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Uniqueness of determining a periodic structure from discrete far field observations

Gottfried Bruckner[1], Jin Cheng[2], Masahiro Yamamoto[3]

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[1] Weierstraß-Institut
für
Angewandte Analysis und Stochastik
Mohrenstr. 39
D-10117 Berlin
Germany
E-Mail: bruckner@wias-berlin.de

[2] Fudan University
Department of Mathematics
Shanghai 200433
China
&
Gunma University
Departement of Mathematics
Faculty of Engineering
Kiryu 376-8515
Japan
E-mail: jcheng@math.sci.gunma-u.ac.jp

[3] The University of Tokyo
Department of Mathematical Sciences
3-8-1 Komaba,Meguro
Tokyo 153
Japan
E-mail: myama@ms.u-tokyo.ac.jp

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Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper is devoted to the uniqueness in the inverse scattering problem of determining a perfectly reflecting periodic surface from far field observations on a discrete set. Our proof is based on the unique continuation of the solution to the Helmholtz equation from a discrete set.

1 Introduction

We consider the scattering by a perfectly reflecting periodic surface in the 2D case. According to Bao, Doblin and Cox [4], Hettlich and Kirsch [10], we can formulate the problem as follows. Let $f \in C^2(\mathbb{R})$ be 2π-periodic, and let us set

$$\Omega_f = \{(x_1, x_2); x_2 > f(x_1),\ x_1 \in \mathbb{R}\}.$$

Then we regard $\partial\Omega_f = \{(x_1, x_2); x_2 = f(x_1),\ x_1 \in \mathbb{R}\}$ as a periodic surface to be determined by scattering data. To this end, taking a wave number $k \in \mathbb{C}$, we consider an incident field $u^i(x_1, x_2; k)$ given by

$$u^i(x_1, x_2; k) = \exp\{ik(x_1 \sin\theta - x_2 \cos\theta)\}.$$

Here $i = \sqrt{-1}$; $\Re z$ and $\Im z$ denote the real part and imaginary part of $z \in \mathbb{C}$, respectively, and $\overline{z}$ is the complex conjugate. Then the resulting scattered field $u^s(x_1, x_2; k)$ satisfies the Helmholtz equation with perfectly reflecting boundary condition:

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in} \quad \Omega_f, \tag{1.1}$$

and

$$u^s + u^i = 0 \qquad \text{on} \quad \partial\Omega_f. \tag{1.2}$$

Moreover, throughout this paper we set $\alpha = k \sin\theta$ and pose the following α-quasi-periodicity condition for u^s: Let

$$u^s(x_1 + 2\pi, x_2; k) = e^{2\pi\alpha i} u^s(x_1, x_2; k) \tag{1.3}$$

hold for all $(x_1, x_2) \in \mathbb{R}^2$ (see e.g. Hettlich and Kirsch [10]).

Like in [10], we define the solution space of quasi-periodic solutions as

$$\begin{aligned} Q(\alpha) \quad = \quad & \{u \in C^2(\Omega_f) \cap C^1(\overline{\Omega_f}) \mid u \text{ is } \alpha\text{-quasi-periodic, bounded} \\ & \text{and } \Delta u + k^2 u = 0 \text{ in } \Omega_f\}. \end{aligned}$$

Existence and uniqueness of solutions to the Dirichlet problem (1.1) – (1.3) were proved by an integral equation method or a variational method ([1], [14], [15]).

Let $\{t_j\}_{j=1}^{\infty} \subset [0, 2\pi]$ be a discrete set on the real line $\mathbb{R}$. We choose $a > 0$ sufficiently large such that $a > f(x_1)$, $x_1 \in \mathbb{R}$, and fix a as the level for observations.

We state our inverse problem with respect to the discrete observation set as follows:

Inverse Problem of Diffractive Optics: Determine $x_2 = f(x_1)$, $x_1 \in \mathbb{R}$, from the measurements

$$u^s(t_j, a; k), \qquad j = 1, 2, \cdots,$$

where u^s satisfies (1.1) – (1.3).

For the inverse problem of determining the function f from measurements on $x_2 = a$, i.e. $u^s(x_1, a; k)$, $x_1 \in \mathbb{R}$, the uniqueness was proved for a lossy medium (i.e. $\Im k > 0$) by Bao [3], and in the case of $k \in \mathbb{R}$ by Hettlich and Kirsch [10]. We further refer to Ammari [2] and Kirsch [13].

From the practical point of view, in general, one can only measure data on a discrete set, not $u^s(x_1, a; k)$, $x_1 \in \mathbb{R}$, on the whole line. Therefore it is natural to ask whether the uniqueness is still true for the inverse problem with discrete observations.

The purpose of this paper is to give a positive answer to this question. The key of our proof is unique continuation of the solutions to the Helmholtz equation. This is shown in Section 3. Our approach is similar to the method used in Cheng, Hon and Yamamoto [6], Cheng and Yamamoto [7].

Simply speaking, we proceed as follows: We extend the solutions to the Helmholtz equation from real variables to complex variables ([9]) and apply standard results for functions of complex variables so that the values of the solution to the Helmholtz equation on the line $x_2 = a$ can be determined by the values on the discrete set $\{(t_j, a)\}_{j=1}^{\infty}$. Finally, combining with uniqueness results obtained by other authors, we prove the uniqueness of the function f from far field measurements on a discrete set.

It should be remarked here that we can also obtain conditional stability similar to Isakov [11], [12]. This kind of conditional stability is very important for proposing a numerical algorithm for a stable reconstruction. On the basis of this kind of conditional stability, we can apply a principle due to Cheng and Yamamoto [8] and we can guarantee a convergence rate for the Tikhonov regularized solutions. This will be done in a forthcoming paper. For the local conditional stability, we refer to Bao and Friedman [5].

This paper is organized as follows:

- Section 2. Main results.
- Section 3. Unique continuation for a solution to the Helmholtz equation.
- Section 4. Proofs of the main results.
- Section 5. Remarks.

2 Main results

In this paper, we only consider the following two cases: (1) the case of a lossy (i.e. energy absorbing) medium ($\Im k > 0$), (2) the case that k is real and some a priori information on heights of surfaces is available.

Our main results can be stated as follows.

Theorem 2.1 *Suppose that θ, k are fixed and $\Im k > 0$. Let f_1 and f_2 be 2π-periodic functions and $a > \max\{f_1(t), f_2(t)\}$, $t \in I\!R$. If the scattering solutions $u_j^s(x_1, x_2; k)$, $j = 1, 2$, of the problem (1.1) – (1.3) corresponding to $f = f_j$, $j = 1, 2$, satisfy*

$$u_1^s(t_j, a; k) = u_2^s(t_j, a; k), \qquad j = 1, 2, \cdots,$$

then we have

$$f_1(t) = f_2(t), \qquad t \in I\!R.$$

Theorem 2.2 *Suppose that $h > 0$, $k_0 > 0$ and $|\theta| < \frac{\pi}{2}$ are fixed and $N \in I\!N$ satisfies $N > \frac{h}{2}k_0^2 + \frac{hk_0}{\pi}\cos\theta$. We assume that*

$$0 \le f_1(t), f_2(t) \le h, \qquad for \quad t \in I\!R.$$

Let f_1 and f_2 be 2π-periodic functions and $a > \max\{f_1(t), f_2(t)\}$, $t \in I\!R$. Let the scattering solutions $u_j^s(x_1, x_2; k)$, $j = 1, 2$, of the problem (1.1) – (1.3) corresponding to $f = f_j$, $j = 1, 2$, satisfy

$$u_1^s(t_j, a; k_l) = u_2^s(t_j, a; k_l), \qquad j = 1, 2, \cdots; l = 1, 2, \cdots, N,$$

where $k_l \in (0, k_0]$, $l = 1, 2, \cdots, N$. Then we have

$$f_1(t) = f_2(t), \qquad t \in I\!R.$$

3 Unique continuation of the solutions to the Helmholtz equation

In this section, we prove one kind of a unique continuation assertion for solutions to the Helmholtz equation. It should be mentioned that, although our proof is given for the 2D Helmholtz equation, this kind of unique continuation is also true for general elliptic partial differential equations with analytic coefficients in arbitrary dimensions.

Let $\mathcal{D}$ be a simply connected domain in $\mathbb{R}^2$ and L be a line which intersects with $\mathcal{D}$. Without loss of generality, let us assume that $L = \{(x_1, 0) | x_1 \in \mathbb{R}\}$ and $\mathcal{D} \cap L = \{(x_1, 0) | -3\pi < x_1 < 3\pi\}$.

Lemma 3.1 *(Complex extension.)* *Suppose that w satisfies*

$$\Delta w + k^2 w = 0 \qquad in \quad \mathcal{D}.$$

Then for any fixed constant $\delta > 0$, there exists a complex function $G(z_1)$ which is holomorphic in $\mathcal{D}'$ such that

$$G(t) = w(t, 0), \qquad -3\pi + \delta < t < 3\pi - \delta.$$

Here $\mathcal{D}' = \{z | -3\pi + \delta < \Re z < 3\pi - \delta; \quad -\delta_1 < \Im z < \delta_1\}$ and $\delta_1 > 0$ is a constant which depends on δ, $\mathcal{D}$ and L.

Proof. Let $E(\cdot,\cdot,y_1,y_2)$ be the fundamental solution to the Helmholtz equation in $\mathcal{D}$. Then applying the results in [9] (p.93, Corollary 1.1), we have that $E(\cdot,\cdot,y_1,y_2)$ can be extended as a holomorphic function $E(z_1,z_2,y_1,y_2)$ for $(z_1,z_2)\in\mathbb{C}^2\setminus S(y_1,y_2)$. Here $S(y_1,y_2)$ denotes the "solid isotropic cone" with vertex at $(y_1,y_2)\in\mathbb{R}^2$, that is

$$\begin{aligned} S(y_1,y_2) &= \{(z_1,z_2)|\quad (\Re z_1-y_1)\Im z_1+(\Re z_2-y_2)\Im z_2=0;\\ &\qquad |(\Re z_1,\Re z_2)-(y_1,y_2)|\le|(\Im z_1,\Im z_2)|\}.\end{aligned}$$

On the other hand, by the Green formula, we have that

$$\begin{aligned} w(x_1,x_2) &= \int_{\mathcal{D}}\{(\Delta_y+k^2)E(x_1,x_2,y_1,y_2)\}w(y_1,y_2)dy_1dy_2\\ &\quad -\int_{\mathcal{D}}\{(\Delta_y+k^2)w(y_1,y_2)\}E(x_1,x_2,y_1,y_2)dy_1dy_2\\ &= \int_{\partial\mathcal{D}}\{\frac{\partial}{\partial\nu_y}E(x_1,x_2,y_1,y_2)\}w(y_1,y_2)d\sigma_y\\ &\quad -\int_{\partial\mathcal{D}}\{\frac{\partial}{\partial\nu_y}w(y_1,y_2)\}E(x_1,x_2,y_1,y_2)d\sigma_y,\quad (x_1,x_2)\in\mathcal{D},\end{aligned}$$

where ν is the outer unit normal with respect to $\mathcal{D}$.

On the line $L=\{(x_1,0)|x_1\in\mathbb{R}\}$, we have

$$\begin{aligned} w(x_1,0) &= \int_{\partial\mathcal{D}}\{\frac{\partial}{\partial\nu_y}E(x_1,0,y_1,y_2)\}w(y_1,y_2)d\sigma_y\\ &\quad -\int_{\partial\mathcal{D}}\{\frac{\partial}{\partial\nu_y}w(y_1,y_2)\}E(x_1,0,y_1,y_2)d\sigma_y.\end{aligned}\tag{3.1}$$

Next, we can directly verify that there exists a positive constant δ_1, which depends on δ, $\mathcal{D}$ and L, such that

$$\{(z_1,0)|z_1\in\mathcal{D}'\}\subset\mathbb{C}^2\setminus\bigcup_{(y_1,y_2)\in\partial\mathcal{D}}S(y_1,y_2).$$

We define the complex function

$$\begin{aligned} G(z) &= \int_{\partial\mathcal{D}}\{\frac{\partial}{\partial\nu_y}E(z,0,y_1,y_2)\}w(y_1,y_2)d\sigma_y\\ &\quad -\int_{\partial\mathcal{D}}\{\frac{\partial}{\partial\nu_y}w(y_1,y_2)\}E(z,0,y_1,y_2)d\sigma_y,\end{aligned}\tag{3.2}$$

where $E(z,0,y_1,y_2)$ is the holomorphic extension of the fundamental solution $E(\cdot,0,y_1,y_2)$. By (3.2), the function $G(z_1)$ is holomorphic in $\mathcal{D}'$. By (3.1) and (3.2), we have

$$G(t)=w(t,0),\qquad -3\pi+\delta<t<3\pi-\delta.$$

∎

Lemma 3.2 *(Unique continuation.)* *Suppose that w satisfies*

$$\Delta w+k^2w=0\qquad in\quad \mathcal{D}.$$

If

$$w(t_j, 0) = 0, \qquad j = 1, 2, \cdots,$$

then we have that

$$w(t, 0) = 0, \qquad t \in (-3\pi, 3\pi).$$

Proof. Suppose that the conclusion is not true. Then there exists a point $t^* \in (-3\pi, 3\pi)$ such that

$$w(t^*, 0) \neq 0. \tag{3.3}$$

Let $\delta = \frac{1}{2}\min\{|t^* - 3\pi|, |t^* + 3\pi|, \pi\}$. Then by Lemma 3.1, we have that, for the solution w to the Helmholtz equation, there exists a holomorphic function in $\mathcal{D}'$ such that

$$w(t, 0) = G(t), \qquad -3\pi + \delta < t < 3\pi - \delta. \tag{3.4}$$

Since $\{t_j\}_{j=1}^{\infty} \subset [0, 2\pi] \subset (-3\pi + \delta, 3\pi - \delta)$, we have

$$\text{closure of } \{t_j\}_{j=1}^{\infty} \subset \mathcal{D}'$$

and

$$G(t_j) = w(t_j, 0) = 0, \qquad j = 1, 2, \cdots.$$

By the unicity theorem for holomorphic functions, we conclude that

$$G(z_1) = 0, \qquad z_1 \in \mathcal{D}'.$$

By (3.4), we obtain that

$$w(t, 0) = G(t) = 0, \qquad -3\pi + \delta < t < 3\pi - \delta. \tag{3.5}$$

On the other hand, from (3.3), we have

$$w(t^*, 0) \neq 0, \qquad t^* \in (-3\pi + \delta, 3\pi - \delta).$$

This is a contradiction to (3.5). ∎

Remark 3.1 *This kind of unique continuation for a solution to the Helmholtz equation is only valid on the line L. Outside this line, we have no information about the solution as the following example shows.*

Example: We consider the function

$$w(x_1, x_2) = x_2 e^{ikx_1}.$$

Then $w(x_1, x_2)$ satisfies $\Delta w + k^2 w = 0$ and $w(x_1, 0) = 0$. However it does not vanish outside $L \equiv \{(x_1, 0) | x_1 \in \mathbb{R}\}$.

Remark 3.2 *Similar unique continuation holds true with conditional stability for a solution of the Laplace equation ([6], [7]).*

4 Proof of the main results

Now we can give the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1: We set $u_j = u_j^s + u_j^i$, $j = 1, 2$ and note that u_j is $(k \sin\theta)$-periodic. Then we discuss in terms of u_1 and u_2.

We assume that the conclusion is not true, i.e. $f_1(t_0) \neq f_2(t_0)$ for some $t_0 \in \mathbb{R}$. Let

$$w(x_1, x_2; k) = u_1(x_1, x_2; k) - u_2(x_1, x_2; k).$$

Then w satisfies:

$$\Delta w + k^2 w = 0 \qquad in \quad \Omega'$$

and

$$w(t_j, a; k) = 0, \qquad j = 1, 2, \cdots.$$

Here $\Omega' = \{(x_1, x_2) | x_2 > \max\{f_1(x_1), f_2(x_1)\},\ x_1 \in \mathbb{R}\}$.

Since $\{(x_1, a) | x_1 \in \mathbb{R}\} \subset \Omega'$, by Lemma 3.2, we have that

$$w(x_1, a; k) = 0, \qquad x_1 \in \mathbb{R}.$$

Therefore we see that $w(x_1, x_2; k) = 0$, $x_2 \geq a$, by the uniqueness of the Dirichlet boundary value problem for the Helmholtz equation in $\{x_2 > a\}$ with quasi-periodicity ([1], [10]).

By the standard unique continuation for elliptic equations, we have that

$$w(x_1, x_2; k) = 0, \qquad x_2 > \max\{f_1(x_1), f_2(x_1)\}.$$

Let us consider the domain

$$\Omega_0 = \{(x_1, x_2) \quad | \quad \min\{f_1(x_1), f_2(x_1)\} < x_2 < \max\{f_1(x_1), f_2(x_1)\}; \\ 0 < x_1 < 2\pi\}.$$

Then we know that $\Omega_0 \neq \emptyset$ and Ω_0 contains some open set O.

We set

$$\Gamma_1 = \{(0, x_2) | \min\{f_1(0), f_2(0)\} < x_2 < \max\{f_1(0), f_2(0)\}\}$$

and

$$\Gamma_2 = \{(2\pi, x_2) | \min\{f_1(2\pi), f_2(2\pi)\} < x_2 < \max\{f_1(2\pi), f_2(2\pi)\}\}.$$

Note that Γ_1, Γ_2 may be empty.

Let

$$W(x_1, x_2) = \begin{cases} u_1(x_1, x_2; k), & f_1(x_1) < f_2(x_1) \\ u_2(x_1, x_2; k), & f_1(x_1) > f_2(x_1) \\ 0, \quad \text{otherwise}. \end{cases} \tag{4.1}$$

It is obvious that W satisfies $\Delta W + k^2 W = 0$ in Ω_0.

By the periodicity of the functions f_j, $j = 1,2$ and the quasi-periodicity of u_j, $j = 1,2$, we have $\Gamma_1 + (2\pi, 0) = \Gamma_2$, and

$$\int_{\Gamma_1} \overline{e^{-i\alpha x_1} W} \frac{\partial (e^{-i\alpha x_1} W)}{\partial x_1} ds - \int_{\Gamma_2} \overline{e^{-i\alpha x_1} W} \frac{\partial (e^{-i\alpha x_1} W)}{\partial x_1} ds = 0.$$

Here we regard the integrals over Γ_1, Γ_2 as 0 if $\Gamma_1 = \emptyset$. Therefore by (4.1) we obtain

$$\int_{\partial\Omega_0} \overline{W} \frac{\partial W}{\partial \nu} ds = 0.$$

Applying the Green formula, we have

$$\|\nabla W\|^2_{L^2(\Omega_0)} - k^2 \|W\|^2_{L^2(\Omega_0)} = \int_{\partial\Omega_0} \overline{W} \frac{\partial W}{\partial \nu} ds = 0.$$

Using $\Im k > 0$, we easily verify that

$$W = 0 \qquad \text{in} \quad \Omega_0.$$

Since Ω_0 contains some open set O, by the standard unique continuation for the Helmholtz equation, we have

$$u_j(x_1, x_2; k) = u_j^s(x_1, x_2; k) + u_j^i(x_1, x_2; k) = 0, \qquad x_2 > f_j(x_1),\ j = 1,2.$$

Since f_1 and f_2 are not identically equal, this is a contradiction to that u^i is an incoming and u^s is an outgoing wave. The proof is complete.

Proof of Theorem 2.2:

Since this proof is similar to the previous one, we just give a sketch of it. First applying Lemma 3.2, we have

$$u_1(x_1, a; k) = u_2(x_1, a; k), \qquad x_1 \in \mathbb{R}.$$

Then, by Theorem 3.2 in [10], we obtain the conclusion.

5 Some remarks

We give some remarks about the inverse problem and our method.

Remark 5.1 *In this paper, we only consider the two-dimensional problem. For the three-dimensional inverse problem of diffractive optics, our method still works under the condition that the set* $\overline{\{(x_1^j, x_2^j, a)\}_{j=1}^\infty}$ *contains an open set on the plane* $x_3 = a$.

Remark 5.2 *It would be interesting to discuss the inverse problems of diffractive optics from far field measurements on finitely many points. In general, the uniqueness fails. However we can expect the conditional stability which is important for constructing the Tikhonov regularization functional and proving a convergence rate of the regularized solutions ([8]). This will be our future topic.*

Remark 5.3 *The unique continuation in Lemma 3.2 is also true for a discrete set on an analytic curve and for general elliptic partial differential equations with analytic coefficients.*

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