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Hysteresis filtering in the space of bounded measurable functions

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Abstract

We define a mapping which with each function $u \in L^\infty(0,T)$ and an admissible value of $r > 0$ associates the function ξ with a prescribed initial condition ξ^0 which minimizes the total variation in the r-neighborhood of u in each subinterval $[0,t]$ of $[0,T]$. We show that this mapping is non-expansive with respect to u, r and ξ^0, and coincides with the so-called play operator if u is a regulated function.

Introduction

The subject of the paper is motivated by applications of one-dimensional hysteresis operators in damage evaluation algorithms based on the classical *rainflow counting method*, see [2, 3] for further references. The original engineering problem consists in estimating the material fatigue caused by a large number of medium amplitude oscillations. The input signal has the form of a very long sequence of real numbers representing, say, successive measurements of one stress component. The rainflow method picks out and counts closed loops of each given amplitude and, according to the so-called *Palmgren-Miner rule*, gives the instantaneous value of the damage functional as a linear superposition of individual contributions of each closed loop obtained from the *Wöhler diagram*. According to the experimental evidence, there exists a number $r > 0$ such that loops of amplitude smaller than r contribute only negligibly to the total damage. Such loops can therefore be filtered out of the input string in order to reduce the computational complexity. It was shown in [3] that the rainflow filtering procedure coincides with what is called the *play operator* in the literature devoted to the mathematical theory of hysteresis, see [2, 6, 7, 9].

It is convenient to represent the input string as a (piecewise constant) *function of time* and to consider the play operator in a suitable space of (possibly discontinuous) functions defined in a time interval $[0,T]$. A natural candidate seems to be the space of left-continuous regulated functions as the closure of the set of left-continuous piecewise constant functions with respect to the uniform convergence. For a given parameter $r > 0$, a given left-continuous piecewise monotone input function $u : [0,T] \to \mathbb{R}$ with ℓ monotonicity intervals $[t_{k-1}, t_k]$, $0 = t_0 < t_1 < \ldots < t_\ell = T$, and a given initial condition $\xi^0 \in \mathbb{R}$ we define the output $\xi = \mathcal{F}_r[\xi^0, u]$ of the play operator $\mathcal{F}_r$ by the recurrent formula

$$\xi(t) \;=\; \max\{u(t) - r\,,\, \min\{u(t) + r\,,\, \xi(t_{k-1})\}\} \tag{0.1}$$

for $t \in \,]t_{k-1}, t_k]$, $k = 1, \ldots, \ell$. It was shown in [2] that the operator $\mathcal{F}_r$ thus defined is Lipschitz continuous with respect to the supremum norm: it can therefore be extended to a Lipschitz continuous operator in the whole space of left-continuous regulated functions.

Alternatively, the play operator for continuous inputs and $\xi^0 \in [u(0) - r, u(0) + r]$ can be considered as the solution operator of the evolution variational inequality in the Stieltjes integral form

$$
(0.2) \qquad \left\{ \begin{array}{rcl} |u(t) - \xi(t)| & \le & r \quad \forall t \in [0,T]\,, \\ \displaystyle\int_0^T (u(t) - \xi(t) - y(t))\, d\xi(t) & \ge & 0 \end{array} \right.
$$

for every continuous test function y such that $|y(t)| \le r$ for every t, see [7]. The integral is meaningful due to the remarkable fact pointed out in [6] that the play operator maps continuous functions into continuous functions of bounded variation.

Our aim here is to construct a further extension of the play operator beyond the spaces of continuous or regulated functions. Since both these spaces are closed in $L^\infty(0,T)$, a simple density argument based either on the explicit formula (0.1) or on the variational inequality (0.2) cannot work. We make use of another particular property of the play discovered more than ten years ago by A. Vladimirov and V. Chernorutskii for continuous inputs, namely that it associates with each function u the function of *minimal total variation* within the r-neighborhood of u in each subinterval $[0,t]$ of $[0,T]$. This also illustrates the hidden meaning of hysteresis filtering in the original engineering problem: the play operator minimizes the amount of relevant information which has to be stored. The original result has been published only recently in [8, Section 4], it had been however mentioned earlier as private communication in [7] and, in another form, in [9] (cf. also the related concept of ε-*variation* of regulated functions introduced in Sect. 3 of [4]).

If u is only in $L^\infty(0,T)$, there exists still a critical value $\varrho(u) > 0$ such that the r-neighborhood of u does contain functions of bounded variation for $r > \varrho(u)$ and does not for $r < \varrho(u)$. We thus state the problem the other way round using the Vladimirov-Chernorutskii property as another definition of the play: given $u \in L^\infty(0,T)$ and $r > \varrho(u)$, we look for the function of minimal variation in the r-neighborhood of u with a prescribed initial condition.

Our main results (Theorems 1.2, 1.3) state that the play operator is well defined and Lipschitz continuous in $L^\infty(0,T)$ for $r > \varrho(u)$. As corollaries, we prove that (0.1) holds for left-continuous piecewise monotone inputs (and hence our definition coincides with the classical one on the space of left-continuous regulated functions), a superposition formula (Brokate's identity) holds, and that there exists a unique extension up to $r = \varrho(u)$.

The paper is organized as follows. In Section 1 we state the problem and list our main results. The well-posedness of the play operator in $L^\infty(0,T)$ is established in Section 2. The following Section 3 is devoted to the Lipschitz estimate. The corollaries are proved in Section 4.

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1 Main results

Let $T \in \,]0,\infty[\,$. We consider the space $L^\infty(0,T)$ endowed with the system of seminorms

$$
(1.1) \qquad \|v\|_{[a,b]} \; := \; \sup\operatorname{ess}\,\{|v(t)|\,;\, t \in \,]a,b[\,\}
$$

for $0 \le a < b \le T$. Indeed, $\|\cdot\|_{[0,T]}$ is a norm.

We further denote by $G(a,b)$ the space of regulated functions $u : [a,b] \to \mathbb{R}$, that is, functions for which both one-sided finite limits $u(t+)$, $u(t-)$ exist for all $t \in [a,b]$, with the convention $u(a-) := u(a)$, $u(b+) := u(b)$, and by $BV(a,b)$ the subset of $G(a,b)$ of functions of bounded variation. The space $G(a,b)$ endowed with the norm $\|\cdot\|_{[a,b]}$ is a Banach space and $BV(a,b)$ is dense in $G(a,b)$ (see e. g. [1]). By $LG(a,b)$ and $LBV(a,b)$ we denote the space of left-continuous functions from $G(a,b)$ and $BV(a,b)$, respectively.

For any function $u \in L^\infty(0,T)$ we define the number

$$\varrho(u) := \inf\left\{r > 0\,;\, \exists \eta \in BV(0,T)\,,\, \|u-\eta\|_{[0,T]} \le r\right\}. \tag{1.2}$$

Obviously, $\varrho(u)$ is always finite as $\varrho(u) \le \|u\|_{[0,T]}$, and $\varrho(u) = 0$ if and only if $u \in G(0,T)$.

For $u \in L^\infty(0,T)$, $\xi^0 \in \mathbb{R}$ and $r > \varrho(u)$ we define the set

$$\mathcal{B}_r(\xi^0,u) := \left\{\eta \in LBV(0,T)\,;\, \|u-\eta\|_{[0,T]} \le r\,,\, \eta(0) = \xi^0\right\}. \tag{1.3}$$

Note that $\mathcal{B}_r(\xi^0,u)$ is non-empty. Indeed, as $r > \varrho(u)$, it follows from (1.2) that there is $\eta \in BV(0,T)$ such that $\|u-\eta\|_{[0,T]} \le r$. We now introduce the function $\hat\eta$ defined by

$$\hat\eta(0) := \xi^0\,,\ \hat\eta(t) := \eta(t-)\,,\ t \in]0,T]\,. \tag{1.4}$$

Then η and $\hat\eta$ coincide except on a countable subset of $[0,T]$. We therefore have that $\|u-\hat\eta\|_{[0,T]} \le r$ and $\hat\eta \in LBV(0,T)$ with $\hat\eta(0) = \xi^0$, hence $\hat\eta \in \mathcal{B}_r(\xi^0,u)$.

Definition 1.1 For given $u \in L^\infty(0,T)$, $\xi^0 \in \mathbb{R}$ and $r > \varrho(u)$ we define the subset $\mathcal{P}_r(\xi^0,u)$ of $\mathcal{B}_r(\xi^0,u)$ as the set of all functions $\xi \in \mathcal{B}_r(\xi^0,u)$ such that

$$\operatorname*{Var}_{[0,t]} \xi \;=\; \inf\left\{\operatorname*{Var}_{[0,t]} \eta\,;\, \eta \in \mathcal{B}_r(\xi^0,u)\right\} \qquad \text{for every } \ t \in [0,T]\,. \tag{1.5}$$

Our main results can be stated as follows.

Theorem 1.2 (Existence and uniqueness) *Let $u \in L^\infty(0,T)$, $\xi^0 \in \mathbb{R}$ and $r > \varrho(u)$ be given. Then the set $\mathcal{P}_r(\xi^0,u)$ contains a unique element denoted by $\mathfrak{p}_r[\xi^0,u]$. Moreover, there exists a partition $0 = t_0 < t_1 < \ldots < t_\ell = T$ such that the function $\xi := \mathfrak{p}_r[\xi^0,u]$ is monotone in each closed interval $[t_{k-1},t_k]$, $k = 1,\ldots,\ell$, and non-monotone in each interval $[t_{k-1},t_{k+1}]$, $k = 1,\ldots,\ell-1$.*

A typical diagram of the dependence of $\xi = \mathfrak{p}_r[\xi^0,u]$ on u for a special choice $\xi^0 = u(0)$ of the initial condition is shown on Figure 1.

Theorem 1.3 (Lipschitz continuity) *For arbitrary $u,v \in L^\infty(0,T)$, $\xi^0,\eta^0 \in \mathbb{R}$ and $r > \varrho(u)$, $s > \varrho(v)$ there holds*

$$\left\|\mathfrak{p}_r[\xi^0,u] - \mathfrak{p}_s[\eta^0,v]\right\|_{[0,T]} \;\le\; \max\left\{|\xi^0-\eta^0|\,,\, |r-s| + \|u-v\|_{[0,T]}\right\}. \tag{1.6}$$

The identity (1.7) below for the play defined by (0.1) is in this form due to Brokate, see Proposition 2.2.16 of [2]. Similar considerations can be found in Section 34.2 of [6].

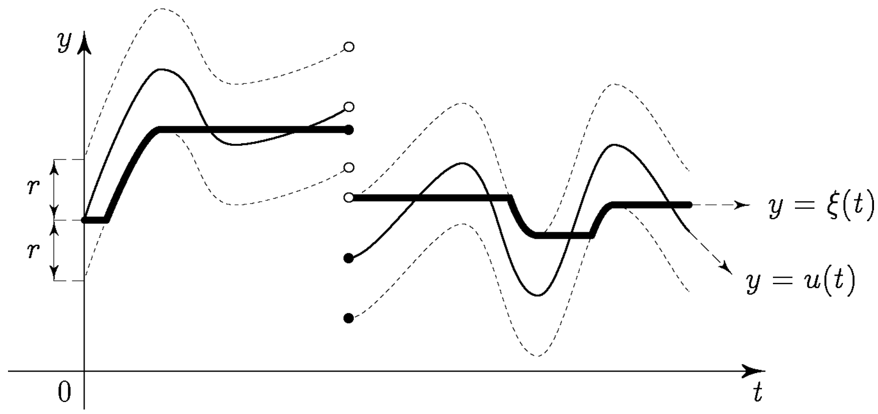

Figure 1: A diagram of the play operator $\xi = \mathfrak{p}_r[u(0), u]$.

Corollary 1.4 *For every* $u \in L^\infty(0,T)$, $r > \varrho(u)$, $h > 0$ *and* $\xi^0, \eta^0 \in \mathbb{R}$ *such that* $|\xi^0 - \eta^0| \le h$ *we have*

$$\mathfrak{p}_h[\xi^0, \mathfrak{p}_r[\eta^0, u]] \;=\; \mathfrak{p}_{r+h}[\xi^0, u]\,. \tag{1.7}$$

The next result enables us to extend our construction up to the limit case $r = \varrho(u)$. In typical cases, the function $\mathfrak{p}_{\varrho(u)}[\xi^0, u]$ will no longer be of bounded variation, but the main analytical properties are preserved.

Corollary 1.5 *Consider* $\eta^0 \in \mathbb{R}$ *and* $u \in L^\infty(0,T)$. *Then there exists a function* $\mathfrak{p}_{\varrho(u)}[\eta^0, u] \in LG(0,T)$ *such that*

$$\lim_{h\to 0+} \left\| \mathfrak{p}_{\varrho(u)+h}[\eta^0, u] \;-\; \mathfrak{p}_{\varrho(u)}[\eta^0, u] \right\|_{[0,T]} \;=\; 0\,. \tag{1.8}$$

Moreover, for every $h > 0$ *and* $\xi^0 \in [\eta^0 - h, \eta^0 + h]$ *we have*

$$\mathfrak{p}_{\varrho(u)+h}[\xi^0, u] \;=\; \mathfrak{p}_h[\xi^0, \mathfrak{p}_{\varrho(u)}[\eta^0, u]]\,. \tag{1.9}$$

To conclude, we show that $\mathfrak{p}_r$ coincides with the classical play operator on inputs $u \in LG(0,T)$ by proving the following statement.

Corollary 1.6 *Let* $0 = t_0 < t_1 < \ldots < t_\ell = T$ *be a partition of* $[0,T]$. *Consider a function* $u \in LG(0,T)$ *which is monotone in* $[t_{k-1}, t_k]$ *for every* $k = 1, \ldots, \ell$. *Let* $\xi^0 \in \mathbb{R}$ *and* $r > 0$ *be given. Then the function* $\xi = \mathfrak{p}_r[\xi^0, u]$ *satisfies (0.1).*

We prove Theorem 1.2 in Section 2, Theorem 1.3 in Section 3, and the proofs of Corollaries 1.4, 1.5, 1.6 are given in Section 4.

2 Existence and uniqueness

With the notation from Section 1, the aim of this section is to prove Theorem 1.2. As we are working with piecewise monotone functions in $\mathcal{B}_r(\xi^0, u)$, we first look for subintervals of $[0,T]$ on which the function u is within a distance (in L^∞) at most r from a monotone function. Before passing to the proof of Theorem 1.2 itself, we formulate this as an auxiliary statement.

Lemma 2.1 *Consider $a \in [0,T[$ and $\eta^0 \in \mathbb{R}$. For $t \in]a,T]$ we introduce the set $M(t)$ defined by*

$$M(t) = \left\{\eta \in LBV(a,t) \text{ monotone}\,;\, \eta(a) = \eta^0\,,\ \|u-\eta\|_{[a,t]} \le r\right\}. \tag{2.1}$$

Assume that the set

$$A := \{t \in]a,T]\,,\, M(t) \neq \emptyset\} \tag{2.2}$$

is non-empty and put $b := \sup A$. Then there exists a function $\xi \in M(b)$ such that for every $t \in [a,b]$ and $\eta \in LBV(a,b)$ satisfying

$$\eta(a) = \eta^0\,,\ \|u-\eta\|_{[a,b]} \le r \tag{2.3}$$

there holds

$$\operatorname*{Var}_{[a,t]} \xi \ \le \operatorname*{Var}_{[a,t]} \eta\,. \tag{2.4}$$

Proof of Lemma 2.1. We first prove that $M(b)$ is non-empty. By definition of b, there exists an increasing sequence $\{t_n\}$, $t_n \to b$ as $n \to \infty$ and a sequence $\{\eta^{(n)}\}$, $\eta^{(n)} \in M(t_n)$ for $n \in \mathbb{N}$. We extend the functions $\eta^{(n)}$ onto $[a,b]$ by putting $\eta^{(n)}(t) = \eta^{(n)}(t_n)$ for $t \in]t_n,b]$, and for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
|\eta^{(n)}(t)| &\le \|u\|_{[0,T]} + r \qquad \forall\, t \in [a,b]\\
\operatorname*{Var}_{[a,b]} \eta^{(n)} &= |\eta^{(n)}(b) - \eta^{(n)}(a)| \ \le\ \|u\|_{[0,T]} + r + |\eta^0|\,.
\end{aligned}$$

By the Helly Selection Principle (see [5]) there is a subsequence of $\{\eta^{(n)}\}$ (not relabeled) which converges pointwise to a function $\eta \in BV(a,b)$. Moreover, the functions $\eta^{(n)}$ being monotone, the sequence $\{\eta^{(n)}\}$ contains either an infinite number of non-decreasing functions or an infinite number of non-increasing functions. We may therefore assume that η is monotone in $[a,b]$. Finally, since $\left\|u-\eta^{(n)}\right\|_{[a,t_n]} \le r$ for every $n \in \mathbb{N}$, we have for every non-negative test function $f \in L^1(a,b)$

$$\int_a^b f(t)\,(|u(t)-\eta^{(n)}(t)| - r)\,dt \le \int_{t_n}^b f(t)\,(|u(t)-\eta^{(n)}(t)| - r)\,dt \le 2\,\|u\|_{[0,T]} \int_{t_n}^b f(t)\,dt\,,$$

and the Fatou lemma yields

$$\int_a^b f(t)\,(|u(t)-\eta(t)| - r)\,dt \ \le\ \liminf_{n\to\infty} \int_a^b f(t)\,(|u(t)-\eta^{(n)}(t)| - r)\,dt \ \le\ 0\,,$$

hence $\|u-\eta\|_{[a,b]} \le r$. We now put $\bar\eta(t) := \eta(t-)$ for $t \in]a,b]$, $\bar\eta(a) := \eta^0$. Then $\bar\eta(t) \in LBV(a,b)$ is monotone and coincides almost everywhere with η, hence $\bar\eta \in M(b)$ and $M(b)$ is thus non-empty.

We next denote by $M^+(b)$ ($M^-(b)$) the set of of non-decreasing (non-increasing, respectively) functions in $M(b)$. Assume first that both $M^+(b)$ and $M^-(b)$ are non-empty, and let $\eta^+ \in M^+(b)$, $\eta^- \in M^-(b)$ be arbitrary. Then we have for a. e. $t \in [a,b]$

$$u(t) - r \;\le\; \eta^-(t) \;\le\; \eta^0 \;\le\; \eta^+(t) \;\le\; u(t) + r\,,$$

hence the constant function $\xi(t) \equiv \eta^0$ belongs to $M(b)$ and (2.4) trivially holds.

Assume now that either $M^+(b)$ or $M^-(b)$ is empty, say $M^-(b) = \emptyset$. We then put

$$\xi(t) \;:=\; \inf\{\eta(t)\,;\, \eta \in M^+(b)\} \qquad \text{for } \; t \in [a,b]\,. \tag{2.5}$$

We first claim that

$$\xi \in M^+(b)\,. \tag{2.6}$$

Taking (2.6) for granted, we pick $\eta \in LBV(a,b)$ satisfying (2.3) and put

$$\hat{\xi}(t) \;:=\; \min\left\{\xi(t)\,,\, \eta^0 + \operatorname*{Var}_{[a,t]} \eta\right\} \qquad \text{for } \; t \in [a,b]\,. \tag{2.7}$$

For a. e. $t \in [a,b]$ we have by hypothesis

$$\begin{aligned} u(t) - r \;&\le\; \xi(t) \;\le\; u(t) + r\,,\\ u(t) - r \;&\le\; \eta(t) \;\le\; \eta^0 + \operatorname{Var}_{[a,t]} \eta\,, \end{aligned}$$

hence

$$u(t) - r \;\le\; \hat{\xi}(t) \;\le\; u(t) + r \qquad \text{a. e. in } \; [a,b]\,.$$

Moreover, $\hat{\xi}$ is a non-decreasing left-continuous function on $[a,b]$ satisfying $\hat{\xi}(a) = \eta^0$, hence $\hat{\xi} \in M^+(b)$, and $\hat{\xi}(t) \le \xi(t)$ a. e. as well. Recalling (2.5) we conclude that $\hat{\xi} \equiv \xi$, hence $\operatorname{Var}_{[a,t]} \xi = \xi(t) - \eta^0 \le \operatorname{Var}_{[a,t]} \eta$ for $t \in [a,b]$ and (2.4) holds.

It remains to check (2.6). It is easy to see that $\xi(a) = \eta^0$ and ξ is non-decreasing on $[a,b]$. We next prove that there is a sequence $\{\xi^{(n)}\}$ in $M^+(b)$ converging pointwisely to ξ. To this end we argue as in the proof of the Helly Selection Principle (see [5], pp. 372 – 374). Recalling that ξ has only a countable number of discontinuity points in $[a,b]$ we choose an arbitrary dense countable subset $K := \{z_j\}_{j\ge 1}$ of $[a,b]$ containing a, b and all discontinuity points of ξ. Fix $n \ge 1$. For each $j \in \{1,\dots,n\}$ there is $\eta^{(j)} \in M^+(b)$ such that $0 \le \eta^{(j)}(z_j) - \xi(z_j) \le 1/n$. The function

$$\xi^{(n)} := \min\left\{\eta^{(j)}\,,\, 1 \le j \le n\right\}$$

then clearly belongs to $M^+(b)$ and satisfies

$$0 \le \xi^{(n)}(z_j) - \xi(z_j) \le 1/n\,,\; 1 \le j \le n.$$

Therefore, for every $t \in K$ we have

$$\lim_{n\to\infty} \xi^{(n)}(t) \;=\; \xi(t)\,. \tag{2.8}$$

Each point $s \in [a,b] \setminus K$ is a continuity point of ξ, and for $t \in K$, $t \ge s$ we have

$$0 \;\le\; \xi^{(n)}(s) - \xi(s) \;\le\; (\xi^{(n)}(t) - \xi(t)) + (\xi(t) - \xi(s))\,,$$

hence (2.8) holds for every $t \in [a,b]$.

We are now in a position to complete the proof of (2.6). For each $n \in \mathbb{N}$ there exists a set $Z_n \subset [a,b]$ of measure zero such that

$$(2.9) \qquad u(t) - r \;\leq\; \xi^{(n)}(t) \;\leq\; u(t) + r \qquad \text{for } \; t \in [a,b] \setminus Z_n \, .$$

Then $Z := \cup_{n=1}^{\infty} Z_n$ is a set of measure zero and passing to the limit in (2.9) as $n \to \infty$ we obtain

$$u(t) - r \;\leq\; \xi(t) \;\leq\; u(t) + r \qquad \text{for } \; t \in [a,b] \setminus Z \, ,$$

hence $\|u - \xi\|_{[a,b]} \leq r$. It remains to check that ξ is left-continuous. Indeed, put

$$\bar{\xi}(t) \;:=\; \xi(t-) \quad \text{for } \; t \in]a,b] \, , \qquad \bar{\xi}(a) \;:=\; \eta^0 \, .$$

Then, as $\bar{\xi}$ and ξ coincide except on a countable set, we conclude that $\bar{\xi} \in M^+(b)$ and $\bar{\xi} \leq \xi$. From (2.5) we obtain that $\bar{\xi} = \xi$, hence (2.6) holds.

Finally, if $M^+(b) = \emptyset$, we introduce

$$\xi(t) \;:=\; \sup\{\eta(t)\, ; \, \eta \in M^-(b)\} \qquad \text{for } \; t \in [a,b]$$

and proceed similarly as in the previous case to complete the proof of Lemma 2.1. ■

We now pass to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $u \in L^\infty(0,T)$, $\xi^0 \in \mathbb{R}$ and $r > \varrho(u)$ be given. For some $r' \in]\varrho(u), r[$ we choose $\eta' \in \mathcal{B}_{r'}(\xi^0, u)$. According to Assertion 7.3.2.1.(3) of [1], every regulated function can be uniformly approximated by piecewise constant functions. Since η' is left-continuous, the approximations can be chosen to be left-continuous as well, hence there exists a partition $0 = s_0 < s_1 < \ldots < s_m = T$ of $[0,T]$ and real numbers $w_1, \ldots, w_m$ such that the function w of the form

$$w(0) = \xi^0 \, , \quad w(t) = w_i \quad \text{for } \; t \in]s_{i-1}, s_i] \, , \; i = 1, \ldots, m$$

satisfies $\|w - \eta'\|_{[0,T]} \leq r - r'$. This means in particular that

$$(2.10) \qquad w \;\in\; \mathcal{B}_r(\xi^0, u) \, .$$

We next introduce the set

$$(2.11) \qquad M_1(t) = \left\{ \eta \in LBV(0,t) \text{ monotone}\, ; \, \eta(0) = \xi^0 \, , \; \|u - \eta\|_{[0,t]} \leq r \right\}$$

for $t \in [0,T]$. Owing to (2.10), the function $w|_{[0,s_1]}$ belongs to $M_1(s_1)$, hence the set

$$A_1 \;:=\; \{t \in]0,T] \, , \, M_1(t) \neq \emptyset\}$$

contains s_1. We apply Lemma 2.1 and putting $t_1 := \sup A_1 \geq s_1$ we find a function $\xi_1 \in M_1(t_1)$ such that

$$(2.12) \qquad \operatorname*{Var}_{[0,t]} \xi_1 \;\leq\; \operatorname*{Var}_{[0,t]} \eta$$

for every $t \in [0,t_1]$ and $\eta \in LBV(0,t_1)$ with $\eta(0) = \xi^0$, $\|u-\eta\|_{[0,t_1]} \le r$. Note that ξ_1 is non-constant in $[0,t_1]$; otherwise, for $i_1 := \max\{j\,;\, s_j \le t_1\} \ge 1$, the function $\tilde\xi_1 : [0, s_{i_1+1}] \to \mathbb{R}$ defined by

$$\tilde\xi_1(t) \;:=\; \begin{cases} \xi^0 & \text{if } \; t \in [0,t_1] \\ w(t) & \text{if } \; t \in]t_1, s_{i_1+1}] \end{cases}$$

would belong to $M_1(s_{i_1+1})$ which contradicts the definition of t_1.

We next continue by induction. Assume that we have already constructed a partition $0 = t_0 < t_1 < \ldots < t_k < T$ for some $k \ge 1$ and a sequence $\{\xi_1, \ldots, \xi_k\}$ of functions $\xi_j : [0,t_j] \to \mathbb{R}$, $1 \le j \le k$, such that

(2.13) $\quad t_j \;\ge\; s_j \quad \text{for } \; 0 \le j \le k\,,$

(2.14) $\quad \xi_k|_{[0,t_j]} \;=\; \xi_j \quad \text{for } \; 1 \le j \le k\,,$

(2.15) $\quad \xi_k(0) = \xi^0 \quad \text{and} \quad \|u - \xi_k\|_{[0,t_k]} \;\le\; r\,,$

(2.16) $\quad \xi_k$ is monotone and non-constant in $[t_{j-1}, t_j]$, $\quad 1 \le j \le k$,

(2.17) $\quad \xi_j$ does not admit any monotone extension onto $[t_{j-1}, t_j + \varepsilon]$ such that $\|u - \xi_j\|_{[0,t_j+\varepsilon]} \le r$ for any $\varepsilon > 0$ and $1 \le j \le k$,

(2.18) $$\operatorname*{Var}_{[0,t]} \xi_k \;\le\; \operatorname*{Var}_{[0,t]} \eta$$

for each $t \in [0,t_k]$ and $\eta \in LBV(0,t_k)$ with $\eta(0) = \xi^0$, $\|u-\eta\|_{[0,t_k]} \le r$.

We now proceed to the induction step. Assume for instance that ξ_k is non-decreasing in $[t_{k-1}, t_k]$ and put $\xi^k := \xi_k(t_k)$. As before we introduce the set

(2.19) $$M_{k+1}(t) = \left\{\eta \in LBV(t_k, t) \text{ monotone}\,;\, \eta(t_k) = \xi^k\,,\ \|u-\eta\|_{[t_k,t]} \le r\right\}$$

for $t \in]t_k, T]$. Put $i_k := \max\{j\,;\, s_j \le t_k\} \ge k$. Then the function $\hat w : [t_k, s_{i_k+1}] \to \mathbb{R}$ defined by

$$\hat w(t_k) := \xi^k\,, \quad \hat w(t) := w(t) = w_{i_k+1} \quad \text{for } \; t \in]t_k, s_{i_k+1}]$$

belongs to $M_{k+1}(s_{i_k+1})$, hence the set

$$A_{k+1} \;:=\; \{t \in]t_k, T]\,,\ M_{k+1}(t) \ne \emptyset\}$$

contains s_{i_k+1}. As before, we apply Lemma 2.1 and putting $t_{k+1} := \sup A_{k+1} \ge s_{i_k+1} \ge s_{k+1}$ we find a function $\hat\xi_{k+1} \in M_{k+1}(t_{k+1})$ such that

(2.20) $$\operatorname*{Var}_{[t_k,t]} \hat\xi_{k+1} \;\le\; \operatorname*{Var}_{[t_k,t]} \eta$$

for every $t \in [t_k, t_{k+1}]$ and $\eta \in LBV(t_k, t_{k+1})$ with $\eta(t_k) = \xi^k$, $\|u-\eta\|_{[t_k,t_{k+1}]} \le r$.

Observe that for every $t \in]t_k, t_{k+1}]$, the set $M_{k+1}(t)$ contains only non-increasing non-constant functions. Indeed, if there would exist a non-decreasing function $\tilde\eta \in M_{k+1}(\tilde t\,)$ for some $\tilde t \in]t_k, t_{k+1}]$, then putting

$$\tilde\xi(t) \;:=\; \begin{cases} \xi_k(t) & \text{if } \; t \in [0,t_k]\,, \\ \tilde\eta(t) & \text{if } \; t \in]t_k, \tilde t\,] \end{cases}$$

we would obtain a non-decreasing extension $\tilde{\xi}$ of ξ_k onto $[0, \tilde{t}\,]$ which would satisfy $\left\|u - \tilde{\xi}\right\|_{[0,\tilde{t}]} \le r$ in contradiction with (2.17). This implies in particular that $w(t_k+) = w_{i_k+1} < \xi^k$.

We now define the function ξ_{k+1} by

$$\xi_{k+1}(t) \; := \; \begin{cases} \xi_k(t) & \text{if } \; t \in [0, t_k] \\ \hat{\xi}_{k+1}(t) & \text{if } \; t \in \,]t_k, t_{k+1}] \,. \end{cases}$$

By construction, the properties (2.13) – (2.17) are fulfilled at the level $k+1$. It remains to check that (2.18) holds for ξ_{k+1}.

Let $\eta \in LBV(0, t_{k+1})$ be given with $\eta(0) = \xi^0$, $\|u - \eta\|_{[0,t_{k+1}]} \le r$. The induction step will be complete if we prove that

$$\text{(2.21)} \qquad \operatorname*{Var}_{[0,t]} \xi_{k+1} \;\le\; \operatorname*{Var}_{[0,t]} \eta \quad \text{for every } \; t \in \,]t_k, t_{k+1}] \,.$$

We first notice that (2.17) yields

$$\text{(2.22)} \qquad \eta(t_k+) \;\le\; \xi^k \,.$$

Indeed, we have $w(t_k+) < \xi^k$, and if $\eta(t_k+) > \xi^k$, there holds $w(t) \le \xi^k \le \eta(t)$ for $t \in [t_k, t_k + \delta]$ for some $\delta > 0$. Consequently $|\xi^k - u(t)| \le r$ a.e. in $[t_k, t_k + \delta]$ and we could extend ξ_k by the constant value ξ^k beyond t_k in contradiction with (2.17).

We fix the number

$$\text{(2.23)} \qquad \bar{t} \;:=\; \inf\{t \in [t_{k-1}, t_k]\,;\, \xi_k(t) = \xi^k\} \,,$$

and check that

$$\text{(2.24)} \qquad \max\{\eta(\bar{t}), \eta(\bar{t}+)\} \;\ge\; \xi^k \,.$$

Assume for contradiction that (2.24) does not hold. Then there exist $\varepsilon > 0$, $\delta > 0$ such that

$$\text{(2.25)} \qquad \eta(t) \;\le\; \xi^k - \varepsilon$$

for $t \in \,]\bar{t} - \delta, \bar{t} + \delta] \,\cap\, [t_{k-1}, t_k]$. Put $a := \max\{t_{k-1}, \bar{t} - \delta\}$, $b := \min\{t_k, \bar{t} + \delta\}$. Then (2.16) yields $\xi_k(a) < \xi^k$, $\xi_k(b) = \xi^k$. Taking a smaller value of ε if necessary we may assume that $\xi_k(a) \le \xi^k - \varepsilon$. Put

$$\text{(2.26)} \qquad \tilde{\xi}(t) \;:=\; \begin{cases} \min\{\xi_k(t), \xi^k - \varepsilon\} & \text{for } \; t \in \,]a, b] \,, \\ \xi_k(t) & \text{for } \; t \in [0, a] \cup \,]b, t_k] \,. \end{cases}$$

Then, owing to (2.25), we have

$$u(t) + r \;\ge\; \xi_k(t) \;\ge\; \tilde{\xi}(t) \;\ge\; \min\{\xi_k(t), \eta(t)\} \;\ge\; u(t) - r$$

for a.e. $t \in \,]a, b]$, and $\tilde{\xi}$ is non-decreasing in $[t_{k-1}, t_k]$ with $\tilde{\xi}(b) < \xi_k(b) = \xi^k$, hence

$$\operatorname*{Var}_{[0,b]} \tilde{\xi} \;<\; \operatorname*{Var}_{[0,b]} \xi_k \,,$$

which contradicts (2.18).

We thus proved that (2.24) holds. Now we distinguish two cases:

(i) $\bar{t} < t_k$:

Let us introduce an auxiliary function η^* by the formula

$$\eta^*(t) := \begin{cases} \xi^k & \text{for } \; t \in \,]\bar{t}, t_k]\,, \\ \eta(t) & \text{for } \; t \in [0, \bar{t}\,] \cup \,]t_k, t_{k+1}]\,. \end{cases} \tag{2.27}$$

Then (2.20) yields

$$\operatorname*{Var}_{[t_k,t]} \xi_{k+1} \;\leq\; \operatorname*{Var}_{[t_k,t]} \eta^* \tag{2.28}$$

for every $t \in [t_k, t_{k+1}]$ and from (2.18) we obtain

$$\operatorname*{Var}_{[0,t_k]} \xi_{k+1} \;\leq\; \operatorname*{Var}_{[0,t_k]} \eta^*\,. \tag{2.29}$$

On the other hand, for $t \in \,]t_k, t_{k+1}]$ we have

$$\operatorname*{Var}_{[0,t]} \eta - \operatorname*{Var}_{[0,t]} \eta^* \;=\; \operatorname*{Var}_{[\bar{t},t_k]} \eta + |\eta(t_k+) - \eta(t_k)| - |\eta(\bar{t}) - \xi^k| - |\xi^k - \eta(t_k+)|\,, \tag{2.30}$$

where, by (2.22) and (2.24), either $\eta(\bar{t}\,) \geq \xi^k$ and

$$|\eta(\bar{t}) - \xi^k| + |\xi^k - \eta(t_k+)| \;=\; \eta(\bar{t}) - \eta(t_k+) \;\leq\; \operatorname*{Var}_{[\bar{t},t_k]} \eta \,+ |\eta(t_k) - \eta(t_k+)|\,,$$

or $\eta(\bar{t}\,) < \xi^k$, $\eta(\bar{t}+) \geq \xi^k$ and

$$\begin{aligned} |\eta(\bar{t}\,) - \xi^k| + |\xi^k - \eta(t_k+)| \quad &= \quad 2\xi^k - \eta(\bar{t}\,) - \eta(t_k+) \;\leq\; 2\eta(\bar{t}+) - \eta(\bar{t}\,) - \eta(t_k+) \\ &\leq \quad \operatorname*{Var}_{[\bar{t},t_k]} \eta \,+ |\eta(t_k) - \eta(t_k+)|\,, \end{aligned}$$

hence $\operatorname{Var}_{[0,t]} \eta \geq \operatorname{Var}_{[0,t]} \eta^*$ and (2.21) follows from (2.28) and (2.29).

(ii) $\bar{t} = t_k$:

Put $\eta^{**}(t) := \eta(t)$ for $t \in [0, t_k[\, \cup \,]t_k, t_{k+1}]$, $\eta^{**}(t_k) := \xi^k$. Then (2.18) and (2.20) yield

$$\operatorname*{Var}_{[0,t_k]} \xi_{k+1} \;\leq\; \operatorname*{Var}_{[0,t_k]} \eta\,, \qquad \operatorname*{Var}_{[t_k,t]} \xi_{k+1} \;\leq\; \operatorname*{Var}_{[t_k,t]} \eta^{**} \qquad \text{for } \; t \in \,]t_k, t_{k+1}]\,. \tag{2.31}$$

By (2.22) and (2.24) we have $\eta^{**}(t_k+) = \eta(t_k+) \leq \xi^k$, $\eta(t_k) \geq \xi^k = \eta^{**}(t_k)$, hence

$$\begin{aligned} \operatorname*{Var}_{[t_k,t]} \eta^{**} \quad &= \quad \operatorname*{Var}_{[t_k,t]} \eta - |\eta(t_k+) - \eta(t_k)| + |\eta^{**}(t_k+) - \eta^{**}(t_k)| \\ &= \quad \operatorname*{Var}_{[t_k,t]} \eta + \xi^k - \eta(t_k) \;\leq\; \operatorname*{Var}_{[t_k,t]} \eta\,, \end{aligned}$$

and we obtain (2.21) from (2.31). The induction step is complete.

Owing to (2.13), after a finite number of steps we obtain $t_\ell = T$ for some $\ell \leq m$. Putting $\xi := \xi_\ell$ we have found a function $\xi \in \mathcal{P}_r(\xi^0, u)$ satisfying the conditions of Theorem 1.2, and the existence part is proved.

To prove uniqueness, we consider an arbitrary function $\eta \in \mathcal{P}_r(\xi^0, u)$ and put

$$V(t) \;:=\; \operatorname*{Var}_{[0,t]} \xi \;=\; \operatorname*{Var}_{[0,t]} \eta$$

for $t \in [0,T]$, where ξ is the element of $\mathcal{P}_r(\xi^0, u)$ we have just constructed above. Assume that

$$\hat{t} \;:=\; \max\{t \in [0,T]\,;\, \xi(t) = \eta(t)\} \;<\; T\,.$$

Observe that the maximum exists by the left-continuity of ξ and η. We find $k \in \{1,\dots,\ell\}$ such that $\hat{t} \in [t_{k-1}, t_k[$ and assume for instance that ξ is non-decreasing in $[t_{k-1}, t_k]$. The function $t \mapsto \xi(t) - \eta(t) = \xi(\hat{t}) - V(\hat{t}) + V(t) - \eta(t)$ is non-decreasing and positive in $]\hat{t}, t_k]$. As ξ and η have the same total variation on every subinterval of $[0,T]$ and coincide on $[0,\hat{t}\,]$, there holds $\xi(\hat{t}+) - \xi(\hat{t}\,) = |\eta(\hat{t}+) - \eta(\hat{t})| = |\eta(\hat{t}+) - \xi(\hat{t})|$, hence either $\xi(\hat{t}+) = \eta(\hat{t}+)$, or $\xi(\hat{t}+) - \eta(\hat{t}+) = 2(\xi(\hat{t}+) - \xi(\hat{t}))$. Assume first that $\xi(\hat{t}+) - \eta(\hat{t}+) = \varepsilon > 0$. Then $\xi(\hat{t}+) - \xi(\hat{t}) = \varepsilon/2$, and putting

$$\hat{\xi}(t) \;:=\; \begin{cases} \xi(t) & \text{for } \; t \in [0,\hat{t}\,] \cup\,]t_k, T]\,, \\ \xi(t) - \varepsilon/2 & \text{for } \; t \in\,]\hat{t}, t_k]\,, \end{cases}$$

we see that $\hat{\xi}$ is non-decreasing in $[t_{k-1}, t_k]$, and for a.e. $t \in [t_{k-1}, t_k]$ we have $u(t) + r \geq \xi(t) \geq \hat{\xi}(t) \geq \eta(t) \geq u(t) - r$ with $\operatorname{Var}_{[0,t_k]} \hat{\xi} = V(t_k) - \varepsilon/2$, which is a contradiction.

Therefore we necessarily have $\xi(\hat{t}+) = \eta(\hat{t}+)$. We fix some $\kappa \in]0, \xi(t_k) - \eta(t_k)[$ and put

$$s \;:=\; \inf\{t \in [\hat{t}, t_k]\,;\, \xi(t) - \eta(t) \geq \kappa\}\,.$$

Clearly $s > \hat{t}$ and for $t \in]\hat{t}, s[$ we have

$$0 < \xi(t) - \eta(t) < \kappa \leq \xi(s+) - \eta(s+). \tag{2.32}$$

We next define the number

$$\varepsilon \;:=\; \xi(s) - \eta(s) \;\in\,]0, \kappa]\,, \tag{2.33}$$

and choose $\delta > 0$ in such a way that

$$|\xi(s) - \xi(t)| \;\leq\; \varepsilon/2\,, \quad |\eta(s) - \eta(t)| \;\leq\; \varepsilon/2 \qquad \forall\, t \in [s-\delta, s] \,\subset\,]t_{k-1}, s]\,.$$

We have $\xi(s+) - \xi(s-\delta) = V(s+) - V(s-\delta) \geq |\eta(s+) - \eta(s-\delta)|$, hence

$$\sigma \;:=\; \min\{\varepsilon\,,\, \xi(s+) - \xi(s-\delta)\} \;>\; 0\,.$$

Indeed, $\sigma = 0$ would imply $\xi(s+) - \xi(s-\delta) = \eta(s+) - \eta(s-\delta) = 0$, hence $\xi(s+) - \eta(s+) = \xi(s-\delta) - \eta(s-\delta)$ which contradicts (2.32).

We define the function

$$\hat{\xi}(t) \;:=\; \begin{cases} \xi(t) & \text{for } \; t \in [0, s-\delta] \cup\,]t_k, T]\,, \\ \min\{\xi(t)\,,\, \xi(s+) - \sigma\} & \text{for } \; t \in]s-\delta, s]\,, \\ \xi(t) - \sigma & \text{for } \; t \in]s, t_k]\,. \end{cases} \tag{2.34}$$

Then $\hat{\xi}$ is non-decreasing in $[t_{k-1}, t_k]$, $\hat{\xi}(t) \leq \xi(t)$ for every $t \in]s-\delta, t_k]$. On the other hand,

$$\begin{aligned} \hat{\xi}(t) - \eta(t) \;&\geq\; \xi(s-\delta) - \eta(t) \;=\; (\xi(s) - \eta(s)) + (\xi(s-\delta) - \xi(s)) + (\eta(s) - \eta(t)) \\ &\geq\; 0 \qquad \text{for every } \; t \in]s-\delta, s] \; \text{ and} \end{aligned}$$

$$\hat{\xi}(t) - \eta(t) \;\geq\; \xi(t) - \eta(t) - \varepsilon \;\geq\; 0 \qquad \text{for every } \; t \in]s, t_k]\,,$$

hence $\left\| u - \hat{\xi} \right\|_{[0,T]} \leq r$, while $\operatorname{Var}_{[0,t_k]} \hat{\xi} \;=\; \operatorname{Var}_{[0,t_k]} \xi - \sigma \;<\; V(t_k)$, which is a contradiction. This completes the proof of Theorem 1.2. ■

3 Lipschitz continuity

In this section we give the proof of Theorem 1.3. It will be based on the following two lemmas.

Lemma 3.1 *Consider* $u \in L^\infty(0,T)$, $r > \varrho(u)$, $\xi^0 \in \mathbb{R}$ *and* $\xi = \mathfrak{p}_r[\xi^0, u]$. *Assume that there is a subinterval* $[\tau_0, \tau_1]$ *of* $[0,T]$ *on which there holds*

$$\xi(t) \ > \ \xi(\tau_0) \qquad \forall\, t \in]\tau_0, \tau_1]\ . \tag{3.1}$$

Then for each $\varepsilon > 0$ *and* $t \in]\tau_0, \tau_1]$ *the set*

$$M^+_{\xi,\varepsilon}(t) \ := \ \{\tau \in]\tau_0, t[\ ,\ \xi(\tau) \le u(\tau) - r + \varepsilon\} \tag{3.2}$$

has positive measure.

Lemma 3.2 *Consider* $u \in L^\infty(0,T)$, $r > \varrho(u)$, $\xi^0 \in \mathbb{R}$ *and* $\xi = \mathfrak{p}_r[\xi^0, u]$. *Assume that there is a subinterval* $[\tau_0, \tau_1]$ *of* $[0,T]$ *on which there holds*

$$\xi(t) \ < \ \xi(\tau_0) \qquad \forall\, t \in]\tau_0, \tau_1]\ . \tag{3.3}$$

Then for each $\varepsilon > 0$ *and* $t \in]\tau_0, \tau_1]$ *the set*

$$M^-_{\xi,\varepsilon}(t) \ := \ \{\tau \in]\tau_0, t[\ ,\ \xi(\tau) \ge u(\tau) + r - \varepsilon\} \tag{3.4}$$

has positive measure.

We prove only the assertion of Lemma 3.1; Lemma 3.2 is completely analogous.

Proof of Lemma 3.1. We first check that under the hypotheses of Lemma 3.1, the set

$$A_\xi(t) \ = \ \{\tau \in]\tau_0, t[\ ,\ |u(\tau) - \xi(\tau_0)| > r\}$$

has positive measure for every $]\tau_0, \tau_1]$. Indeed, assume for contradiction that there is $t^* \in]\tau_0, \tau_1]$ such that $\operatorname{meas} A_\xi(t^*) = 0$. Putting

$$\xi^*(t) \ := \ \begin{cases} \xi(t) & \text{if } \ t \in [0, \tau_0] \cup]t^*, T]\,, \\ \xi(\tau_0) & \text{if } \ t \in]\tau_0, t^*]\,, \end{cases}$$

we have, on the one hand, $\xi^* \in \mathcal{B}_r(\xi^0, u)$. On the other hand, (3.1) yields

$$\operatorname*{Var}_{[0,t^*]} \xi \ \ge \ \operatorname*{Var}_{[0,\tau_0]} \xi \ + \ \xi(t^*) - \xi(\tau_0) \ > \ \operatorname*{Var}_{[0,\tau_0]} \xi \ = \ \operatorname*{Var}_{[0,t^*]} \xi^*$$

which contradicts Definition 1.1 and Theorem 1.2.

We next proceed to the proof of (3.2). Arguing by contradiction again we assume that there is $\varepsilon > 0$ and $t^* \in]\tau_0, \tau_1]$ such that

$$\operatorname{meas} M^+_{\xi,\varepsilon}(t^*) \ = \ 0\,.$$

Recalling Theorem 1.2, we find $k \in \{1, \dots, \ell\}$ such that $\tau_0 \in [t_{k-1}, t_k[$ and put $\tau^* := \min\{t^*, t_k\}$. Then

$$\text{meas } M^+_{\xi,\varepsilon}(\tau^*) \;=\; 0\,, \tag{3.5}$$

$$\xi \quad \text{is non-decreasing on} \quad [\tau_0, \tau^*] \qquad \text{(recall (3.1))}\;. \tag{3.6}$$

Consider $t \in]\tau_0, \tau^*[$. It follows from (3.1) that, for almost every $\tau \in A_\xi(t)$, there holds

$$\xi(\tau_0) \;<\; \xi(\tau) \;\le\; u(\tau) + r \quad \text{and} \quad |u(\tau) - \xi(\tau_0)| > r\,,$$

consequently

$$\xi(\tau_0) \;<\; u(\tau) - r \quad \text{for a.e.} \quad \tau \in A_\xi(t)\,.$$

Combining this inequality with (3.5) yields

$$\xi(\tau_0) \;<\; \xi(\tau) - \varepsilon \quad \text{for a.e.} \quad \tau \in A_\xi(t)\,. \tag{3.7}$$

As the set $A_\xi(t)$ has positive measure for every $t \in]\tau_0, \tau^*[$, we may let τ tend to τ_0+ in (3.7) and conclude that

$$\xi(\tau_0) \;\le\; \xi(\tau_0+) - \varepsilon\,. \tag{3.8}$$

We now define a function $\hat{\xi} \in LBV(0,T)$ by

$$\hat{\xi}(t) \;:=\; \begin{cases} \xi(t) & \text{if} \;\; t \in [0, \tau_0] \cup]\tau^*, T]\,, \\ \xi(t) - \varepsilon & \text{if} \;\; t \in]\tau_0, \tau^*]\,. \end{cases}$$

Owing to (3.5), we have $\left\|u - \hat{\xi}\right\|_{[\tau_0, \tau^*]} \le r$ and thus $\hat{\xi} \in \mathcal{B}_r(\xi^0, u)$. On the other hand, (3.6), (3.8) yield that $\hat{\xi}$ is non-decreasing in $[\tau_0, \tau^*]$ and

$$\operatorname*{Var}_{[0,\tau^*]} \hat{\xi} \;=\; \operatorname*{Var}_{[0,\tau_0]} \xi + \hat{\xi}(\tau^*) - \hat{\xi}(\tau_0) \;=\; \operatorname*{Var}_{[0,\tau^*]} \xi - \varepsilon$$

which contradicts Theorem 1.2. Consequently meas $M^+_{\xi,\varepsilon}(t) > 0$ for every $t \in]\tau_0, \tau_1]$ and the proof of Lemma 3.1 is complete. ∎

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $u, v \in L^\infty(0,T)$, $\xi^0, \eta^0 \in \mathbb{R}$ and $r > \varrho(u)$, $s > \varrho(v)$ be given, and put $\xi := \mathfrak{p}_r[\xi^0, u]$, $\eta := \mathfrak{p}_s[\eta^0, v]$. We first prove that for every $t \in [0,T]$ we have

$$\xi(t) - \eta(t) \;\le\; \max\left\{|\xi^0 - \eta^0|\,,\, |r - s| + \|u - v\|_{[0,T]}\right\}\,. \tag{3.9}$$

Interchanging the roles of ξ and η and taking the maximum over $t \in [0,T]$ we then obtain the assertion of Theorem 1.3.

To prove (3.9), we put

$$d_0 \;:=\; \max\left\{|\xi^0 - \eta^0|\,,\, |r - s| + \|u - v\|_{[0,T]}\right\}$$

and consider an arbitrary $d > d_0$. Assume that there is $\tau_1 \in]0,T]$ such that

$$\xi(\tau_1) - \eta(\tau_1) \;>\; d\,, \tag{3.10}$$

and put

$$\tau_0 := \sup\{t \in [0, \tau_1]\,;\, \xi(t) - \eta(t) \leq d\}\,. \tag{3.11}$$

Clearly $\tau_0 \geq 0$ and the left continuity of ξ and η entails that $\xi(\tau_0) - \eta(\tau_0) \leq d$, hence $\tau_0 < \tau_1$. We have thus found a subinterval $[\tau_0, \tau_1]$ of $[0, T]$ such that

$$\xi(\tau_0) - \eta(\tau_0) \leq d \quad \text{and} \quad \xi(t) - \eta(t) > d \quad \text{for} \quad t \in]\tau_0, \tau_1]\,. \tag{3.12}$$

According to Theorem 1.2, we may assume that both ξ and η are monotone in $[\tau_0, \tau_1]$ by taking a smaller τ_1 if necessary. Now we have either

$$\xi(t) \;>\; \xi(\tau_0) \quad \text{for} \quad t \in]\tau_0, \tau_1]\,, \tag{3.13}$$

or there is $t^* \in]\tau_0, \tau_1]$ such that

$$\xi(t^*) \;\leq\; \xi(\tau_0)\,. \tag{3.14}$$

If (3.13) holds true, we put $\varepsilon := d - d_0 > 0$. By Lemma 3.1 the set $M^+_{\xi,\varepsilon}(\tau_1)$ has positive measure and we have for a. e. $t \in M^+_{\xi,\varepsilon}(\tau_1)$ that

$$\begin{aligned} \xi(t) - \eta(t) \;&\leq\; u(t) - r + \varepsilon - (v(t) - s) \;\leq\; \|u - v\|_{[0,T]} + \varepsilon + s - r \\ &\leq\; d_0 + \varepsilon \;\leq\; d\,, \end{aligned}$$

hence a contradiction with (3.12).

On the contrary, if (3.14) holds, then ξ is non-increasing in $[\tau_0, t^*]$. From (3.12) it then follows for $t \in]\tau_0, t^*]$ that

$$\eta(t) \;<\; \xi(t) - d \;\leq\; \xi(\tau_0) - d \;\leq\; \eta(\tau_0)\,.$$

We now apply Lemma 3.2 to conclude that $M^-_{\eta,\varepsilon}(t^*)$ has positive measure for the same ε as above. But for almost every $t \in M^-_{\eta,\varepsilon}(t^*)$ there holds

$$\begin{aligned} \eta(t) - \xi(t) \;&\geq\; v(t) + s - \varepsilon - (u(t) + r) \;\geq\; -\|u - v\|_{[0,T]} - \varepsilon + s - r \\ &\geq\; -d_0 - \varepsilon \;\geq\; -d\,, \end{aligned}$$

which again contradicts (3.12). In other words, (3.10) cannot hold for any $\tau_1 \in]0, T]$, hence

$$\xi(t) - \eta(t) \;\leq\; d \qquad \text{for all} \;\; t \in [0, T]\,.$$

As this is valid for each $d > d_0$, we obtain (3.9) and the proof is complete. ∎

4 Further properties

In this section, we give the proofs of Corollaries 1.4 – 1.6.

Proof of Corollary 1.4. Put $\xi_r := \mathfrak{p}_r[\eta^0, u]$ and $\xi_{r+h} := \mathfrak{p}_{r+h}[\xi^0, u]$. Since ξ_r belongs to $LBV(0, T)$, we have $\varrho(\xi_r) = 0$ and $\eta_h := \mathfrak{p}_h[\xi^0, \xi_r]$ is well defined. By Theorem 1.2 we have $\eta_h \in LBV(0, T)$ and

$$\|u - \eta_h\|_{[0,T]} \;\leq\; \|u - \xi_r\|_{[0,T]} + \|\xi_r - \eta_h\|_{[0,T]} \;\leq\; r + h\,,$$

hence $\eta_h \in \mathcal{B}_{r+h}(\xi^0, u)$. Moreover, $\xi_{r+h} \in LBV(0,T)$ satisfies $\xi_{r+h}(0) = \xi^0$ and (1.6) entails that

$$\|\xi_r - \xi_{r+h}\|_{[0,T]} \;\leq\; h\,.$$

Consequently, $\xi_{r+h} \in \mathcal{B}_h(\xi^0, \xi_r)$ and (1.5) guarantees that

$$\operatorname*{Var}_{[0,t]} \eta_h \;\leq\; \operatorname*{Var}_{[0,t]} \xi_{r+h}$$

for every $t \in [0,T]$, hence $\eta_h \in \mathcal{P}_{r+h}(\xi^0, u)$. By Theorem 1.2 we readily conclude that $\eta_h = \xi_{r+h}$ and Corollary 1.4 is proved. ■

Proof of Corollary 1.5. Consider $u \in L^\infty(0,T)$, $\eta^0 \in \mathbb{R}$ and a sequence $\{h_n\}$ of positive real numbers such that $h_n \to 0$ as $n \to \infty$. For every n we can define $\xi_n := \mathfrak{p}_{\varrho(u)+h_n}[\eta^0, u]$ and we have by (1.6) that

$$\|\xi_n - \xi_m\|_{[0,T]} \;\leq\; |h_n - h_m|\,. \tag{4.1}$$

Consequently, $\{\xi_n\}$ is a Cauchy sequence in $L^\infty(0,T)$ and its limit which we denote by ξ_∞ is independent of the specific choice of the sequence $\{h_n\}$. Putting $\mathfrak{p}_{\varrho(u)}[\eta^0, u] := \xi_\infty$ we thus obtain (1.8). Moreover, as a uniform limit of functions from $LBV(0,T)$, the function $\mathfrak{p}_{\varrho(u)}[\eta^0, u]$ belongs to $LG(0,T)$. Finally, passing to the limit in (1.7) as $r \to \varrho(u)+$ we obtain (1.9) and the proof is complete. ■

Proof of Corollary 1.6. Let $k \in \{1, \ldots, \ell\}$ be arbitrary and assume for instance that u is non-decreasing in $[t_{k-1}, t_k]$. Put

$$\eta_k(t) \;:=\; \begin{cases} \xi(t) & \text{for } \; t \in [0, t_{k-1}]\,, \\ \max\{\xi(t_{k-1})\,,\, u(t) - r\} & \text{for } \; t \in\,]t_{k-1}, t_k]\,, \end{cases}$$

and

$$\bar{t} \;:=\; \max\{t \in [t_{k-1}, t_k]\,;\, u(t) - r \leq \xi(t_{k-1})\}\,.$$

Since both ξ and u are left-continuous, we have $u(t) - r \leq \xi(t) \leq u(t) + r$ for every $t \in [0,T]$, in particular $\xi(t_{k-1}) \leq u(t_{k-1}) + r \leq u(t) + r$ for $t \in\,]t_{k-1}, t_k]$. Consequently $u(t) - r \leq \eta_k(t) \leq u(t) + r$ for every $t \in [0, t_k]$. Moreover,

$$\begin{aligned}
\operatorname*{Var}_{[0,t]} \xi \;&\geq\; \operatorname*{Var}_{[0,t_{k-1}]} \xi \;=\; \operatorname*{Var}_{[0,t]} \eta_k && \text{for } \; t \in [t_{k-1}, \bar{t}]\,, \\
\operatorname*{Var}_{[0,t]} \xi \;&\geq\; \operatorname*{Var}_{[0,t_{k-1}]} \xi + \xi(t) - \xi(t_{k-1}) && \\
&\geq\; \operatorname*{Var}_{[0,t_{k-1}]} \xi + u(t) - r - \xi(t_{k-1}) \;=\; \operatorname*{Var}_{[0,t]} \eta_k && \text{for } \; t \in\,]\bar{t}, t_k]\,,
\end{aligned}$$

and Theorem 1.2 implies that $\eta_k = \xi$ in $[0, t_k]$. The argument is similar if u is non-increasing in $[t_{k-1}, t_k]$ and the assertion follows easily. ■

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