds\, X_r^i(dx_1) \right]^2 L_{\mathbf{X}}(dr, dx_2) + \eta_T^i \right] \\
&\le c(p, \gamma, \sigma) \left(\log T\right)^2 \left(\sum_{i=1,2} X_T^{i^*}(1)^2 \right) L_{\mathbf{X}}(T)(\mathbb{R}^2) \\
&+ \sum_{i=1,2} c(\gamma, \sigma) \int_0^T \int \int \left[\int_0^{T^{-p}} \mathrm{p}_{s_1}(x_1, x_2) \int_{s_1}^{T^{-p}} \frac{ds}{s}\, ds_1 \right] X_r^i(dx_1') X_r^i(dx_1) L_{\mathbf{X}}(dr, dx_2) \\
&\le c_2 (\log T)^2 \left[\sum_{I=1,2} X_T^{i^*}(1)^2 \right] A_T + c_2 \sum_{i=1,2} X_T^{i^*}(1) \delta_T^i, \qquad (179)
\end{aligned}$$

recalling $A_T \equiv L_{\mathbf{X}}(T)(\mathbb{R}^2)$.

Lemma 56 shows that if $0 < p' < 1/2$,

$$\begin{aligned}
E\left(\delta_T^i\right) &\le c_{56} \bar{\mathcal{E}}_{p'}(\mathbf{X}_0) \int_0^{T^{-p}} \left(\log(T^{-p}/s)\, s^{-1/2}\, ds \right. \\
&= c_{56} \bar{\mathcal{E}}_{p'}(\mathbf{X}_0) \left[\int_0^1 (\log 1/u)\, u^{-1/2} du \right] T^{-p/2} \\
&\le c_3 \bar{\mathcal{E}}_{p'}(\mathbf{X}_0) T^{-p/2}. \qquad (180)
\end{aligned}$$

Assume $X_0^1(1) X_0^2(1) > 0$ and $T_1(\mathbf{X}_0)$ is chosen large enough so that for $T \ge T_1(\mathbf{X}_0)$,

$$c_1(\log T) X_0^1(1) X_0^2(1) \ge 2 \quad \text{and} \quad T \ge 2 \vee M^4. \qquad (181)$$

Then (177) and (178) imply that for $T \ge T_1(\mathbf{X}_0)$

$$
\begin{aligned}
P\left(A_T \le 1\right) &\le P\left(c_1(\log T)X_0^1(1)X_0^2(1) + N_T \le 1, A_T \le 1\right) \\
&\le P\Bigg(N_T \le -\frac{c_1}{2}(\log T)X_0^1(1)X_0^2(1), A_T \le 1, X_T^{1^*}(1) \vee X_T^{2^*}(1) \le R, \\
&\qquad\qquad \delta_T^1 \vee \delta_T^2 \le R\,(\log T)^2 \Bigg) \\
&\quad + \sum_{i=1,2} P\left(X_T^{i\,*}(1) > R, A_T \le 1\right) + P\left(\delta_T^i > R(\log T)^2\right) \qquad \text{by (181)} \\
&\le P\left(N_T \le -\frac{c_1}{2}(\log T)X_0^1(1)X_0^2(1), \langle N\rangle_T \le 4c_2(\log T)^2R^2\right) \qquad (182) \\
&\quad + \left(\sum_{i=1,2} P_{X_0^i(1)}\left(\sup_{s\le 1} B_s \ge R\right)\right) + 2R^{-1}(\log T)^{-2}c_3\bar{\mathcal{E}}_{p'}(\mathbf{X}_0)T^{-p/2},
\end{aligned}
$$

where in the last line B is a linear Brownian motion starting at x under P_x, and we have used (179), (180), and the Dubins-Schwarz theorem to write $X_t^i(\mathbb{R}^2)$ as $B(A_t)$. Assume

$$
R \ge \max\left(2\left(\langle X_0^1,1\rangle + \langle X_0^2,1\rangle\right),\ \langle X_0^1,1\rangle\langle X_0^2,1\rangle\right),
$$

then an elementary calculation with Brownian motion, again using Dubins-Schwarz (see [DP98, (3.12) and (3.13)]) shows that the first term on the right side of (182) is at most

$$
1 - c_4\langle X_0^1,1\rangle\langle X_0^2,1\rangle R^{-1}
$$

($c_4 > 0$ universal) and the second term is at most

$$
\frac{8}{R}\exp(-R^2/8).
$$

Now set

$$
\begin{aligned}
R &= R\left(\langle X_0^1,1\rangle, \langle X_0^2,1\rangle\right) \\
&= \max\left(2\left(\langle X_0^1,1\rangle + \langle X_0^2,1\rangle\right), \langle X_0^1,1\rangle\langle X_0^2,1\rangle, \left[8\left|\log\left[32\left(c_4\langle X_0^1,1\rangle\langle X_0^2,1\rangle\right)^{-1}\right]\right|\right]^{1/2}\right)
\end{aligned}
$$

and then assume $T \in \mathbb{N}$, in addition to (181), also satisfies $T \ge T_2(\mathbf{X}_0)$ to ensure the last term on the right side of (182) is at most $\frac{c_4}{4}X_0^1(1)X_0^2(1)R^{-1}$. In fact define $T(\mathbf{X}_0)$ to be the smallest such T in $\mathbb{N}$. Set $T \equiv \infty$ if $X_0 \notin M_{\mathrm{f,se}}$ or $X_0^1(1)X_0^2(1) = 0$. Combining the above bounds and using them in (182), we get

$$
\begin{aligned}
P(A_T > 1) &\ge c_4X_0^1(\mathbb{R}^2)X_0^2(\mathbb{R}^2)R^{-1} - \frac{8}{R}e^{-R^2/8} - \frac{c_4}{4}X_0^1(\mathbb{R}^2)X_0^2(\mathbb{R}^2)R^{-1} \\
&\ge \frac{c_4}{4}X_0^1(\mathbb{R}^2)X_0^2(\mathbb{R}^2)R(X_0^1(\mathbb{R}^2), X_0^2(\mathbb{R}^2))^{-1} \text{ (by the choice of } R\text{)} \qquad (183) \\
&\equiv q(X_0^1(\mathbb{R}^2), X_0^2(\mathbb{R}^2)).
\end{aligned}
$$

Set $q(0,x) = q(x,0) = 0$ so that (183) remains valid if $\langle X_0^1, 1\rangle\langle X_0^2, 1\rangle = 0$. Note that

$$\inf\{q(u,v) : u \geq \delta, v \geq \delta\} = \varepsilon(\delta) > 0 \qquad \forall \delta > 0. \tag{184}$$

Inductively define $\mathbb{N}$–valued stopping times by $T_{n+1} = T(X^1_{T_n}, X^2_{T_n}) + T_n \leq \infty$. By the Markov property for X if $\mathcal{F}_t^X = \sigma(\mathbf{X}_r : r \leq t)$,

$$\begin{aligned} &P_{\mathbf{X}_0}\left(A_{T_{n+1}} - A_{T_n} \geq 1 \mid \mathcal{F}^X_{T_n}\right) \\ &= P_{\mathbf{X}_{T_n}}\left(A\left(T_1(\mathbf{X}_0)\right) \geq 1\right) 1(T_n < \infty) \geq q\left(\mathbf{X}_{T_n}(1)\right) 1(T_n < \infty) \quad \text{(by (183))}. \end{aligned}$$

Now use the conditional version of the Borel-Cantelli Lemma and the fact that $\lim_{t\to\infty} A_t = A_\infty < \infty$ a.s. (because $X_t^i(\mathbb{R}^2) \xrightarrow{\text{a.s.}} X_\infty^i(\mathbb{R}^2) < \infty$) as in (3.18) of [DP98] to conclude that

$$\sum_{n=1}^{\infty} q\left(\mathbf{X}_{T_n}(\mathbb{R}^2)\right) 1(T_n < \infty) < \infty \qquad \text{a.s.} \tag{185}$$

If $T_n < \infty$ for all n then (184) and (185) imply $\lim_{n\to\infty} X^1_{T_n}(\mathbb{R}^2) X^2_{T_n}(\mathbb{R}^2) = 0$ a.s. and so $\lim_{t\to\infty} X_t^1(\mathbb{R}^2)X_t^2(\mathbb{R}^2) = 0$ a.s. by martingale convergence. If $T_n = \infty$ for some n, then let n_0 be the first such n. Since $X_k \in M_{\mathrm{f,se}}$ for all $k \in \mathbb{Z}_+$ a.s. by Proposition 24(a), this implies $X^1_{T_{n_0-1}}(\mathbb{R}^2) X^2_{T_{n_0-1}}(\mathbb{R}^2) = 0$ and therefore, $X_t^1(\mathbb{R}^2)X_t^2(\mathbb{R}^2) = 0$ for all $t \geq T_{n_0-1}$. The required result is established in either case. ■

6. Existence of Densities and Segregation of Types

We start with a general result giving the existence of densities for a class of measure-valued martingale problems based on a conformal martingale argument. Write $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ for the space of all Radon measures on $\mathbb{R}^d$ equipped with the topology of vague convergence and let $\mathcal{C}^\infty_{\mathrm{com}}(\mathbb{R}^d)$ be the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

Theorem 57. *Let Q_t denote a Feller semigroup on $\mathbb{R}^d$, let $T > 0$, and assume $\mathbf{X}_t = (X_t^1, X_t^2)$, $0 \leq t \leq T$ is an adapted continuous $\mathcal{M}^2$ valued process on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Suppose that for some $c > 0$, for all non-negative $\varphi^j \in \mathcal{C}^\infty_{\mathrm{com}}(\mathbb{R}^d)$, $j = 1, 2$,*

$$\begin{aligned} &N_t^j(\varphi^j) = \left\langle X_t^j, Q_{T-t}\varphi^j \right\rangle,\ t \leq T,\ j = 1,2 \text{ are orthogonal } \mathcal{F}_t\text{-martingales} \\ &\qquad\qquad \text{whose predictable square functions satisfy} \\ &\langle\!\langle N^1(\varphi^1)\rangle\!\rangle_t = c\langle\!\langle N^2(\varphi^1)\rangle\!\rangle_t, \ ;0 \leq t \leq T. \end{aligned} \tag{186}$$

Then $X_T^j \ll \ell$, P-a.s. for $j = 1, 2$ if and only if $Q_T X_0^j \ll \ell$, P-a.s. for $j = 1, 2$.

Proof. By working with the regular conditional probability for $\mathbf{X}$ given $\mathbf{X}_0$ we may assume that $\mathbf{X}_0$ is deterministic (it suffices to assume the above for a countable supnorm dense set of φ^j's).
Step 1°. First we assume that $c = 1$. Fix a non-negative $\varphi \in \mathcal{C}^\infty_{\mathrm{com}}(\mathbb{R}^d)$. Set $X := X^1 + iX^2$ and $N(\varphi) := N^1(\varphi) + iN^2(\varphi)$. Then $N(\varphi)$ is a conformal

martingale (see, e.g., [RY91, § V.2], V.2), and Itô's lemma shows that the bounded process $t \mapsto \mathrm{e}^{-N_t(\varphi)}$ is a continuous $\mathcal{F}_\cdot$–martingale. We therefore have

$$E\langle X_T\,,\varphi\rangle \;=\; EN_T(\varphi) \;=\; N_0(\varphi) \;=\; \langle X_0\,,Q_T\varphi\rangle \tag{187}$$

and

$$E\mathrm{e}^{-\langle X_T\,,\varphi\rangle} \;=\; E\mathrm{e}^{-N_T(\varphi)} \;=\; \mathrm{e}^{-N_0(\varphi)} \;=\; \mathrm{e}^{-\langle X_0\,,Q_T\varphi\rangle}. \tag{188}$$

Let $\{\varphi_n : n \geq 1\}$ denote a (non-negative) radially symmetric approximate identity (that is approximating the δ_0–function) in $\mathcal{C}^\infty_{\text{com}}(\mathbb{R}^d)$. Set $\varphi^x_n(y) := \varphi_n(y-x)$, $x,y \in \mathbb{R}^d$. Since $B \mapsto \big\langle X_0\,,Q_T(\mathbf{1}_B\varphi)\big\rangle =: \mu(B)$ is a *finite* complex measure, we may apply standard differentiation theory of measures (see e.g. Theorem 8.6 in [Rud74]). From the identity (187) we conclude that

(189)
$$EN_T(\varphi^x_n\varphi) \;=\; \big\langle X_0\,,Q_T(\varphi^x_n\varphi)\big\rangle \underset{n\uparrow\infty}{\longrightarrow} f(x) =: f^1(x) + if^2(x) \quad \text{for } \ell\text{–a.a. } x,$$

where f is the density of the absolutely continuous part of μ :

$$\mu(\cdot) \;=\; \int_{(\,\cdot\,)} \ell(\mathrm{d}x)\; f(x) \;+\; \nu(\cdot), \quad \nu \perp \ell. \tag{190}$$

Note that $f^j \geq 0,\; j = 1,2,$ and

$$\int \ell(\mathrm{d}x)\; f^j(x) \;\leq\; \big\langle X^j_0\,,Q_T\varphi\big\rangle \;=\; N^j_0(\varphi) \;<\; \infty, \tag{191}$$

hence $f^j(x) < \infty$ for ℓ–almost all x. Applying the same argument to the random finite complex measure $B \mapsto \langle X_T\,,\mathbf{1}_B\varphi\rangle$, we see that

$$\langle X_T\,,\varphi^x_n\varphi\rangle \underset{n\uparrow\infty}{\longrightarrow} \eta(x) \;=:\; \eta^1(x) + i\eta^2(x) \quad \text{for } \ell \times P\text{–a.a. } (x,\omega), \tag{192}$$

where η is the density of the absolutely continuous part of $\big\langle X_T\,,\mathbf{1}_{(\,\cdot\,)}\varphi\big\rangle$. Fatou's lemma gives

$$\begin{aligned} E\eta^j(x) \;&\leq\; \liminf_{n\uparrow\infty} E\big\langle X^j_T\,,\varphi^x_n\varphi\big\rangle \;=\; \liminf_{n\uparrow\infty} \big\langle X^j_0\,,Q_T\,(\varphi^x_n\varphi)\,\big\rangle \\ &=\; f^j(x) \;<\; \infty, \quad \text{for } \ell\text{–a.a. } x. \end{aligned} \tag{193}$$

Now (189) shows that for ℓ–a.a. x and for $\theta \geq 0$,

$$\mathrm{e}^{-\theta f(x)} \;=\; \lim_{n\uparrow\infty} \exp\Big[-\theta\big\langle X_0\,,Q_T\,(\varphi^x_n\varphi)\,\big\rangle\Big]\,,$$

which by (188), (192), and bounded convergence, equals

$$\lim_{n\uparrow\infty} E\exp\big[-\theta\langle X_T\,,\varphi^x_n\varphi\rangle\big] \;=\; E\mathrm{e}^{-\theta\eta(x)}.$$

We use the finiteness in (193) to differentiate $P\mathrm{e}^{-\theta\eta(x)}$ with respect to θ at $\theta = 0+$ and conclude

$$E\eta(x) \;=\; f(x) \;<\; \infty, \quad \text{for } \ell\text{–a.a. } x. \tag{194}$$

Step 2°. Assume now that $\mathbf{X}_0 * \mathsf{Q}_T \ll (\ell,\ell)$. Then, by (187) and since $\nu = 0$ in the decomposition (190),

$$E\big\langle X^j_T\,,\varphi\big\rangle \;=\; \big\langle X^j_0\,,Q_T\varphi\big\rangle \;=\; \int \ell(\mathrm{d}x)\; f^j(x) \;=\; E\int \ell(\mathrm{d}x)\; \eta^j(x)$$

where in the last step we used (194). This shows the singular part of $B \mapsto \langle X_T^j, 1_B\varphi\rangle$ is a.s. 0 and as φ is an arbitrary smooth non-negative function with compact support, we may conclude that $X_T^j \ll \ell$ P-a.s.

Step 3°. Conversely, assume that $X_T^j \ll \ell$, P–a.s., $j = 1, 2$. Then, if B is a Lebesgue null set in $\mathbb{R}^2$, we get $X_T^j(B) = 0$, P–a.s., and so

$$\langle X_0^j, Q_T\mathbf{1}_B\rangle \;=\; E\langle X_T^j, \mathbf{1}_B\rangle \;=\; 0.$$

In fact, in the first equality we have extended (187) from $\varphi \in \mathcal{C}^\infty_{\text{com}}$ to bounded measurable φ by a standard monotone class argument.

Step 4°. The result for general c now follows by applying the above to $(c^{-1/2}X^1, X^2)$. ■

Although the above result may appear to be fairly general, a bit of thought will convince the reader that these hypotheses are not readily satisfied. Of course we have just worked rather hard to find at least one case where they are satisfied.

Proof of Theorem 17(a). Corollary 23 shows that the hypothesis of Theorem 57 holds with $Q_t = S_T$, the Brownian semigroup, and $c = 1$. The absolute continuity of the Brownian semigroup and Theorem 57 completes the proof. ■

Remark 58. *Note that the proof of Theorem 17(a) only relied on a result (Corollary 23), which was established for any solution of* $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ *independently of our uniqueness results, and on the general Theorem 57, which is independent of the other results in this paper. This will allow us to use the above existence of densities in the derivation of uniqueness in law and the strong Markov property in* [DFMPX00a].

The proof of the segregation of types result, Theorem 17 (b), will be an adaptation of the method of Cox-Klenke-Perkins [CKP00] which was designed to prove convergence to equilibria from more general initial conditions once it is established from uniform initial measures, and will be used for precisely this purpose in [DEFMPX00b]. Given the close links between the local and longtime behaviors (cf. [DEFMPX00b]), this connection is not surprising.

Proof of Theorem 17(b). (b2) is clearly immediate from (b1).

(i) Assume first that $\mathbf{X}_0 \in \mathcal{M}_{f,se}$. Write $p_{t,x}(y) = p_t(x,y)$, let $a_1,\ a_2 \geq 0$ and set $a = a_1 + a_2$, $b = a_1 - a_2$. We let $x_t = x_t^1 + x_t^2$, $y_t = x_t^1 - x_t^2$, $X_t = X_t^1 + X_t^2$, $Y_t = X_t^1 - X_t^2$, $\tilde{X}_t = \tilde{X}_t^1 + \tilde{X}_t^2$ and $\tilde{Y}_t = \tilde{X}_t^1 - \tilde{X}_t^2$, where $\tilde{X}_t^i$ are the dual solutions in Proposition 13. By this latter result and standard differentiation theory, for ℓ-a.a. x

$$\begin{aligned} E_{\mathbf{X}_0}\left(e^{-aX_t(x)+ibY_t(x)}\right) &= \lim_{\delta\downarrow 0} E\left(e^{-a\langle X_t, p_{\delta,x}\rangle + ib\langle Y_t, p_{\delta,x}\rangle}\right) \\ &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} E_{a_1 p_{\delta,x}, a_2 p_{\delta,x}}\left(e^{-\langle X_0, S_\varepsilon \tilde{X}_t\rangle + i\langle Y_0, S_\varepsilon \tilde{Y}_t\rangle}\right), \end{aligned} \tag{195}$$

where the subscript now denotes the initial densities.

Let $k = \delta^{-1}$, fix x so that (195) holds, let $t > 0$, and note that

$$\langle \tilde{X}_t^{i,k,x}, \phi\rangle = \langle \tilde{X}^i_{t/k}, \phi\left((\cdot - x)\sqrt{k}\right)\rangle, \quad i = 1, 2$$

also defines a solution to $(\mathbf{MP})^{\sigma,x}_{\tilde{\mathbf{X}}^{k,x}_0}$ with initial conditions $\tilde{X}^{i,k,x}_0 = a_i\mu_0$, where μ_0 is the normal law on $\mathbb{R}^2$ with mean zero and covariance matrix $\sigma^2 I$. Now use

$$\begin{aligned}\langle X^i_0, S_\varepsilon \tilde{X}^j_t\rangle &= \int \left[\int p_\varepsilon(zk^{-1/2} - y + x) X^i_0(dy)\right] \tilde{X}^{j,k,x}_{kt}(dz)\\ &= \langle \tilde{X}^{j,k,x}_{kt}, S_\varepsilon X^i_0(\cdot\ k^{-1/2} + x)\rangle\end{aligned}$$

in (195) to see

$$\begin{aligned}(196)\qquad & E_{\mathbf{X}_0}\left(e^{-ax_t(x)+iby_t(x)}\right)\\ &= \lim_{k\to\infty}\lim_{\varepsilon\downarrow 0} E_{a_1\mu_0,a_2\mu_0}\left(\exp\left\{-\left\langle \tilde{X}_{kt}, S_\varepsilon X_0(\cdot k^{-1/2}+x)\right\rangle\right.\right.\\ &\qquad\left.\left. +i\left\langle \tilde{Y}_{kt}, S_\varepsilon Y_0(\cdot k^{-1/2}+x\right\rangle\right\}\right).\end{aligned}$$

Let

$$\Delta^i_{k,\varepsilon}(y) = S_\varepsilon X^i_0\left(yk^{-1/2}+x\right) - S_t X^i_0(x)$$

and note by Corollary 23, under $P_{a_1\mu_0,a_2\mu_0}$,

$$(197)\qquad \langle \tilde{X}^j_{kt}, \Delta^i_{k,\varepsilon}\rangle = a_j\langle \mu_0, S_{kt}\Delta^i_{k,\varepsilon}\rangle + \int_0^{kt}\int_{\mathbb{R}^2} S_{kt-r}\Delta^i_{k,\varepsilon}(y)\, d\tilde{M}^j(r,y)\quad \text{a.s.}$$

Fix $\eta\in(0,t/2)$ and consider $r\in[0,kt)$. Then

$$\begin{aligned}(198)\quad S_{kt-r}\Delta^i_{k,\varepsilon}(z) &= \iint p_{tk-r}(y-z)p_\varepsilon(yk^{-1/2}+x-w)X^i_0(dw)dy - S_tX^i_0(x)\\ &= \int\left[p_{t+\varepsilon-rk^{-1}}(w-x-zk^{-1/2}) - p_t(w-x)\right]X^i_0(dw).\end{aligned}$$

As $\varepsilon\downarrow 0$ and $k\to\infty$, the integrand converges pointwise to 0, and for $r\in[0,k(t-\eta))$ is uniformly bounded by c/η. Therefore

$$\begin{aligned}(199)\qquad & 1(r<kt)S_{kt-r}\Delta^i_{k,\varepsilon} \to 0 \text{ pointwise as } \varepsilon\to 0,\ k\to\infty \text{ and}\\ & \sup\left\{S_{kt-r}\Delta^i_{k,\varepsilon}(z) : r\le k(t-\eta),\ z\in\mathbb{R}^2, k\in\mathbb{N}, \varepsilon>0\right\} \le \frac{c}{\eta}X^i_0(\mathbb{R}^2).\end{aligned}$$

By Dominated Convergence, the first term on the right-hand side of (197) approaches 0 as $\varepsilon\downarrow 0$ and $k\to\infty$. Turning to the second term, let

$$N^{i,j}_{k,\varepsilon}(s) = \int_0^s\int_{\mathbb{R}^2} S_{kt-r}\Delta^i_{k,\varepsilon}(y)d\tilde{M}^j(r,y),\ s\le kt.$$

Then

$$\langle N^{i,j}_{k,\varepsilon}\rangle\left(k(t-\eta)\right) = \gamma\int_0^{k(t-\eta)}\int_{\mathbb{R}^2}\left(S_{kt-r}\Delta^i_{k,\varepsilon}(y)\right)^2 L_{\tilde{\mathbf{X}}}(dr,dy)$$

which approaches 0 a.s. as $\varepsilon\downarrow 0$ and $k\to\infty$ by (199), Dominated Convergence, and the fact that $L_{\tilde{\mathbf{X}}}(t,\mathbb{R}^2)\to L_{\tilde{\mathbf{X}}}(\infty,\mathbb{R}^2)<\infty$ a.s. as $t\to\infty$. The latter is true because $\gamma L_{\tilde{\mathbf{X}}}(t,\mathbb{R}^2)$ is the square function of the non-negative martingale $\tilde{X}^i_t(\mathbb{R}^2)$ which therefore must converge a.s. Now use Proposition 15 (c) to see that

$$
\begin{aligned}
&E_{a_1\mu_0,a_2\mu_0}\left(\langle N^{i,j}_{k,\varepsilon}\rangle(kt)-\langle N^{i,j}_{k,\varepsilon}\rangle\left(k(t-\eta)\right)\right)\\
&\le \gamma\int_{k(t-\eta)}^{kt}\int_{\mathbb{R}^2}\left(S_{kt-r}\Delta^i_{k,\varepsilon}(y)\right)^2 a_1a_2S_r\mu_0(y)^2\,dy\,dr\\
&\le c(t,\mathbf{X}_0)\left[\int_{k(t-\eta)}^{kt}\int_{\mathbb{R}^2}\left[\int p_{t+\varepsilon-rk^{-1}}(w-x-yk^{-1/2})X^i_0(dw)\right]^2 p_{r+1}(y)^2\,dy\,dr\right.\\
&\qquad\qquad\left.+\int_{k(t-\eta)}^{kt}\int_{\mathbb{R}^2}p_{r+1}(y)^2dy\,dr\right],
\end{aligned}
$$

where we have used (198) in the last line.
This in turn is bounded by

$$
\begin{aligned}
&c(t,\mathbf{X}_0)\left[\int_{k(t-\eta)}^{kt}\int p_{t+\varepsilon-rk^{-1}}(w_1-x-yk^{-1/2})p_{t+\varepsilon-rk^{-1}}(w_2-x-yk^{-1/2})\,dy(r+1)^{-2}\right.\\
&\qquad\qquad\left.\times X^i_0(dw_1)X^i_0(dw_2)\,dr+\int_{k(t-\eta)}^{kt}p_{2(r+1)}(0)\,dr\right]\\
&\le c(t,\mathbf{X}_0)\left[\int_{k(t-\eta)}^{kt}\int kp_{2(t+\varepsilon-rk^{-1})}(w_1-w_2)X^i_0(dw_1)X^i_0(dw_2)(r+1)^{-2}\,dr+\log(t/t-\eta)\right]\\
&\le c(t,\mathbf{X}_0)\left[\int_{k(t-\eta)}^{kt}(t+\varepsilon-rk^{-1})^{-1/2}(r+1)^{-2}kdr+\log(t/t-\eta)\right]\quad(\text{since }\mathbf{X}_0\in M_{f,se})\\
&\le c'(t,\mathbf{X}_0)\left[\eta^{1/2}+\eta\right]\to 0\quad\text{as}\quad\eta\downarrow 0.
\end{aligned}
$$

It follows from the above results that

$$\langle N^{i,j}_{k,\varepsilon}\rangle(kt)\overset{P_{a_1\mu_0,a_2\mu_0}}{\longrightarrow}0\quad\text{a.s.}\quad\varepsilon\downarrow 0\quad\text{and}\quad k\to\infty,$$

and so by a standard martingale inequality, the second term on the right-hand side of (197) (i.e. $N^{i,j}_{k,\varepsilon}(kt)$) also converges to 0 in $P_{a_1\mu_0,a_2\mu_0}$-probability as $\varepsilon\downarrow 0$ and $k\to\infty$. We have proved $\langle\tilde X^j_{kt},\Delta^i_{k,\varepsilon}\rangle\overset{P_{a_1\mu_0,a_2\mu_0}}{\longrightarrow}0$ as $\varepsilon\downarrow 0$ and $k\to\infty$ and so (196) now gives

$$
\begin{aligned}
E_{\mathbf{X}_0}\left(e^{-aX_t(x)+ibY_t(x)}\right)&=\lim_{k\to\infty}E_{a_1\mu_0,a_2\mu_0}\left(\left\{-\left\langle\tilde X_{kt},1\right\rangle S_tX_0(x)+i\left\langle\tilde Y_{kt},1\right\rangle S_tY_0(x)\right\}\right)\\
&=E^0_{a_1,a_2}\left(e^{-S_tX_0(x)(B^1_{\tau_{ex}}+B^2_{\tau_{ex}})+iS_tY_0(x)(B^1_{\tau_{ex}}-B^2_{\tau_{ex}})}\right)\quad(\text{ by Theorem 20})\qquad(200)\\
&=E^0_{S_tX^1_0(x),S_tX^2_0(x)}\left(e^{-a(B^1_{\tau_{ex}}+B^2_{\tau_{ex}})+ib(B^1_{\tau_{ex}}-B^2_{\tau_{ex}})}\right).
\end{aligned}
$$

The last equality is an easy exercise on harmonic functions which may be found in the proof of Theorem 1.5 in [DP98]. An easy application of the Stone-Weierstrass Theorem, as in the proof of Lemma 2.3(b) in [DP98], shows that the above joint Laplace-Fourier transforms for $a_1, a_2 \geq 0$ uniquely determine the law of $(x_t(x), y_t(x))$ and the result follows for $\mathbf{X}_0 \in \mathcal{M}_{f,se}$.

Assume now that $\mathbf{X}_0 \in \mathcal{M}_{f,e}$. Let $\delta_n \in (0,t)$ decrease to 0. By Proposition 24(a), $\mathbf{X}_{\delta_n} \in \mathcal{M}_{f,se}$ a.s. and so the Markov property and (200) imply

$$\begin{aligned} E_{\mathbf{X}_0}\left(e^{-aX_t(x)+ibY_t(x)}\right) &= E_{\mathbf{X}_0}\left(E_{\mathbf{X}_{\delta_n}}\left(e^{-ax_{t-\delta_n}(x)+iby_{t-\delta_n}(x)}\right)\right) \\ &= E_{\mathbf{X}_0}\left(E^0_{a_1,a_2}\left(e^{-S_{t-\delta_n}X_{\delta_n}(x)(B^1_{\tau_{ex}}+B^2_{\tau_{ex}})+iS_{t-\delta_n}Y_{\delta_n}(x)(B^1_{\tau_{ex}}-B^2_{\tau_{ex}})}\right)\right) \\ &\to E^0_{a_1,a_2}\left(e^{-S_tX_0(x)(B^1_{\tau_{ex}}+B^2_{\tau_{ex}})+iS_tY_0(x)(B^1_{\tau_{ex}}-B^2_{\tau_{ex}})}\right) \quad \text{as } n\to\infty. \end{aligned}$$

In the last line we have used Dominated Convergence, the a.s. continuity of $\mathbf{X}_t$, and the uniform convergence of $p_{t-\delta_n}(\cdot)$ to $p_t(\cdot)$. This establishes (200) for $\mathbf{X}_0 \in \mathcal{M}_{f,e}$ and the proof may be completed just as in the previous case. ■

Proof of Corollary 19. Let $\{B_k : k \in \mathbb{N}\}$ be the set of open balls in $\mathbb{R}^2$ with rational centers and radii. Choose non-negative $\{\phi_k\} \subset \mathcal{C}_{\text{com}}(\mathbb{R}^2)$ such that $\{\phi_k > 0\} = B_k$. We may fix $\varepsilon_n \downarrow 0$ such that

$$\langle L^{*,\varepsilon_n}_{\mathbf{X}}(t), \phi_k\rangle \to \langle L_{\mathbf{X}}(t), \phi_k\rangle \quad \forall t \in \mathbb{Q}_+ \quad \forall k \quad \text{a.s.} \tag{201}$$

By Theorem 17 we may fix ω outside a null set such that (201) holds,

$$X^i_s(dx) = X^i_s(x)dx \quad \text{for Lebesgue a.a.} \quad s > 0, \tag{202}$$

and

$$\int_0^\infty \int_{\mathbb{R}^2} X^1_s(x)X^2_s(x)dx\,ds = 0. \tag{203}$$

It clearly suffices to show that for this fixed choice of ω, the desired conclusion holds for $U = (r_1, r_2) \times B_k$ for a fixed k and fixed rationals $0 \leq r_1 < r_2$. Assume

$$L_{\mathbf{X}}(r_2)(B_k) - L_{\mathbf{X}}(r_1)(B_k) > 0$$

and, say, $\|x^1\|_U < \infty$. Clearly $\exists B_{k'} \subset \overline{B}_{k'} \subset B_k$ such that $\langle L_{\mathbf{X}}(r_2), \phi_{k'}\rangle - \langle L_{\mathbf{X}}(r_1), \phi_{k'}\rangle > 0$. Then by (201)

$$\lim_{n\to\infty} \varepsilon_n^{-1}\int_0^{\varepsilon_n} dr \int_{r_1}^{r_2} ds\left[\int \phi_{k'}(y)S_rX^1_s(y)S_rX^2_s(y)dy\right] = \langle L_{\mathbf{X}}(r_2), \phi_{k'}\rangle - \langle L_{\mathbf{X}}(r_1), \phi_{k'}\rangle > 0,$$

which by Fubini implies

$$\lim_{n\to\infty}\int_{r_1}^{r_2}\int_{\mathbb{R}^2}\left[\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} S_rX^1_s(y)S_rX^2_s(y)dr\right]\phi_{k'}(y)\,dy\,ds = \langle L_{\mathbf{X}}(r_2), \phi_{k'}\rangle - \langle L_{\mathbf{X}}(r_1), \phi_{k'}\rangle > 0. \tag{204}$$

On the other hand (202) implies

$$S_rX^i_s(y) \to X^i_s(y) \quad \text{for Lebesgue a.a.} \quad (s,y) \quad \text{a.s.} \quad r \downarrow 0. \tag{205}$$

Let $d(B_{k'}, B^c_k) = \eta_k > 0$. Recall $\|\ \|_U$ denotes the essential supremum with respect to Lebesgue measure on the space-time open set U. We abuse this notation slightly and let $\|\ \|_{B_k}$ denote the essential sup with respect to Lebesgue measure on B_k.

We may fix $s \in (r_1, r_2)$ outside a Lebesgue null set so that (202) holds and $\|x_s^1\|_{B_k} \le \|x^1\|_U < \infty$. If $y \in B_{k'}$, then

$$\begin{aligned} S_r X_s^1(y) &\le \int_{B_k} p_r(z-y) x_s^1(z)\,dz + \int_{B_k^c} p_r(z-y) X_s^1(dz) \\ &\le \|x_s^1\|_{B_k} + p_r(\eta_k) X_s^1(\mathbb{R}^2) \\ &\le \|x^1\|_U + X_s^1(\mathbb{R}^2) \end{aligned}$$

providing $r < r(k)$, where $r(k) > 0$. This implies that for $\varepsilon_n < r(k)$, $y \in B_{k'}$, and Lebesgue a.a. $s \in (r_1, r_2)$, we have

$$\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} S_r X_s^1(y) S_r X_s^2(y)\,dr \le \left[\|x^1\|_U + \sup_{s \le r_2} X_s^1(\mathbb{R}^2)\right] \frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} S_r X_s^2(y)\,dr. \tag{206}$$

Assume for the moment that

$$\begin{aligned} &H_n(s,y) = \frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} S_r X_s^2(y)\,dr \quad (n \in \mathbb{N}) \text{ is a uniformly integrable family on} \\ &((r_1, r_2) \times \mathbb{R}^2, \phi_{k'}(y) ds\,dy) \end{aligned} \tag{207}$$

Then (206) allows us to take the limit in (204) through the first two integrals and conclude that the limit on the left side of (204) equals

$$\begin{aligned} &\int_{r_1}^{r_2}\int_{\mathbb{R}^2} \left[\lim_{n\to\infty} \frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} S_r X_s^1(y) S_r X_s^2(y) dr\right] \phi_{k'}(y)\,dy\,ds \\ =&\int_{r_1}^{r_2}\int_{\mathbb{R}^2} X_s^1(y) X_s^2(y) \phi_{k'}(y)\,dy\,ds \quad \text{by (205)} \\ =&\ 0 \quad \text{by (203)}. \end{aligned}$$

This contradicts (204) and so shows that for ω as above $L_{\mathbf{X}}(U) > 0$ implies $\|x^1\|_U = \infty$. By symmetry the proof is complete except for the verification of (207). To this end note that by (205), $\lim_{n\to\infty} H_n(s,y) = x_s^2(y)$ for Lebesgue a.a. (s, y) and

$$\begin{aligned} \lim_{n\to\infty}\int_{r_1}^{r_2}\int H_n(s,y)\phi_{k'}(y) dy\,ds &= \lim_{n\to\infty}\int_{r_1}^{r_2}\int \frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} S_r\phi_{k'}(y) dr\, x_s^2(y) dy\,ds \\ &= \int_{r_1}^{r_2}\int \phi_{k'}(y) x_s^2(y) dy\,ds \\ &\quad \text{(by Dominated Convergence).} \end{aligned}$$

Since $H_n \ge 0$ (207) follows, and the proof is complete.

7. Some Open Questions

An intriguing feature of this process is the volatile nature of its densities. There are a number of interesting open problems about the qualitative nature of the densities but, after spending three papers just to get existence, uniqueness and the basic features of the process straight, we will leave these for another day and perhaps another bunch of authors. Throughout this Section (X_t^1, X_t^2) denotes the unique solution of $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ starting at $\mathbf{X}_0 \in \mathcal{M}_{f,e}$.

We know from Theorem 17 that at a fixed time the densities segregate and the measures are mutually singular. This does not however say anything about their closed supports. Let $S(X_t^i)$ denote the closed support of X_t^i and let

$$G(X^i) = \mathrm{cl}\{(t,x) : x \in S(X_t^i), t > 0\}$$

(cl denotes closure in $\mathbb{R}_+ \times \mathbb{R}^2$) be the closed graph of X^i for $i = 1, 2$.

Conjecture 1. The interface $I = G(X^1) \cap G(X^2)$ is a.s. Lebesgue null in $\mathbb{R}_+ \times \mathbb{R}^2$ and there are versions of the densities $x^i(\cdot,\cdot)$ which are smooth on I^c and satisfy $\frac{\partial x^i}{\partial t} = \frac{\sigma^2 \Delta x^i}{2}$ on I^c.

Conjecture 2. For $t > 0$, the fixed time interface $S(X_t^1) \cap S(X_t^2)$ is a.s. Lebesgue null.

Assuming the second conjecture is correct we have

Question 3 What is the Hausdorff dimension of $S(X_t^1) \cap S(X_t^2)$?

Uniform in time behavior leads to an even more difficult set of problems. Even the simplest kind of uniform in t non-singularity (membership in $\mathcal{M}_{f,e}$ for all $t \geq 0$ a.s.) led to some non-trivial arguments in Proposition 24 (b) and we were never able to decide if in fact $X_t^i \in \mathcal{M}_{f,se}$ for all $t > 0$ a.s. The fact that the existence of the densities at a fixed time is rather delicate means the existence for all t is uncertain.

Question 4. Is $X_t^i(dx) \ll dx$ for all $t > 0$ a.s.? Is $S(X_t^1) \cap S(X_t^2)$ Lebesgue null for all $t > 0$ a.s.?

We showed in Corollary 19 that the densities blow up at typical points in the interface.

Question 5. Can one find a canonical rate of explosion of $x^i(t,x)$ as x approaches x_0 for $L_{\mathbf{X}}$ a.a. (t, x_0)?

As mentioned in Section 1.2 we feel that the results of this paper should hold for any (γ, σ^2).

Problem 6. Prove this.

Having done this, the reader may then want to turn to higher dimensions. Recall for super-Brownian motion branching in a super-Brownian medium, the process will only exist in 3 or fewer dimensions as it is critical that a typical Brownian path collides with the time-dependent catalyst [DF97a]. The situation for mutually catalytic branching is less clear and, depending on the time of day, you may be able to convince yourself that it should exist in any dimension, or only for $d \leq 3$, or only for $d \leq 2$.

Problem 7. Construct a solution to $(\mathbf{MP})_{\mathbf{X}_0}^{\sigma,\gamma}$ in higher dimensions or prove they cannot exist for sufficiently high dimensions.

8. Appendices

8.1. Appendix A. Random Walk Kernels.

In this Appendix we gather together the results we need for the transition kernel of our continuous time random walk ${}^\varepsilon\xi$ on $\varepsilon\mathbb{Z}^2$ which jumps to a randomly chosen nearest neighbor with rate $2\varepsilon^{-2}\sigma^2$. One would have thought that references containing proofs of Lemma 8 are commonplace but we could not locate one. Recall that

$${}^\varepsilon p_t(x) = \varepsilon^{-2}\Pi({}^\varepsilon\xi_t = x),\ x \in \varepsilon\mathbb{Z}^2 \text{ and } p_t(x) = (2\pi t\sigma^2)^{-1} e^{-|x|^2/2\sigma^2 t}.$$

Let ${}^{\varepsilon}q_t(x)$ and $q_t(x)$ be the one-dimensional counterparts of ${}^{\varepsilon}p_t(x)$ and $p_t(x)$, respectively, so that ${}^{\varepsilon}p_t(x_1,x_2) = {}^{\varepsilon}q_t(x_1)\,{}^{\varepsilon}q_t(x_2)$ and $p_t(x_1,x_2) = q_t(x_1)q_t(x_2)$. Lemma 8 then is immediate from its one-dimensional version which we now prove.

Lemma 59. *(a)* $\forall s>0 \lim_{\varepsilon\to 0} \sup_{x\in\varepsilon\mathbb{Z}} |{}^{\varepsilon}q_s(x) - q_s(x)| = 0$

(b) There is a universal constant $c_{A.1}<\infty$ such that for all $\varepsilon>0$

$$\sup_{s\geq 0, x\in\varepsilon\mathbb{Z}} {}^{\varepsilon}q_s(x)\sqrt{s}\sigma = \sup_{s\geq 0} {}^{\varepsilon}q_s(0)\sqrt{s}\sigma = c_{A.1}.$$

Proof. The characteristic function of ${}^{\varepsilon}q_s(x)$ is given by

$$\begin{aligned}{}^{\varepsilon}\widehat{q}_s(\theta) &= e^{-\sigma^2 s/\varepsilon^2}\sum_n \frac{(\sigma^2 s/\varepsilon^2)^n}{n!}(cos\theta\varepsilon)^n \\ &= \exp\big(-\sigma^2 s\big[\frac{1-cos\theta\varepsilon}{\varepsilon^2}\big]\big).\end{aligned}$$

Then by Fourier inversion (see p. 511 of [Fel71]) we have

$$ {}^{\varepsilon}q_s(x) = (2\pi)^{-1}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} \cos x\theta \exp\left(-\sigma^2 s\left[\frac{1-\cos\theta\varepsilon}{\varepsilon^2}\right]\right)\,d\theta \tag{208}$$

and

$$ q_s(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} \cos x\theta \exp\left(-\sigma^2 s\theta^2/2\right)\,d\theta. \tag{209}$$

Let $K>1$ and assume $\varepsilon < \frac{\pi}{K}s^{1/2}$. Then

$$\begin{aligned}|{}^{\varepsilon}q_s(x) - q_s(x)| &\leq \pi^{-1}\int_0^{Ks^{-1/2}} \left|\exp\left(\frac{-\sigma^2 s\theta^2}{2}\right) - \exp\left(-\sigma^2 s\left[\frac{1-\cos(\theta\varepsilon)}{\varepsilon^2}\right]\right)\right| d\theta \\ &\quad + \pi^{-1}\int_{Ks^{-1/2}}^{\pi/\varepsilon} \exp(-\sigma^2 s\theta^2/2) + \exp\left(-\sigma^2 s\left[\frac{1-\cos(\theta\varepsilon)}{\varepsilon^2}\right]\right) d\theta \\ &\equiv I_1 + I_2.\end{aligned}$$

Let $\eta>0$ and define $c_0 = \inf_{|x|\leq\pi}(1-\cos x)x^{-2} \in (0,\frac{1}{2}]$. Then

$$\begin{aligned}I_2 \leq 2\pi^{-1}\int_{Ks^{-1/2}}^{\infty}\exp(-\sigma^2 sc_0\theta^2)d\theta &\leq 2\pi^{-1}\int_{Ks^{-1/2}}^{\infty}\frac{\theta}{Ks^{-1/2}}e^{-\sigma^2 sc_0\theta^2}d\theta \\ &= (\pi\sigma^2 c_0)^{-1}s^{-1/2}e^{-\sigma^2 c_0 K^2} \\ &\leq \eta s^{-1/2}, \end{aligned}\tag{210}$$

where the last line is valid provided $K\geq K_0(\sigma,\eta)$. For I_1 use a second order Taylor expansion to write

$$\frac{1-\cos\theta\varepsilon}{\varepsilon^2} = \frac{\theta^2}{2}\cos X_\theta \text{ for some } X_\theta\in(0,\theta\varepsilon),$$

and conclude that

$$\begin{aligned}I_1 &= \pi^{-1}\int_0^{Ks^{-1/2}}\left|\exp(-\sigma^2 s\theta^2/2)\left[1-\exp\left(\frac{-\sigma^2 s\theta^2}{2}(\cos X_\theta - 1)\right)\right]\right| d\theta \\ &\leq \pi^{-1}\int_0^{Ks^{-1/2}}\exp(-\sigma^2 s\theta^2/2)\left|\exp(\sigma^2 s\theta^4\varepsilon^2/4)-1\right| d\theta.\end{aligned}$$

The elementary inequality $1-\cos x \le x^2/2$ is used in the last line.
For $0 \le \theta \le Ks^{-1/2}$, our assumed bound on ε gives

$$\sigma^2 s\theta^4\varepsilon^2 \le \sigma^2 s^{-1}K^4\varepsilon^2 < \pi^2\sigma^2K^2,$$

and so

$$\exp(\sigma^2 s\theta^4\varepsilon^2/4) - 1 \le \exp(\pi^2\sigma^2K^2/4)\sigma^2 s\theta^4\varepsilon^2/4.$$

This gives

$$\begin{aligned} I_1 &\le \pi^{-1}\exp(\pi^2\sigma^2K^2/4)\int_0^{Ks^{-1/2}} \exp(-\sigma^2 s\theta^2/2)\,\sigma^2 s\theta^4\varepsilon^2/4\,d\theta \\ &\le \pi^{-1}\exp(\pi^2\sigma^2K^2/4)\frac{\varepsilon^2}{\sqrt{2}\sigma^2 s^{3/2}}\int_0^\infty e^{-u}u^{3/2}\,du \\ &\le c(K,\sigma)\varepsilon^2 s^{-3/2}. \end{aligned}$$

Combine this with (210) and set $K = K_0(\sigma,\eta)$ to see that

$$\sup_{x\in\mathbb{Z}} |{}^\varepsilon q_s(x) - q_s(x)| \le \eta s^{-1/2} + c(K_0,\sigma)\varepsilon^2 s^{-3/2} \quad \text{for } \varepsilon < \frac{\pi}{K_0}s^{1/2}. \tag{211}$$

(a) is immediate from the above.

The first equality in (b) is clear from (208). For the second note that (211) implies that for $\varepsilon < \frac{\pi}{K_0}s^{1/2}$,

$${}^\varepsilon q_s(0) \le q_s(0) + \eta s^{-1/2} + c'(K_0,\sigma)s^{-1/2} \le c_1 s^{-1/2}.$$

For $\varepsilon \ge \frac{\pi}{K_0}s^{1/2}$,

$${}^\varepsilon q_s(0) \le \varepsilon^{-1} \le K_0\pi^{-1}s^{-1/2},$$

and (b) follows. ■

Proof of Lemma 44. We may consider $x = (x_1,x_2) \in \varepsilon\mathbb{Z}^2$ such that $x_1 = |x_1| \ge |x_2|$. By scaling, Lemma 59(b), and (208)

$$\begin{aligned} \int_0^\delta {}^\varepsilon p_s(x_1,x_2)ds &= \int_0^{\delta\varepsilon^{-2}} {}^1p_u(x_1\varepsilon^{-1},x_2\varepsilon^{-1})du \\ &\le c_1(\sigma)\int_0^{\delta\varepsilon^{-2}} {}^1q_u(x_1\varepsilon^{-1})u^{-1/2}du \\ &= c_1(\sigma)\pi^{-1}\int_0^{\delta\varepsilon^{-2}}\int_0^\pi \cos(x_1\theta/\varepsilon)\exp(-\sigma^2(1-\cos\theta)u)d\theta u^{-1/2}du \\ &= c_2(\sigma)\int_0^\pi \cos(x_1\theta/\varepsilon)g_{\delta,\varepsilon}(\theta)d\theta, \end{aligned} \tag{212}$$

where $g_{\delta,\varepsilon}(\theta) = \int_0^{\delta\varepsilon^{-2}} \exp(-\sigma^2(1-\cos\theta)u)u^{-1/2}\,du$. Note that $g_{\delta,\varepsilon}$ is a decreasing function on $[0,\pi]$ and if $c(\theta) = \sigma\sqrt{1-\cos\theta}$, then

$$g_{\delta,\varepsilon}(\theta) = \int_0^{c(\theta)^2\delta\varepsilon^{-2}} e^{-v}v^{-1/2}\,dv c(\theta)^{-1} \le c_3\left[c(\theta)^{-1} \wedge \frac{\sqrt{\delta}}{\varepsilon}\right]. \tag{213}$$

This implies

$$
\begin{aligned}
&\int_0^{(\varepsilon/x_1)\wedge\pi} |\cos(x_1\theta/\varepsilon)g_{\delta,\varepsilon}(\theta)|d\theta \\
&\le c_3\int_0^{(\varepsilon/x_1)\wedge(\varepsilon/\sqrt{\delta})} \frac{\sqrt{\delta}}{\varepsilon}d\theta + c_3\int_{(\varepsilon/\sqrt{\delta})\wedge(\varepsilon/x_1)}^{\varepsilon/x_1} c(\theta)^{-1}1(\theta\le\pi)d\theta \\
&\le c_4(\sigma)\left[1\wedge\frac{\sqrt{\delta}}{x_1}+\int_{(\varepsilon/\sqrt{\delta})\wedge(\varepsilon/x_1)}^{\varepsilon/x_1}\theta^{-1}d\theta\right] \\
&\le c_4(\sigma)\left[1\wedge\frac{\sqrt{\delta}}{x_1}+\log^+\left(\frac{\sqrt{\delta}}{x_1}\right)\right]. \qquad (214)
\end{aligned}
$$

An integration by parts shows that if $x_1>\varepsilon/\pi$, then

$$
\begin{aligned}
&\int_{\varepsilon/x_1}^{\pi}\cos(x_1\theta/\varepsilon)g_{\delta,\varepsilon}(\theta)d\theta \\
&= \frac{\varepsilon}{x_1}\sin(x_1\theta/\varepsilon)g_{\delta,\varepsilon}(\theta)\Big|_{\varepsilon/x_1}^{\pi} - \int_{\varepsilon/x_1}^{\pi}\frac{\varepsilon}{x_1}\sin(x_1\theta/\varepsilon)g'_{\delta,\varepsilon}(\theta)d\theta \\
&\le \frac{\varepsilon}{x_1}g_{\delta,\varepsilon}(\pi)+\frac{\varepsilon}{x_1}(g_{\delta,\varepsilon}(\varepsilon/x_1)-g_{\delta,\varepsilon}(\pi)) = \frac{\varepsilon}{x_1}g_{\delta,\varepsilon}(\varepsilon/x_1). \qquad (215)
\end{aligned}
$$

In the last line we bounded the integrand in absolute value by $\frac{\varepsilon}{x_1}(-g'_{\delta,\varepsilon}(\theta))$. Now use (213) in (215) to conclude that for $x_1>\varepsilon/\pi$,

$$
\begin{aligned}
\int_{\varepsilon/x_1}^{\pi}\cos(x_1\theta/\varepsilon)g_{\delta,\varepsilon}(\theta)d\theta &\le \frac{\varepsilon}{x_1}g_{\delta,\varepsilon}(\varepsilon/x_1) \\
&\le \frac{\varepsilon}{x_1}c_3\left[c(\varepsilon/x_1)^{-1}\wedge\sqrt{\delta}/\varepsilon\right] \\
&\le c_5(\sigma)\frac{\varepsilon}{x_1}\frac{x_1\wedge\sqrt{\delta}}{\varepsilon} = c_5(\sigma)\left(1\wedge\frac{\sqrt{\delta}}{x_1}\right).
\end{aligned}
$$

Combine this with (214) in (212) to see that

$$
\int_0^{\delta} {}^{\varepsilon}p_s(x_1,x_2)ds \le c_6(\sigma)\left[1\wedge\frac{\sqrt{\delta}}{x_1}+\log^+\frac{\sqrt{\delta}}{x_1}\right].
$$

Recalling our assumption that $x_1=|x_1|\ge|x_2|$, we see that the result follows. ■

Proof of Lemma 34. By Lemma 59 (b) and the fact that $|(x_1,x_2)|>s^{r/2}+\varepsilon^r$ implies $|x_i|>\frac{s^{r/2}+\varepsilon^r}{2}$ for $i=1$ or 2, the result follows from

$$
\sup\{s^{-1/2}\,{}^{\varepsilon}q_s(x): 0<s,\varepsilon,\ |x|>\frac{s^{r/2}+\varepsilon^r}{2},\ x\in\varepsilon\mathbb{Z}\}<\infty. \qquad (216)
$$

Another application of Lemma 59(b) shows that we need only consider $s\le1$. If τ_1 is the first jump time of the one-dimensional random walk ${}^{\varepsilon}\xi$ then for $x\neq0$,

$$
{}^{\varepsilon}q_s(x)\le\varepsilon^{-1}P(\tau_1<s)=\varepsilon^{-1}(1-\exp(-\sigma^2s\varepsilon^{-2}))\le\sigma^2s\varepsilon^{-3}
$$

and so

$$
\sup\{s^{-1/2}\,{}^{\varepsilon}q_s(x): s^{1/6}\le\varepsilon,\ x\neq0\}\le\sigma^2.
$$

These observations show that it now suffices to prove

$$\sup\{s^{-1/2}\,{}^{\varepsilon}q_s(x) : |x| > \frac{s^{r/2}+\varepsilon^r}{2},\ 0<\varepsilon\le s^{1/6}\le 1\} < \infty. \tag{217}$$

To get bounds for larger values of s we first use some exponential bounds. Let S_n be a simple symmetric random walk on $\varepsilon\mathbb{Z}$ and let $N^\varepsilon(s)$ be a Poisson process with rate $\sigma^2\varepsilon^{-2}$ which is independent of $\{S_n\}$. Then we may take ${}^{\varepsilon}\xi(s) = S_{N^\varepsilon(s)}$ and a simple calculation shows that if $0<\lambda\le\varepsilon^{-1}$, then

$$E(e^{\lambda\,{}^{\varepsilon}\xi(s)}) = \exp(\sigma^2\varepsilon^{-2}s(\cosh\lambda\varepsilon - 1)) \le e^{c_1 s\lambda^2}$$

for some $c_1 = c_1(\sigma^2) > 0$. If $\lambda = \varepsilon^{-1}\wedge s^{-1/2}$ and $x \ge (\varepsilon^r + s^{r/2})/2$, then

$$\begin{aligned}{}^{\varepsilon}q_s(x) \le \varepsilon^{-1}P(e^{\lambda\,{}^{\varepsilon}\xi(s)} \ge e^{\lambda x}) &\le \varepsilon^{-1}\exp(-\lambda x + c_1 s\lambda^2)\\ &\le \varepsilon^{-1}\exp(-(\varepsilon^{-1}\wedge s^{-1/2})(\varepsilon^r+s^{r/2})/2 + c_1)\\ &\le \varepsilon^{-1}\exp(-(\varepsilon\vee s^{1/2})^{r-1}/2 + c_1).\end{aligned}$$

By symmetry in x this gives

$$\begin{aligned}&\sup\{s^{-1/2}\,{}^{\varepsilon}q_s(x) : |x| \ge \frac{\varepsilon^r + s^{r/2}}{2}, 0<s^9\le\varepsilon\le s^{1/6}\le 1\}\\ &\quad\le \sup\{s^{-1/2}\varepsilon^{-1}\exp(-(\varepsilon\vee s^{1/2})^{r-1}/2 + c_1) : 0<s^9\le\varepsilon\le s^{1/6}\le 1\}\\ &\quad\le \sup\{s^{-19/2}\exp(-s^{(r-1)/6}/2 + c_1) : 0<s\le 1\} = c_2 < \infty.\end{aligned}$$

To obtain (217) it therefore now suffices to show

$$\sup\{s^{-1/2}\,{}^{\varepsilon}q_s(x) : |x| \ge \frac{\varepsilon^r + s^{r/2})}{2}, 0<\varepsilon<s^9\le 1\} < \infty. \tag{218}$$

For this use (208) to see that

$$\begin{aligned}{}^{\varepsilon}q_s(x) &= \pi^{-1}\int_0^{\pi/\varepsilon}\cos x\theta\exp\left(-\sigma^2 s\left[\frac{1-\cos\theta\varepsilon}{\varepsilon^2}\right]\right)d\theta\\ &= \pi^{-1}\int_0^{s^{-1}}\cos x\theta\left[\exp\left(-\sigma^2 s\left[\frac{1-\cos\theta\varepsilon}{\varepsilon^2}\right]\right) - \exp(-\sigma^2 s\theta^2/2)\right]d\theta\\ &+ \pi^{-1}\int_{s^{-1}}^{\pi/\varepsilon}\exp(-\varepsilon^{-2}\sigma^2 s[1-\cos\theta\varepsilon])d\theta - \pi^{-1}\int_{s^{-1}}^{\infty}\exp(-\sigma^2 s\theta^2/2)d\theta\\ &+ \pi^{-1}\int_0^{\infty}\cos\theta x\exp(-\sigma^2 s\theta^2/2)d\theta\\ &\le \pi^{-1}\left|\int_0^{s^{-1}}\cos x\theta\left[\exp\left(-\sigma^2 s\left[\frac{1-\cos\theta\varepsilon}{\varepsilon^2}\right]\right) - \exp(-\sigma^2 s\theta^2/2)\right]d\theta\right|\\ &+ \pi^{-1}\int_{s^{-1}}^{\pi/\varepsilon}\exp(-\varepsilon^{-2}\sigma^2 s[1-\cos\theta\varepsilon])d\theta + \pi^{-1}\int_{s^{-1}}^{\infty}\exp(-\sigma^2 s\theta^2/2)d\theta\\ &+ \pi^{-1}\left|\int_0^{\infty}\cos\theta x\exp(-\sigma^2 s\theta^2/2)d\theta\right|\\ &\equiv I + II + III + IV.\end{aligned} \tag{219}$$

By Fourier inversion we see that for $|x|\ge s^{r/2}$ and $s\in(0,1]$,

$$IV = p_s(x) \le c(\sigma^2)s^{-1/2}\exp(-s^{r-1}/2\sigma^2) \le c(\sigma^2)s^{1/2}. \tag{220}$$

Use the fact that $(1-\cos\theta\varepsilon)\varepsilon^{-2} \geq c_2\theta^2$ for all $|\theta| \leq \pi/\varepsilon$ and some $c_2 \in (0, 1/2]$, to see that

$$
\begin{aligned}
II + III &\leq \int_{s^{-1}}^{\infty} \exp(-\sigma^2 c_2 s\theta^2)d\theta \\
&\leq \int_{s^{-1}}^{\infty} \exp(-\sigma^2 c_2 s\theta^2)\theta s d\theta \\
&= (2\sigma^2 c_2)^{-1} \exp(-\sigma^2 c_2 s^{-1}) \leq c(\sigma^2)s^{1/2}. \qquad (221)
\end{aligned}
$$

To bound I, use Taylor's formula to write $1 - \cos\theta\varepsilon = \frac{\theta^2\varepsilon^2}{2} - \frac{(\cos X)\theta^4\varepsilon^4}{4!}$ for some $|X| < \theta\varepsilon$ and note that for $0 \leq \theta \leq s^{-1}$and $\varepsilon \leq s^9 \leq 1$,

$$\sigma^2 s\theta^4\varepsilon^2 \leq \sigma^2. \qquad (222)$$

Therefore for $\varepsilon < s^9 \leq 1$,

$$
\begin{aligned}
I &\leq \int_0^{s^{-1}} \exp(-\sigma^2 s\theta^2/2)|\exp((\cos X)\theta^4\sigma^2 s\varepsilon^2/24) - 1|d\theta \\
&\leq \int_0^{s^{-1}} c(\sigma^2)s\varepsilon^2\theta^4 d\theta \quad \text{by (222)} \\
&\leq c(\sigma^2)s^{1/2}. \qquad (223)
\end{aligned}
$$

We use the condition on ε and s in the last line. Now use (220),(221) and (223) in (219) to derive (218) and complete the proof. ■

8.2. **Appendix B. Integration Lemmas.**

Lemma 60. *Let $p \in (0,1)$ and $I_{n,p}(s) = \int_{\mathbb{R}_+^n} \frac{1}{s+u_1} \prod_1^{n-1} \frac{1}{u_i+u_{i+1}} \frac{1}{u_n^p} du_1 \ldots du_n$. Then $I_{n,p}(s) = \left(\frac{\pi}{\sin((1-p)\pi)}\right)^n s^{-p}$ for all $n \in \mathbb{N}$.*

Proof. Let $z = (u_n/u_{n-1})^{1-p}$ to see that $(u_n = u_{n-1}z^{1/(1-p)})$

$$
\begin{aligned}
\int_0^\infty \frac{1}{u_{n-1}+u_n}\frac{1}{u_n^p}du_n &= \int_0^\infty \frac{1}{u_{n-1}(1+z^{1/(1-p)})}\frac{u_{n-1}z^{p/(1-p)}}{u_{n-1}^p z^{p/(1-p)}(1-p)}dz \\
&= \frac{u_{n-1}^{-p}}{1-p}\int_0^\infty \frac{dz}{1+z^{1/(1-p)}} \\
&= \frac{\pi}{\sin(\pi(1-p))}u_{n-1}^{-p}
\end{aligned}
$$

by a standard residue calculation. This shows

$$
\begin{aligned}
I_{n,p}(s) &= \int_{\mathbb{R}_+^{n-1}} \frac{1}{s+u_1}\prod_1^{n-2}\frac{1}{u_i+u_{i+1}}\left[\int_0^\infty \frac{1}{u_{n-1}+u_n}\frac{1}{u_n^p}du_n\right]du_1\ldots du_{n-1} \\
&= \frac{\pi}{\sin(\pi(1-p))}I_{n-1,p}(s)
\end{aligned}
$$

and $I_{1,p}(s) = \frac{\pi}{\sin(\pi(1-p))}s^{-p}$. The obvious induction completes the proof. ■

Corollary 61. *Let $0 < p < 1$ and for $s, T > 0$, define*

$$J_n(s,T) = \int_{\mathbb{R}_+^n} \frac{1}{s+u_1}1(u_n \leq T)\prod_{i=1}^{n-1}\frac{1}{u_i+u_{i+1}}du_1\ldots du_n.$$

Then there is a constant $c_{61}(p)$ *such that*
$J_n(s,T) \le c_{61}(p)\left(\frac{\pi}{\sin(\pi(1-p))}\right)^{n-1}\left(\frac{T}{s}\right)^p$ *for all* $n \in \mathbb{N}$.

Proof.

$$\int_0^T \frac{1}{u_{n-1}+u_n}du_n = \log\left(1+\frac{T}{u_{n-1}}\right) \le c_{61}(p)\left(\frac{T}{u_{n-1}}\right)^p$$

because $\log(1+x) \le c_{61}x^p$ for all $x \ge 0$. Therefore by Lemma 60

$$\begin{aligned} J_n(s,T) &\le c_{61}\int_{\mathbb{R}_+^{n-1}} \frac{1}{s+u_1}\prod_{i=1}^{n-2}\frac{1}{u_i+u_{i+1}}\left(\frac{T}{u_{n-1}}\right)^p du_1\dots du_{n-1} \\ &= c_{61}T^pI_{n-1}(s) \le c_{225}\left(\frac{\pi}{\sin((1-p)\pi)}\right)^{n-1}\left(\frac{T}{s}\right)^p. \blacksquare \end{aligned}$$

Corollary 62. *Assume* $h:(0,\infty)\to[0,\infty)$ *satisfies* $h(u)\le c(1+u^{-p})$ *for* $u \in [0,T]$ *and some* $p\in(0,1)$. *Then*

$$\begin{aligned} J_n(s,h) &\equiv \int_{\mathbb{R}_+^n}\frac{1(u_n\le T)}{s+u_1}\left(\prod_1^{n-1}\frac{1}{u_i+u_{i+1}}\right)h(u_n)du_1\dots du_n \\ &\le cc_{62}(p)\left(\frac{\pi}{\sin((1-p)\pi)}\right)^n s^{-p}(T^p+1) \end{aligned}$$

Proof. Immediate from the previous two results. $\blacksquare$

Proof of Lemma 49. (a) Let $u=(1-w)/(x-1)$ in the integral defining ϕ_p to see that

$$\begin{aligned} \phi_p(x) &= \frac{x}{1+(x-1)^{-p}}\Big[\int_0^{1/(x-1)}\frac{x-1}{(x-1)(1+u)}(1-(x-1)u)^{-p}du \\ &\quad + \int_0^{1/(x-1)}\frac{x-1}{(x-1)(1+u)}(x-1)^{-p}u^{-p}du\Big] \end{aligned}$$

(224)
$$= \frac{x}{(x-1)^p+1}\left[\int_0^{1/(x-1)}\frac{((x-1)^{-1}-u)^{-p}}{1+u}du + \int_0^{1/(x-1)}(1+u)^{-1}u^{-p}du\right].$$

If $x \ge 2$, then

$$\begin{aligned} \phi_p(x) &\le \frac{2x}{(x-1)^p+1}\int_0^{1/(x-1)}u^{-p}du \\ &= \left(\frac{2}{1-p}\right)\frac{x}{x-1+(x-1)^{1-p}} \le \frac{2}{1-p}. \end{aligned} \tag{225}$$

Assume now that $x\in(1,2)$. If $R=(x-1)^{-1}\ge 1$ and we set $w=R-u$, then

$$\begin{aligned} \int_0^R (R-u)^{-p}(1+u)^{-1}du &\le \int_0^{R/2}u^{-p}(1+u)^{-1}du + \int_{R/2}^R (R-u)^{-p}(1+R-u)^{-1}du \\ &= \int_0^{R/2}u^{-p}(1+u)^{-1}du + \int_0^{R/2}w^{-p}(1+w)^{-1}dw \\ &\le 2\int_0^\infty u^{-p}(1+u)^{-1}du. \end{aligned}$$

Use this together with the fact $(x-1)^p + 1 \geq x - 1 + 1 = x$ for $x < 2$, to see that (224) implies

$$\phi_p(x) \leq 3 \int_0^\infty u^{-p}(1+u)^{-1} du = \frac{3\pi}{\sin(1-p)\pi}, \tag{226}$$

the last by a standard contour integration. As $\frac{3\pi}{\sin(1-p)\pi} > \frac{2}{1-p}$, the result follows from (225) and (226).
(b) If $w = \frac{s_2}{s_1}$, then

$$\begin{aligned} K_2^{(p)}(s_0, s_1) &= \int_0^{s_1} (s_0 - s_2)^{-1} s_1^{-1} s_2^{-p} (1 + ((s_1 - s_2)/s_2)^{-p}) ds_2 \\ &= s_1^{-1-p} \int_0^1 ((s_0/s_1) - w)^{-1} (w^{-p} + (1-w)^{-p}) dw \\ &\leq c_{49} s_1^{-p} s_0^{-1} (1 + ((s_0/s_1) - 1)^{-p}). \end{aligned}$$

In the last line we used (a). This gives the result for $n = 2$. Assume the result for $n \geq 2$. Then

$$\begin{aligned} K_{n+1}^{(p)}(s_0, s_1) &= \int_0^{s_1} K_n(s_1, s_2)(s_0 - s_2)^{-1} ds_2 \\ &\leq c_{49}^{n-1} s_1^{-1} \int_0^{s_1} (s_0 - s_2)^{-1} s_2^{-p} (1 + ((s_1 - s_2)/s_2)^{-p}) ds_2 \\ &= c_{49}^{n-1} K_2^{(p)}(s_0, s_1). \end{aligned}$$

Use the result derived for $n = 2$ to obtain the required bound for $n + 1$ and hence complete the induction. ■

Lemma 63. *Let $\{X_n\}$ be a sequence of non-negative random variables on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Assume for some $p > 1$, $\{E(X_n^p|\mathcal{G}) : n \in \mathbb{N}\}$ is bounded in probability and X_n converges in probability to X_∞. Then*

$$E(X_n|\mathcal{G}) \text{ converges in probability to } E(X_\infty|\mathcal{G}) < \infty \text{ a.s.}$$

Proof. This may be shown by making the obvious changes in the standard proof of the unconditional version of this result. ■

9. Acknowledgement

Thanks go to Carl Mueller who suggested studying this model several years ago.

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version of November 15, 2000 mutfin.tex typeset by LaTeX

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