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Control variational methods for differential equations

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Abstract

We review recent results established in the literature via the optimal control approach to differential equations, and we show that a systematic study of general variational inequalities associated to fourth-order operators can be performed by similar methods.

1 Introduction

Optimal control approaches associated to domain decomposition methods or to fictitious domains methods are well-known in the scientific literature devoted to numerical methods for differential equations. They may be viewed as applications of the general least squares minimization procedure, and we quote the works of Lions and Pironneau [11], Glowinski, Lions and Pironneau [6], Neittaanmäki and Tiba [13], for recent advances in this area.

It turns out that in certain important examples, arising for instance in mechanics, standard variational formulations based on the minimization of energy can be advantageously replaced by appropriate optimal control formulations that yield the existence and the uniqueness of the solution under low regularity conditions on the coefficients, i.e. on the geometric parameters of the problem. Other useful consequences of this new approach concern general results on the continuous/differentiable dependence of the solution on these parameters and, even, explicit solutions (in the case of arches) obtained via duality theory in optimal control. These theoretical developments are important in the setting of shape optimization problems in structural mechanics.

In Section 2, we shall give a brief presentation of some recently obtained results along these lines, following the works of Sprekels and Tiba [14], [15], [16], Ignat, Sprekels and Tiba [10], Arnăutu, Langmach, Sprekels and Tiba [1]. Although complete proofs are not included for the sake of brevity, most relevant arguments are carefully described, and precise quotations of the literature are indicated.

In Section 3, we shall study variational inequalities associated to fourth-order differential operators, emphasizing the applications to obstacle-type problems for clamped arches and plates. We underline that our approach is constructive and easily implemented using piecewise linear finite elements in the computations.

Finally, we notice that our optimal control variational formulation for differential systems provides, via the corresponding Pontryagin maximum principle, a nonstandard decomposition of the original equations, which is at the core of our argument.

2 The optimal control approach

We start with a simplified model (Bendsoe [3]) of a clamped plate, with variable thickness $u \in L^\infty(\Omega)_+$, and with normalized mechanical constants,

$$\Delta(u^3 \Delta y) = f \quad \text{in } \Omega, \tag{2.1}$$

$$y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Here, Ω is a bounded Lipschitzian domain in $I\!R^N$ (for $N = 2$, the plate model is obtained), $f \in L^2(\Omega)$ denotes the load and $y \in H_0^2(\Omega)$ the deflection.

We also consider the distributed control problem ($\varepsilon > 0$ is a "small" parameter),

$$\text{Min} \left\{ \frac{1}{2\varepsilon} \int_{\partial\Omega} \left(\frac{\partial y}{\partial n} \right)^2 d\sigma + \frac{1}{2} \int_{\Omega} \ell h^2 \, dx \right\}, \tag{2.3}$$

subject to

$$\Delta y = \ell g + \ell h \quad \text{in } \Omega, \tag{2.4}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Here $\ell = u^{-3} \in L^\infty(\Omega)_+$, and g is defined by $\Delta g = f$ in Ω, $g = 0$ in $\partial\Omega$.

If $0 < m \le u \le M$ a.e. in Ω then $\ell \ge M^{-3}$ a.e. in Ω, and the coercivity of (2.3) gives the existence of a unique optimal pair $[y_\varepsilon, h_\varepsilon] \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$. It is unique by the strict convexity.

The Pontryagin maximum principle for the unconstrained optimal control problem (2.3)–(2.5) is given by (2.4), (2.5), and, for some $p_\varepsilon \in H^1(\Omega)$, by:

$$\Delta p_\varepsilon = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$p_\varepsilon = \frac{1}{\varepsilon} \frac{\partial y_\varepsilon}{\partial n} \quad \text{on } \partial\Omega, \tag{2.7}$$

$$p_\varepsilon + h_\varepsilon = 0 \quad \text{a.e. in } \Omega. \tag{2.8}$$

The pair $[0, -g]$ gives the cost $\frac{1}{2} \int_\Omega \ell g^2 \, dx$ in (2.3), independently of $\varepsilon > 0$. This shows that $[y_\varepsilon, h_\varepsilon]$ are bounded with respect to $\varepsilon > 0$ since $\ell \ge M^{-3} > 0$ a.e. in Ω, as noticed before. Moreover, again due to (2.3), $\frac{\partial y_\varepsilon}{\partial n} \to 0$ strongly in $L^2(\partial\Omega)$. From (2.6), (2.8), we get that h_ε is harmonic, which is preserved by passing to the limit in the weak topology of $L^2(\Omega)$. A simple limitting argument in (2.3)–(2.5), and the definitions of ℓ, g, give:

Theorem 2.1 *The solution of* (2.1), (2.2) *is the limit of the optimal states* y_ε *in* $H^2(\Omega) \cap H_0^1(\Omega)$ *weak, for* $\varepsilon \to 0$.

This result was established in the paper of Arnăutu, Langmach, Sprekels and Tiba [1]. It is also valid for simply supported plates, i.e. with (2.2) replaced by

$$y = \Delta y = 0 \quad \text{on } \partial\Omega\,. \tag{2.2'}$$

The above discussion shows that the original fourth-order boundary value problem (2.1), (2.2) is equivalent with (2.4), (2.5), with h some harmonic mapping in $L^2(\Omega)$, and with the extra condition $\dfrac{\partial y}{\partial n} = 0$ on $\partial\Omega$.

Assume that $\ell_n \to \ell$ weakly* in $L^\infty(\Omega)$, and denote by y_n the solution of (2.1), (2.2) associated to $u_n = \ell_n^{-\frac{1}{3}}$, and by h_n the corresponding harmonic mappings appearing in (2.4). It is a standard argument to see that $\{y_n\}$ is bounded in $H_0^2(\Omega)$ and that $\{h_n\}$ is bounded in $L^2(\Omega)$; moreover, they weakly converge, on a subsequence, to the limits $y \in H_0^2(\Omega)$, respectively $h \in L^2(\Omega)$. The difficulty to pass to the limit in the equations is related to the products $u_n^3\,\Delta y_n$ or $\ell_n\,h_n$ appearing in (2.1), respectively (2.4), and to the weak convergence. However, as h_n, h are harmonic, the solid mean property gives that $h_n(x) \to h(x)$, $\forall\, x \in \Omega$, and the Egorov theorem shows that $h_n \to h$ strongly in $L^s(\Omega)$, $\forall\, s < 2$. Then, we clearly get that $\ell_n\,h_n \to \ell\,h$ weakly in $L^2(\Omega)$, and we can pass to the limit in (2.4). Notice that the weak limit of u_n is in general different from $\ell^{-\frac{1}{3}}$, but we have:

Theorem 2.2 *Assume that $\ell_n \to \ell$ weakly* in $L^\infty(\Omega)$. If $y = \lim y_n$ in $H_0^2(\Omega)$ weak, then it satisfies the equation*

$$\Delta(\ell^{-1}\,\Delta y) = f \quad \text{in } \Omega\,. \tag{2.1'}$$

This result gives the "continuous" dependence of the solution on the coefficient in (2.1), in the weak* topology of $L^\infty(\Omega)$. It was established in Sprekels and Tiba [15] and has important consequences in the existence theory for shape optimization problems or in homogenization problems for plates.

We now consider the differentiability with respect to the coefficient ℓ:

Theorem 2.3 *The mappings $\ell \mapsto y$ and $\ell \mapsto h$ are Gâteaux differentiable from $L^\infty(\Omega)$ into $H^2(\Omega)$ and $L^2(\Omega)$, respectively, and the directional derivatives at ℓ in the direction $v \in L^\infty(\Omega)$ satisfy:*

$$\Delta\bar{y} = \ell\,\bar{h} + v(h + g) \quad \text{in } \Omega\,, \tag{2.9}$$

$$\bar{y} = \frac{\partial\bar{y}}{\partial n} = 0 \quad \text{on } \partial\Omega\,, \tag{2.10}$$

$$\Delta\bar{h} = 0 \quad \text{in } \Omega\,. \tag{2.11}$$

The solution $[\bar{y}\,,\,\bar{h}]$ of (2.9)–(2.11) is unique in $H_0^2(\Omega) \times L^2(\Omega)$.

This result was established in Ignat, Sprekels and Tiba [10], and an essential ingredient in the necessary estimates is the observation that the decomposition of

(2.1) provided by (2.4) has the orthogonality property $\Delta y \perp h$, in the $L^2(\Omega)$-inner product. **Theorem 2.3** allows the writing of the optimality conditions in shape optimization problems for plates, without differentiability assumptions on the coefficients. The obtained gradient can be used in numerical experiments. We also stress that the approximation of (2.1), (2.2) via (2.3)–(2.5) is a simple and efficient method for the computation of solutions to plate equations. Numerical examples related to **Theorems 2.1–2.3** can be found in the work of Arnăutu, Langmach, Sprekels and Tiba [1].

Remark. The variant of the control variational approach given by (2.3) includes the penalization in the cost of one of the boundary conditions (2.2). In the sequel, we briefly describe another variant based on the use of constrained control problems.

With this aim, we now consider the Kirchhoff–Love model for clamped arches:

$$\int_0^1 \left[\frac{1}{\delta}(v_1' - c\,v_2)\,(u_1' - c\,u_2) + (v_2' + c\,v_1)'\,(u_2' + c\,u_1)'\right] ds$$
$$= \int_0^1 (f_1\,u_1 + f_2\,u_2)\,ds\,, \quad \forall\, u_1 \in H_0^1(0,1)\,, \quad \forall\, u_2 \in H_0^2(0,1)\,. \tag{2.12}$$

Above, $\varphi : [0,1] \to I\!R^2$ is the parametrization of a smooth clamped arch with the curvature denoted by c, and with the (constant) thickness given by $\sqrt{\delta}$. The mappings $v_1 \in H_0^1(0,1)$, $v_2 \in H_0^2(0,1)$ are the tangential and the normal components of the deformation, while $[f_1\,,\,f_2]$ is a similar notation for the load, in the local system of axes. A thorough presentation of the model for $\varphi \in C^3(0,1)$ via Dirichlet's principle and Korn's inequality may be found in Ciarlet [5].

Let $\theta : [0,1] \to I\!R$ denote the angle between the tangent vector to the arch (given by φ') and the horizontal axis. If φ is smooth, then $\theta' = c$. We also consider the orthogonal matrix

$$W(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix} \tag{2.13}$$

and the functions ℓ, h, g_1, g_2 constructed from $f_1, f_2 \in L^2(0,1)$ as follows:

$$g_1 = \delta\,\ell, \quad -g_2'' = h, \quad g_2(0) = g_2(1) = 0\,, \tag{2.14}$$

$$\begin{bmatrix} \ell \\ h \end{bmatrix}(t) = -\int_0^t W(t)\,W^{-1}(s)\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds\,. \tag{2.15}$$

We define the constrained control problem

$$\text{Min}\left\{\frac{1}{2\,\delta}\int_0^1 u^2\,ds + \frac{1}{2}\int_0^1 (z')^2\,ds\right\}, \tag{2.16}$$

subject to $u \in L^2(0,1)$, $z \in H_0^1(0,1)$, such that the mappings $[v_1\,,\,v_2]$ given by

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(t) \;=\; \int_0^t W(t)\,W^{-1}(s)\begin{bmatrix} u+g_1 \\ z+g_2 \end{bmatrix}(s)\,ds \tag{2.17}$$

satisfy $v_1(1) = v_2(1) = 0$ in the sense that

$$\int_0^1 W^{-1}(s)\begin{bmatrix} u(s)+g_1(s) \\ z(s)+g_2(s) \end{bmatrix} ds \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.18}$$

We underline that relations (2.13)–(2.18) are meaningful under the mere assumption that $\theta \in L^\infty(0,1)$. Then, $[v_1\,,\,v_2] \in L^\infty(0,1)^2$ represent the mild solution of the Cauchy problem (written formally)

$$v_1' \;-\; c\,v_2 \;=\; u \;+\; g_1 \quad \text{in } [0,1]\,, \tag{2.19}$$

$$v_2' \;+\; c\,v_1 \;=\; z \;+\; g_2 \quad \text{in } [0,1]\,, \tag{2.20}$$

$$v_1(0) \;=\; v_2(0) \;=\; 0\,. \tag{2.21}$$

The constraint (2.18) is a terminal state constraint, expressed as a control constraint, since the state system (2.17) is in explicit form and the matrix $W(t)$ is nonsingular.

We denote by $[u_\delta\,,\,z_\delta] \in L^2(0,1) \times H_0^1(0,1)$ the unique optimal control associated to (2.16)–(2.18). It exists due to the coercivity of the cost functional and since the pair $[-g_1\,,\,-g_2]$ is clearly admissible. We also denote by $[v_1^\delta\,,\,v_2^\delta] \in L^\infty(0,1)^2$ the optimal state corresponding to $[u_\delta\,,\,z_\delta]$ via (2.17).

Theorem 2.4 *If $\theta \in W^{2,\infty}(0,1)$, then $[v_1^\delta\,,\,v_2^\delta]$ is the solution of* (2.12).

We briefly indicate the argument:

We get $c \in W^{1,\infty}(0,1)$, and (2.17) can be written in the strong form (2.19)–(2.21). The same holds for (2.15).

The Euler equation associated to (2.16)–(2.18) is

$$\frac{1}{\delta}\int_0^1 u_\delta\,\mu\,ds \;+\; \int_0^1 z_\delta'\,\zeta'\,ds \;=\; 0\,, \tag{2.22}$$

for any $[\mu\,,\,\zeta] \in L^2(0,1) \times H_0^1(0,1)$ such that

$$\int_0^1 W^{-1}(s)\begin{bmatrix} \mu(s) \\ \zeta(s) \end{bmatrix} ds \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.23}$$

For any $u_1 \in H_0^1(0,1)$, $u_2 \in H_0^2(0,1)$, we introduce

$$\tilde{\mu} \;=\; u_1' \;-\; c\,u_2 \;\in\; L^2(0,1)\,, \tag{2.24}$$

$$\tilde{\zeta} = u_2' + c\,u_1 \in H_0^1(0,1)\,, \tag{2.25}$$

and we have, consequently, that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(t) = \int_0^t W(t)\,W^{-1}(s) \begin{bmatrix} \tilde{\mu}(s) \\ \tilde{\zeta}(s) \end{bmatrix} ds\,. \tag{2.26}$$

As $u_1(1) = u_2(1) = 0$, we see that $[\tilde{\mu}\,,\,\tilde{\zeta}]$ given by (2.24), (2.25), satisfy (2.23) and may be used in (2.22), whence

$$\begin{aligned} 0 \;&=\; \frac{1}{\delta}\int_0^1 \Big((v_1^\delta)' - c\,v_2^\delta - g_1\Big)\Big(u_1' - cu_2\Big)\,ds + \int_0^1 \Big((v_2^\delta)' + c\,v_1^\delta - g_2\Big)'\Big(u_2' + c\,u_1\Big)'\,ds \\ &=\; \frac{1}{\delta}\int_0^1 \Big((v_1^\delta)' - c\,v_2^\delta\Big)\Big(u_1' - c\,u_2\Big)\,ds + \int_0^1 \Big((v_2^\delta)' + c\,v_1^\delta\Big)'\Big(u_2' + c\,u_1\Big)'\,ds \\ &\quad - \int_0^1 \ell\Big(u_1' - c\,u_2\Big)\,ds - \int_0^1 h\Big(u_2' + c\,u_1\Big)\,ds\,, \end{aligned} \tag{2.27}$$

where we have used (2.19), (2.20) and (2.14). By partial integration in the last two terms in (2.27), and by (2.15), we recover from (2.27) the equation (2.12).

Remark. **Theorem 2.4** shows that the constrained control problem (2.16)–(2.18) is a weak formulation of the Kirchhoff–Love model under very low geometric regularity assumptions. Other arguments along these lines can be found in Sprekels and Tiba [16], Ignat, Sprekels and Tiba [10].

We also notice that the constraint (2.18) is affine and finite dimensional. This allows a complete solution of the control problem via duality theory, Barbu and Precupanu [2]. In the work of Ignat, Sprekels and Tiba [10] the dual control problem giving the (two-dimensional) Lagrange multiplier is explicitly derived, and the results are used for numerical experiments with Lipschitzian arches, i.e. for $\varphi \in W^{1,\infty}(0,1)^2$. Moreover, by writing a Pontryagin-type maximum principle for the problem (2.16)–(2.18), continuity and differentiability of the solution $[v_1^\delta\,,\,v_2^\delta]$ with respect to the parameter $\theta \in L^\infty(0,1)$ can be studied. In this way, in Ignat, Sprekels and Tiba [10], a complete theoretical and numerical analysis of shape optimization problems associated with Kirchhoff–Love arches is performed.

Remark. For the plate equation (2.1), (2.2), the corresponding constrained control problem is

$$\text{Min}\left\{ \frac{1}{2}\int_\Omega \ell\,h^2\,dx \right\},$$

subject to (2.4), (2.5) and to the constraint

$$\frac{\partial y}{\partial n} = 0 \quad \text{on } \partial\Omega\,. \tag{2.28}$$

We notice that (2.28) is affine, but infinite dimensional. A dual control problem (unconstrained!) can be obtained, but it remains an infinite dimensional optimization problem. Consequently, it is not possible to find an explicit solution, in general, and a standard treatment is to employ the penalization of (2.28) in the cost, as in (2.3).

3 Variational inequalities for fourth-order differential operators

We shall show that the technique presented in the previous section can also be applied to establish existence results for general variational inequalities.

We examine first the case of clamped plates subjected to unilateral conditions, since it is more intuitive. We define the control problem with state constraints,

$$\operatorname{Min}\left\{\frac{1}{2\varepsilon}\int_{\partial\Omega}\left(\frac{\partial y}{\partial n}\right)^2 d\sigma + \frac{1}{2}\int_{\Omega}\ell\, h^2\, dx\right\}, \tag{3.1}$$

subject to

$$\Delta y = \ell\, g + \ell\, h \quad \text{in } \Omega, \tag{3.2}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{3.3}$$

$$y \in \mathcal{K}. \tag{3.4}$$

The notations are the same as in Section 2, and $\mathcal{K} \subset H^2(\Omega)$ is a closed convex subset such that $\mathcal{K} \cap H_0^2(\Omega) \neq \emptyset$. Then, the pair $[\hat{y}\,,\, \ell^{-1}\Delta\hat{y} - g]$, with $\hat{y} \in \mathcal{K} \cap H_0^2(\Omega)$, is clearly admissible, and the corresponding cost is independent of $\varepsilon > 0$.

We denote by $[y_\varepsilon\,,\, h_\varepsilon] \in H^2(\Omega) \times L^2(\Omega)$, the unique optimal pair of (3.1)–(3.4) (recall that $\ell \geq M^{-3} > 0$ in Ω). We have:

$$\frac{1}{2\varepsilon}\int_{\partial\Omega}\left(\frac{\partial y_\varepsilon}{\partial n}\right)^2 d\sigma + \frac{1}{2}\int_{\Omega}\ell\, h_\varepsilon^2\, dx \leq \frac{1}{2}\int_{\Omega}\ell\,(\ell^{-1}\Delta\hat{y} - g)^2\, dx. \tag{3.5}$$

Let $[z\,,\, v] \in H^2(\Omega) \times L^2(\Omega)$ be another admissible pair, i.e. satisfying (3.2)–(3.4). We consider admissible variations of the type

$$[y_\varepsilon\,,\, h_\varepsilon] + \lambda\,[z - y_\varepsilon\,,\, v - h_\varepsilon] \in \mathcal{K} \tag{3.6}$$

with $\lambda \in [0,1]$. By comparing the optimal cost with that associated to (3.6), we get the inequality

$$0 \leq \frac{1}{\varepsilon}\int_{\partial\Omega}\frac{\partial y_\varepsilon}{\partial n}\left(\frac{\partial z}{\partial n} - \frac{\partial y_\varepsilon}{\partial n}\right) d\sigma + \int_{\Omega}\ell\, h_\varepsilon\,(v - h_\varepsilon)\, dx. \tag{3.7}$$

We introduce again the auxiliary function $p_\varepsilon \in H^1(\Omega)$ given by (2.6), (2.7), and we underline that this is not the adjoint mapping from control theory, since it does not take into account the state constraint (3.4). A general discussion about this approach in state-constrained control problems, in a different setting, may be found in Bergounioux and Tiba [4].

Inequality (3.7) may be rewritten as

$$0 \leq \int_{\partial\Omega} p_\varepsilon \left(\frac{\partial z}{\partial n} - \frac{\partial y_\varepsilon}{\partial n} \right) d\sigma + \int_\Omega \ell \, h_\varepsilon \, (v - h_\varepsilon) \, dx \, . \tag{3.8}$$

Multiplying by p_ε in the equation for $z - y_\varepsilon$ and integrating by parts, we obtain

$$- \int_\Omega p_\varepsilon \, \ell \, (v - h_\varepsilon) \, dx = - \int_\Omega p_\varepsilon \, \Delta(z - y_\varepsilon) \, dx = - \int_{\partial\Omega} p_\varepsilon \frac{\partial}{\partial n}(z - y_\varepsilon) \, d\sigma \, . \tag{3.9}$$

Combining (3.8) and (3.9), we can infer that

$$0 \leq \int_\Omega \ell \, (p_\varepsilon + h_\varepsilon) \, (v - h_\varepsilon) \, dx \tag{3.10}$$

for any control v admissible for (3.1)–(3.4). Relation (3.10) corresponds to the Pontryagin maximum principle and can be reformulated as

$$0 \leq \int_\Omega (p_\varepsilon + h_\varepsilon) \, (\Delta z - \Delta y_\varepsilon) \, dx \, . \tag{3.11}$$

Consider now the special case $z \in H_0^2(\Omega)$. We notice:

$$\begin{aligned}
- \int_\Omega p_\varepsilon \Delta(z - y_\varepsilon) \, dx &= \int_\Omega \nabla p_\varepsilon \nabla(z - y_\varepsilon) \, dx - \int_{\partial\Omega} p_\varepsilon \frac{\partial}{\partial n}(z - y_\varepsilon) \, d\sigma \\
&= \frac{1}{\varepsilon} \int_{\partial\Omega} \left(\frac{\partial y_\varepsilon}{\partial n} \right)^2 d\sigma - \int_\Omega \Delta p_\varepsilon (z - y_\varepsilon) \, dx + \int_{\partial\Omega} \frac{\partial p_\varepsilon}{\partial n}(z - y_\varepsilon) \, d\sigma \\
&= \frac{1}{\varepsilon} \int_{\partial\Omega} \left(\frac{\partial y_\varepsilon}{\partial n} \right)^2 d\sigma \geq 0 \, .
\end{aligned} \tag{3.12}$$

By (3.11), (3.12), we infer that

$$0 \leq \int_\Omega h_\varepsilon (\Delta z - \Delta y_\varepsilon) \, dx \tag{3.13}$$

for $z \in H_0^2(\Omega)$ admissible. From (3.2), (3.13), and the definitions of ℓ, g, one easily obtains that

$$\int_\Omega u^3 \, \Delta y_\varepsilon \, \Delta(y_\varepsilon - z) \, dx = \int_\Omega f \, (y_\varepsilon - z) \, dx \quad \forall \, z \in \mathcal{K} \cap H_0^2(\Omega) \, . \tag{3.14}$$

From (3.5), it is obvious that $\{h_\varepsilon\}$ is bounded in $L^2(\Omega)$, and by virtue of (3.2), $\{y_\varepsilon\}$ is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. Again (3.5) shows that

$$\frac{\partial y_\varepsilon}{\partial n} \to 0 \quad \text{strongly in } L^2(\partial\Omega). \tag{3.15}$$

Then, we have $y_\varepsilon \to y^*$ weakly in $H^2(\Omega)$, and $y^* \in \mathcal{K} \cap H_0^2(\Omega)$. By using the weak lower semicontinuity of quadratic forms, we can take $\varepsilon \to 0$ in (3.14) and finally arrive at the result:

Theorem 3.1 *The mapping $y^* \in \mathcal{K} \cap H_0^2(\Omega)$ is the unique solution to the variational inequality*

$$\int_\Omega u^3 \, \Delta y^* \, \Delta(y^* - z) \, dx \le \int_\Omega f \, (y^* - z) \, dx \quad \forall \, z \in \mathcal{K} \cap H_0^2(\Omega). \tag{3.16}$$

Remark. The above argument yields the existence of the solution to (3.16) and its approximation by the control problem (3.1)–(3.4). Uniqueness is obtained immediately, by contradiction.

Remark. Important examples entering into the formulation (3.16) are the obstacle problem, obtained for

$$\mathcal{K} = \{z \in H^2(\Omega) \cap H_0^1(\Omega) \, ; \, a \le z \le b \text{ a.e. } \Omega\},$$

or the variational inequality studied by Glowinski et al. [7] via a direct method, corresponding to

$$\mathcal{K} = \{z \in H^2(\Omega) \cap H_0^1(\Omega) \, ; \, a \le \Delta z \le b \text{ a.e. in } \Omega\}.$$

Here, a, b are some given mappings such that $\mathcal{K}$ is nonvoid. If the boundary conditions are changed, or if unilateral conditions on the boundary are considered, then other subspaces of $H^2(\Omega)$ have to be taken into account, and the argument proceeds similarly. A variational inequality for a partially clamped plate was studied by an ad-hoc method in Sprekels and Tiba [14].

Remark. Variational inequalities are obtained by imposing constraints in the variational formulation of the corresponding equation, Lions and Stampacchia [12]. By comparing **Theorem 3.1** with **Theorem 2.1**, we see that this remains valid for the control variational method, as well.

We now continue the study of variational inequalities associated to Kirchhoff–Love arches. We consider the state constrained control problem given by (2.16), (2.17), and

$$[v_1 \, , \, v_2] \in \mathcal{C}, \tag{3.17}$$

where $\mathcal{C} \subset L^\infty(0,1)^2$ is a closed convex set, compatible with the null initial conditions. Notice that (2.18) is no longer imposed and that relations (2.16), (2.17) correspond to a partially clamped arch (in $t = 0$), while (3.17) will yield the unilateral conditions on the arch, as we shall see in the sequel.

All the notations have the same significance as in Section 2; however, the control space for z is $V = \{w \in H^1(0,1)\,;\, w(0) = 0\}$, and the definitions of g_1, g_2 are replaced by

$$g_1 = \delta\,\ell, \quad -g_2'' = h, \quad g_2(0) = g_2'(1) = 0, \tag{3.18}$$

$$\begin{bmatrix}\ell\\ h\end{bmatrix}(t) = \int_t^1 W(t)\,W^{-1}(s)\begin{bmatrix}f_1(s)\\ f_2(s)\end{bmatrix} ds\,. \tag{3.19}$$

As we have no constraints on the control variables u, z, admissibility may be assumed in connection with (3.17), and we obtain again the existence of a unique optimal quadruple denoted by u_δ, z_δ, v_1^δ, v_2^δ, in $L^2(0,1) \times V \times \mathcal{C}$.

We take admissible control variations of the type

$$[u_\delta\,,\, z_\delta] + \lambda[u - u_\delta\,,\, z - z_\delta]\,, \quad \lambda \in [0,1]\,, \tag{3.20}$$

with $[u\,,\,z]$ any admissible control. A simple argument yields the Euler inequality

$$0 \le \frac{1}{\delta}\int_0^1 u_\delta(u - u_\delta)\,ds + \int_0^1 z_\delta'(z - z_\delta)'\,ds\,. \tag{3.21}$$

Under the regularity assumption $W \in W^{2,\infty}(0,1)^4$ (as in **Theorem 2.4**), we shall show that v_1^δ, v_2^δ are the solutions of a general variational inequality.

Fix any $[w_1\,,\,w_2] \in \mathcal{C} \cap [V \times U]$, with $U = \{z \in H^2(0,1)\,;\, z(0) = z'(0) = 0\}$. The corresponding controls, generating w_1, w_2 via (2.16), (2.17), are

$$\mu = w_1' - c\,w_2 - g_1 \in L^2(0,1)\,, \tag{3.22}$$

$$\zeta = w_2' + c\,w_1 - g_2 \in V\,, \tag{3.23}$$

(due to the regularity of W). Moreover, v_1^δ, v_2^δ satisfy (2.19)–(2.21).

Using (3.22), (3.23), and (2.19)–(2.21), in (3.21), we obtain

$$\begin{aligned} 0 \le\ & \frac{1}{\delta}\int_0^1 \Big(-g_1 + (v_1^\delta)' - c\,v_2^\delta\Big)\Big(w_1' - c\,w_2 - (v_1^\delta)' + c\,v_2^\delta\Big)\,ds \\ & + \int_0^1 \Big(-g_2 + (v_2^\delta)' - c\,v_1^\delta\Big)'\Big(w_2' + c\,w_1 - (v_2^\delta)' - c\,v_1^\delta\Big)'\,ds\,. \end{aligned} \tag{3.24}$$

We compute first the terms:

$$- \frac{1}{\delta} \int_0^1 g_1 \Big(w_1' - c\, w_2 - (v_1^\delta)' + c\, v_2^\delta \Big)\, ds + \int_0^1 g_2'' \Big(w_2' + c\, w_1 - (v_2^\delta)' - c\, v_1^\delta \Big)\, ds$$

$$= - \int_0^1 \ell \Big(w_1' - c\, w_2 - (v_1^\delta)' + c\, v_2^\delta \Big)\, ds - \int_0^1 h \Big(w_2' + c\, w_1 - (v_2^\delta)' - c\, v_1^\delta \Big)\, ds$$

$$= - \int_0^1 f_1\, (w_1 - v_1^\delta)\, ds - \int_0^1 f_2 (w_2 - v_2^\delta)\, ds\,. \tag{3.25}$$

In (3.25), we have repeatedly integrated by parts, and we have made use of (3.18), (3.19). Combining (3.24) and (3.25), we have proved the following result:

Theorem 3.2 *If* $W \in W^{2,\infty}(0,1)^4$, *then* v_1^δ, v_2^δ *given by* (2.16), (2.17), (3.17) *satisfy*

$$\frac{1}{\delta} \int_0^1 \Big((v_1^\delta)' - c\, v_2^\delta \Big) \Big((v_1^\delta)' - c\, v_2^\delta - w_1' + c\, w_2 \Big)\, ds$$

$$+ \int_0^1 \Big((v_2^\delta)' + c\, v_1^\delta \Big)' \Big((v_2^\delta)' + c\, v_1^\delta - w_2' - c\, w_1 \Big)'\, ds$$

$$\leq \int_0^1 f_1 (v_1^\delta - w_1)\, ds + \int_0^1 f_2 (v_2^\delta - w_2)\, ds\,, \tag{3.26}$$

for a.e. $[w_1\,,\, w_2] \in \mathcal{C} \cap [V \times U]$.

Remark. If the convex $\mathcal{C}$ includes null conditions at the point $t = 1$, then we obtain a variational inequality for a clamped arch. **Theorem 3.2**, compared with **Theorem 2.4**, is an example of how the spaces (for the state and for the control) should be adapted when different boundary conditions are imposed. The method introduced in this section allows for general unilateral conditions and various boundary conditions. We conjecture that it also allows the extension of **Theorem 2.2** to the case of variational inequalities. Concerning the differentiability properties discussed in **Theorem 2.3**, it is known that, generally, they are not valid for variational inequalities.

Remark. In the case of fourth-order ordinary differential equations, the works of Hlavacek, Bock and Lovisek [8], [9], Sprekels and Tiba [16] discussed variational inequalities associated to beam models. **Theorem 3.2** seems to be a first result in the literature related to arches submitted to unilateral conditions. The problem (2.16), (2.17), (3.17) is a new weak formulation of the variational inequality (3.26), valid for $W \in L^\infty(0,1)^4$.

We close this presentation with some short examples. If $\mathcal{C}$ has the form:

$$\mathcal{C} = L^\infty(0,1) \times \{v_2 \in L^\infty(0,1)\,;\, \alpha \le v_2 \le \beta \text{ a.e. in } (0,1)\}\ ,$$

then we have an obstacle problem for the normal component of the deflection (α and β are some given mappings such that $\mathcal{C}$ allows null initial conditions for v_2). Obstacle problems for the tangential component or for both components are obtained similarly.

Under the smoothness hypothesis $W \in W^{2,\infty}(0,1)^4$, the solution $[v_1^\delta\,,\, v_2^\delta]$ is in $V \times U$, and we can impose from the beginning that $\mathcal{C}$ is a closed convex subset of $V \times U$. One situation of interest is given by:

$$\mathcal{C} = \Big\{[v_1\,,\, v_2] \in V \times U\,;\, v_1(1) \ge r\Big\}$$

with $r \in I\!R$ a given constant. This represents a partially clamped arch with a unilateral condition on the tangential component in the end point $t = 1$. Similar formulations may easily be written for the normal component or for both, or in other points, and so on.

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