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Forward and Backward Lagrangian stochastic models of turbulent transport

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Abstract

The key question analysed in the paper is: under what condition are the Lagrangian stochastic models stochastically reversible? We show that this property is deeply related to Thomson's well-mixed condition. Direct and backward in time Monte Carlo algorithms are suggested. A consistency principle in the turbulent transport problems and in the financial mathematics is analysed to find out an analogy in constructing Lagrangian stochastic models in these two different fields.

1 Introduction and formulation of the problem

In modern Monte Carlo simulation algorithms, one often uses stochastic Lagrangian models to simulate individual Lagrangian trajectories and estimate different Lagrangian statistical characteristics (e.g., see [18]-[24]). We mention, for instance, the dispersion of particles in turbulent flows and the dynamics of the bond and stock prices in financial mathematics (e.g., see [21], [11]). The Lagrangian stochastic models have a form of Ito type stochastic differential equation, and they are constructed often in a quite heuristical way. There exists a more rigorous approach, which is based on an accurate description of the Eulerian dynamics. This dynamics however includes random fields, and there arises a nontrivial problem of consistency between the Lagrangian stochastic models and the stochastic dynamic models generated by the Eulerian random velocity fields. The Lagrangian description allows us to analyze directly the motion of material fluid elements. Importance of the Lagrangian trajectories is that the quantities of practical interest are expressed through the n-particle statistical characteristics. In particular, the mean concentration of a passive scalar and its covariance are defined through the one-particle and two-particle statistical characteristics, respectively, and similarly, a financial derivative can be expressed as an expectation over solutions to a large system of stochastic differential equations (SDE) (see [21]). Another example where Lagrangian stochastic models are used is the *Footprint problem*. As formulated in the literature (e.g., see [1]-[3], [22]), this problem essentially deals with the calculation of the contribution to the mean concentration and its flux at a fixed point from a given source of particles.

There are mainly two different approaches: (1) conventional deterministic methods based on the semi-empirical turbulent diffusion equation and closure assumptions (e.g., see [22]) and (2), stochastic approach which utilizes trajectory simulations (e.g., see [16], [17]-[24] and [12]). The deterministic approach directly deals with the equation governing the mean concentration, but it is restricted by the use of the Boussinesq hypothesis whose applicability should be additionally studied. For instance, this hypothesis cannot be true if the concentration is calculated close to the sources [14]. More generally, the high order closure methods are developed, but different closure hypotheses also should be made [14]. Stochastic models do not require any closure hypotheses, and the main difficulty is to construct adequate Lagrangian trajectories with the desired statistical characteristics [15].

There are two main approaches in constructing stochastic methods. The first one is based on Monte Carlo simulation of the Eulerian random velocity fields (e.g., see [8], [16], [5]). Second approach treats the stochastic Lagrangian trajectories as solutions to the stochastic generalized Langevin equation (e.g., see [23], [19], [17]). The first approach is more rigorous, but generally it requires a lot of computer time. In addition, it needs a detailed information about statistical characteristics of the whole velocity field. In contrast, the second approach needs only one-point probability density function (pdf) of the Eulerian velocity field, and is much more efficient in numerical calculations. It should be noted however, that this approach is rigorously justified only in the case of stationary isotropic turbulent flow. Even in the case of homogeneous but unisotropic turbulence the justification problem remains unsolved; in particular there are several different stochastic models which satisfy the well-mixed condition [23], [10],[18].

In this paper we concentrate around the consistency principle for the stochastic Lagrangian models in the turbulent transport problems, and extend it to more general SDE, e.g., to the evaluation of derivatives in the financial mathematics. We emphasis a deep interrelation between the well-mixed condition due to Thomson and the Girsanov transformation. A backward theoretically zero-variance stochastic model for evaluation of the solution at a fixed point is constructed. We remark that Thomson's backward algorithm leads to a control variate estimator. It is interesting to note that recently, when our paper was in its almost final form, we have found two remarkable papers where special cases were studied. The first paper [7] was written by Kolmogorov in 1937, and a generalization of Kolmogorov's result [25] was published by Yaglom in 1949. The authors are answering the question: when a diffusion process is reversible provided there exists a stationary distribution of the diffusion process under study.

Let us consider a passive scalar dispersed by the turbulent incompressible velocity field in the surface layer of the atmosphere. The passive scalar is assumed to follow the streamlines of the flow We assume that the source of particles is quite arbitrary, for instance, it might be situated on the surface or in the space, or even at given points. Let us denote by $q(\mathbf{x}, t)$ the spatial-temporal density distribution function of the source, i.e., the number of emitted particles per unit volume in a unit time interval at the phase point $(\mathbf{x}, t)$. For simplicity, we assume that initially, the spatial density of particles is zero. The particles are transported by a 3D turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ in the surface layer $D = \{\mathbf{x} = (x_1, x_2, x_3) : x_3 \geq 0\}$. Let us denote by $\mathbf{X}(t; \mathbf{x}_0, t_0)$ and $\mathbf{V}(t; \mathbf{x}_0, t_0)$ the Lagrangian spatial coordinates and the velocity, respectively. Here and throughout the paper we use the boldface characters for vectors, e.g., $\mathbf{a} = (a_1, a_2, a_3)$.

The mean concentration at $(\mathbf{x}, t)$ is defined by [14]:

$$\langle c(\mathbf{x}, t)\rangle = \int_0^t dt_0 \int_D d\mathbf{x}_0 \, q(\mathbf{x}_0, t_0) p_L(\mathbf{x}, t; \mathbf{x}_0, t_0), \tag{1.1 }$$

where $p_L(\mathbf{x}, t; \mathbf{x}_0, t_0) = \langle \delta(\mathbf{x} - \mathbf{X}(t; \mathbf{x}_0, t_0))\rangle$ is the probability density function (pdf) of the particle's coordinate at the time t which was started in the point $\mathbf{x}_0$ at the time t_0, $\delta(\cdot)$ is the Dyrac delta-function. Here and throughout the paper we use the notation $\langle\cdot\rangle$ for the averaging over the samples of the turbulent velocity field.

Let $p_L(\mathbf{x},\mathbf{u},t;\mathbf{x}_0,t_0) = \langle \delta(\mathbf{x}-\mathbf{X}(t;\mathbf{x}_0,t_0))\delta(\mathbf{u}-\mathbf{V}(t;\mathbf{x}_0,t_0))\rangle$ be the pdf of the spatial-velocity phase point. In the analysis, it is convenient to deal with a general quantity, the spatial-velocity distribution of an ensemble of particles:

$$p(\mathbf{x},\mathbf{u},t) = \int_0^t dt_0 \int_D d\mathbf{x}_0\, q(\mathbf{x}_0,t_0) p_L(\mathbf{x},\mathbf{u},t;\mathbf{x}_0,t_0) \ . \tag{1.2 }$$

From (1.1) and (1.2) we find

$$\langle c(\mathbf{x},t)\rangle = \int_{\mathbb{R}^3} p(\mathbf{x},\mathbf{u},t) d\mathbf{u} \ .$$

It is of practical interest to calculate the mean concentration and relevant fluxes for arbitrarily situated surface sources. In the literature, this problem is called a footprint problem (e.g., see [22], [1], [3]). Note that in this problem, the mean concentration and fluxes are evaluated at a fixed point. We consider a more general quantity:

$$(p,h) = \int_{\mathbb{R}^3} d\mathbf{u} \int_0^T dt \int_D d\mathbf{x}\ p(\mathbf{x},\mathbf{u},t) h(\mathbf{x},\mathbf{u},t), \tag{1.3 }$$

where $h(\mathbf{x},\mathbf{u},t)$ is an arbitrary function which can be chosen relevant to the quantity of interest. For instance, in the case $h(\mathbf{y},\mathbf{u},s) = \delta(\mathbf{y}-\mathbf{x})\delta(s-t)$ we have $(p,h) = \langle c(\mathbf{x},t)\rangle$. If $h(\mathbf{y},\mathbf{u},s) = u_i\delta(\mathbf{y}-\mathbf{x})\delta(s-t)$, then $(p,h) = F_i(\mathbf{x},t)$ is the concentration flux in direction i. Thus we concentrate on the problem of calculation of the quamtity (p,h).

2 Stochastic Lagrangian algorithm

To construct algorithms based on the representations given above, we need samples of the Lagrangian trajectories $\mathbf{X}(t) = \mathbf{X}(t;\mathbf{x}_0,t_0)$, $t \geq t_0$. Ideally, if we had samples of the Eulerian velocity $\mathbf{u}(\mathbf{x},t)$, the trajectories could be simulated by solving the problem

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{u}(\mathbf{X}(t),t), \quad t > t_0 \quad \mathbf{X}(t_0) = \mathbf{x}_0 \ . \tag{2.1 }$$

In practice one uses approximate models of the Eulerian velocity field. For instance, randomized models of the Gaussian velocity fields are used (e.g., see [16]). This approach is well developed and justified only in the case of homogeneous turbulence while inhomogeneous case requires further development. In general inhomogeneous case one uses another approach based on stochastic differential equation of Langevin type governing directly the Lagrangian trajectory. This equation has the form (see, for instance, [23], [19], [18]):

$$\begin{aligned} d\mathbf{Y}(t) &= \mathbf{V}(t)dt, \\ d\mathbf{V}(t) &= \mathbf{a}(t,\mathbf{Y}(t),\mathbf{V}(t))dt + \sqrt{C_0\bar{\varepsilon}(\mathbf{Y}(t),t)}\, d\mathbf{W}(t), \end{aligned} \tag{2.2 }$$

where the function $\mathbf{a} = (a_1,a_2,a_3)$ is to be defined in each specific situation, C_0 is the universal Kolmogorov constant ($C_0 \approx 4 \div 6$), and $\bar{\varepsilon}(\mathbf{x},t)$ is the mean dissipation rate of the kinetic energy of turbulence, and $\mathbf{W}(t)$ is the standard 3D Wiener process. In this section, we deal with the general scheme.

Note that to complete the description of the Lagrangian stochastic model, we need to define the behaviour of $(\mathbf{Y}(t), \mathbf{V}(t))$ in the neighbourhood of the boundary $\Gamma = \{\mathbf{x} = (x_1, x_2, x_3) : x_3 = 0\}$, see for details [12].

Obviously, $(\mathbf{Y}(t), \mathbf{V}(t))$, the solution to (2.2) is only an approximation of the solution to (2.1): it is impossible to exactly satisfy $\mathbf{V}(t) = \mathbf{u}(\mathbf{X}(t), t)$, $\mathbf{X}(t) = \mathbf{Y}(t)$. However certain consistency between the Eulerian and Lagrangian description should be satisfied.

Two consistency criteria used in the literature are:
(A) Consistency with the Kolmogorov similarity theory,
(B) Consistency with Novikov's integral relation.

Here (A) reads

$$\langle (dV_i)^2 \rangle = C_0 \bar{\varepsilon} dt, \ (i = 1, 2, 3), \langle dV_1\, dV_2 \rangle = \langle dV_1\, dV_3 \rangle = \langle dV_3\, dV_2 \rangle = 0,$$

where dV_i are the components of the increments of the Lagrangian velocity, $\bar{\varepsilon}$ is the mean rate of the dissipation of turbulence energy, C_0 is the universal constant (e.g., see [14], [19], [23]).

Note that (A) implies (e.g., see [23]) that in (2.1), all the velocity components have the same intensity of the fluctuated part.

Novikov's integral relation has the form [15]

$$p_E(\mathbf{u}; \mathbf{x}, t) = \int\limits_{R^3} p_L(\mathbf{x}, \mathbf{u}, t; \mathbf{x}_0, t_0) d\mathbf{x}_0. \tag{2.3}$$

Here p_E is the probability density function of the Eulerian velocity $\mathbf{u}$, in the fixed point $\mathbf{x}$, at the time t, and p_L is the joint pdf of the true Lagrangian phase point $(\mathbf{Y}, \mathbf{V})$ defined by the trajectory started at $\mathbf{x}_0$. We recall that the flow is assumed to be incompressible.

Thus the consistency with the Novikov relation (2.3) means that the pdf of the model trajectory governed by (2.2), say $\hat{p}_L$, satisfies

$$p_E(\mathbf{u}; \mathbf{x}, t) = \int\limits_{R^3} \hat{p}_L(\mathbf{x}, \mathbf{u}, t; \mathbf{x}_0, t_0) d\mathbf{x}_0. \tag{2.4}$$

Note that (2.4) and the Fokker-Planck-Kolmogorov equation for $\hat{p}_L$ lead to the well-mixed condition due to D. Thomson [23]:

$$\frac{\partial p_E}{\partial t} + u_i \frac{\partial p_E}{\partial x_i} + \frac{\partial}{\partial u_i}(a_i p_E) = \frac{C_0 \bar{\varepsilon}}{2} \left\{ \frac{\partial^2 p_E}{\partial u_1^2} + \frac{\partial^2 p_E}{\partial u_2^2} + \frac{\partial^2 p_E}{\partial u_3^2} \right\}. \tag{2.5}$$

Here we use the summation convention. In [18], the authors study a horizontally homogeneous turbulent flow which implies that p_E does not depend on x_1, x_2. Generally, the main problem is that (2.5) does not define the coefficients a_i of the model (2.2) uniquely. Indeed, even for the homogeneous turbulence, in [20] two different choices of a_i are presented, both satisfying the well-mixed condition (2.5) but whose statistical characteristics are different. For the Gaussian form of p_E, one of appropriate techniques of getting the coefficients a_i is given in [23]. In the nongaussian 3D case there is no appropriate choice of these coefficients. In 2D a nongaussian case was treated by Flesch and Wilson in [1]. These authors mentioned that the two different models do not lead to essentially different results in the case of Gaussian p_E. As reported in [1], the same is true for two models considered in [20].

In [18], a proper choice of the coefficients a_i in a general case of pdf p_E is suggested. The derivation is based on some assumptions which ensure a unique choice of the model.

3 Backward trajectory estimators

In this section we treat the problem of evaluation of the integral:

$$I_{h,q} = \int\limits_D d\mathbf{y} \int\limits_0^T dt \int\limits_D d\mathbf{y}_0 \int\limits_0^t dt_0 h(\mathbf{y},t) q(\mathbf{y}_0,t_0) p^f(\mathbf{y},t;\mathbf{y}_0,t_0), \qquad (3.1\ )$$

where D is a domain in $\mathbb{R}^n$, $T > 0$, h and q are functions defined in $D \times [0,T]$, and $p^f(\mathbf{y},t;\mathbf{y}_0,t_0) = \langle \delta(\mathbf{y} - \mathbf{Y}_t^{\mathbf{y}_0,t_0}) \rangle$ is the transition density of the n-dimensional diffusion process $\mathbf{Y}_t^{\mathbf{y}_0,t_0}$, the solution to

$$dY_i(t) = A_i(\mathbf{Y}(t),t)dt + \sigma_{ij}(\mathbf{Y}(t),t)dW_j(t),\ t > t_0,\ \ \mathbf{Y}(t)|_{t=t_0} = \mathbf{y}_0. \qquad (3.2\ )$$

Here and throughout the paper we keep to use the notation $\langle\ \rangle$ for the averaging over solutions of stochastic differential equations.

We assume that the boundary of D is impenetrable, i.e., the trajectories determined by (3.2) do not reach the boundary. The Direct Monte Carlo estimator for evaluating the integral (3.1) is straightforward:

$$\begin{aligned} I_{h,q} &= \int\limits_D d\mathbf{y}_0 \int\limits_0^T dt_0 \int\limits_D d\mathbf{y} \int\limits_{t_0}^T dt\, h(\mathbf{y},t) q(\mathbf{y}_0,t_0) p^f(\mathbf{y},t;\mathbf{y}_0,t_0) \\ &= \mathbb{E}\Big\{ \frac{q(\tilde{\mathbf{y}}_0,\tilde{t}_0)}{p_0(\tilde{\mathbf{y}}_0,\tilde{t}_0)} \int\limits_{\tilde{t}_0}^T h(\mathbf{Y}_t^{\tilde{\mathbf{y}}_0,\tilde{t}_0},t)\, dt \Big\}. \end{aligned}$$

Here $p_0(\mathbf{y}_0,t_0)$ is an arbitrary pdf in $D \times [0,T]$ consistent with the function $q(\mathbf{y}_0,t_0)$ in the sense that $p_0(\mathbf{y}_0,t_0) > 0$ if $q_0(\mathbf{y}_0,t_0) \neq 0$, and the expectation is taken over all sample points $(\tilde{\mathbf{y}}_0,\tilde{t}_0)$ and sample trajectories $\mathbf{Y}_t^{\tilde{\mathbf{y}}_0,\tilde{t}_0}$, $\tilde{t}_0 \le t \le T$; the random points $\tilde{\mathbf{y}}_0, \tilde{t}_0$ are distributed with $p_0(\mathbf{y}_0,t_0)$.

In the evaluation of the linear functional (p,h) defined in (1.3), two functions are involved: the source $q(\mathbf{x}_0,t_0)$ and the detector function $h(\mathbf{x},\mathbf{u},t)$. In the case when the detector is a delta function, the direct algorithm described above cannot be practically used. Then, a backward in time stochastic differential equation comes in play.

A backward estimator can be obtained by a generalization of Thomson's approach [23]. Assume that we have a positive function $\rho(\mathbf{y},t)$ defined on $D \times [0,T]$ as a solution to the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial y_i}(A_i \rho) = \frac{1}{2}\frac{\partial^2 (B_{ij}\rho)}{\partial y_i \partial y_j}, \tag{3.3 }$$

where $\sigma_{ik}\sigma_{kj} = B_{ij}$. Let $p^b(\mathbf{y}_0, t_0; \mathbf{y}, t) = \langle \delta(\mathbf{y}_0 - \mathbf{Z}_{t_0}^{\mathbf{y},t})\rangle$ be the transition density of the diffusion process $\mathbf{Z}_{t_0}^{\mathbf{y},t}$, $0 \le t_0 \le t$ which is defined by

$$dZ_i = A_i^*(\mathbf{Z}, t_0)\, dt_0 + \sigma_{ij}(\mathbf{Z}, t_0) \overset{\leftarrow}{d} W_j(t_0), \quad t_0 < t, \quad \mathbf{Z}(t) = \mathbf{y}. \tag{3.4 }$$

Here

$$A_i^*(\mathbf{y}, t) = A_i(\mathbf{y}, t) - \frac{1}{\rho(\mathbf{y}, t)}\frac{\partial}{\partial y_j}(B_{ij}(\mathbf{y}, t)\rho(\mathbf{y}, t)). \tag{3.5 }$$

In (3.4), the differential $\overset{\leftarrow}{d} W$ means that here the backward Ito integral is taken[1].

It is convenient to rewrite (3.4) in an integral form:

$$Z_i(t_0) = y_i - \int_{t_0}^{t} A_i^*(\mathbf{Z}, s) ds - \int_{t_0}^{t} \sigma_{ij}(\mathbf{Z}, s) \overset{\leftarrow}{d} W_j(s) \ .$$

We assume, that the solutions to (3.4) do never reach the boundary of D. Then the following statement is true.

Theorem 1. *Let $\rho(\mathbf{y}, t)$ be a positive solution to (3.3). The equality*

$$\rho(\mathbf{y}_0, t_0) p^f(\mathbf{y}, t; \mathbf{y}_0, t_0) = \rho(\mathbf{y}, t) p^b(\mathbf{y}_0, t_0; \mathbf{y}, t) \tag{3.6 }$$

is a necessary and sufficient condition for the unique definition of the diffusion process $\mathbf{Z}_{t_0}^{\mathbf{y},t}$ through (3.4) with the shift term given by (3.5).

Proof. First remark that the function p^b and

$$F(\mathbf{y}_0, t_0; \mathbf{y}, t) = \frac{\rho(\mathbf{y}_0, t_0)}{\rho(\mathbf{y}, t)} p^f(\mathbf{y}, t; \mathbf{y}_0, t_0)$$

satisfy the equations

$$\mathcal{L}_{\mathbf{y}_0, t_0} F = 0, \quad \mathcal{L}_{\mathbf{y}_0, t_0} p^b = 0$$

where the operator $\mathcal{L}_{\mathbf{y}_0, t_0}$ acts on a function $g(\mathbf{y}_0, t_0)$ as follows:

$$\mathcal{L}_{\mathbf{y}_0, t_0} g = \frac{\partial g}{\partial t_0} + \frac{\partial}{\partial y_{0i}}(A_i^* g) + \frac{1}{2}\frac{\partial^2 (B_{ij} g)}{\partial y_{0i}\partial y_{0j}}.$$

[1] The backward Ito integral is defined by

$$\int_s^t \xi(\tau) \overset{\leftarrow}{d} W(\tau) := \int_{T-t}^{T-s} \xi(T-\tau)\, dW_T(\tau),$$

$s \le t \le T$, $W_T(\tau) := W(T) - W(T-\tau)$ is a standard Wiener process. This integral does not depend on the choice of T. For details see, e.g., [9].

Since the values of the functions F and p^b at $t_0 = t$ coincide:

$$p^b(\mathbf{y}_0, t; \mathbf{y}, t) = F(\mathbf{y}_0, t; \mathbf{y}, t) = \delta(\mathbf{y} - \mathbf{y}_0),$$

we conclude that $F \equiv p^b$, provided that the equation $\mathcal{L}_{\mathbf{y}_0,t_0} g = 0$ with initial condition $g(t, \mathbf{y}_0) = \delta(\mathbf{y}_0 - \mathbf{y})$ has a unique solution. This implies that the equality (3.6) is true. Now we note that $\mathcal{L}_{\mathbf{y}_0,t_0} p^b = 0$ is true. Indeed, p^b satisfies the first Kolmogorov equation in backward coordinates $t'_0 = T - t_0,\ \mathbf{y}'_0 = -\mathbf{y}_0$, which results in $\mathcal{L}_{\mathbf{y}_0,t_0} p^b = 0$ when back to the direct coordinates.

The equality $\mathcal{L}_{\mathbf{y}_0,t_0} F = 0$ then follows from (3.3), the inverse Kolmogorov's equation

$$\frac{\partial p^f(\mathbf{y}, t; \mathbf{y}_0, t_0)}{\partial t_0} + A_i(\mathbf{y}_0, t_0)\frac{\partial p^f}{\partial y_{0i}} + \frac{1}{2}B_{ij}(\mathbf{y}_0, t_0)\frac{\partial^2 p^f}{\partial y_{0i}\partial y_{0j}} = 0\ ,$$

and the expression for $A^*_i(\mathbf{y}_0, t_0)$ through A_i and B_{ij} given above. To find that (3.5) is the unique right choice, just apply the operator $\mathcal{L}_{\mathbf{y}_0,t_0}$ to both sides of (3.6) and rearrange the terms. In more details, this kind of transformation is presented in Theorem 2 (formula (4.8)) in a similar situation. Theorem is proved.

Now, we present the backward Monte Carlo algorithm based on the property (3.6). We proceed as follows: substitute the expression for p^f from (3.6) into (3.1), then

$$I_{h,q} = \int\limits_D d\mathbf{y} \int\limits_0^T dt \int\limits_D d\mathbf{y}_0 \int\limits_0^t dt_0 h(\mathbf{y}, t) q(\mathbf{y}_0, t_0)\frac{\rho(\mathbf{y}, t)}{\rho(\mathbf{y}_0, t_0)} p^b(\mathbf{y}_0, t_0; \mathbf{y}, t). \tag{3.7}$$

Let $r(\mathbf{y}, t)$ be a probability density in $D \times [0, T]$ consistent with $h\rho$, i.e., $r > 0$ if $h\rho \neq 0$. Then from (3.7) we get

$$I_{h,q} = \mathbf{E}\left\{\frac{h(\tilde{\mathbf{y}}, \tilde{t})\rho(\tilde{\mathbf{y}}, \tilde{t})}{r(\tilde{\mathbf{y}}, \tilde{t})} \int\limits_0^{\tilde{t}} \frac{q(\mathbf{Z}^{\tilde{\mathbf{y}},\tilde{t}}_{t_0}, t_0)}{\rho(\mathbf{Z}^{\tilde{\mathbf{y}},\tilde{t}}_{t_0}, t_0)}\, dt_0\right\}.$$

Here the expectation is taken over the random points $(\tilde{\mathbf{y}}, \tilde{t})$ distributed in $D \times [0, T]$ with density $r(\mathbf{y}, t)$, and backward trajectories $\mathbf{Z}^{\tilde{\mathbf{y}},\tilde{t}}_{t_0}$, $0 \le t_0 \le \tilde{t}$.

4 Transformed direct trajectories

Let us introduce two differential operators, $\mathcal{L}^A_{\mathbf{y},t}$ and $\mathcal{L}^{*A}_{\mathbf{y},t}$, acting on a function $g(\mathbf{y}, t)$, $\mathbf{y} = (y_1, \ldots, y_n)$ through

$$\mathcal{L}^A_{\mathbf{y},t}\, g = \frac{\partial g}{\partial t} + \frac{\partial}{\partial y_i}(A_i g) - \frac{1}{2}\frac{\partial^2(B_{ij} g)}{\partial y_i \partial y_j}, \tag{4.1}$$

and

$$\mathcal{L}^{*A}_{\mathbf{y},t}\, g = \frac{\partial g}{\partial t} + A_i(\mathbf{y}, t)\frac{\partial g}{\partial y_i} + \frac{1}{2}B_{ij}(\mathbf{y}, t)\frac{\partial^2 g}{\partial y_i \partial y_j}\ . \tag{4.2}$$

Let us introduce, along with the forward in time diffusion process $\mathbf{Y}$ governed by (3.2), a new forward in time diffusion process $\hat{\mathbf{Y}}$ defined by the following stochastic differential equation

$$d\hat{Y}_i(t) = \hat{A}_i(\hat{\mathbf{Y}}(t), t)dt + \hat{\sigma}_{ij}(\hat{\mathbf{Y}}(t), t)dW_j(t), \ t > t_0, \ \ \hat{\mathbf{Y}}(t)\Big|_{t=t_0} = \mathbf{y}_0, \tag{4.3}$$

and let $p_*^f(\mathbf{y}, t; \mathbf{y}_0, t_0) = \langle \delta(\mathbf{y} - \hat{\mathbf{Y}}_t^{\mathbf{y}_0, t_0}) \rangle$ be the transition density of the n-dimensional diffusion process $\hat{\mathbf{Y}}_t^{\mathbf{y}_0, t_0}$, the solution to (4.3).

Our goal is to derive the form of the coefficients $\hat{A}_i$ and $\hat{\sigma}_{ij}$ from the following assumption.

We suppose that at the time t, the "particles" are distributed in the phase space with a probability density function proportional to $\rho^*(\mathbf{y}, t)$. Our assumption then reads:

$$p_*^f(\mathbf{y}, t; \mathbf{y}_0, t_0)\rho^*(\mathbf{y}_0, t_0) = p^f(\mathbf{y}, t; \mathbf{y}_0, t_0)\rho^*(\mathbf{y}, t). \tag{4.4}$$

Below we show that there exists such a choice of the function $\rho^*(\mathbf{y}, t)$ and coefficients $\hat{A}_i$, $\hat{\sigma}_{ij}$. Indeed, let ρ^* be a positive solution to

$$\mathcal{L}_{\mathbf{y},t}^{*A} \, \rho^*(\mathbf{y}, t) = 0. \tag{4.5}$$

Theorem 2. *The equality (4.4) with the function $\rho^*(\mathbf{y}, t)$ satisfying (4.5) is a necessary and sufficient condition for the unique definition of the diffusion process $\hat{Y}_i$ through (4.3) with*

$$\hat{A}_i(\mathbf{y}, t) = A_i(\mathbf{y}, t) + \frac{B_{ij}(\mathbf{y}, t)}{\rho^*(\mathbf{y}, t)} \frac{\partial \rho^*}{\partial y_j}, \quad \hat{\sigma}_{ij}(\mathbf{y}, t) = \sigma_{ij}(\mathbf{y}, t). \tag{4.6}$$

Proof. Assume (4.4) is true. We prove that (4.6) is the unique right choice.

Let us apply the operator $\mathcal{L}_{\mathbf{y},t}^{\hat{A}}$ to both sides of (4.4). Since, by definition,

$$\mathcal{L}_{\mathbf{y},t}^{\hat{A}} \, p_*^f(\mathbf{y}, t; \mathbf{y}_0, t_0) = 0, \tag{4.7}$$

we conclude

$$\mathcal{L}_{\mathbf{y},t}^{\hat{A}} \left[p^f(\mathbf{y}, t; \mathbf{y}_0, t_0)\rho^*(\mathbf{y}, t) \right] = 0.$$

Equivalent transformations yield

$$\begin{aligned} \mathcal{L}_{\mathbf{y},t}^{\hat{A}}[\, p^f(\mathbf{y}, t; \mathbf{y}_0, t_0)\rho^*(\mathbf{y}, t)] \; &= \; \rho^*(\mathbf{y}, t)\mathcal{L}_{\mathbf{y},t}^{A} \, p^f(\mathbf{y}, t; \mathbf{y}_0, t_0) + \, p^f \mathcal{L}_{\mathbf{y},t}^{*A} \rho^*(\mathbf{y}, t) \\ &+ \; \frac{\partial}{\partial y_i}\Big[\rho^* \, p^f(\hat{A}_i - A_i) - \hat{B}_{ij} \, p^f \frac{\partial \rho^*}{\partial y_j}\Big] \\ &- \; \frac{\rho^*}{2} \frac{\partial^2 (p^f(\hat{B}_{ij} - B_{ij}))}{\partial y_i \partial y_j} + \frac{p^f}{2}(\hat{B}_{ij} - B_{ij}) \frac{\partial^2 \rho^*}{\partial y_i \partial y_j} = 0. \end{aligned} \tag{4.8}$$

By the definition, $\mathcal{L}_{\mathbf{y},t}^{A} \, p^f(\mathbf{y}, t; \mathbf{y}_0, t_0) = 0$ and $\mathcal{L}_{\mathbf{y},t}^{*A} \rho^*(\mathbf{y}, t) = 0$. Therefore the appropriate choice of the coefficients $\hat{A}_i$ and $\hat{B}_{ij}$ is given by (4.6). Note that this choice is unique because the relation (4.4) implies that the coefficients $\hat{A}_i$ and $\hat{B}_{ij}$ are uniquely related to the diffusion process whose transition density is p_*^f.

We prove now that (4.4) is true if the differential equation (4.3) governing the diffusion process $\hat{\mathbf{Y}}$ is defined by the coefficients (4.6) provided the function ρ^* is a positive solution to (4.5). Let

$$F(\mathbf{y},t;\mathbf{y}_0,t_0)=\frac{p^f(\mathbf{y},t;\mathbf{y}_0,t_0)\rho^*(\mathbf{y},t)}{\rho^*(\mathbf{y}_0,t_0)}.$$

We now prove that the $F\equiv p^f_*$. Indeed, the function $p^f_*(\mathbf{y},t;\mathbf{y}_0,t_0)$ satisfies the equation (4.7). The same equation is satisfied by the function $F(\mathbf{y},t;\mathbf{y}_0,t_0)$ which can be seen from the transformations (4.8). Therefore, $F\equiv p^f_*$, since these functions coincide at $t=t_0$. Of course it is assumed that the problem $\mathcal{L}^{\hat{A}}_{\mathbf{y},t}\,\psi(\mathbf{y},t)=0$, $\psi(\mathbf{y},t_0)=\delta(\mathbf{y}-\mathbf{y}_0)$ has a unique solution.

Remark 1. Note that it is possible to derive different relations similar to (4.4). To this end, we use a backward diffusion process. Let us define a transition density function $p^b_1(\mathbf{y}_0,t_0;\mathbf{y},t)$ by

$$\frac{\partial p^b_1}{\partial t_0}+\frac{\partial}{\partial y_{0i}}(A'_i p^b_1)+\frac{1}{2}\frac{\partial^2(B_{ij}p^b_1)}{\partial y_{0i}\partial y_{0j}}=0,\quad p^b_1(\mathbf{y}_0,t;\mathbf{y},t)=\delta(\mathbf{y}_0-\mathbf{y}),$$

where

$$A'_i(\mathbf{y},t)=A_i(\mathbf{y},t)-\frac{\partial B_{ij}(\mathbf{y},t)}{\partial y_j},$$

i.e., $p^b_1(\mathbf{y}_0,t_0;\mathbf{y},t)$ is a pdf of a diffusion process $\hat{\mathbf{Z}}^{\mathbf{y},t}(t_0)$ defined as the solution to the problem

$$\begin{aligned}&d\hat{Z}_i(t_0)=A'_i(\hat{\mathbf{Z}}(t_0),t_0)dt_0+\sigma_{ij}(\hat{\mathbf{Z}}(t_0),t_0)\overleftarrow{d}W_j(t_0),\\&0\le t_0\le t,\quad \hat{\mathbf{Z}}(t_0)\Big|_{t_0=t}=\mathbf{y}.\end{aligned}$$

The functions $p^f(\mathbf{y},t;\mathbf{y}_0,t_0)$ and $p^b_1(\mathbf{y}_0,t_0;\mathbf{y},t)$ are related by

$$p^f(\mathbf{y},t;\mathbf{y}_0,t_0)=p^b_1(\mathbf{y}_0,t_0;\mathbf{y},t)\exp\Big(-\int_{t_0}^{t}\beta(s)ds\Big),\tag{4.9}$$

where

$$\beta=\frac{\partial A_i}{\partial y_i}-\frac{1}{2}\frac{\partial^2 B_{ij}}{\partial y_i\partial y_j}.$$

The proof is similar to that of Theorem 2. Indeed, let

$$F=p^b_1(\mathbf{y}_0,t_0;\mathbf{y},t)\exp\Big(-\int_{t_0}^{t}\beta(s)ds\Big).$$

The functions p^f and F satisfy the same equation

$$\mathcal{L}^{*A}_{\mathbf{y}_0,t_0}\,p^f(\mathbf{y},t;\mathbf{y}_0,t_0)=0,\quad \mathcal{L}^{*A}_{\mathbf{y}_0,t_0}\,F=0.$$

Since the values of the functions F and p^b_1 coincide at $t_0=t$ (that is, they equal to $\delta(\mathbf{y}_0-\mathbf{y})$), we conclude that $F=p^b_1$.

Analogous to (4.9), we can get another equality which relates the forward and backward pdf's:

$$p_1^b(\mathbf{y}_0, t_0; \mathbf{y}, t)\rho_1(\mathbf{y}, t) = p_*^f(\mathbf{y}, t; \mathbf{y}_0, t_0)\rho_1(\mathbf{y}_0, t_0). \tag{4.10 }$$

Here ρ_1 is a positive solution to the equation

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial y_i}(\rho_1 A_i') + \frac{1}{2}\frac{\partial^2(B_{ij}\rho_1)}{\partial y_i \partial y_j} = 0.$$

In conclusion let us note that the relation (4.4) follows from (4.10) and (4.9).

5 Applications in stochastic finance theory

5.1 Valuation of financial derivatives

In many cases, for instance, in financial mathematics, the functional of interest are averages of the direct trajectories, but the variance of the estimators is too large. An approach to decrease the variance based on the Girsanov transformation can be used (e.g., see [13]). We will show that the approach presented in the previous sections leads to the same transformed stochastic differential equation obtained by Girsanov transformation to ensure a zero variance. However we suggest a special constructive choice of the function in the Girsanov transformation which leads to a control variate estimators.

In the stochastic finance theory (e.g., see [21]), the valuation of wide class of derivatives is based on the calculation of the expectation

$$u(\mathbf{y}_0, t_0) = \mathbf{E}\, f(\mathbf{Y}_t^{\mathbf{y}_0, t_0})$$

over the solutions to a forward in time stochastic differential equation of type (3.2); here $f(\mathbf{y})$ is a payoff function given at a time t.

Thus, the function $u(\mathbf{y}_0, t_0)$ solves the problem

$$\mathcal{L}^{*A}_{\mathbf{y}_0, t_0} u = 0, \quad u(\mathbf{y}, t) = f(\mathbf{y}). \tag{5.1 }$$

As we have shown above, a different probabilistic representation of this problem can be obtained using the relation (4.4):

$$u(\mathbf{y}_0, t_0) = \rho^*(\mathbf{y}_0, t_0)\mathbf{E}\left\{ \frac{f(\hat{\mathbf{Y}}_t^{t_0, \mathbf{y}_0})}{\rho^*(\hat{\mathbf{Y}}_t^{t_0, \mathbf{y}_0}, t)} \right\} . \tag{5.2 }$$

Remarkably, the variance of this estimator can be made zero if we assume that the solution $u(\mathbf{y}_0, t_0)$ is known to within a constant factor c not depending on $(\mathbf{y}_0, t_0)$. Indeed, this follows from the choice $\rho^*(\mathbf{y}_0, t_0) = cu(\mathbf{y}_0, t_0)$. This choice is possible since the condition (4.5) is satisfied. Since $u(\mathbf{y}, t) = f(\mathbf{y})$ we put $c = \rho^*(\mathbf{y}, t)/f(\mathbf{y})$.

Thus theoretically, the variance can be made zero. This can be used in Monte Carlo calculations in a standard manner, by constructing an approximation of ρ^*. Note that analogous results can be obtained by applying the Girsanov transformation, however more interesting is to find a constructive choice of the functions ρ and ρ^* which lead to a reduction of the variance.

Indeed, we will show below that the function $u(\hat{\mathbf{Y}}_t^{t_0,\mathbf{y}_0},t)/\rho^*(\hat{\mathbf{Y}}_t^{t_0,\mathbf{y}_0},t)$ is a martingale, thus presenting a reasonable choice of the free function $\mathbf{h}(\mathbf{y},t)$ in the Girsanov transformation.

The same can be done in the backward algorithm. Indeed, let $v(\mathbf{y},t)$ be the solution to

$$\mathcal{L}^A_{\mathbf{y},t} v = 0, \quad v(\mathbf{y}_0,t_0) = f(\mathbf{y}_0). \tag{5.3 }$$

In this case we have by (3.6)

$$v(\mathbf{y},t) = \int d\mathbf{y}_0\, f(\mathbf{y}_0)\, p^f(\mathbf{y},t;\mathbf{y}_0,t_0) = \int d\mathbf{y}_0\, v(\mathbf{y}_0,t_0)\, \frac{\rho(\mathbf{y},t)}{\rho(\mathbf{y}_0,t_0)}\, p^b(\mathbf{y}_0,t_0;\mathbf{y},t).$$

Consequently,

$$v(\mathbf{y},t) = \mathbf{E}\xi(\mathbf{y},t), \quad \xi(\mathbf{y},t) = \rho(\mathbf{y},t)\frac{v(\mathbf{Z}^{\mathbf{y},t}_{t_0},t_0)}{\rho(\mathbf{Z}^{\mathbf{y},t}_{t_0},t_0)}. \tag{5.4 }$$

Similarly, if we choose $\rho(\mathbf{y},t) = cv(\mathbf{y},t)$ $(c = \rho(\mathbf{y}_0,t_0)/f(\mathbf{y}_0))$, we get the zero variance.

5.2 Use of Girsanov Transformation

Note that the crucial point in the backward algorithm is the relation (3.6) which is proven in Theorem 1 for the special choice of the drift coefficient (3.5).

It is convenient to work here with the inverse time, so let us introduce the variables: $t' = T - t$, $\mathbf{y}' = -\mathbf{y}$. In these variables the problem (5.3) reads

$$\frac{\partial v}{\partial t'} + \frac{\partial}{\partial y'_i}(A_i v) + \frac{1}{2}\frac{\partial^2 (B_{ij} v)}{\partial y'_i \partial y'_j} = 0, \quad v(\mathbf{y}',t')|_{t'=T_b} = f(\mathbf{y}'),$$

$$0 \le t' \le T_b \equiv T - t_0.$$

It will not cause confusion, if we omit, in what follows, the prime sign of the variables $\mathbf{y}$ and t. Simple algebra gives:

$$\mathcal{L}^{*A'}_{\mathbf{y},t} v(\mathbf{y},t) + \beta v(\mathbf{y},t) = 0, \tag{5.5 }$$

where

$$A'_i(\mathbf{y},t) = A_i(\mathbf{y},t) + \frac{\partial B_{ij}}{\partial y_j}, \quad \beta(\mathbf{y},t) = \frac{\partial A_i}{\partial y_i} + \frac{1}{2}\frac{\partial^2 B_{ij}}{\partial y_i \partial y_j}.$$

Here the operator $\mathcal{L}^{*A'}_{\mathbf{y},t}$, adjoint to $\mathcal{L}^{A'}_{\mathbf{y},t}$ is defined as in (4.2). The probabilistic representation for the solution to (5.5) has the form [4]:

$$v(\mathbf{y},t) = \mathbf{E}_{(\mathbf{y},t)}\left\{ f(\hat{\mathbf{Z}}^{\mathbf{y},t}(T_b))\eta_1(\hat{\mathbf{Z}}^{\mathbf{y},t},T_b)\right\} \tag{5.6 }$$

where

$$\eta_1(\hat{\mathbf{Z}}^{\mathbf{y},t},\tau) = \exp\Big(\int_t^\tau \beta(\hat{\mathbf{Z}}^{\mathbf{y},t}(s),s)ds\Big),$$

and the stochastic process $\hat{\mathbf{Z}}^{\mathbf{y},t}(s)$ is defined as the solution to the problem

$$d\hat{Z}_i(s) = A_i'(\hat{\mathbf{Z}}(s), s)ds + \sigma_{ij}(\hat{\mathbf{Z}}(s), s)dW_j(s),$$
$$0 \le s \le T_b, \quad \hat{\mathbf{Z}}(s)\Big|_{s=0} = \mathbf{y}.$$

We now use the Girsanov transformation [6] to turn to a different stochastic differential equation with a free chosen vector function $\mathbf{h}(\mathbf{y}, t)$. Let us introduce a random process $\tilde{\mathbf{Z}}^{\mathbf{y},t}(s)$ as the solution to the problem

$$d\,\tilde{Z}_i(s) = \Big(A_i'(\tilde{\mathbf{Z}}(s), s) - \sigma_{ij}(\tilde{\mathbf{Z}}(s), s)h_j(\tilde{\mathbf{Z}}(s), s)\Big)ds + \sigma_{ij}(\tilde{\mathbf{Z}}(s), s)dW_j(s),$$
$$0 \le s \le T_b, \quad \tilde{\mathbf{Z}}(s)\Big|_{s=0} = \mathbf{y}, \tag{5.7 }$$

where $h_j(\mathbf{y}, t)$ is an arbitrary function entering the Girsanov transformation of one standard Wiener process to another one. Let us define a random process $\eta_2(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)$ as the solution to the problem

$$d\eta_2(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s) = \eta_2(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)h_j(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\,dW_j(s), \quad \eta_2(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\Big|_{s=0} = 1,$$

and the random process

$$\eta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s) = \eta_1(\tilde{\mathbf{Z}}^{\mathbf{y},t}, s)\eta_2(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s).$$

Note that this process solves the problem

$$d\eta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s) = \eta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\Big(\beta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)ds + h_j(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\,dW_j(s)\Big)$$
$$\eta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\Big|_{s=0} = 1.$$

We show now that $v(\tilde{\mathbf{Z}}(s), s)\eta(\tilde{\mathbf{Z}}(s), s)$ is a martingale. Indeed, it is sufficient to show that the shift term in $d\,[v(\tilde{\mathbf{Z}}(s), s)\eta(\tilde{\mathbf{Z}}(s), s)]$ vanishes. The Ito formula yields:

$$\begin{aligned} d\,[v(\tilde{\mathbf{Z}}, s)\eta(\tilde{\mathbf{Z}}, s)] &= \eta(\tilde{\mathbf{Z}}, s) \\ &\times \left\{\frac{\partial v}{\partial s} + A_i'(\tilde{\mathbf{Z}}, s)\frac{\partial v}{\partial z_i} + \frac{1}{2}B_{ij}(\tilde{\mathbf{Z}}, s)\frac{\partial^2 v}{\partial z_i \partial z_j} + v(\tilde{\mathbf{Z}}, s)\beta(\tilde{\mathbf{Z}}, s)\right\} ds \\ &+ \eta(\tilde{\mathbf{Z}}, s)\left\{\frac{\partial v}{\partial z_i}\sigma_{ij}(\tilde{\mathbf{Z}}, s) + v(\tilde{\mathbf{Z}}, s)h_j(\tilde{\mathbf{Z}}, s)\right\} dW_j(s). \end{aligned} \tag{5.8 }$$

In view of (5.5) the shift term in this expression vanishes.

It is proven that $v(\tilde{\mathbf{Z}}(s), s)\eta(\tilde{\mathbf{Z}}(s), s)$ is a martingale, hence

$$v(\mathbf{y}, t) = \mathbf{E}_{(\mathbf{y},t)}\left\{v(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\eta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(s), s)\right\} \tag{5.9 }$$

for any $s \in [0, T_b]$. Here we used the property $\eta(\tilde{\mathbf{Z}}, s)\Big|_{s=0} = 1$.

Thus (5.9) gives us a series of probabilistic representations depending on the arbitrary function $\mathbf{h}(\mathbf{y}, t)$. It is now of interest to find an appropriate choice of this function. One of possible approach would be to find $\mathbf{h}(\mathbf{y}, t)$ for which the unbiased random estimator

$$\zeta(\mathbf{y}, t) = f(\tilde{\mathbf{Z}}^{\mathbf{y},t}(T_b))\eta(\tilde{\mathbf{Z}}^{\mathbf{y},t}(T_b), T_b); \quad v(\mathbf{y}, t) = \mathbf{E}_{(\mathbf{y},t)}\zeta(\mathbf{y}, t)$$

has a zero variance. We will show now that such a choice exists. Indeed, note that the variance of $\zeta(\mathbf{y},t)$ is zero if we force the coefficient at the Wiener increment in (5.8) to be zero:

$$\eta(\tilde{\mathbf{Z}},s)\frac{\partial v}{\partial z_i}\sigma_{ij}(\tilde{\mathbf{Z}},s)+v(\tilde{\mathbf{Z}},s)\eta(\tilde{\mathbf{Z}},s)h_j(\tilde{\mathbf{Z}},s)=0.$$

From this we get

$$h_i(\tilde{\mathbf{Z}},s)=-\frac{1}{v(\tilde{\mathbf{Z}},s)}\frac{\partial v(\tilde{\mathbf{Z}},s)}{\partial z_j}\sigma_{ij}(\tilde{\mathbf{Z}},s). \tag{5.10}$$

Remark 2. Note that for this special choice of $\mathbf{h}(\mathbf{y},t)$ and for $\rho(\mathbf{y},t)=const\cdot v(\mathbf{y},t)$ the process $\tilde{\mathbf{Z}}^{\mathbf{y},t}(s)$ coincides with the process $\mathbf{Z}^{\mathbf{y},t}(s)$ defined by (3.4). Indeed, in the coordinates $(\mathbf{y}',t')$, these processes are governed by stochastic differential equations whose drift terms coincide and are equal to

$$\begin{aligned}A_i^*(\mathbf{y}',t') &= A_i'(\mathbf{y}',t')-\sigma_{ij}h_j(\mathbf{y}',t')\\ &= A_i(\mathbf{y}',t')+\frac{1}{\rho(\mathbf{y}',t')}\frac{\partial}{\partial y_j'}\Big(B_{ij}(\mathbf{y}',t')\rho(\mathbf{y}',t')\Big).\end{aligned}$$

Statement 1. *The random processes $v(\mathbf{Z}^{\mathbf{y},t}(s),s)/\rho(\mathbf{Z}^{\mathbf{y},t}(s),s)$ and $u(\hat{\mathbf{Y}},t)/\rho^*(\hat{\mathbf{Y}},t)$ are martingales.*

Indeed,

$$\begin{aligned}d\Big[v/\rho\Big](\mathbf{Z},s) &= \frac{1}{\rho}\Big[\mathcal{L}^A_{\mathbf{z},s}v(\mathbf{Z},s)\Big]\,dt-\frac{v}{\rho^2}\Big[\mathcal{L}^A_{\mathbf{z},s}\rho(\mathbf{Z},s)\Big]\,dt\\ &+ \frac{1}{\rho}\Big[\frac{\partial v}{\partial z_i}-\frac{v}{\rho}\frac{\partial\rho}{\partial z_i}\Big]\,\sigma_{ij}\,dW_j\ .\end{aligned}$$

Since ρ is a positive solution to (3.3), and v solves the problem (5.3), we find that the shift term vanishes, hence, the martingale property is proven.

Analogously, u/ρ^* is a martingale. Indeed, by definitions of the functions u and ρ^* we conclude that the shift term in

$$\begin{aligned}d\Big[u/\rho^*\Big](\hat{\mathbf{Y}},t) &= \frac{1}{\rho^*}\Big[\mathcal{L}^{*A}_{\mathbf{y},t}u(\hat{\mathbf{Y}},t)\Big]\,dt-\frac{u}{\rho^{*2}}\Big[\mathcal{L}^{*A}_{\mathbf{y},t}\rho^*(\hat{\mathbf{Y}},t)\Big]\,dt\\ &+ \frac{1}{\rho^*}\Big[\frac{\partial u}{\partial y_i}-\frac{u}{\rho^*}\frac{\partial\rho*}{\partial y_i}\Big]\,\sigma_{ij}\,dW_j\ .\end{aligned} \tag{5.11}$$

vanishes, hence, u/ρ^* is a martingale.

It is interesting to note that if we use in the Girsanov transformation the function $\mathbf{h}$ as in (5.10) but replacing the unknown function v with the known function ρ, then we come to a control variate estimator in (5.9). Indeed, by (5.11) and by the definition of the random process $\mathbf{Z}^{\mathbf{y},t}$ given in (3.4) we get:

$$\begin{aligned}v(\mathbf{y},t) &= \mathbf{E}_{(\mathbf{y},t)}\left\{\Big[\rho(\mathbf{Z}^{\mathbf{y},t},s)+v(\mathbf{Z}^{\mathbf{y},t}(s),s)-\rho(\mathbf{Z}^{\mathbf{y},t},s)\Big]\frac{\rho(\mathbf{y},t)}{\rho(\mathbf{Z}^{\mathbf{y},t},s)}\right\}\\ &= \rho(\mathbf{y},t)+\mathbf{E}_{(\mathbf{y},t)}\left\{\Big[v(\mathbf{Z}^{\mathbf{y},t}(s),s)-\rho(\mathbf{Z}^{\mathbf{y},t},s)\Big]\frac{\rho(\mathbf{y},t)}{\rho(\mathbf{Z}^{\mathbf{y},t},s)}\right\}\end{aligned}$$

for any $s \in [t, T]$. This is the standard control variate estimator which leads to a variance reduction if the function ρ is a reasonable approximation to the function v. This shows that Thomson's algorithm results in the control variate estimator with the function ρ equal to the pdf p_E.

6 Conclusion

Lagrangian stochastic models for the particle's transport and financial derivatives are constructed on the basis of statistically reversible stochastic differential equations. Direct and backward in time Monte Carlo algorithms are suggested.

The backward algorithm originally presented by Thomson is extended to more general case when the transport in the phase space is described by a general stochastic differential equation. It is shown that a special choice of the shift term leads to a zero variance Monte Carlo estimators both for the concentration and financial derivatives. We suggest a special choice of the Girsanov transformation which leads to the control variate estimator which present a constructive variance reduction technique.

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