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Mathematical Theory of Porous Media - Lecture Notes

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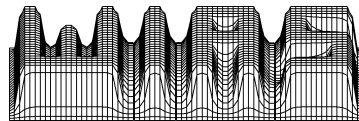
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1 Phenomenology

1.1 Diffusion and thermodiffusion in systems with a solid phase

Theories of porous and granular materials can be constructed on different levels of observation. *Microscopic* models rely on Newton equations of motion of material points or molecules and use methods of molecular dynamics. Such models can be transferred on a *semimacroscopic* level by multiscaling and averaging procedures. Finally on a *macroscopic* level continuous field models are constructed. These may either follow from semimacroscopic models by homogenizing, averaging over Representative Elementary Volumes (REV), construction of moments of kinetic distribution functions or they may be constructed by means of a phenomenological macroscopic approach. In these notes we present solely the latter type of models with a marginal reference to averaging procedures.

The construction of macroscopic continuous models of systems with a solid component in its most sophisticated form stems from models of multicomponent systems. Differences are primarily connected with an art of interactions within the solid component. Models must be clearly different in the cases of suspensions, of granular materials or of porous materials. In the first case solid particles interact with each other either through the fluid or through collisions and there is no permanent contact between them. In the second case a granular solid component may behave as a solid which cannot carry a tensile loading (unilateral constraint on constitutive relations) or it may fluidize and then behave as a suspension. Finally a porous material behaves in average as a usual solid and it forms a deformable carrier for fluid components. We limit our attention in these notes to the last case.

The most important feature of *porous materials* is the appearance of different kinematics for the solid component - the *skeleton*, and fluid components in channels of the skeleton. This yields *diffusion processes* characterized by relative velocities of components. In most cases of a practical bearing the dependence on the relative velocity is reduced to a linear contribution to momentum balance equations (momentum sources) or even to a simpler form called the *Darcy's law*. In these notes we present solely some elements of a more general approach which leads to a set of hyperbolic field equations reminding the hierarchy appearing in the extended thermodynamics [1,2]. Some details can be found in references quoted in the sequel.

The problem of *thermodiffusion* within such models is still very much open. This is related to difficulties with an appropriate definition of the temperature on the macroscopic level of description. The most important property of the classical *thermodynamical temperature*, its continuity on ideal thermal walls and, consequently, its experimental measurability, is not fulfilled in porous materials ([3], p. 76). In addition such processes as phase transitions or chemical reactions in porous materials are characterized by real thermodynamical temperatures (e.g. melting and freezing points, evaporation, etc.) of components on a semimacroscopic level of description.

It means that even if we have introduced a macroscopic notion of temperature we would have to know a rule of transformation of this quantity to the semimacroscopic level. This is mathematically an ill-posed problem. Even though one can formally work with notions such as partial heat fluxes, specific heats etc. their operational meaning is not clarified yet. This seems to be one of the most important open issues of modelling porous and granular materials.

In addition we have to deal frequently with the problem of different temperatures for different components. In contrast to gases a local thermodynamical equilibrium is reached in porous and granular materials after macroscopically long relaxation times. For instance a hot water flowing through a cold porous material does not reach locally a common temperature with the skeleton within seconds or minutes. Consequently we should construct thermodynamical models with different temperatures of components. Such a construction is missing even in the case of fluid mixtures. One of the reasons is again the problem of measurability.

Let us mention in passing that in theories of granular materials stemming from a kinetic equation it is common to work with a *kinetic temperature* rather than a thermodynamical temperature. It is defined in a way similar to this of the kinetic theory of gases as a mean kinetic energy of granulae. There are numerous difficulties connected with such a notion. For example a natural equilibrium state of a granular material in which particles do not move would have a temperature equal to zero. Consequently deviations from the equilibrium state which are used in the construction of macroscopic moment equations of the kinetic theory would have to be constructed by means of a trivial distribution function. Certainly this cannot give any reasonable physical results. For this reason moment equations are constructed by a reference to a Maxwell-like distribution describing processes of simple shearing flows rather than real equilibrium states. In contrast to - say - Grad 13 moment method of rarified gases such procedures are not justified in any way. Moreover the questions of measurability of kinetic temperature, a relation to the thermodynamical temperature etc. are not even asked as yet.

1.2 Mass exchange, chemical reactions, adsorption

Within multicomponent continuous models an exchange of mass is described by mass sources in partial mass balance equations. However these contributions must contain additional *microstructural* variables. For instance in the case of chemical reactions this is the vector of extent of chemical reaction (e.g. [4,5]). This requires an extension of the set of field equations. In many cases additional equations for microstructural variables have the form of evolution equations. Then there is no need to introduce additional boundary conditions. Such microstructural variables cannot be controlled, they develop spontaneously from initial data. On the other hand the latter can be usually easily formulated because many microstructural variables are defined in such a way that they vanish in thermodynamical equilibria.

Further in these notes we present in some details a model of processes of exchange

of mass called *adsorption*. These processes appear in cases of components which, in contrast to chemical bindings, form weak van der Waals bindings solely with the skeleton. Such are, for example, processes of transport of many pollutants in soils. According to the simplest model of these processes, developed by Langmuir (see: [6,7] for references) they are described by an additional field of the so-called number of bare sites. In the case of materials with very small diameters of channels adsorption processes possess a hysteresis loop in the relation between the partial pressure of adsorbate in the fluid phase and an amount of mass adsorbed by the skeleton and this plays a very important role in controlling technological processes in such materials. Such loops are caused by capillary effects. For this reason they do not appear in materials with moderate and large channels which is characteristic for usual soils.

1.3 Dynamics; additional modes of bulk and surface waves

As already mentioned above multicomponent models of porous materials contain more than one velocity field. This yields field equations following from partial momentum balance equations with a corresponding number of partial accelerations. Consequently one expects that in such models additional modes of weak discontinuity waves have to appear. This is indeed the case. One of these modes was predicted by M. A. Biot in 1941. Due to the tradition stemming from geophysics this mode is called P2 compressional (Biot's) wave as the usual longitudinal wave registered in seismograms was called P1. Existence of this mode was confirmed in numerous experiments. It has been found out that it is the slowest of three modes P1, S (transversal wave) and P2. It is also very strongly attenuated.

As consequence of existence of additional modes there exist as well additional modes of surface waves. Apart from the classical Rayleigh wave there exist the so-called Stoneley waves, various leaky waves and, in general, a number of possible modes of surface waves depends on properties of neighbouring systems, i.e. on the structure of boundary conditions. We discuss this problem further in these notes. Let us mention that surface waves are much weaker attenuated as bulk waves and for this reason they are easier attainable in measurements. In recent years one can observe a vehement progress in these measuring techniques [8].

1.4 Coupled problems (combustion, explosions)

Couplings of dynamical properties of porous and granular materials with mass exchange between components play an important role in various combustion and explosion problems. These are connected with the propagation of strong discontinuities such as shock waves and combustion fronts in combustion of solid fuels or deformations of soils due to impacts of meteorites. Models for such processes are still rather weakly developed. Most important contributions are based on the model proposed by M. A. Goodman and C. Cowin [9] (see also [10]) which refers to some additional

microstructural properties called the principle of equilibrated pressures. The model leads to a quasilinear hyperbolic set of equations which admits the existence of shock waves. However, apart from some rather simple properties of propagation conditions results are rather scarce.

Some elementary properties of one-dimensional Rankine-Hugoniot conditions have been also investigated within the frame of the model with the porosity balance equation which we discuss later. However a comprehensive theory of shock waves is still missing and one of the reasons is lack of a selection (entropy) criterion. We return to this problem in the discussion of Riemann problems.

1.5 Instabilities (e.g. eruptions during earthquakes)

Many processes in porous and granular materials are connected with the development of instabilities. They lead to fluidization of saturated sands, to the creation of patterns in porous materials and to some instabilities, such as Saffmann-Taylor, in flows of fluid components. As usual they are connected with nonlinearities appearing in the model.

One of the most spectacular phenomena accompanying earthquakes is the fountain-like explosion of water from the sand. It has been found that prior to this phenomenon the character of permeability of the soil changes in an unstable manner. In the first stage the homogeneity of the system breaks down and a pattern of chimney-like channels with a very high permeability is formed. In the second stage one of these channels becomes dominant and this leads to an explosin-like eruption of water from the ground. This behavior seems to be connected with a nonlinear coupling of the diffusion velocity with the gradient of porosity but theoretical results are still preliminary.

Another class of instabilities appears in the model with the balance equation of porosity. These are connected with the coupling between dynamical changes of porosity and partial stresses in components. Let us mention one of those instabilities. In a case of a Riemann problem the system develops soliton-like waves of porosity. These are connected with the loss of symmetry of the front of propagation if the two-dimensional front is concave. Most likely in the vicinity of the symmetry axis the system develops a mashy region. We present some aspects of this problem further in these notes.

2 Continuous models

2.1 Review of some models (incompressibility), some new contributions

Multicomponent modeling of porous materials is based on the assumption that additionally to usual fields of theories of fluid mixtures there exists a microstructure which is reflected in the simplest case by a single additional field of porosity and by solid-like properties of one of the components. In some models this microstructural extension is even broader and corresponding models contain, for instance, the so-called volume fractions of all components, double porosity, tortuosity as a simplest measure of complexity of geometrical structure of channels, couple stresses etc.

We present here an example of such a model for a two-component system with an assumption of incompressibility of components. Models of this art appear quite frequently in applications to soil mechanics or glaciology [11].

As in all continuum models we define in *Eulerian description* fields on a common domain \mathcal{B}_t which is time dependent and corresponds to a part of the three-dimensional space of motion occupied in a current instant of time by all components. In the case of semipermeable boundary $\partial\mathcal{B}_t$ parts of components which flow out of this domain are considered separately and one has to solve contact problems. We return to the problem of configurations of multicomponent system further in this work.

In a purely mechanical model which we want to consider in this Section processes are described by two current partial mass densities $\rho_t^S(\mathbf{x}, t)$, $\rho_t^F(\mathbf{x}, t)$, $\mathbf{x} \in \mathcal{B}_t \subset \mathfrak{R}^3$, $t \in \mathcal{T} \subset \mathfrak{R}$, for the skeleton and the fluid component, respectively, and by two velocity fields $\mathbf{v}^S(\mathbf{x}, t)$, $\mathbf{v}^F(\mathbf{x}, t)$ for these two components. All these fields are *macroscopic* which means that they are defined on the common domain \mathcal{B}_t and, for instance the fluid mass and the skeleton mass contained in a subdomain $\mathcal{P}_t \subset \mathcal{B}_t$ are given by the Lebesgue integrals

$$M^F(\mathcal{P}_t) = \int_{\mathcal{P}_t} \rho_t^F dV, \quad M^S(\mathcal{P}_t) = \int_{\mathcal{P}_t} \rho_t^S dV. \quad (2.1)$$

In the definition of incompressible components one uses a "pseudomesoscopic" quantities which are called *realistic mass densities*. We denote them by ρ_t^{SR} and ρ_t^{FR} . They are also defined in each point of the domain \mathcal{B}_t and not in points of the skeleton or of the fluid, respectively. They may be related to mesoscopic (or semimacroscopic) quantities ρ_m^{SR} , ρ_m^{FR} by the following formulae

$$\rho_t^{SR} = \frac{1}{V(\mathcal{P}_t^S)} \int_{\mathcal{P}_t} \rho_m^{SR} (1 - \chi^F) dV, \quad \rho_t^{FR} = \frac{1}{V(\mathcal{P}_t^F)} \int_{\mathcal{P}_t} \rho_m^{FR} \chi^F dV, \quad (2.2)$$

where

$$\begin{aligned} \mathcal{P}_t &= \mathcal{P}_t^S \cup \mathcal{P}_t^F, \quad \mathcal{P}_t^S \cap \mathcal{P}_t^F = \emptyset, \\ V(\mathcal{P}_t^F) &:= \int_{\mathcal{P}_t} \chi^F dV, \quad V(\mathcal{P}_t^S) := \int_{\mathcal{P}_t} (1 - \chi^F) dV, \end{aligned} \quad (2.3)$$

and χ^F is the characteristic function for \mathcal{P}_t^F . \mathcal{P}_t is a neighbourhood of a generic point \mathbf{x} and it is usually chosen to be identical for all points of the porous body, i.e. it is obtained by a shift of a chosen domain over the whole current configuration \mathcal{B}_t . This domain should be small enough to deliver a good approximation, for instance, in the vicinity of the boundary. Then it is called the *Representative Elementary Volume (REV)*. In contrast to ρ_t^{SR}, ρ_t^{FR} which do not possess any physical interpretation in points of the real fluid for the first quantity and in points of the skeleton for the second one, the mass densities ρ_m^{SR}, ρ_m^{FR} are defined solely in points of the real skeleton, and of the real fluid, respectively. Consequently, they possess a usual physical interpretation. For instance, ρ_m^{FR} is equal to $1000 \frac{\text{kg}}{\text{m}^3}$ for water in normal conditions.

It is easy to check the following relations

$$V(\mathcal{P}_t^S) = (1 - n) V(\mathcal{P}_t), \quad V(\mathcal{P}_t^F) = n V(\mathcal{P}_t), \quad (2.4)$$

$$n := \frac{1}{V(\mathcal{P}_t)} \int_{\mathcal{P}_t} \chi^F dV, \quad V(\mathcal{P}_t) := \int_{\mathcal{P}_t} dV.$$

The quantity n defined in (2.4)₃ is called the *porosity*.

Consequently

$$\rho_t^S = (1 - n) \rho_t^{SR}, \quad \rho_t^F = n \rho_t^{FR}. \quad (2.5)$$

By means of these relations we are now in the position to introduce the notion of incompressibility appearing in some theories of porous and granular materials. Namely it is assumed for such models that

$$\rho_t^{SR} = \text{const.}, \quad \rho_t^{FR} = \text{const.} \quad (2.6)$$

Consequently the current mass densities ρ_t^F, ρ_t^S are not independent fields. They can be reduced to the single field of porosity n . In such a case partial mass balance equations (without mass exchange!) reduce to the following form

$$\begin{aligned} \frac{1}{\rho_t^{SR}} \left\{ \frac{\partial \rho_t^S}{\partial t} + \operatorname{div} (\rho_t^S \mathbf{v}^S) \right\} &\equiv -\frac{\partial n}{\partial t} + \operatorname{div} ((1-n) \mathbf{v}^S) = 0, \\ \frac{1}{\rho_t^{FR}} \left\{ \frac{\partial \rho_t^F}{\partial t} + \operatorname{div} (\rho_t^F \mathbf{v}^F) \right\} &\equiv \frac{\partial n}{\partial t} + \operatorname{div} (n \mathbf{v}^F) = 0. \end{aligned} \quad (2.7)$$

We can also combine these two equations to the following one

$$\operatorname{div} (n \mathbf{v}^F + (1-n) \mathbf{v}^S) = 0. \quad (2.8)$$

If we consider equation (2.7)₁ as a candidate for the field equation for the porosity n then equation (2.8) is a *constraint condition* of the model. This condition yields certain limitations on constitutive relations appearing in the phenomenological model which are not always physically and mathematically acceptable (see: [12]). We show here two examples of models which are thermodynamically admissible. The first one is used frequently in soil mechanics and in the description of suspensions. In order to obtain field equations we need momentum balance equations which have the following form in the Eulerian description

$$\begin{aligned} \frac{\partial (\rho_t^S \mathbf{v}^S)}{\partial t} + \operatorname{div} (\rho_t^S \mathbf{v}^S \otimes \mathbf{v}^S - \mathbf{T}^S) &= \hat{\mathbf{p}}^S, \\ \frac{\partial (\rho_t^F \mathbf{v}^F)}{\partial t} + \operatorname{div} (\rho_t^F \mathbf{v}^F \otimes \mathbf{v}^F - \mathbf{T}^F) &= \hat{\mathbf{p}}^F, \quad \hat{\mathbf{p}}^S + \hat{\mathbf{p}}^F = 0, \end{aligned} \quad (2.9)$$

where $\mathbf{T}^S, \mathbf{T}^F$ denote symmetric partial Cauchy stress tensors, $\hat{\mathbf{p}}^S, \hat{\mathbf{p}}^F$ are momentum sources. We make the assumption that these quantities satisfy the following constitutive relations

$$\begin{aligned} \mathbf{T}^S &= \mathbf{T}^S (n, \operatorname{grad} n, \mathbf{e}^S, \mathbf{w}), \quad \mathbf{T}^F = \mathbf{T}^F (n, \operatorname{grad} n, \mathbf{e}^S, \mathbf{w}), \\ \hat{\mathbf{p}}^S &= \hat{\mathbf{p}}^S (n, \operatorname{grad} n, \mathbf{e}^S, \mathbf{w}), \end{aligned} \quad (2.10)$$

where the symmetric deformation tensor of the skeleton \mathbf{e}^S satisfies the equation

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S, \quad (2.11)$$

and $\mathbf{w} := \mathbf{v}^F - \mathbf{v}^S$ is the relative velocity of components.

By means of the second law of thermodynamics one can show that the constraint (2.8) is thermodynamically admissible. This would not be the case were constitutive relations (2.10) independent of $\operatorname{grad} n$ [12]. In this sense we deal with a *higher gradient model*. If we assume in addition the isotropy and linearity with respect

to both vector variables $gradn$ and \mathbf{w} then we obtain a relatively explicit form of constitutive relations

$$\begin{aligned}\mathbf{T}^S &= -(1-n)p\mathbf{1} + \mathbf{T}_{eff}^S(n, \mathbf{e}^S), & \mathbf{T}^F &= -np\mathbf{1} + \mathbf{T}_{eff}^F(n, \mathbf{e}^S), \\ \hat{\mathbf{p}}^S &= -\hat{\mathbf{p}}^F = \pi(n, \mathbf{e}^S) \mathbf{w} - p gradn,\end{aligned}\quad (2.12)$$

where $\mathbf{T}_{eff}^S, \mathbf{T}_{eff}^F$ are the so-called *effective partial stress tensors*. The second one is frequently assumed to be zero and the first one, if it is linear (small deformation of the skeleton $\|\mathbf{e}^S\| \ll 1$), is given by a Hooke's law with material coefficients depending on the porosity n . The *permeability coefficient* π is also usually assumed to be constant. The *pore pressure* p is the reaction force on the constraint.

In order to account for instabilities of the microstructure one can try to extend the above model by accounting for nonlinear dependence on the relative velocity \mathbf{w} . This is justified because such instabilities appear by flows of a high intensity of the fluid component. In a continuum model the latter corresponds to contributions $\rho_t^F \mathbf{w}$. Such an extension yields in the lowest approximation the following constitutive relations

$$\begin{aligned}\mathbf{T}^S &= -(1-n)p\mathbf{1} + \mathbf{T}_{eff}^S(n, \mathbf{e}^S) + \delta \mathbf{w} \otimes \mathbf{w}, \\ \mathbf{T}^F &= -np\mathbf{1} + \mathbf{T}_{eff}^F(n, \mathbf{e}^S) - \delta \mathbf{w} \otimes \mathbf{w},\end{aligned}\quad (2.13)$$

$$\hat{\mathbf{p}}^S = -\hat{\mathbf{p}}^F = \pi(n, \mathbf{e}^S) \mathbf{w} - (p + \Gamma \mathbf{w} \cdot \mathbf{w}) gradn,$$

where δ and Γ are additional material parameters.

Such a model seems to be appropriate to describe, for instance, instabilities leading to fluidization and eruption in water saturated sands by earthquakes¹.

The above described class of models shall not be discussed any further in this work. In spite of their important role in some problems of soil mechanics these models have some faults which do not seem to be acceptable in cases of wave processes. Most important of them is the lack of hyperbolicity (the part of the operator connected with the constraint is elliptic). This leads to a reduced number of real eigenvalues corresponding to speeds of propagation and, consequently, to the lack of certain modes of propagation of weak discontinuity waves. In particular the P2-wave and some important surface waves cannot be described by such models. We discuss the structure of those modes following from a different model presented in the next Section.

¹see: a forthcoming paper of Theo Wilhelm and K. Wilmanski and the forthcoming PhD Thesis of Theo Wilhelm at the University of Innsbruck, Institut für Geotechnik und Tunnelbau of Prof. D. Kolymbas

2.2 The model with the porosity balance equation

2.2.1 Thermodynamical structure (Lagrangian description, extended thermodynamics)

In this Section we present the model of porous materials developed in the recent years [13-16] for an elastic skeleton and ideal fluid components. We present its nonlinear foundations without thermodynamical details. These are rather cumbersome and can be found in my book [17] as well as in the paper [16].

Geometric nonlinearities connected with possibly large deformations of the skeleton indicate that the convenient way to describe processes is to define fields on a *reference configuration* of the skeleton. For such a configuration the deformation gradient of the skeleton is defined as the unit matrix: $\mathbf{F}^S = \mathbf{1}$. Hence we formulate the *Lagrangian description* of motion of the porous material.

The aim of the model is to find the following fields defined in points \mathbf{X} of the reference domain $\mathcal{B}_0 \subset \mathfrak{R}^3$ and in instances t of the time interval $\mathcal{T} \subset [0, \infty)$:

1. mass density of the skeleton referred to a unit reference volume: $\rho^S(\mathbf{X}, t)$,
2. mass densities of the fluid components referred to a unit reference volume: $\rho^\alpha(\mathbf{X}, t)$, $\alpha = 1, \dots, A$,
3. velocity of the skeleton: $\dot{\mathbf{x}}^S(\mathbf{X}, t)$,
4. deformation gradient of the skeleton: $\mathbf{F}^S(\mathbf{X}, t)$, $J^S := \det \mathbf{F}^S > 0$,
5. velocities of fluid components $\dot{\mathbf{x}}^\alpha(\mathbf{X}, t)$, $\alpha = 1, \dots, A$,
6. porosity: $n(\mathbf{X}, t)$,
7. temperature common for all components $T(\mathbf{X}, t)$.

Consequently a *thermomechanical process* is described by the mapping

$$\mathbf{u} : (\mathbf{X}, t) \mapsto \mathfrak{R}^{4A+15}, \quad \mathbf{u} := \left\{ \rho^S, \rho^\alpha, \dot{\mathbf{x}}^S, \mathbf{F}^S, \dot{\mathbf{x}}^\alpha, n, T \right\}, \quad \alpha = 1, \dots, A. \quad (2.14)$$

Field equations for these fields follow from balance equations which we proceed to formulate.

Balance equations are formulated in their global form on material domains of components. For porous materials in the Lagrangian description the family of material domains for the skeleton is defined as a class of subsets of \mathcal{B}_0 which is time independent and satisfies conditions identical with those of the classical continuum mechanics [18]. Material domains of the skeleton are time independent because the reference configuration \mathcal{B}_0 is defined with respect to the deformation gradient of skeleton \mathbf{F}^S .

It is not the case any more for fluid components. They have different kinematics than the skeleton which means that domains in the space of motion containing during the motion the same particles of a particular fluid component move with respect to material domains of the skeleton. In the Eulerian description the velocity field for this motion is given by the difference $\mathbf{v}^\alpha(\mathbf{x}, t) - \mathbf{v}^S(\mathbf{x}, t)$ for the fluid component α , where $\mathbf{v}^\alpha(\mathbf{x}, t)$ is the velocity of the fluid and $\mathbf{v}^S(\mathbf{x}, t)$ is the velocity of the skeleton at the same spatial position \mathbf{x} and at the same instant of time t . This relative motion yields the time dependence of material domains of fluid components projected on the reference configuration \mathcal{B}_0 . The projection is carried by the function of motion of the skeleton

$$\begin{aligned} \forall \mathbf{X} \in \mathcal{B}_0, t \in \mathcal{T} : \quad \mathbf{x} = \chi^S(\mathbf{X}, t) &\implies \\ \implies \mathbf{F}^S = \text{Grad}\chi^S(\mathbf{X}, t), \quad \dot{\mathbf{x}}^S = \frac{\partial \chi^S(\mathbf{X}, t)}{\partial t}, \end{aligned} \quad (2.15)$$

whose existence is assumed in the model. The condition for the existence of the function of motion χ^S shall be formulated later.

It is easy to check that the Lagrangian fields of velocities of material domains of fluid components are given by the following relation

$$\begin{aligned} \forall \mathbf{X} \in \mathcal{B}_0 : \quad \dot{\mathbf{X}}^\alpha(\mathbf{X}, t) &:= \mathbf{F}^{S-1}(\dot{\mathbf{x}}^\alpha - \dot{\mathbf{x}}^S), \\ \dot{\mathbf{x}}^S(\mathbf{X}, t) = \mathbf{v}^S(\chi^S(\mathbf{X}, t), t), \quad \dot{\mathbf{x}}^\alpha(\mathbf{X}, t) &= \mathbf{v}^\alpha(\chi^S(\mathbf{X}, t), t). \end{aligned} \quad (2.16)$$

In order to appreciate the operational meaning of this transformation we formulate balance equations of mass for all components. They have the following form

$$\frac{d}{dt} \int_{\mathcal{P}^S} \rho^S dV = \int_{\mathcal{P}^S} \hat{\rho}^S dV, \quad (2.17)$$

for every material domain of the skeleton $\mathcal{P}^S \subset \mathcal{B}_0$, and

$$\frac{d}{dt} \int_{\mathcal{P}^\alpha(t)} \rho^\alpha dV = \int_{\mathcal{P}^\alpha(t)} \hat{\rho}^\alpha dV, \quad (2.18)$$

for every material domain of the α fluid component $\mathcal{P}^\alpha(t) \subset \mathcal{B}_0$, $\alpha = 1, \dots, A$. In the above relations $\hat{\rho}^S, \hat{\rho}^\alpha$ are the mass sources which satisfy the following bulk conservation law

$$\forall \mathbf{X} \in \mathcal{B}_0, t \in \mathcal{T} : \quad \hat{\rho}^S + \sum_{\alpha=1}^A \hat{\rho}^\alpha = 0. \quad (2.19)$$

Time dependence of material domains for fluid components yields the following rules of time differentiation

$$\begin{aligned}\frac{d}{dt} \int_{\mathcal{P}^S} \rho^S dV &= \int_{\mathcal{P}^S} \frac{\partial}{\partial t} \rho^S dV, \\ \frac{d}{dt} \int_{\mathcal{P}^\alpha(t)} \rho^\alpha dV &= \int_{\mathcal{P}^\alpha(t)} \frac{\partial}{\partial t} \rho^\alpha dV + \oint_{\partial \mathcal{P}^\alpha(t)} \rho^\alpha \mathbf{N} \cdot \dot{\mathbf{X}}^\alpha dS,\end{aligned}\quad (2.20)$$

where \mathbf{N} denotes the unit normal vector field of the boundary $\partial \mathcal{P}^\alpha(t)$.

These relations yield the following local form of mass balance equations for $\alpha = 1, \dots, A$,

$$\frac{\partial \rho^S}{\partial t} = \hat{\rho}^S, \quad \frac{\partial \rho^\alpha}{\partial t} + Div(\rho^\alpha \dot{\mathbf{X}}^\alpha) = \hat{\rho}^\alpha, \quad (2.21)$$

in regular points (almost everywhere) of \mathcal{B}_0 , and

$$U [[\rho^S]] = 0, \quad [[[\rho^\alpha (\dot{\mathbf{X}}^\alpha \cdot \mathbf{N} - U)]]] = 0, \quad (2.22)$$

in points of singular surfaces moving through the reference configuration \mathcal{B}_0 with the local speed U . The brackets $[[\cdot \cdot \cdot]]$ denote the difference of finite limits of quantities in these brackets on the positive and negative side of the surface.

In a similar manner we obtain the following partial momentum balance equations in their local form. For the regular points of the reference configuration \mathcal{B}_0

$$\begin{aligned}\frac{\partial (\rho^S \dot{\mathbf{x}}^S)}{\partial t} - Div \mathbf{P}^S &= \hat{\mathbf{p}}^S + \rho^S \mathbf{b}^S, \\ \frac{\partial (\rho^\alpha \dot{\mathbf{x}}^\alpha)}{\partial t} + Div(\rho^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{X}}^\alpha - \mathbf{P}^\alpha) &= \hat{\mathbf{p}}^\alpha + \rho^\alpha \mathbf{b}^\alpha, \quad \hat{\mathbf{p}}^S + \sum_{\alpha=1}^A \hat{\mathbf{p}}^\alpha = 0,\end{aligned}\quad (2.23)$$

and for points on singular surfaces

$$\begin{aligned}\rho^S U [[\dot{\mathbf{x}}^S]] + [[[\mathbf{P}^S]]] \mathbf{N} &= 0, \\ \rho^\alpha (\dot{\mathbf{X}}^\alpha \cdot \mathbf{N} - U) [[[\dot{\mathbf{x}}^\alpha]]] - [[[\mathbf{P}^\alpha]]] \mathbf{N} &= 0.\end{aligned}\quad (2.24)$$

In these relations $\mathbf{P}^S, \mathbf{P}^\alpha$ denote partial Piola-Kirchhoff stress tensors, $\mathbf{b}^S, \mathbf{b}^\alpha$ are partial mass forces, and $\hat{\mathbf{p}}^S, \hat{\mathbf{p}}^\alpha$ denote the momentum sources. Relation (2.23)₃ expresses the bulk conservation of momentum.

We do not need to present details of partial energy balance equations. Under the assumption of a single field of temperature we need solely the bulk energy conservation law. Details concerning partial energy balance equations and the derivation of the bulk equation can be found in the book [17]. The derivation is based on a principle of the theory of mixtures that bulk quantities must be defined in such a way that balance equations for these quantities have the form of classical conservation laws of the single component continuum thermodynamics.

Bearing this principle in mind we define the following bulk quantities

$$\rho := \rho^S + \sum_{\alpha=1}^A \rho^\alpha, \quad \rho \dot{\mathbf{x}} := \rho^S \dot{\mathbf{x}}^S + \sum_{\alpha=1}^A \rho^\alpha \dot{\mathbf{x}}^\alpha, \quad \rho \dot{\mathbf{X}} := \sum_{\alpha=1}^A \rho^\alpha \dot{\mathbf{X}}^\alpha, \quad (2.25)$$

which are the bulk mass density, the bulk momentum, and an objective relative momentum connected with the reference of the motion to the skeleton rather than to local centers of gravity;

$$\begin{aligned} \mathbf{P} &: = \mathbf{P}_I - \mathbf{F}^S \left\{ \rho^S \dot{\mathbf{X}} \otimes \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \otimes (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \right\}, \\ \mathbf{P}_I &: = \mathbf{P}^S + \sum_{\alpha=1}^A \mathbf{P}^\alpha; \end{aligned} \quad (2.26)$$

this is the bulk Piola-Kirchhoff stress tensor with the so-called intrinsic part \mathbf{P}_I ;

$$\begin{aligned} \rho \varepsilon &: = \rho \varepsilon_I + \frac{1}{2} \left\{ \rho^S \mathbf{C}^S \cdot (\dot{\mathbf{X}} \otimes \dot{\mathbf{X}}) + \sum_{\alpha=1}^A \rho^\alpha \mathbf{C}^S \cdot (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \otimes (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \right\}, \\ \rho \varepsilon_I &: = \rho^S \varepsilon^S + \sum_{\alpha=1}^A \rho^\alpha \varepsilon^\alpha, \quad \mathbf{C}^S := \mathbf{F}^{ST} \mathbf{F}^S, \end{aligned} \quad (2.27)$$

this is the bulk specific internal energy with the intrinsic part $\rho \varepsilon_I$. The symmetric tensor \mathbf{C}^S is the *right Cauchy-Green deformation tensor* of the skeleton;

$$\begin{aligned} \mathbf{Q} &:= \mathbf{Q}_I + \frac{1}{2} \left\{ -\rho^S \dot{\mathbf{X}} \otimes \dot{\mathbf{X}} \otimes \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \otimes (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \otimes (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) \right\} \mathbf{C}^S, \\ \mathbf{Q}_I &: = \mathbf{Q}^S + \sum_{\alpha=1}^A \mathbf{Q}^\alpha - \rho^S \varepsilon^S \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha \varepsilon^\alpha (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}) + \\ &\quad + \mathbf{P}^{ST} \mathbf{F}^S \dot{\mathbf{X}} - \sum_{\alpha=1}^A \mathbf{P}^{\alpha T} \mathbf{F}^S (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}), \end{aligned} \quad (2.28)$$

and this vector describes the bulk heat flux in the Lagrangian description. Again the intrinsic part \mathbf{Q}_I was separated.

The bulk balance equation of energy can be now written in the following form

$$\frac{\partial}{\partial t} \rho \left(\varepsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) + Div \left\{ \rho \left(\varepsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) \dot{\mathbf{X}} + \mathbf{Q} - \mathbf{P}^T \dot{\mathbf{x}} \right\} = \rho \mathbf{b} \cdot \dot{\mathbf{x}} + \rho r, \quad (2.29)$$

where

$$\begin{aligned}\rho \mathbf{b} &:= \rho^S \mathbf{b}^S + \sum_{\alpha=1}^A \rho^\alpha \mathbf{b}^\alpha, \\ \rho r &:= \rho^S r^S + \sum_{\alpha=1}^A \rho^\alpha r^\alpha - \rho^S \mathbf{b}^S \cdot \mathbf{F}^S \dot{\mathbf{X}} + \sum_{\alpha=1}^A \rho^\alpha \mathbf{b}^\alpha \cdot \mathbf{F}^S (\dot{\mathbf{X}}^\alpha - \dot{\mathbf{X}}),\end{aligned}\quad (2.30)$$

and r^S, r^α denote the partial radiations.

We skip the presentation of the energy condition on a singular surface because it shall not be used in these notes.

In the Lagrangian description and with the choice of fields (2.14) we have at disposal the following integrability condition

$$\frac{\partial \mathbf{F}^S}{\partial t} = \text{Grad} \dot{\mathbf{x}}^S. \quad (2.31)$$

This condition yields the existence of the function of motion (2.15). By the choice (2.14) of unknown fields this relation plays the role of the field equation for the deformation gradient \mathbf{F}^S .

It is useful to write equation (2.31) in the following weaker form

$$\frac{d}{dt} \int_{\mathcal{P}^S} \mathbf{F}^S dV = \oint_{\partial \mathcal{P}^S} \dot{\mathbf{x}}^S \otimes \mathbf{N} dV, \quad (2.32)$$

for every material domain of the skeleton $\mathcal{P}^S \subset \mathcal{B}_0$. This balance equation yields the following condition in points of singular surfaces

$$U [[\mathbf{F}^S]] = - [[\dot{\mathbf{x}}^S]] \otimes \mathbf{N}. \quad (2.33)$$

This relation is usually derived by means of the Hadamard Theorem for singular surfaces.

Before we present remaining equations of the model let us discuss some properties of the objects which we have introduced above. It is easy to notice a striking similarity of the structure of bulk quantities to that appearing in the classical theory of mixtures [4]. This concerns terms with explicit contributions of velocities. However in contrast to the mixture theory all velocities of the present model are objective because $\dot{\mathbf{X}}^\alpha$ and $\dot{\mathbf{X}}$ are *relative velocities*. Due to constitutive relations these velocities may be also present in the implicit form in intrinsic parts of stress tensors, internal energy and heat flux vector. It is also important to notice that the explicit dependence is at least quadratic. If we consider processes with small deviations from the thermodynamical equilibrium these contributions can be neglected.

In order to turn over mass balance equations (2.21), momentum balance equations (2.23), energy balance equation (2.29) and compatibility condition (2.31) into field

equations for fields (2.14) we need constitutive relations for partial stress tensors, momentum sources, the bulk internal energy and the bulk heat flux. If we had these relations we would have $14(A + 1)$ equations. Consequently we would be missing one equation. This is connected with the fact that the porosity n is the additional microstructural variable and this requires an additional equation. We proceed to formulate this equation.

We have seen in Section 2.1 that changes of porosity may be described by a balance equation (2.7). This was the consequence of incompressibility of components. If the components are compressible we are missing this equation. In addition the porosity equation following from the mass conservation law does not contain a source. Such a source would describe a spontaneous relaxation of porosity. We know from experience with other microstructural variables that this is an important property yielding evolution equations for such variables. All these arguments can be made more precise if we derive an equation for porosity from a semimacroscopic model. This was done by a multiscaling technique in the work [16].

Bearing the above remarks in mind we introduce in regular points the *balance equation of porosity* in the following general form

$$\frac{\partial n}{\partial t} + \text{Div} \mathbf{J} = \hat{n}, \quad (2.34)$$

where the flux of porosity \mathbf{J} and the source of porosity \hat{n} must be given by constitutive relations. In addition we expect that n tends to an equilibrium under constant external conditions. The equilibrium value of porosity n_E satisfies the equation (2.34) with the flux and source equal to zero. The latter as we show later follows indeed from the second law of thermodynamics. Such a condition for n_E does not determine this quantity. Consequently we have to add also a constitutive relation for the equilibrium porosity.

Making an assumption that sources of porosity do not carry surface singularities we can write the following compatibility condition for porosity on such surfaces

$$U [[n]] - [[\mathbf{J}]] \cdot \mathbf{N} = 0, \quad (2.35)$$

which may suggest the form of natural boundary conditions for porosity. We discuss this problem further in the Section on Riemann problems. We show there that in some particular models such conditions are not needed. This seems to be an important conclusion due to practical problems with a control of porosity on boundaries.

Let us collect balance equations which we have discussed in this Section. They are shown in the Tables 1 and 2.

Table 1: Balance equations for $A + 1$ -component porous material in regular points

| | |
|---------------------------------|--|
| mass of S | $\frac{\partial \rho^S}{\partial t} = \hat{\rho}^S$ |
| mass of α | $\frac{\partial \rho^\alpha}{\partial t} + Div \rho^\alpha \dot{\mathbf{X}}^\alpha = \hat{\rho}^\alpha$ |
| momentum of S | $\frac{\partial(\rho^S \dot{\mathbf{x}}^S)}{\partial t} - Div \mathbf{P}^S = \hat{\mathbf{p}}^S + \rho^S \mathbf{b}^S$ |
| momentum of α | $\frac{\partial(\rho^\alpha \dot{\mathbf{x}}^\alpha)}{\partial t} + Div (\rho^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{X}}^\alpha - \mathbf{P}^\alpha) = \hat{\mathbf{p}}^\alpha + \rho^\alpha \mathbf{b}^\alpha$ |
| bulk energy | $\frac{\partial}{\partial t} \rho \left(\varepsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) + Div \left\{ \rho \left(\varepsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) \dot{\mathbf{X}} + \mathbf{Q} - \mathbf{P}^T \dot{\mathbf{x}} \right\} = \rho \mathbf{b} \cdot \dot{\mathbf{x}} + \rho r$ |
| integrability of \mathbf{F}^S | $\frac{\partial \mathbf{F}^S}{\partial t} = Grad \dot{\mathbf{x}}^S$ |
| porosity | $\frac{\partial n}{\partial t} + Div \mathbf{J} = \hat{n}$ |

Table 2: Balance equations (dynamic compatibility conditions) in points of the singular surface

| | |
|----------------------|---|
| mass of S | $U \llbracket \rho^S \rrbracket = 0$ |
| mass of α | $\llbracket \rho^\alpha (\dot{\mathbf{X}}^\alpha \cdot \mathbf{N} - U) \rrbracket = 0$ |
| momentum of S | $\rho^S U \llbracket \dot{\mathbf{x}}^S \rrbracket + \llbracket \mathbf{P}^S \rrbracket \cdot \mathbf{N} = 0,$ |
| momentum of α | $\rho^\alpha (\dot{\mathbf{X}}^\alpha \cdot \mathbf{N} - U) \llbracket \dot{\mathbf{x}}^\alpha \rrbracket - \llbracket \mathbf{P}^\alpha \rrbracket \cdot \mathbf{N} = 0$ |
| integrability | $U \llbracket \mathbf{F}^S \rrbracket = - \llbracket \dot{\mathbf{x}}^S \rrbracket \otimes \mathbf{N}$ |
| porosity | $U \llbracket n \rrbracket - \llbracket \mathbf{J} \rrbracket \cdot \mathbf{N} = 0$ |

As already indicated we do not quote here the dynamic compatibility relation for the bulk energy.

In order to construct field equations for the fields \mathbf{u} listed in relation (2.14) we have to solve the *closure problem*, i.e. we have to add to balance equations of the Table 1 constitutive relations. Let us define the following vectors

$$\begin{aligned}
 \mathbf{F}_0 &:= \left\{ \rho^S, \rho^\alpha, \rho^S \dot{\mathbf{x}}^S, \rho^\alpha \dot{\mathbf{x}}^\alpha, \rho \left(\varepsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right), \mathbf{F}^S, n \right\} \in \mathfrak{R}^{4A+15}, \\
 \mathbf{F}_K &:= \left\{ 0, \rho^\alpha \dot{\mathbf{X}}^\alpha \cdot \mathbf{G}_K, -\mathbf{P}^S \mathbf{G}_K, \left(\rho^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{X}}^\alpha - \mathbf{P}^\alpha \right) \mathbf{G}_K, \right. \\
 &\quad \left. \left(\rho \left(\varepsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) \dot{\mathbf{X}} + \mathbf{Q} - \mathbf{P}^T \dot{\mathbf{x}} \right) \cdot \mathbf{G}_K, -\dot{\mathbf{x}}^S \otimes \mathbf{G}_K, \mathbf{J} \cdot \mathbf{G}_K \right\} \in \mathfrak{R}^{4A+15}, \\
 \mathbf{f} &:= \left\{ \hat{\rho}^S, \hat{\rho}^\alpha, \hat{\mathbf{p}}^S, \hat{\mathbf{p}}^\alpha, 0, \mathbf{0}, \hat{n} \right\} \in \mathfrak{R}^{4A+15}, \tag{2.36} \\
 \mathbf{f}_{ext} &:= \left\{ 0, 0, \rho^S \mathbf{b}^S, \rho^\alpha \mathbf{b}^\alpha, \rho \mathbf{b} \cdot \dot{\mathbf{x}} + \rho r, \mathbf{0}, 0 \right\} \in \mathfrak{R}^{4A+15},
 \end{aligned}$$

where \mathbf{G}_K denote unit basis vectors of Lagrangian coordinates. Then the balance equations can be written in the following compact form

$$\frac{\partial F_0}{\partial t} + \frac{\partial F_K}{\partial X^K} = \mathbf{f} + \mathbf{f}_{ext}, \quad (2.37)$$

where $\{X^K\}_{K=1,2,3}$ denote Lagrangian coordinates. For convenience we have chosen a Cartesian coordinate system.

In procedures of *extended thermodynamics* [19] it is assumed that the vectors defined by (2.36) are sufficiently smooth functions of the vector \mathbf{u} of unknown fields, i.e.

$$\mathbf{F}_0 = \mathbf{F}_0(\mathbf{u}), \quad \mathbf{F}_K = \mathbf{F}_K(\mathbf{u}), \quad \mathbf{f} = \mathbf{f}(\mathbf{u}). \quad (2.38)$$

If we do the same in the case under considerations we obtain the model of *poroelastic materials* without heat conduction. In order to incorporate the heat conduction we would have to introduce either an equation for the heat flux or a dependence on $\text{Grad}T$ as a constitutive variable. Similarly we would have to extend the model if we wanted to describe viscous or plastic effects - we would have to add equations for partial stresses or a dependence on gradients of partial velocities. Some aspects of the last problem are investigated in the works [20,21].

We shall not do so in this work. Neither of the above mentioned extensions has been investigated.

Let us mention in passing that the structure of constitutive relations (2.38) is one of the most characteristic features of extended thermodynamics. Namely the constitutive variables are fields themselves but not their derivatives as it is always the case in ordinary nonequilibrium thermodynamics.

In the case of thermal processes difficulties have been already mentioned earlier in these notes. The problem of balance equations of stresses seems to be easier from the physical point of view but technically it is very complicated indeed.

For these reasons we consider further solely the closure in the form (2.38). In addition we assume processes to be *isothermal*. This means that we leave out the temperature in the definition (2.14) of \mathbf{u} and denote it then \mathbf{u}_{is} as well as we ignore the energy balance equation.

The set of equations (2.38) for \mathbf{u}_{is} without energy balance has thermodynamically the same structure as the corresponding set of extended thermodynamics. For this reason we can apply the same principles connected with the **thermodynamical admissibility**. They can be formulated as follows

- *entropy inequality*: there exist a nontrivial entropy function h_0 and a flux $\mathbf{H} = H_K \mathbf{G}_K$ such that for each thermodynamical process (i.e. for each solution of field equations) the following inequality is satisfied

$$\frac{\partial h_0}{\partial t} + \text{Div} \mathbf{H} \geq 0, \quad h_0 = h_0(\mathbf{u}_{is}) \in \mathfrak{R}, \quad \mathbf{H} = \mathbf{H}(\mathbf{u}_{is}) \in \mathfrak{R}^3, \quad (2.39)$$

- *convexity and causality*: the entropy function $h_0 = h_0(\mathbf{u})$ is concave, i.e.

$$\forall \mathbf{v} \in \mathfrak{R}^{14(A+1)}, \mathbf{v} \neq \mathbf{0} : \quad \frac{\partial^2 h_0}{\partial \mathbf{u}_{is} \partial \mathbf{u}_{is}} \cdot (\mathbf{v} \otimes \mathbf{v}) < 0, \quad (2.40)$$

- *principle of relativity (Galilean invariance of field equations)*.

The last principle yields a decomposition of all quantities of the model into two parts: a convective part which depends explicitly on the absolute velocity fields and a nonconvective part which does not depend on absolute velocities at all. This principle is satisfied identically in the case of Lagrangian description because we deal solely with relative velocities.

Entropy inequality (2.39) is exploited by means of Lagrange multipliers which eliminate the limitation of this inequality to thermodynamical processes. According to this procedure requirement (2.39) is equivalent to the following inequality for all fields, and not only for solutions of field equations

$$\begin{aligned} \forall \mathbf{u}_{is} \in \mathfrak{R}^{14(A+1)} : \quad & \frac{\partial h_0}{\partial t} + \frac{\partial H_K}{\partial X^K} - \Lambda \cdot \left(\frac{\partial \tilde{\mathbf{F}}_0}{\partial t} + \frac{\partial \tilde{\mathbf{F}}_K}{\partial X^K} - \tilde{\mathbf{f}} \right) \geq 0, \quad \Lambda \in \mathfrak{R}^{14(A+1)}, \\ \mathbf{H} = H_K \mathbf{G}_K, \end{aligned} \quad (2.41)$$

where Λ are the Lagrange multipliers, and functions of \mathbf{u}_{is} . As mentioned above $\tilde{\mathbf{F}}_0, \tilde{\mathbf{F}}_K, \tilde{\mathbf{f}}$ are truncations of functions (2.36) to the subspace $\mathfrak{R}^{14(A+1)}$ without the energy balance equation.

The solution of the above problem has the following form

$$dh_0 = \Lambda \cdot d\tilde{\mathbf{F}}_0, \quad dH_K = d\tilde{\mathbf{F}}_K \cdot \Lambda \quad \implies \quad \frac{\partial \Lambda}{\partial \mathbf{u}_{is}} = \frac{\partial^2 h_0}{\partial \mathbf{u}_{is} \partial \mathbf{u}_{is}}, \quad (2.42)$$

i.e. according to (2.40), the map $\mathbf{u}_{is} \rightarrow \Lambda$ is globally invertible. Hence after Legendre transformation

$$\begin{aligned} h'_0(\Lambda) &= \Lambda \cdot \tilde{\mathbf{F}}_0 - h_0, \quad H'_K(\Lambda) = \Lambda \cdot \tilde{\mathbf{F}}_K - H_K \quad \implies \\ \implies \quad \tilde{\mathbf{F}}_0 &= \frac{\partial h'_0}{\partial \Lambda}, \quad \tilde{\mathbf{F}}_K = \frac{\partial H'_K}{\partial \Lambda}, \quad \Lambda \cdot \tilde{\mathbf{f}}(\Lambda) \geq 0. \end{aligned} \quad (2.43)$$

Consequently, the functions $\tilde{\mathbf{F}}_0$, and $\tilde{\mathbf{F}}_K$ which determine the left-hand side of the field equations are given if the *four-potential* $(h'_0(\Lambda), H'_K(\Lambda))$ is known. It leaves

unspecified but restricted by the *dissipation inequality* (2.43)₅ solely the sources $\tilde{\mathbf{f}}(\mathbf{\Lambda})$ of the field equations. This is one of the most important consequences of the second law within the rational extended thermodynamics. Moreover relations (2.43) yield the symmetry of field equations for the unknown vector $\mathbf{\Lambda}$:

$$\frac{\partial^2 h_0}{\partial \mathbf{\Lambda} \partial \mathbf{\Lambda}} \frac{\partial \mathbf{\Lambda}}{\partial t} + \frac{\partial^2 H'_K}{\partial \mathbf{\Lambda} \partial \mathbf{\Lambda}} \frac{\partial \mathbf{\Lambda}}{\partial X^K} = \tilde{\mathbf{f}}. \quad (2.44)$$

Consequently under the second part of the condition of thermodynamical admissibility (i.e. nonsingularity of matrix $\frac{\partial^2 h_0}{\partial \mathbf{\Lambda} \partial \mathbf{\Lambda}}$) the system is *hyperbolic*.

It remains to invert the variables, i.e. to find the map $\mathbf{\Lambda} \rightarrow \mathbf{u}_{is}$. This is usually a very difficult technical problem. We show the results for two models of porous materials in the next Sections of these notes.

Let us mention that the residual inequality (2.43)₅ defines the *dissipation*. This function vanishes in states called *the thermodynamical equilibrium states*. Consequently the necessary and sufficient condition for the thermodynamical equilibrium within the model discussed in this work have the form

$$\begin{aligned} \hat{\rho}^\alpha|_E &= 0 \quad \text{for } \alpha = 1, \dots, A, \\ \hat{\mathbf{p}}^\alpha|_E &= 0 \quad \text{for } \alpha = 1, \dots, A, \\ \hat{n}|_E &= 0. \end{aligned} \quad (2.45)$$

They follow directly from definition (2.36)₃ of the vector \mathbf{f} truncated to $\tilde{\mathbf{f}}$.

Some additional details concerning thermodynamical properties of such a model can be found in the article [2].

We proceed to discuss some particular models following from the above thermodynamical scheme.

2.2.2 Simplified models, poroelastic two-component materials

We begin our considerations with the two-component model without mass sources. The set of fields reduces in such a case to the following one

$$\mathbf{u} := \{ \rho^S, \rho^F, \dot{\mathbf{x}}^S, \dot{\mathbf{x}}^F, n \}. \quad (2.46)$$

where $\rho^F, \dot{\mathbf{x}}^F$ denote the mass density of the fluid components related to the reference configuration of the skeleton, and the velocity of the fluid component, respectively. The partial mass density of the fluid can be readily transformed to the usual Eulerian current mass density by the following transformation

$$\rho_t^F = \rho^F J^S, \quad J^S := \det \mathbf{F}^S. \quad (2.47)$$

The fields (2.46) have to satisfy the following partial balance equations

$$\begin{aligned} \frac{\partial \rho^F}{\partial t} + \text{Div}(\rho^F \dot{\mathbf{X}}^F) &= 0, & \frac{\partial \rho^S}{\partial t} &= 0, \\ \dot{\mathbf{X}}^F &= \mathbf{F}^{S-1}(\dot{\mathbf{x}}^F - \dot{\mathbf{x}}^S), & J^S &:= \det \mathbf{F}^S > 0, \\ \rho^F \left(\frac{\partial \dot{\mathbf{X}}^F}{\partial t} + \dot{\mathbf{X}}^F \cdot \text{Grad} \dot{\mathbf{X}}^F \right) &= \text{Div} \mathbf{P}^F - \hat{\mathbf{p}}, & \rho^S \frac{\partial \dot{\mathbf{X}}^S}{\partial t} &= \text{Div} \mathbf{P}^S + \hat{\mathbf{p}}, \end{aligned} \quad (2.48)$$

$$\frac{\partial n}{\partial t} + \text{Div} \mathbf{J} = \hat{n}, \quad \mathbf{F}^S := \text{Grad} \chi^S(\mathbf{X}, t), \quad \dot{\mathbf{x}}^S := \frac{\partial \chi^S}{\partial t}(\mathbf{X}, t).$$

These equations become field equations if we specify constitutive relations for partial Piola-Kirchhoff stress tensors $\mathbf{P}^F, \mathbf{P}^S$, the source of momentum $\hat{\mathbf{p}}$, the flux and source of the porosity, \mathbf{J} and \hat{n} , respectively.

We make now an assumption that processes deviate little from the thermodynamical equilibrium. Consequently we account solely for linear contributions of the relative velocity $\dot{\mathbf{x}}^F - \dot{\mathbf{x}}^S$, and linear contributions of the deviation of porosity from its equilibrium value $\Delta = n - n_E$.

For such processes in poroelastic materials filled with an ideal fluid the constitutive relations have the form [16]

$$\begin{aligned} \mathbf{J} &= \Phi_0 \dot{\mathbf{X}}^F, & \hat{\mathbf{p}} &= \pi (\dot{\mathbf{x}}^F - \dot{\mathbf{x}}^S), & \hat{n} &= -\frac{\Delta}{\tau}, & \Delta &:= n - n_E, \\ \mathbf{P}^F &= -p^F \mathbf{F}^{S-T}, & p^F &= \rho^F \left(\rho^S \frac{\partial \psi^S}{\partial \rho^F} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} \right) + \beta' \Phi_0 \Delta, \end{aligned} \quad (2.49)$$

$$\mathbf{P}^S = \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} + \beta' \frac{\partial \Phi_0}{\partial \mathbf{F}^S} \Delta,$$

where the Helmholtz free energy functions ψ^F, ψ^S , the flux coefficient Φ_0 , and the equilibrium porosity n_E are functions of equilibrium variables $\{\mathbf{F}^S, \rho^S, \rho^F\}$. Obviously the constitutive relations (2.2) do not contain any viscous effects². We have

²The lack of viscosity in the macroscopic model does not mean, however, that the real fluid component is not viscous. W. Gray has constructed a Gedankenexperiment (a remark which he made during his Udine CISM-Course in 1998) which shows that a real viscous fluid may lead macroscopically to smeared-out properties of the fluid component which are described by laws of ideal inviscid fluids. Namely if we consider a semimacroscopic flow through pipe-like channels then a semimacroscopic distribution of velocity in channels is parabolic due to the sticking boundary conditions on the walls of channels. However the macroscopic average flow does not reflect these conditions and contains only a macroscopic velocity. The magnitude of this velocity is smaller when the viscosity of real fluid is larger but otherwise there are no macroscopic indications of the semimacroscopic viscosity. Solely the coefficient of permeability in momentum balance equations must be correspondingly corrected because the influence of the relative velocity must be increased for larger viscosities ("internal friction forces").

neglected them in order to expose better effects connected with dissipation effects caused by the microstructure itself.

We make further simplifications of the model which seem to be justified for many porous materials undergoing large deformations.

The first simplification concerns the constitutive relation for the equilibrium porosity. We assume that it has the following isotropic form

$$n_E = n_E(\rho_t^F, \rho_t^S), \quad \rho_t^F = \rho^F J^{S-1}, \quad \rho_t^S = \rho^S J^{S-1}, \quad (2.50)$$

i.e. the equilibrium porosity depends solely on current partial mass densities.

Certainly in equilibrium the balance equation of porosity reduces to the following one

$$\dot{\mathbf{x}}^F|_E = \dot{\mathbf{x}}^S|_E =: \mathbf{v}, \quad \hat{n}|_E = 0 \quad \implies \quad \frac{\partial n_E}{\partial t} = 0, \quad (2.51)$$

in the Lagrangian description. Substitution of (2.50) leads to the differential equation

$$\rho_t^F \frac{\partial n_E}{\partial \rho_t^F} + \rho_t^S \frac{\partial n_E}{\partial \rho_t^S} = 0, \quad n_E|_{t=0} = n_0(\rho_0^F, \rho^S) \quad (2.52)$$

where mass balance equations (2.48)_{1,2} in equilibrium have been used. After integration along characteristics

$$\frac{d\rho_t^F}{d\lambda} = \rho_t^F, \quad \frac{d\rho_t^S}{d\lambda} = \rho_t^S, \quad \frac{dn_E}{d\lambda} = 0,$$

we obtain easily the solution

$$n_E = n_0\left(\frac{\rho_t^F}{\rho_t^S}, \rho^S\right) \equiv n_0\left(\frac{\rho^F}{\rho^S}, \rho^S\right), \quad (2.53)$$

where n_0 is a given initial porosity determined by initial mass densities (ρ_0^F, ρ^S) , and ρ^S has been chosen to select a characteristic. For instance in the case of linear dependence on the first argument and a constant reference mass density of the skeleton ρ^S we have

$$n_E = n_0 \frac{\rho^F}{\rho_0^F} \equiv n_0 \frac{\rho_t^F}{\rho_0^F} J^S, \quad (2.54)$$

where n_0 is a constant initial porosity. This example shows well the qualitative behavior of equilibrium porosity. For instance the skeleton undergoing large equilibrium deformations and filled with a gas whose changes of mass density can be neglected (almost a vacuum) changes its porosity proportionally to macroscopic changes of the volume which is measured by the determinant J^S . If $J^S < 1$ (compression) then $n_E < n_0$, and this is, of course, the expected behavior.

The second simplification concerns the coupling of partial stresses appearing in (2.49) with the coupling coefficient β' . We assume that the contribution to both partial stresses has the same form, i.e.

$$\frac{\partial \Phi_0}{\partial \mathbf{F}^S} = \Phi_0 \mathbf{F}^{S-T}. \quad (2.55)$$

Simultaneously we strengthen the isotropy assumption and require that the constitutive law for the flux coefficient Φ_0 has the form

$$\Phi_0 = \Phi_0(n_E, J^S). \quad (2.56)$$

The first argument is consistent with the assumption on small deviations from the thermodynamical equilibrium, and the second one follows from the strong form of isotropy with respect to deformations of the skeleton.

Bearing the result (2.53) in mind we can write condition (2.54) in the following form

$$\frac{\partial \Phi_0}{\partial \mathbf{F}^S} = \frac{\partial \Phi_0}{\partial J^S} J^S \mathbf{F}^{S-T} = \Phi_0 \mathbf{F}^{S-T} \quad \implies \quad \Phi_0 = \varphi(n_E) J^S. \quad (2.57)$$

We shall rely on relations (2.53), and (2.57) in the rest of these notes.

Substitution of these results in constitutive relations for partial stresses leads to the following relations for partial Cauchy stresses which we use frequently further in this work

$$\begin{aligned} \mathbf{T}^F & : = J^{S-1} \mathbf{P}^F \mathbf{F}^{ST} = -p_t^F \mathbf{1}, \quad p_t^F = \rho_t^F \left(\rho_t^S \frac{\partial \psi^S}{\partial \rho_t^F} + \rho_t^F \frac{\partial \psi^F}{\partial \rho_t^F} \right) + \beta \Delta, \\ \beta & : = \beta' \varphi = \beta(n_E), \\ \mathbf{T}^S & := J^{S-1} \mathbf{P}^S \mathbf{F}^{ST} = \rho_t^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} \mathbf{F}^{ST} + \beta \Delta \mathbf{1}. \end{aligned} \quad (2.58)$$

The contribution $\beta \Delta$ reminds the contribution of the so-called *dynamical pressure* in the extended thermodynamics of ideal gases [19]. It has the fundamental property of being zero in thermodynamical equilibria.

According to thermodynamical arguments which we presented earlier the above constitutive relations lead to hyperbolic field equations.

Very little has been done for poroelastic materials under large deformations. A simple example of a boundary value problem has been solved in [22,23] for simplified constitutive relations similar to those of the Signorini material of the classical nonlinear elasticity.

3 Waves in poroelastic materials

3.1 Bulk waves

In the next three Subsections we investigate dynamical properties of the simplest fully linear model of poroelastic materials saturated by a single ideal fluid component. This limited model is sufficient to show most fundamental properties of weak discontinuity waves in porous materials. Some more general properties will be included in the Subsection on boundary conditions.

First of all we assume that deformations of the skeleton are small. In such a case there is no difference between Lagrangian and Eulerian description of the motion of skeleton. Namely for the Almansi-Hamel (Euler) deformation tensor of the skeleton defined by the relation

$$\mathbf{e}^S := \frac{1}{2} \left(\mathbf{1} - \mathbf{F}^{S-T} \mathbf{F}^{-1} \right), \quad (3.1)$$

we assume

$$t \in \mathcal{T} \sup \mathbf{x} \in \mathcal{B}_t \sup \left| \mathbf{e}^S \cdot \mathbf{n} \otimes \mathbf{n} \right| \ll 1 \quad \text{for all } \mathbf{n}, \quad |\mathbf{n}| = 1. \quad (3.2)$$

We have for this tensor the following integrability conditions

$$\frac{\partial \mathbf{e}^S}{\partial t} = \text{symgrad} \left(\mathbf{v}^S \right), \quad J^S \approx 1 + \text{tr} \mathbf{e}^S \approx 1. \quad (3.3)$$

In the Eulerian description we denote the partial velocities in the classical manner $\mathbf{v}^S = \dot{\mathbf{x}}^S \left(\chi^{S-1}(\mathbf{x}, t), t \right) = \mathbf{v}^S(\mathbf{x}, t)$, $\mathbf{v}^F = \dot{\mathbf{x}}^F \left(\chi^{S-1}(\mathbf{x}, t), t \right) = \mathbf{v}^F(\mathbf{x}, t)$.

Now we proceed to investigate the propagation of weak discontinuity waves. A weak discontinuity wave is defined as a singular surface moving with a normal speed U on which the following conditions are satisfied

$$\left[\left[\rho_t^F \right] \right] = 0, \quad \left[\left[\mathbf{v}^F \right] \right] = 0, \quad \left[\left[\mathbf{v}^S \right] \right] = 0, \quad \left[\left[\mathbf{e}^S \right] \right] = 0, \quad \left[\left[n \right] \right] \equiv \left[\left[\Delta \right] \right] = 0, \quad (3.4)$$

where, as before, $\left[\left[\dots \right] \right] = (\dots)^+ - (\dots)^-$ is the difference of limits on both sides of this surface. The derivatives of the fields may possess finite discontinuities and we denote

$$r := \left[\left[\frac{\partial \rho_t^F}{\partial t} \right] \right], \quad \mathbf{a}^F := \left[\left[\frac{\partial \mathbf{v}^F}{\partial t} \right] \right], \quad \mathbf{a}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t} \right] \right], \quad D := \left[\left[\frac{\partial \Delta}{\partial t} \right] \right]. \quad (3.5)$$

The balance equations appropriate for the linear model (Eulerian description) follow from equations (2.48) and have the form

$$\begin{aligned}
\frac{\partial \rho_t^F}{\partial t} + \operatorname{div}(\rho_t^F \mathbf{v}^F) &= 0, & \frac{\partial \rho_t^S}{\partial t} + \operatorname{div}(\rho_t^S \mathbf{v}^S) &= 0, \\
\rho_t^F \left(\frac{\partial \mathbf{v}^F}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} \mathbf{v}^F \right) &= \operatorname{div} \mathbf{T}^F - \pi (\mathbf{v}^F - \mathbf{v}^S), & (3.6) \\
\rho_t^S \left(\frac{\partial \mathbf{v}^S}{\partial t} + \mathbf{v}^S \cdot \operatorname{grad} \mathbf{v}^S \right) &= \operatorname{div} \mathbf{T}^S + \pi (\mathbf{v}^F - \mathbf{v}^S), \\
\frac{\partial n}{\partial t} + \operatorname{div} [\varphi(n_E) (\mathbf{v}^F - \mathbf{v}^S)] &= -\frac{\Delta}{\tau}, & \Delta := n - n_E, \quad \varphi(n_E) \approx n_E,
\end{aligned}$$

where, for small deformations of the skeleton, we have $\rho_t^S \approx \text{const.}$ instead of the equation (3.6)₂ and we can leave out the kinematic contribution to the acceleration in (3.6)₄.

The field equations are constructed by means of the following constitutive relations for partial Cauchy stress tensors

$$\begin{aligned}
\mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S (\operatorname{tr} \mathbf{e}^S) \mathbf{1} + 2\mu^S \mathbf{e}^S + \beta \Delta \mathbf{1}, & (3.7) \\
\mathbf{T}^F &= -p_0^F \mathbf{1} - \kappa (\rho_t^F - \rho_0^F) \mathbf{1} - \beta \Delta \mathbf{1}.
\end{aligned}$$

These relations follow from (2.58) by linearization. The above relations are linear not only with respect to the deformation of the skeleton but also with respect to changes of the mass density of the fluid. Simultaneously it is assumed that the equilibrium porosity n_E is constant. Inspection of relation (2.53) shows that changes of n_E contribute to at least second order effects which are neglected in this part of the work.

Evaluation of jumps of the equations (3.6) on the wave front yields the algebraic homogeneous set of equations for the amplitudes (3.5). As usual one has to make use of kinematic (Hadamard) compatibility conditions which are standard [24]. We obtain the following relations

$$r = \frac{1}{U} \mathbf{a}^F \cdot \mathbf{n}, \quad D = \frac{1}{U} \varphi (\mathbf{a}^F - \mathbf{a}^S) \cdot \mathbf{n}, \quad \mathbf{a}^F = \mathbf{a}^F \cdot \mathbf{n} \mathbf{n}, \quad (3.8)$$

$$\begin{aligned}
\mathbf{a}_\perp^S \neq \mathbf{0} &\implies \left(U^2 - \frac{\mu^S}{\rho^S} \right) = 0, & \mathbf{a}_\perp^S &:= \mathbf{a}^S - \mathbf{a}^S \cdot \mathbf{n} \mathbf{n}, \\
\mathbf{a}^S \cdot \mathbf{n} \neq \mathbf{0}, \quad \mathbf{a}^F \cdot \mathbf{n} \neq \mathbf{0} &\implies \\
\implies \left(U^2 - \kappa - \frac{\varphi \beta}{\rho^F} \right) \left(U^2 - \frac{\lambda^S + 2\mu^S}{\rho^S} - \frac{\varphi \beta}{\rho^S} \right) - \left(\frac{\varphi \beta}{\rho^F} \right)^2 \frac{\rho^F}{\rho^S} &= 0. & (3.9)
\end{aligned}$$

Clearly we obtain three modes of propagation. Two speeds of the so-called P1-, and P2-mode follow from the equation (3.9)₂. The slower mode P2 is called also the

Biot wave and according to the relation (3.8)₃ it is longitudinal with respect to the amplitude of the fluid component \mathbf{a}^F . The third mode is transversal with respect to the amplitude \mathbf{a}^S and its speed is identical with the speed of the shear wave in the solid component (formula (3.9)₁). The weak discontinuities of the mass density in the fluid and of the porosity are carried by the P1- and P2-modes. These results are confirmed by numerous laboratory experiments on rocks and sintered glass (e.g. [25]).

In order to expose the most important features of bulk waves in poroelastic materials we consider the propagation of monochromatic waves described by the following set of linear field equations following from (3.6) in the 1-D case (*plane waves*)

$$\begin{aligned}
\frac{\partial \rho_t^F}{\partial t} + \rho_0^F \frac{\partial v^F}{\partial x} &= 0, & \rho_0^F \frac{\partial v^F}{\partial t} &= -\frac{\partial p^F}{\partial x} - \pi (v^F - v^S), \\
\rho^S \frac{\partial v^S}{\partial t} &= \frac{\partial \sigma^S}{\partial x} + \pi (v^F - v^S), & \frac{\partial \Delta}{\partial t} + n_E \frac{\partial (v^F - v^S)}{\partial x} &= -\frac{\Delta}{\tau}, \\
p^F &= p_0^F + \kappa (n_E) (\rho^F - \rho_0^F) + \beta (n_E) \Delta, \\
\sigma^S &= \sigma_0^S + (\lambda^S (n_E) + 2\mu^S (n_E)) e^S + \beta (n_E) \Delta, \\
\frac{\partial e^S}{\partial t} &= \frac{\partial v^S}{\partial x}.
\end{aligned} \tag{3.10}$$

In these relations $\rho_0^F, \rho^S, n_E, p_0^F, \sigma_0^S$ denote constant reference values of partial mass densities, porosity, partial pressure and normal stress in the skeleton, respectively. Material parameters $\kappa, \lambda^S, \mu^S, \beta, \pi, \tau$ depend parametrically on the equilibrium porosity n_E which, as before, is assumed to be constant. Consequently the essential coupling between components is due to changes of porosity Δ (parameter β) and the relative motion $v^F - v^S$ (coefficient of permeability π).

We seek the solution in the following form

$$\begin{aligned}
\rho_t^F &= \rho_0^F + R \exp i(kx - \omega t), \\
v^F &= V^F \exp i(kx - \omega t), & v^S &= V^S \exp i(kx - \omega t), \\
e^S &= E^S \exp i(kx - \omega t), & \Delta &= D \exp i(kx - \omega t),
\end{aligned} \tag{3.11}$$

where R, V^F, V^S, E^S, D are constant amplitudes of the wave. We solve the problem of monochromatic vibrations with a given *frequency* ω . It means that we have to find the *wave number* k as a function of the frequency.

As usual we obtain from (3.10) the dispersion relation as the condition for existence of bulk waves. It has the following form

$$\begin{aligned} & \left(\omega^2 - U^{S2} k^2 - \frac{i\omega\tau}{1+i\omega\tau} \frac{n_E\beta}{\rho^S} k^2 - i\frac{\pi\omega}{\rho^S} \right) \times \\ & \times \left(\omega^2 - U^{F2} k^2 - \frac{i\omega\tau}{1+i\omega\tau} \frac{n_E\beta}{\rho_0^F} k^2 - i\frac{\pi\omega}{\rho_0^F} \right) + \left(\frac{i\omega\tau}{1+i\omega\tau} \frac{n_E\beta}{\rho_0^F} k^2 + \frac{\pi\omega}{\rho_0^F} \right)^2 \frac{\rho_0^F}{\rho^S} = 0. \end{aligned} \quad (3.12)$$

In this relation the speeds U^S, U^F are defined as follows

$$U^{S2} := \frac{\lambda^S + 2\mu^S}{\rho^S}, \quad U^{F2} := \kappa. \quad (3.13)$$

In the one-dimensional case one mode of propagation - the shear wave cannot appear. Consequently the relation (3.12) yields to branches of the dispersion relation connected with the P1- and P2-wave alone.

The above equation is biquadratic with respect to an unknown wave number k . The solution is complex in contrast to solutions for each component taken separately. If this was the case we would have

$$\begin{aligned} & \beta = 0, \quad \pi = 0 \quad \implies \\ \implies & \left(\omega^2 - U^{S2} k^2 \right) \left(\omega^2 - U^{F2} k^2 \right) = 0. \end{aligned} \quad (3.14)$$

Hence the solution of the dispersion relation yields classical results for nondispersive monochromatic waves in the skeleton, and in the fluid, respectively

$$\frac{\omega^2}{k^2} = U^{S2} \quad \text{or} \quad \frac{\omega^2}{k^2} = U^{F2}. \quad (3.15)$$

Certainly they check with solutions of equation (3.9) for $\beta = 0$.

In the general case solutions of the dispersion relation are complex which means that waves are attenuated. In Figure 1 we show a numerical example of this solution. We have chosen the following numerical data

$$U^S = 3.1 \frac{km}{s}, \quad U^F = 0.9 \frac{km}{s}, \quad n = 0.23, \quad \rho^S = 2400 \frac{kg}{m^3}, \quad \rho_0^F = 230 \frac{kg}{m^3},$$

$$\pi = 2.602 \times 10^7 \frac{kg}{m^3 s}, \quad \beta = 0.313 \times 10^9 \frac{kg}{m s^2}, \quad \tau = 3.7 \times 10^6 s.$$

They are typical for rocks saturated with water (e.g. [24,26]).

The curves on the left picture illustrate the behavior of speeds of propagation as functions of the frequency, and the curves on the right picture illustrate the attenuations also as functions of the frequency. They follow from the following definitions

$$U_{ph} := \frac{\omega}{Rek}, \quad U_g := \frac{d\omega}{dRek}, \quad \gamma := Imk, \quad (3.16)$$

where

$$k^2 = \frac{1}{2A} (-B \pm \sqrt{D}),$$

$$A : = 1 - i\frac{\pi}{\omega} \left(\frac{1}{\rho^S} + \frac{1}{\rho_0^F} \right), \quad (3.17)$$

$$B : = - \left[\left(1 - i\frac{\pi}{\rho^S \omega} \right) \left(U^{F2} + \frac{i\omega\tau}{1 + i\omega\tau} \frac{n_E \beta}{\rho_0^F} \right) + \left(1 - i\frac{\pi}{\rho_0^F \omega} \right) \left(U^{S2} + \frac{i\omega\tau}{1 + i\omega\tau} \frac{n_E \beta}{\rho^S} \right) - 2 \frac{i\omega\tau}{1 + i\omega\tau} \frac{n_E \beta \pi}{\rho_0^F \rho^S \omega} \right],$$

$$C : = \left(U^{S2} + \frac{i\omega\tau}{1 + i\omega\tau} \frac{n_E \beta}{\rho^S} \right) \left(U^{F2} + \frac{i\omega\tau}{1 + i\omega\tau} \frac{n_E \beta}{\rho_0^F} \right) + \left(\frac{i\omega\tau n_E \beta}{1 + i\omega\tau} \right)^2 \frac{1}{\rho_0^F \rho^S},$$

$$D : = B^2 - 4AC.$$

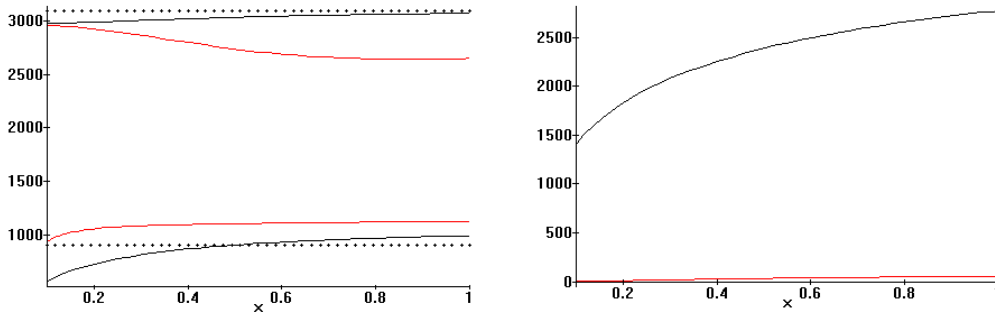


Figure 1: Phase, group velocities in $\frac{m}{s}$ and attenuation in $\frac{1}{m}$ of monochromatic 1D waves ($x = \omega\tau$)

The phase speed U_{ph} has two branches: the most upper one corresponding to monochromatic P1 waves, and the lowest one corresponding to monochromatic P2 waves. The limits $\omega \rightarrow \infty$ are equal, respectively, U^S and U^F . Let us observe that the speed of the P2 wave behave for low frequencies similarly to the speed of monochromatic waves in parabolic models ($U \sim \sqrt{\omega}$). However for high frequencies it reaches an asymptotic limit rather than going to infinity. This is due to hyperbolicity of the set of field equations. Two other curves of the left Figure correspond to the group velocity U_g . This velocity is usually identified with a ratio of the mean flux of energy of the wave to the mean value of the energy itself. It has a particular bearing in the estimates of wave interactions.

The right curves of Figure 1 show the attenuation γ : the upper one corresponds to P2 waves, and the lower one to P1 waves. It is clearly seen that P2 waves are attenuated with the intensity app. 3 orders of magnitude higher than P1 waves. For this reason they are very difficult to be observed in experiments.

3.2 Boundary and compatibility conditions

As usual in theories of multicomponent systems with diffusion the problem of formulation of boundary conditions is connected with many additional physical problems stemming primarily from the fact that a relative motion of components leads to the creation of boundary layers. For isothermal processes in two-component models one expects two vectorial conditions on the boundary $\partial\mathcal{B}_0$. If the boundary is impermeable we have either purely kinematical conditions

$$\chi^S(\mathbf{X}, t) - \chi^S(\mathbf{X}, 0) \Big|_{\mathbf{x} \in \partial\mathcal{B}_0} = 0, \quad \dot{\mathbf{X}}^F(\mathbf{X}, t) \Big|_{\mathbf{x} \in \partial\mathcal{B}_0} = 0, \quad (3.18)$$

or mixed conditions

$$(\mathbf{P}^S + \mathbf{P}^F) \mathbf{N} \Big|_{\mathbf{x} \in \partial\mathcal{B}_0} = \mathbf{p}_{ext}, \quad \dot{\mathbf{X}}^F(\mathbf{X}, t) \Big|_{\mathbf{x} \in \partial\mathcal{B}_0} = 0. \quad (3.19)$$

The physical interpretation is obvious. In the first case we require that the boundary which is identified with the boundary of the skeleton does not move (the clamped condition), and that the velocity of the fluid component is identical with the velocity of the skeleton (lack of permeability). In the second case the second condition is identical, and the first one follows from the dynamic compatibility conditions quoted in Table 2 if we add momentum relations and substitute $U = 0$ (the condition for the surface to be material with respect to the skeleton). The right hand side of (3.19)₁ contains either an external, and given boundary loading \mathbf{p}_{ext} (in its Lagrangian form) or a total stress vector of the other body (continuity of stress vector following from dynamical compatibility conditions, see: Table 2).

In the Eulerian description the conditions (3.19) have the form

$$(\mathbf{T}^S + \mathbf{T}^F) \mathbf{n} \Big|_{\mathbf{x} \in \partial\mathcal{B}_t} = \mathbf{t}_{ext}, \quad \mathbf{v}^F - \mathbf{v}^S \Big|_{\mathbf{x} \in \partial\mathcal{B}_t} = 0. \quad (3.20)$$

The situation is different in the case of permeable boundary. It was demonstrated by von Terzaghi in a Gedankenexperiment (see: quotations and the description of the experiment in [7,27]) that the external loading cannot be distributed among components *a priori* on a permeable boundary, and we can solely require the condition of the form (3.19)₁ to hold. Consequently we need an additional vectorial condition in the case of two-component systems.

According to considerations connected with this Gedankenexperiment it is assumed that the flow of the fluid component through the permeable boundary is proportional to the real pressure difference on both sides of a boundary or interface between two different porous materials. According to the mass compatibility condition (see: Table 2) we have

$$\rho^F \dot{\mathbf{X}}^F \cdot \mathbf{N} \Big|^- = \rho^F \dot{\mathbf{X}}^F \cdot \mathbf{N} \Big|^+. \quad (3.21)$$

For this reason we do not use any signature for the boundary value of the mass flow. According to the above described assumption we have

$$\rho^F \dot{\mathbf{X}}^F \cdot \mathbf{N} = \alpha_0 \left[\left[\frac{\mathbf{P}^F}{n} \right] \right] \cdot \mathbf{F}^S. \quad (3.22)$$

In this relation the product $-\frac{1}{3}J^{S-1}\mathbf{P}^F \cdot \mathbf{F}^S \equiv -\frac{1}{3}J^{S-1}\text{tr}\mathbf{P}^F\mathbf{F}^{ST}$ is identical with the partial pressure in Eulerian description. This justifies the structure of relation (3.22). The coefficient α_0 denotes the *surface permeability coefficient* related to the Lagrangian image of the boundary or interface. Its existence is connected with the fact that the flow of the fluid component from the negative side of the surface to its positive side must overcome obstacles connected with the change of geometrical properties of the neighbouring media. In our model this is reflected primarily by different porosities on both sides of the surface. It is reasonable to assume that such a change of flow conditions yields the existence of a boundary layer which is replaced by a jump condition (3.22) with a phenomenological coefficient α_0 .

It is easier to appreciate the structure of condition (3.22) in the Eulerian description

$$\rho_t^F (\mathbf{v}^F - \mathbf{v}^S) \cdot \mathbf{n} = \alpha \left(p^{F-} - \frac{n^-}{n^+} p^{F+} \right), \quad \alpha := \frac{3\alpha_0}{n^-} (\mathbf{N} \cdot \mathbf{C}^{S-1} \mathbf{N})^{-\frac{1}{2}}, \quad (3.23)$$

where \mathbf{n} is the unit normal vector of the surface in its current configuration.

If the surface is an external boundary we have $n^+ = 1$, and p^{F+} is an external pressure which must be given on the surface in addition to the external loading appearing in (3.19).

Let us mention in passing that this art of boundary conditions can be introduced in multicomponent models based on partial momentum equations in contrast to scalar models following from the application of Darcy's law in mass balance equations alone. Models of the latter type leading to nonlinear parabolic reaction-diffusion equations dominate in the mathematical literature on porous materials, they are frequently considered to be the theories of porous materials *per se*, and usually require rather artificial boundary conditions like a continuity of the pore pressure [28].

The boundary condition (3.23) has been tested in classical problems of consolidation [29] as well as in processes of coupling between diffusion and adsorption. The latter problem shall be presented further in these notes (Section 4.1). It has been found out that it checks very well with expectations. Another class of problems in which this condition plays an important role are surface waves. We discuss surface waves in the next Subsection.

Certainly the condition (3.23) determines only one component of the vector (in the direction of the normal \mathbf{n}) which we have to prescribe on the boundary. We have to add a condition on the tangential component of the relative velocity. In the case of viscous fluids the problem is similar to that which we discussed above and we have to account for the boundary layer. We consider in these notes solely ideal fluids and in this case we deal with a sticking condition

$$(\mathbf{v}^{F\pm} - \mathbf{v}^{F\pm} \cdot \mathbf{nn}) - (\mathbf{v}^S - \mathbf{v}^S \cdot \mathbf{nn}) = 0, \quad (3.24)$$

for the interface. It reduces to the single condition on the external boundary.

Conditions (3.23-24) together with the condition (3.20)₁ form the full set of boundary conditions.

Let us mention that the additional porosity balance equation most likely does not require boundary conditions and it yields effects similar to these of usual evolution equations for internal (microstructural) variables. We demonstrate this property further on an example of a Riemann problem.

3.3 Surface waves for the contacts: porous body/vacuum, porous body/liquid, porous body/porous body; asymptotic properties of the dispersion relation

We have seen in Subsection 3.1 that the two-component model of porous materials considered in these notes yields three modes of propagation of bulk waves. It means that we must expect instead of a single Rayleigh surface wave of a classical single component continuum a number of surface modes of propagation. We show that this is the case.

Let us first briefly recall the construction of classical Rayleigh waves in linear elasticity. These follow from the superposition of longitudinal and transversal waves which are solutions of two wave equations

$$\begin{aligned} \frac{\partial^2 \mathbf{u}_L}{\partial t^2} &= U_L^2 \Delta \mathbf{u}_L, & U_L^2 &:= \frac{\lambda + 2\mu}{\rho}, & \operatorname{rot} \mathbf{u}_L &= 0, \\ \frac{\partial^2 \mathbf{u}_T}{\partial t^2} &= U_T^2 \Delta \mathbf{u}_T, & U_T^2 &:= \frac{\mu}{\rho}, & \operatorname{div} \mathbf{u}_T &= 0, \end{aligned} \quad (3.25)$$

where λ, μ are Lamé elasticity constants, ρ denotes the mass density, and the displacement $\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T$. If we consider now the boundary value problem for the halfspace $y > 0$ with the free boundary $y = 0$ then we can easily show that there exist solutions of the following general form

$$\begin{aligned} \mathbf{u} &= \left(u^L \exp(-\gamma^L y) + u^T \exp(-\gamma^T y) \right) \sin(k(x - U_R t)) \mathbf{e}_x + \\ &+ \left(u^L \frac{\gamma^L}{k} \exp(-\gamma^L y) + u^T \frac{k}{\gamma^T} \exp(-\gamma^T y) \right) \cos(k(x - U_R t)) \mathbf{e}_y, \quad U_R < U_T, \end{aligned} \quad (3.26)$$

where $\mathbf{e}_x, \mathbf{e}_y$ are unit base vectors in directions of x and y axes, respectively, u^L, u^T are real amplitudes and γ^L, γ^T, k with $\frac{\gamma^L}{k} = O(1), \frac{\gamma^T}{k} = O(1)$ are real constants following from the dispersion relations

$$\begin{aligned} \left(\frac{\gamma^L}{k} \right)^2 &= 1 - \frac{U_R^2}{U_L^2}, & \left(\frac{\gamma^T}{k} \right)^2 &= 1 - \frac{U_R^2}{U_T^2}, \\ \left(2 - \frac{U_R^2}{U_T^2} \right)^2 &= 4 \sqrt{\left(1 - \frac{U_R^2}{U_L^2} \right)} \sqrt{\left(1 - \frac{U_R^2}{U_T^2} \right)}, \end{aligned} \quad (3.27)$$

which are, in turn, a result of solvability of equations (3.25) and of the homogeneity of boundary conditions, respectively. It is clearly seen that the wave described by (3.26) propagates with speed U_R , and it attenuates very fast with the depth y . This is the reason for calling it a *surface wave*. Obviously the displacement $\mathbf{u} = u_x \mathbf{e}_x + u_y \mathbf{e}_y$ describes ellipses

$$\frac{u_x^2}{\left(u^L e^{-\gamma^L y} + u^T e^{-\gamma^T y}\right)^2} + \frac{u_y^2}{\left(u^L \frac{\gamma^L}{k} e^{-\gamma^L y} + u^T \frac{k}{\gamma^T} e^{-\gamma^T y}\right)^2} = 1. \quad (3.28)$$

These are local trajectories of materials particles.

It is important that surface waves do not attenuate along the boundary. A simple geometrical argument shows then that their dispersion in the case of the point source is much slower than this of bulk waves. Namely bulk waves are spherical and, consequently, their amplitudes decay as $\frac{1}{r^3}$, r being the distance from the source while surface waves are cylindrical, and for this reason their amplitudes decay as $\frac{1}{r^2}$. This makes surface waves very attractive in practical applications.

We proceed to describe surface waves in poroelastic materials saturated with a fluid. We investigate three particular cases of boundary conditions:

1. a poroelastic material in contact with the vacuum. In this case the surface permeability parameter $\alpha = 0$, and boundary conditions (3.23-24) reduce to the kinematic condition of relative velocity to be zero [30,32],
2. a poroelastic material in contact with an ideal fluid which flows through the boundary [31,32],
3. a poroelastic material in contact with another poroelastic material [31]. In this case we investigate solely a few simplified problems which shall be specified in the sequel.

In all cases we solve the 2-D problem. All constitutive relations are linearized, and initial values ρ_0^F, ρ_0^S, n_E are constant.

For the sake of consistent notation we introduce the following function \mathbf{u}^F

$$\mathbf{v}^F = \frac{\partial \mathbf{u}^F}{\partial t}, \quad (3.29)$$

which is a counterpart of the displacement \mathbf{u}^S in the skeleton. As the function \mathbf{u}^F itself does not appear we do not have to specify undetermined constants.

The set of field equations in Eulerian description has now the form³

³due to the small deformation the jacobian $J^S \approx 1$ and we do not need to add an index t in mass densities

$$\begin{aligned} \frac{\partial \rho^F}{\partial t} + \rho_0^F \operatorname{div} \frac{\partial \mathbf{u}^F}{\partial t} &= 0, & \frac{\partial \rho^S}{\partial t} + \rho_0^S \operatorname{div} \frac{\partial \mathbf{u}^S}{\partial t} &= 0, \\ \rho_0^F \frac{\partial^2 \mathbf{u}^F}{\partial t^2} + \operatorname{grad} (p^F + \beta \Delta) + \pi \frac{\partial}{\partial t} (\mathbf{u}^F - \mathbf{u}^S) &= 0, \end{aligned} \quad (3.30)$$

$$\rho_0^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} - \mu^S \Delta \mathbf{u}^S - (\lambda^S + \mu^S) \operatorname{grad} \operatorname{div} \mathbf{u}^S - \beta \operatorname{grad} \Delta - \pi \frac{\partial}{\partial t} (\mathbf{u}^F - \mathbf{u}^S) = 0,$$

$$\frac{\partial \Delta}{\partial t} + n_E \operatorname{div} \frac{\partial}{\partial t} (\mathbf{u}^F - \mathbf{u}^S) = -\frac{\Delta}{\tau},$$

where the partial stress tensor in the skeleton and the partial pressure are given by the constitutive relations

$$\begin{aligned} \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S \operatorname{div} \mathbf{u}^S \mathbf{1} + 2\mu^S \operatorname{sym} \operatorname{grad} \mathbf{u}^S + \beta \Delta \mathbf{1}, \\ p^F &= p_0^F + \kappa (\rho^F - \rho_0^F) + \beta \Delta. \end{aligned} \quad (3.31)$$

We seek solutions of these equations in a 2-D case in the following form

$$\mathbf{u}^F = \operatorname{grad} \varphi^F + \operatorname{rot} \psi^F, \quad \mathbf{u}^S = \operatorname{grad} \varphi^S + \operatorname{rot} \psi^S, \quad (3.32)$$

i.e.

$$\mathbf{u}_i^F = \frac{\partial \varphi^F}{\partial x^i} + \varepsilon_{ijk} \frac{\partial \psi_k^F}{\partial x^j}, \quad \mathbf{u}_i^S = \frac{\partial \varphi^S}{\partial x^i} + \varepsilon_{ijk} \frac{\partial \psi_k^S}{\partial x^j}, \quad x^1 = x, x^2 = y, x^3 = z,$$

with the following ansatz for the scalar and vector potentials

$$\begin{aligned} \varphi^F &= A^F(y) \exp(i(kx - \omega t)), & \varphi^S &= A^S(y) \exp(i(kx - \omega t)), \\ \psi_z^F &= B^F(y) \exp(i(kx - \omega t)), & \psi_x^F &= \psi_y^F = 0, \\ \psi_z^S &= B^S(y) \exp(i(kx - \omega t)), & \psi_x^S &= \psi_y^S = 0. \end{aligned} \quad (3.33)$$

Simultaneously

$$\begin{aligned} \rho^F - \rho_0^F &= A_\rho^F(y) \exp(i(kx - \omega t)), & \rho^S - \rho_0^S &= A_\rho^S(y) \exp(i(kx - \omega t)), \\ \Delta &= A^\Delta \exp(i(kx - \omega t)). \end{aligned} \quad (3.34)$$

We consider the problem of the wave propagation with a given wave length which in our case is identical with $\frac{1}{k}$. Consequently k is assumed to be real and frequencies ω as a part of the solution will be complex. Hence the phase velocity and attenuation are defined as follows

$$U_{ph} = \frac{\operatorname{Re} \omega}{k}, \quad \gamma_t = \operatorname{Im} \omega. \quad (3.35)$$

Substitution of these relations in field equations (3.30) yields

- from mass balance equations

$$\begin{aligned} A_\rho^F(y) + \rho_0^F \left(\frac{d^2}{dy^2} - k^2 \right) A^F(y) &= 0, \\ A_\rho^S(y) + \rho_0^S \left(\frac{d^2}{dy^2} - k^2 \right) A^S(y) &= 0; \end{aligned} \quad (3.36)$$

- from the momentum balance equation for the fluid

$$\begin{aligned} A_\rho^F &= \rho_0^F k_{\parallel}^{F2} A^F + i \frac{\pi}{U_{\parallel}^F} k_{\parallel}^F (A^F - A^S) - \frac{\beta}{U_{\parallel}^{F2}} A^\Delta, \\ B^F &= \frac{i\pi}{\rho_0^F \omega + i\pi} B^S, \quad U_{\parallel}^{F2} := \kappa, \quad k_{\parallel}^{F2} := \frac{\omega^2}{U_{\parallel}^{F2}}. \end{aligned} \quad (3.37)$$

- from the momentum balance equation for the skeleton

$$\begin{aligned} \rho_0^S \left(\frac{d^2}{dy^2} + k_{\parallel}^{S2} - k^2 \right) A^S - i \frac{\pi}{U_{\parallel}^S} k_{\parallel}^S (A^F - A^S) + \frac{\beta}{U_{\parallel}^{S2}} A^\Delta, \\ \rho_0^S \left(\frac{d^2}{dy^2} + k_{\perp}^{S2} - k^2 \right) B^S + i \frac{\pi}{U_{\perp}^S} k_{\perp}^S (B^F - B^S) = 0, \end{aligned} \quad (3.38)$$

where

$$k_{\parallel}^{S2} = \frac{\omega^2}{U_{\parallel}^{S2}}, \quad U_{\parallel}^{S2} = \frac{\lambda^S + 2\mu^S}{\rho_0^S}, \quad k_{\perp}^{S2} = \frac{\omega^2}{U_{\perp}^{S2}}, \quad U_{\perp}^{S2} = \frac{\mu^S}{\rho_0^S}; \quad (3.39)$$

- from the balance equation of porosity

$$A^\Delta = n_E \frac{i\omega\tau}{1 - i\omega\tau} \left(\frac{d^2}{dy^2} - k^2 \right) (A^F - A^S). \quad (3.40)$$

Consequently we have three ordinary differential equations for amplitudes $A^F(y)$, $A^S(y)$, $B^S(y)$, and four algebraic relations for amplitudes $B^F(y)$, $A^\Delta(y)$, $A_\rho^F(y)$, $A_\rho^S(y)$. We proceed to find their general solutions.

It is easy to find the solution for B^S . We obtain

$$B^S = C^S e^{-\gamma_{\perp} y}, \quad \gamma_{\perp} = \sqrt{k^2 - k_{\perp}^{S2} + \frac{\pi k_{\perp}^S}{\rho_0^S U_{\perp}^S} \frac{i\omega}{\frac{\pi}{\rho_0^F} - i\omega}}. \quad (3.41)$$

We assume that the real part of the expression under the square root is positive. This condition is quite natural, and yields the conclusion similar to this of the classical elasticity that phase velocities of surface waves must be smaller than the speed of the shear wave (see: (3.26)).

Now we solve the remaining two equations. We seek the solution in the generic form

$$\begin{pmatrix} A^F \\ A^S \end{pmatrix} = C \begin{pmatrix} R^F \\ R^S \end{pmatrix} \exp(\pm\gamma y). \quad (3.42)$$

If we eliminate A_ρ^F, A_ρ^S from equations (3.36) by means of the remaining equations we obtain equations for A^F, A^S . Substitution of (3.42) in those relations yields a homogeneous set of equations for R^F, R^S . Hence we obtain the condition of existence of nontrivial solutions

$$d_1^F d_2^S - d_2^F d_1^S = 0, \quad (3.43)$$

where

$$\begin{aligned} d_1^F & : = \left(1 + \frac{\beta n_E k_\parallel^F}{\rho_0^F U_\parallel^F \left(\omega + \frac{i}{\tau} \right)} \right) (\gamma^2 - k^2) + \left(k_\parallel^F + i \frac{\pi}{\rho_0^F U_\parallel^F} \right) k_\parallel^F, \\ d_1^S & = -\frac{\beta n_E k_\parallel^F}{\rho_0^F U_\parallel^F \left(\omega + \frac{i}{\tau} \right)} (\gamma^2 - k^2) + i \frac{\pi}{\rho_0^F U_\parallel^F} k_\parallel^F, \\ d_2^F & = -\frac{\beta n_E k_\parallel^S}{\rho_0^S U_\parallel^S \left(U_\parallel^S k_\parallel^S + \frac{i}{\tau} \right)} (\gamma^2 - k^2) - i \frac{\pi}{\rho_0^S U_\parallel^S} k_\parallel^S, \\ d_2^S & : = \left(1 + \frac{\beta n_E k_\parallel^S}{\rho_0^S U_\parallel^S \left(U_\parallel^S k_\parallel^S + \frac{i}{\tau} \right)} \right) (\gamma^2 - k^2) + \left(k_\parallel^S + i \frac{\pi}{\rho_0^S U_\parallel^S} \right) k_\parallel^S. \end{aligned} \quad (3.44)$$

Equation (3.43) should be solved with respect to γ . In order to be able to obtain this solution analytically we assume $\beta = 0$. This simplification means that we neglect certain couplings between components. In the next Section we argue that this is indeed a reasonable simplification. We obtain in this way a first term leading in an asymptotic approximation. It follows

$$\gamma^2 = \frac{1}{2} (2k^2 - k_\parallel^{S2} - k_\parallel^{F2}) \mp \frac{1}{2} \text{Re} \delta + \frac{i}{2} \left(\mp \text{Im} \delta - \frac{k_\parallel^F \pi}{\rho_0^F U_\parallel^F} - \frac{k_\parallel^S \pi}{\rho_0^S U_\parallel^S} \right), \quad (3.45)$$

where

$$\begin{aligned} \delta & : = \sqrt{\left(2k^2 - k_\parallel^F \left(k_\parallel^F + i \frac{\pi}{\rho_0^F U_\parallel^F} \right) - k_\parallel^S \left(k_\parallel^S + i \frac{\pi}{\rho_0^S U_\parallel^S} \right) \right)^2 - 4D}, \\ D & : = k_\parallel^{F2} (k_\parallel^{F2} - k^2) + i \frac{\pi k_\parallel^S k_\parallel^{F2}}{\rho_0^S U_\parallel^S} + i \frac{\pi k_\parallel^F (k_\parallel^{S2} - k^2)}{\rho_0^S U_\parallel^S} - \\ & \quad - k^2 (k_\parallel^{S2} - k^2) - i \frac{\pi k_\parallel^S k^2}{\rho_0^S U_\parallel^S}. \end{aligned} \quad (3.46)$$

As always in calculations of square roots of functions of complex variables one has to be careful in a choice of a proper Riemann surface. These choices lead to different waves. Namely if we assume

$$\operatorname{Re} \left(2k^2 - k_{\parallel}^{S2} - k_{\parallel}^{F2} \right) \pm \operatorname{Re} \delta > 0, \quad (3.47)$$

then the following choice of γ

$$\gamma_{1,2} = \frac{1}{\sqrt{2}} \sqrt{\left(2k^2 - k_{\parallel}^{S2} - k_{\parallel}^{F2} \right) \mp \frac{1}{2} \operatorname{Re} \delta + i \left(\mp \operatorname{Im} \delta - \frac{k_{\parallel}^F \pi}{\rho_0^F U_{\parallel}^F} - \frac{k_{\parallel}^S \pi}{\rho_0^S U_{\parallel}^S} \right)}, \quad (3.48)$$

$$\operatorname{Re} \gamma_{1,2} > 0,$$

yields the existence of surface waves with the speed of propagation $U_1 < U_{\parallel}^F$.

On the other hand if

$$\operatorname{Re} \left(2k^2 - k_{\parallel}^{S2} - k_{\parallel}^{F2} \right) + \operatorname{Re} \delta > 0, \quad (3.49)$$

$$\operatorname{Re} \left(2k^2 - k_{\parallel}^{S2} - k_{\parallel}^{F2} \right) - \operatorname{Re} \delta < 0,$$

then the same choice of γ_2 , and the following formula for γ_1

$$\gamma_1 = -\frac{i}{\sqrt{2}} \sqrt{\left(k_{\parallel}^{S2} + k_{\parallel}^{F2} - 2k^2 \right) + \frac{1}{2} \operatorname{Re} \delta + i \left(\operatorname{Im} \delta + \frac{k_{\parallel}^F \pi}{\rho_0^F U_{\parallel}^F} + \frac{k_{\parallel}^S \pi}{\rho_0^S U_{\parallel}^S} \right)}, \quad (3.50)$$

yield the existence of the so-called generalized Rayleigh wave with the phase velocity $U_{R'}$ such that $U_{\parallel}^F < U_{R'} < U_{\perp}^S$. We will discuss this problem further in these notes.

Under the above conditions we obtain the following bounded solutions of the problem

$$\begin{pmatrix} A^F \\ A^S \end{pmatrix} = C_1 \begin{pmatrix} R_1^F \\ R_1^S \end{pmatrix} \exp(-\gamma_1 y) + C_2 \begin{pmatrix} R_2^F \\ R_2^S \end{pmatrix} \exp(-\gamma_2 y),$$

$$B^S = C^S \exp(-\gamma_{\perp} y). \quad (3.51)$$

This part of the analysis corresponds to the construction of solutions in the linear elasticity which was leading to relations (3.27)₁. The second part depends on the form of boundary conditions which lead to the specification of constants in the solution (3.51). As we have already mentioned we consider three cases.

We begin with the construction of the dispersion relation in the case of contact between a poroelastic material and vacuum. We follow here the work [30]. By vacuum we mean an external world which does not contribute in any way to processes in the porous material. Consequently we assume that the surface permeability coefficient in condition (3.23) vanishes: $\alpha = 0$. The disturbance is carried solely by the porous material. Such a case appears in practical applications if the so-called

impedances of neighbouring materials differ very considerably. Roughly the order of magnitude of the impedance is equal to the product of the mass density with the speed of sound waves. For instance we have for the air: $I_{air} \approx 1 \frac{kg}{m^3} \times 300 \frac{m}{s}$. For typical porous materials $I_{por} \approx 3 \times 10^3 \frac{kg}{m^3} \times 3 \times 10^3 \frac{m}{s}$. Hence they differ on more than 4 orders of magnitude. In this sense in investigations of surface waves we can consider the air as it was vacuum.

The boundary conditions for $y = 0$ have the form (compare: (3.20))

$$\begin{aligned} \left(\frac{\partial u_x^S}{\partial y} + \frac{\partial u_y^S}{\partial x} \right) \Big|_{y=0} &= 0, \\ \lambda^S \left(\frac{\partial u_x^S}{\partial x} + \frac{\partial u_y^S}{\partial y} \right) + 2\mu^S \frac{\partial u_y^S}{\partial y} - \kappa (\rho^F - \rho_0^F) \Big|_{y=0} &= 0, \\ \left(\frac{\partial u_y^F}{\partial t} - \frac{\partial u_y^S}{\partial t} \right) \Big|_{y=0} &= 0. \end{aligned} \quad (3.52)$$

where indices x and y indicate the corresponding components of displacement vectors.

Substitution of (3.51) in (3.52) yields a homogeneous set of three algebraic relations for three constants C_1, C_2, C^S . We investigate solely the case of short waves, i.e. large values of $|k|$. This asymptotics determines correctly speeds of propagations of fronts of waves because the latter are limits of phase velocities for large frequencies corresponding to short waves.

We skip here the simple derivation and quote the dispersion relation, following from the above boundary conditions

$$\begin{aligned} \left[\frac{(\lambda^S + 2\mu^S) U^2}{\mu^S U_{\parallel}^S} - 2 \right] \gamma_1 \left(2 - \frac{U^2}{U_{\perp}^S} \right) + \rho_0^F \frac{\gamma_2 U^4}{\mu^S U_{\perp}^S} - 4\gamma_{\perp} \frac{\gamma_1 \gamma_2}{k^2} &= 0, \\ U := \frac{\omega}{|k|}, \end{aligned} \quad (3.53)$$

where, due to the above described simplifications,

$$\gamma_1 = |k| \sqrt{1 - \frac{U^2}{U_{\parallel}^F}}, \quad \gamma_2 = |k| \sqrt{1 - \frac{U^2}{U_{\parallel}^S}}, \quad \gamma_{\perp} = |k| \sqrt{1 - \frac{U^2}{U_{\perp}^S}}. \quad (3.54)$$

As usual the radicals γ_1, γ_2 and γ_{\perp} are multi-valued functions. We make them single-valued by selecting Riemann surface with cuts outgoing from the points $\pm U_{\parallel}^F, \pm U_{\parallel}^S, \pm U_{\perp}^S$, and leading to bounded solutions.

After some manipulations we can write the dispersion relation (3.53) in a more convenient form

$$\mathcal{P}_v := \frac{\rho_0^F}{\rho_0^S} \frac{U^4}{U_\perp^{S4}} \sqrt{1 - \frac{U^2}{U_\parallel^{S2}}} + \sqrt{1 - \frac{U^2}{U_\parallel^{F2}}} \mathcal{P}_R = 0, \quad (3.55)$$

where

$$\mathcal{P}_R := \left(2 - \frac{U^2}{U_\perp^{S2}}\right)^2 - 4 \sqrt{\left(1 - \frac{U^2}{U_\parallel^{S2}}\right)} \sqrt{\left(1 - \frac{U^2}{U_\perp^2}\right)}. \quad (3.56)$$

It is easy to see that the contribution \mathcal{P}_R is the Rayleigh expression leading to the dispersion relation (3.27)₂ in the classical elasticity.

It is interesting to observe that relation (3.55) for saturated poroelastic materials is identical with the dispersion relation for surface waves on the interface between a liquid and a solid half-space [33]. We skip here detailed calculations (see [30]) and present solely the final results. In contrast to elastic materials there exist two surface waves

1. the first mode is a true surface wave because it propagates almost without attenuation with the following phase speed

$$U_1^2 \approx U_\parallel^{F2} \left(1 - \frac{1}{4} \frac{\rho_0^{F2}}{\rho_0^{S2}} \left(\frac{U_\parallel^F}{U_\perp^S}\right)^4 \left(1 - \frac{U_\perp^{S2}}{U_\parallel^2}\right)^{-2}\right) < U_\parallel^{F2}, \quad (3.57)$$

2. the second mode is the so-called *leaky wave*. It is a generalization of the Rayleigh wave because it propagates with a phase speed only somewhat different from the solution of the equation $\mathcal{P}_R = 0$ which determines the speed of the classical Rayleigh wave. However the solution of the dispersion relation (3.55) is complex which means that this mode is attenuated. In contrast to the similar mode on the interface between a fluid and a solid this wave radiates its energy not from the solid to the fluid across the interface but rather into the porous material itself. Let us mention in passing that the problem of the definition of energy and energy flux prescribed to different modes of propagation is still not fully solved even though this terminology is commonly used in the literature.

We proceed to investigate the second class of boundary conditions connected with the contact between a saturated porous material and a fluid.

In the porous medium the equations which we have to solve are again given by (3.30), and their solution is sought in the form (3.33). On the other hand the fluid which flows through the boundary of the porous material is assumed to be ideal. Fields describing processes in this fluid, filling the *exterior* where the porosity $n^+ \equiv 1$, are

distinguished from the fields for the porous material by the signature “+”, i.e. they are denoted $\rho^{F+}, \mathbf{v}^{F+} \equiv \frac{\partial \mathbf{u}^{F+}}{\partial t}$. The signature was chosen in agreement with the exterior oriented unit normal vector of the boundary. The above fields satisfy the following linearized field equations

$$\begin{aligned} \frac{\partial \rho^{F+}}{\partial t} + \rho_0^{F+} \operatorname{div} \frac{\partial \mathbf{u}^{F+}}{\partial t} &= 0, \\ \rho_0^{F+} \frac{\partial^2 \mathbf{u}^{F+}}{\partial t^2} + \kappa^+ \operatorname{grad} \rho^{F+} &= 0, \end{aligned} \quad (3.58)$$

where ρ_0^{F+} denotes the initial mass density of the fluid, and κ^+ is its compressibility coefficient appearing in the constitutive relation for the pressure p^{F+}

$$p^{F+} = p_0^{F+} + \kappa^+ (\rho^{F+} - \rho_0^{F+}). \quad (3.59)$$

The solution of this set is sought in the form

$$\begin{aligned} \mathbf{u}^{F+} &= \operatorname{grad} \varphi^{F+}, \\ \varphi^{F+} &= A^{F+}(\mathbf{y}) \exp(i(kx - \omega t)), \\ \rho^{F+} - \rho_0^{F+} &= A_\rho^{F+}(\mathbf{y}) \exp(i(kx - \omega t)). \end{aligned} \quad (3.60)$$

Substitution of (3.60) in (3.58) yields the following bounded solution in the exterior

$$A^{F+} = C_1^+ \exp(\gamma_1^+), \quad \gamma_1^+ = \sqrt{1 - \frac{U^2}{\kappa^+}}, \quad (3.61)$$

$$A_\rho^{F+} = \rho_0^{F+} \frac{U^2}{\kappa^+} k^2 A^{F+}.$$

Constants appearing in solution (3.51) and (3.61) follow from boundary conditions. In the present case we deal with the permeable boundary which means that we have to use the general form of condition (3.23). The full set has the form

- dynamical conditions (continuity of the stress vector)

$$\left(\frac{\partial u_x^S}{\partial y} + \frac{\partial u_y^S}{\partial x} \right) \Big|_{y=0} = 0, \quad (3.62)$$

$$\lambda^S \left(\frac{\partial u_x^S}{\partial x} + \frac{\partial u_y^S}{\partial y} \right) + 2\mu^S \frac{\partial u_y^S}{\partial y} - \kappa (\rho^F - \rho_0^F) \Big|_{y=0} = -\kappa^+ (\rho^{F+} - \rho_0^{F+}) \Big|_{y=0},$$

- continuity of the mass flux through the boundary

$$\rho_0^F \frac{\partial}{\partial t} (u_y^F - u_y^S) \Big|_{y=0} = \rho_0^{F+} \frac{\partial}{\partial t} (u_y^{F+} - u_y^{S+}) \Big|_{y=0}, \quad (3.63)$$

- jump of the partial pressure on the boundary

$$-\rho_0^F \frac{\partial}{\partial t} (u_y^F - u_y^S) \Big|_{y=0} = \alpha (p^F - n_E p^{F+}) \Big|_{y=0}. \quad (3.64)$$

We consider again the approximation $\beta = 0$ and short monochromatic waves (large k). After straightforward calculations we obtain the dispersion relation

$$\begin{aligned} & \left(-\frac{\rho_0^S U_\perp^{S2}}{\tilde{\gamma}_1 \tilde{\gamma}_2} \mathcal{P}_v + \frac{\rho_0^{F+} U^4}{\tilde{\gamma}_1 U_\perp^{S2}} \right) (\tilde{\gamma}_1^+ - i\alpha U (1 - n_E)) - \\ & -\rho_0^{F+} \left(1 + \frac{\tilde{\gamma}_1^+}{\tilde{\gamma}_1} \right) \frac{U^4}{U_\perp^{S2}} \left(1 - \frac{i}{\tilde{\gamma}_1} \alpha U (1 - n_E) \right) = 0, \end{aligned} \quad (3.65)$$

where \mathcal{P}_v is defined by (3.55), and

$$\tilde{\gamma}_1 := \frac{\gamma_1}{|k|}, \quad \tilde{\gamma}_2 := \frac{\gamma_2}{|k|}, \quad \tilde{\gamma}_1^+ := \sqrt{1 - \frac{U^2}{\kappa^+}}. \quad (3.66)$$

Independently of a particular form of solutions $U(k)$ of the dispersion relation it is clear that these solutions depend parametrically on the coefficient α . Consequently measurements of surface waves can determine this material parameter whose observations are otherwise difficult.

We skip the details of the asymptotics of roots of the dispersion relation (3.65) (see: [31] for details), and present solely the final results.

We obtain three modes of propagation.

1. The first mode is the so-called *Stoneley wave* which propagates almost without dispersion. Its speed of propagation is smaller than any speed of bulk waves. Its limit $\alpha \rightarrow 0$ is identical with the first surface mode of the contact problem with vacuum which has been considered earlier.
2. The second mode is the *leaky pseudo-Stoneley wave*. Its speed is a little higher than the speed U_\parallel^F of P2 wave (slow Biot's wave). It is strongly dependent on the permeability coefficient of the surface α and it becomes equal to the speed U_\parallel^F as $\alpha \rightarrow 0$.
3. The third mode is the *leaky generalized Rayleigh wave* with the phase velocity close to the speed of the classical Rayleigh wave U_R . It radiates its energy both into the fluid in exterior as well as into the porous material. The second type of losses seems to dominate.

Let us mention that, in contrast to the analysis of the first case, in the case of contact with the fluid the solution of the dispersion relation becomes a rather involved

algebraic problem for functions of complex variables. It is even more difficult in the third case of boundary conditions which we mentioned at the beginning of this section. We proceed to consider some aspects of this problem.

The contact problem of two different porous materials is described by two sets of field equations of the form (3.30) in which material parameters are different on two sides of the interface. We use the same signature as in the above case of the fluid in the exterior. Then on the positive side of the interface material parameters are given by the set $\{\rho_0^{F+}, \rho_0^{S+}, p_0^{F+}, \mathbf{T}_0^{S+}, n_E^+, \lambda^{S+}, \mu^{S+}, \kappa^+, \pi^+, \beta^+, \tau^+\}$. Boundary conditions have the following form

- dynamical conditions (continuity of the stress vector across the interface)

$$(T_{iy}^S + T_{iy}^F)|_{y=0} = (T_{iy}^{S+} + T_{iy}^{F+})|_{y=0}, \quad i = x, y, \quad (3.67)$$

- continuity of the mass flux across the interface

$$\rho_0^F \frac{\partial}{\partial t} (u_y^F - u_y^S)|_{y=0} = \rho_0^{F+} \frac{\partial}{\partial t} (u_y^{F+} - u_y^{S+})|_{y=0}, \quad (3.68)$$

- continuity of the displacement of the skeleton

$$u_i^S|_{y=0} = u_i^{S+}|_{y=0}, \quad i = x, y, \quad (3.69)$$

- jump of the partial pressure on the interface

$$-\rho_0^F \frac{\partial}{\partial t} (u_y^F - u_y^S)|_{y=0} = \alpha n_E \left(\frac{p^F}{n_E} - \frac{p^{F+}}{n_E^+} \right)|_{y=0}. \quad (3.70)$$

Alltogether we have 6 conditions for 6 constants in contrast to three conditions for three constants in the first problem presented above.

Before we present results for surface waves let us note that the contact problem of two porous materials is essentially different from the two preceding cases. In the first case, as we have already mentioned, the impedance of the porous material is so large in comparison with the exterior that all waves coming to the boundary can be considered as fully reflected. In the second case the impedance of the fluid in the exterior is still about one order of magnitude smaller than the magnitude of the porous material. This means that we cannot neglect an influence of the exterior but still a part of energy of the incoming wave which flows to the exterior is very small. This is not the case any more when we consider the contact of two porous materials whose impedances are almost equal. We have to deal in this case with the transmission of waves, their reflection, and with the propagation of various surface waves created by bulk waves on both sides of the interface. This makes the problem

very complicated indeed. Even in the case of single-component materials the analysis is very complex (e.g. [34]).

Two particular cases of propagation of surface waves on the interface of porous materials have been analyzed up to now in order to establish differences with the above presented cases. In the first case all material parameters of porous materials on both sides of the interface are the same and the only difference appears in the porosity. In the second case materials have different porosities and different material properties but the latter are limited to the case when speeds of P1 wave, S wave, and P2 wave are the same, i.e. $U_{\parallel}^S = U_{\parallel}^{S+}$, $U_{\perp}^S = U_{\perp}^{S+}$, $U_{\parallel}^F = U_{\parallel}^{F+}$. We skip the presentation of details of the analysis and proceed to describe the main results.

Under the above assumptions *four surface modes of propagation* are possible.

Two slow modes, *true Stoneley* and *leaky pseudo-Stoneley waves* propagate with speeds one lower and the other somewhat higher than the speed U_{\parallel}^F of P2 wave. There is an influence of the surface permability coefficient α on the speed of leaky pseudo-Stoneley wave.

The other two modes have speeds lower and higher than the speed U_{\perp}^S of S wave. In the case of different mass densities the speed of one them, the generalized Rayleigh wave is smaller than U_{\perp}^S . The attenuation of this wave is identical with the attenuation of the wave appearing on the boundary with the fluid. On the other hand if the densities are almost equal the generalized Rayleigh wave ceases to exist, and it is replaced by a new mode of propagation whose speed is higher than U_{\perp}^S . This is also a leaky wave.

Pseudo-Stoneley, generalized Rayleigh, and the new surface waves are all transitional modes between surface and bulk waves. Due to attenuation they are the so-called *localized waves*, i.e. they exist in limited domains.

It is obvious that the last case which we presented above solely indicates a complexity of problems connected with the propagation of waves on interfaces and, in general, in stratified porous media. Such questions as the transmission of energy between different modes of propagation, impedances of porous media and reflection and transmission of bulk waves, the range of existence of different modes of surface propagation have not been investigated as yet.

3.4 Riemann problems; soliton porosity waves in 1-D; remarks on 2-D; stability

Analysis of bulk waves presented in Section 3.1 concerns solely speeds of propagation of weak discontinuity waves. We have not investigated a behavior of amplitudes. However in the simple case of fully linear model this problem is not very interesting because amplitudes remain constant during the propagation.

This is not the case any more if we incorporate some nonlinearities into the model. Apart from large deformations of the skeleton there are two natural candidates for

nonlinear extensions. The first one follows from the fact that the fluid component may contain some gaseous parts (e.g. bubbles) which require a generalization of constitutive relation (3.7)₂. A second generalization is connected with the dependence on the equilibrium porosity n_E . If this is not constant but changes, say, according to the law (2.53), the model becomes nonlinear.

One can show in the first case [35] that weak discontinuity waves exist only for a finite time. The time behavior of the amplitude of such waves is described by the Bernoulli equation whose solution tends to infinity as the time $t \rightarrow t_c < \infty$. At this instant of time the weak discontinuity wave becomes a strong discontinuity (shock wave) and classical solutions of the problem do not exist any more. We shall not discuss this problem in these notes.

In the second type of nonlinearity we also deal with strong discontinuities and, consequently, we cannot expect an existence of classical solutions. However the model contains certain small parameters which allow to replace strong discontinuities by smooth solutions in an asymptotic sense. For instance we can investigate soliton-like solutions which possess the property of going in the limit of zero value of the small parameter to the Dirac- δ distribution. One of the most prominent examples of such a behavior is a solution of the classical Korteweg - de Vries equation for an unknown function u

$$\frac{\partial u}{\partial t} + 3 \frac{\partial u^2}{\partial x} + \beta^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \mathfrak{R}^1, \quad t \in [0, T]. \quad (3.71)$$

If $\beta = 0$ this equation, similar to the Burgers equation, is quasi-linear hyperbolic. However limit solutions as $\beta \rightarrow 0$ of the Korteweg - de Vries equation are principally different from solutions of the limit Burgers equation. For example consider the following particular solution of equation (3.71):

$$u(x, t, \beta) = u_0 + A \cosh^{-2}(\gamma\sigma), \quad \sigma := \frac{x - x(t)}{\beta}, \quad (3.72)$$

where $A > 0$, $\gamma = \frac{1}{2}\sqrt{A}$, $x(t) = (A + 6u_0)t$; $a > 0$, u_0 is a constant. The variable σ is called in the asymptotic analysis the *fast variable*.

Indeed the limit in \mathcal{D}' as $\beta \rightarrow 0$ of the function $(u(x, t, \beta) - u_0)/\beta$ is equal to $2\sqrt{A}\delta(x - x(t))$. At the same time the pointwise limit as $\beta \rightarrow 0$ of the function $u(t, x, \beta) - u_0$ for $x \neq x(t)$ is equal to 0. However the limit as $\beta \rightarrow 0$ of the maximum of function $u(t, x, \beta) - u_0$ is equal to A . These solutions are referred to as infinitely thin soliton-like functions.

Motivated by this example we investigate now some properties of the 1D Riemann problem for our model. In order to be able to perform an asymptotic analysis we have to locate small parameters of the model and we do so in the sequel.

First of all in the Table 3 we show typical orders of magnitude of various material parameters appearing in the model.

Table 3: *Typical values of material parameters for rocks*

| | |
|---|--------------------------------|
| reference porosity n_E | 0.3 |
| mass density of the fluid $\frac{\rho_0^F}{n_E}$ | $10^3 \frac{kg}{m^3}$ |
| mass density of the skeleton $\frac{\rho_S}{1-n_E}$ | $3 \times 10^3 \frac{kg}{m^3}$ |
| effective Lamé coefficient λ^S | 40 <i>GPa</i> |
| coefficient of compressibility κ | $10^6 \frac{m^2}{s^2}$ |
| coefficient of permeability π | $10^7 \frac{kg}{m^3 s}$ |
| relaxation time τ | $10^{-6} s$ |
| coupling coefficient β | $10^2 MPa$ |

These values yield the following characteristic orders of magnitude of the fields.

Table 4: *Typical orders of magnitude for the fields of the poroelastic model of rocks*

| | |
|--|---------------------------------------|
| speeds of longitudinal acoustic waves U^S, U^F | $3 \frac{km}{s}$ and $1 \frac{km}{s}$ |
| changes of the fluid mass density $ \rho^F - \rho_0^F $ | $1 \frac{kg}{m^3}$ |
| diffusion velocity $ \mathbf{v}^F - \mathbf{v}^S $ | $0.1 \frac{m}{s}$ |
| partial pressure in the fluid $n_E p^F$ | 0.1 <i>MPa</i> |
| partial stresses in the skeleton $(1 - n_E) tr \mathbf{T}^S$ | 200 <i>MPa</i> |
| changes of porosity $ \Delta $ | 10^{-6} |

We use these estimates to write the field equations in the dimensionless form. Variables and fields shall be replaced by the dimensionless quantities according to the following scheme.

$$\begin{aligned}
t &\rightarrow \frac{t}{t_0}, & x_k &\rightarrow \frac{x_k}{L}, & \mathbf{v}^F &\rightarrow \frac{\mathbf{v}^F}{v_0}, & \mathbf{v}^S &\rightarrow \frac{\mathbf{v}^S}{v_0}, & v_0 &:= \frac{L}{t_0} \\
\rho^F &\rightarrow \frac{\rho^F}{\rho_0^F}, & \rho^S &\rightarrow \frac{\rho^S}{\rho_0^F}, & p^F &\rightarrow \frac{p^F}{p_0^F}, & \mathbf{T}^S &\rightarrow \frac{\mathbf{T}^S}{p_0^F}.
\end{aligned} \tag{3.73}$$

This scheme is chosen in order to preserve the structure of field equations. For instance we normalize both partial mass densities with the same reference mass density ρ_0^F , and both velocities with the same reference value v_0 in order to keep the same form of the diffusion term: $\pi (\mathbf{v}^F - \mathbf{v}^S)$ in both momentum balance equations. Certainly the number of independent reference parameters can be reduced according to methods of dimensional analysis. For instance we can choose according to the momentum balance equation for the fluid (3.6)₃

$$\begin{aligned}
p_0^F &= \frac{\rho_0^F L^2}{t_0^2}, & t_0 &:= \frac{\rho_0^F}{\pi} \sim 10^{-4} s, \\
\implies L &:= \sqrt{\frac{p_0^F \rho_0^F}{\pi^2}} \sim 10^{-3} m, & v_0 &\sim 10^{-1} \frac{m}{s},
\end{aligned} \tag{3.74}$$

where we accounted for the data of Table 3 and we have chosen $\rho_0^F \sim 10^3 \frac{kg}{m^3}$, $p_0^F \sim 0.1MPa$.

Now we have to compare the contributions to the partial pressure in the fluid $\kappa(\rho^F - \rho_0^F)$ and $\beta\Delta$. We have

$$\kappa(\rho^F - \rho_0^F) = p_0^F \left[\frac{\kappa\rho_0^F}{p_0^F} \right] \left(\frac{\rho^F}{\rho_0^F} - 1 \right). \quad (3.75)$$

The quantity in the square brackets defines the dimensionless coefficient of compressibility. For the above quoted data it has the value: $\frac{\kappa\rho_0^F}{p_0^F} = 10^4$. Simultaneously the changes in the round brackets are of the order 10^{-3} (see: Table 2). On the other hand the second contribution has the form

$$\beta\Delta = p_0^F \left[\frac{\beta\Delta_0}{p_0^F} \right] \frac{\Delta}{\Delta_0}. \quad (3.76)$$

Δ_0 is a parameter which normalizes Δ to the same order as changes of mass density in (3.75), i.e. it must be of the order 10^{-3} (see: the last line in Table 4). Consequently the normalized coefficient $\frac{\beta\Delta_0}{p_0^F}$ appearing in the square brackets has the order of magnitude 1. This should be compared with the value 10^4 estimated above for the dimensionless compressibility coefficient $\frac{\kappa\rho_0^F}{p_0^F}$. Hence it is clear that the coupling term is a small correction and we can rescale the problem in such a way that these contributions appear with a small parameter in the field equations. Simultaneously the above scaling yields a similar order of magnitude of dimensionless $\frac{\beta\Delta_0}{p_0^F}$ and $\frac{\tau}{t_0}$ for rocks (compare: τ and the value t_0 in (3.74)). This observation shall be useful in further considerations.

In the sequel **dimensionless quantities are denoted by the same symbols as before but with small indices**.

We normalize the set of field equations (3.6) in such a way that changes of dimensionless mass density of the fluid ρ_f , of the relative elongation ε_s of the skeleton, and changes of relative porosity Π shall be of the order of unity. Then we have

$$\begin{aligned} \frac{\rho^F - \rho_0^F}{\check{\rho}_0^F} &\equiv \rho_f - \rho_0^f = O(1) \Rightarrow \check{\rho}_0^F = 1 \frac{kg}{m^3} \Rightarrow \frac{p^F}{p_0^F} = 1 + K(\rho_f - \rho_0^f) + \beta'^2\Pi, \\ K &: = \frac{\kappa\check{\rho}_0^F}{p_0^F}, \quad \Pi := \frac{\Delta}{\check{\Delta}_0}, \quad \beta'^2 := \frac{\beta\check{\Delta}_0}{p_0^F}, \quad \check{\Delta}_0 = 10^{-6} \Rightarrow \beta' = O(10^{-2}), \\ E &: = \frac{\lambda^S + 2\mu^S}{p_0^F} \check{\varepsilon}^S = O(10^3), \quad \varepsilon_s := \frac{\varepsilon^S}{\check{\varepsilon}^S}, \quad \check{\varepsilon}^S := 10^{-2}. \end{aligned} \quad (3.77)$$

In addition we relate the relaxation time to the small parameter β' . We extend the range of data which are quoted in Table 3, and allow also smaller relaxation times. These are appropriate for porous materials with high rigidity of the skeleton and for

granular materials with almost rigid grains. Hence

$$\frac{\tau}{t_0} = \beta'^\gamma, \quad \gamma = 1, 2. \quad (3.78)$$

It follows for the permeability coefficient

$$\pi' := \frac{\pi t_0}{\rho_0^f} = O(1). \quad (3.79)$$

Analysis of the porosity equation shows that it is convenient to normalize the equilibrium porosity in the following way $n'_E = n_E \frac{\beta'}{\Delta}$.

We consider a one-dimensional problem in the strip $Q_T = \{x \in \mathfrak{R}^{10}, t \in [0, T]\}$. However most of the results can be proven also for 2-D, and 3-D problems. We follow the paper [36].

To simplify the notation **we skip the apostrophe** appearing in quantities β' , n'_E and π' . For the following dimensionless fields

$$\{\rho_f, \rho_s, v_f, v_s, \varepsilon_s, \Pi\}, \quad (3.80)$$

we obtain from the set (3.6) the following dimensionless field equations

$$\begin{aligned} \frac{\partial \rho_f}{\partial t} + \frac{\partial (\rho_f v_f)}{\partial x} &= 0, & \frac{\partial \rho_s}{\partial t} + \frac{\partial (\rho_s v_s)}{\partial x} &= 0, \\ \rho_f \left(\frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial x} \right) + \frac{\partial}{\partial x} [K(n_E) (\rho_f - \rho_0^f)] + \beta^2 \frac{\partial \Pi}{\partial x} + \pi (v_f - v_s) &= 0, \\ \rho_s \left(\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial x} \right) - \frac{\partial}{\partial x} [E(n_E) \varepsilon_s] - \beta^2 \frac{\partial \Pi}{\partial x} - \pi (v_f - v_s) &= 0, \\ \frac{\partial \varepsilon_s}{\partial t} - \frac{\partial v_s}{\partial x} &= 0, \end{aligned} \quad (3.81)$$

$$\beta \frac{\partial \Pi}{\partial t} + \beta v_s \frac{\partial \Pi}{\partial x} + \frac{\partial}{\partial x} [\varphi(n_E) (v_f - v_s)] = -\frac{\Pi}{\beta^{\gamma-1}} - \left(\frac{\partial n_E}{\partial t} + v_s \frac{\partial n_E}{\partial x} \right),$$

with the initial conditions

$$\begin{aligned} \rho_f|_{t=0} &= \rho_0^f(x) + \mathcal{R}_0^f(x, \beta, \tau), & \rho_s|_{t=0} &= \rho_0^s(x) + \mathcal{R}_0^s(x, \beta, \tau), \\ v_f|_{t=0} &= v_0^f(x) + \mathcal{V}_0^f(x, \beta, \tau), & v_s|_{t=0} &= v_0^s(x) + \mathcal{V}_0^s(x, \beta, \tau), \end{aligned} \quad (3.82)$$

$$\Pi|_{t=0} = \Pi^0(x) + \Pi_0(x, \beta, \tau), \quad \varepsilon_s|_{t=0} = \frac{\partial}{\partial x} u_0^s(x) + \mathcal{F}_0^s(x, \beta, \tau).$$

Material parameters E, K, φ are assumed to be smooth functions of n_E . Equilibrium values appearing as the first contribution on the right hand side of these conditions satisfy the stationary limit problem of (3.81) with $\beta = 0$

$$\begin{aligned} \rho_0^f v_0^f \frac{\partial v_0^f}{\partial x} + \pi (v_0^f - v_0^s) &= 0, \\ \rho_0^s v_0^s \frac{\partial v_0^s}{\partial x} - E(n_0) \frac{\partial^2 u_0^s}{\partial x^2} - \pi (v_0^f - v_0^s) &= 0. \end{aligned} \quad (3.83)$$

In the case $\tau = \beta^2$ initial data (3.82) are a small perturbation of order $O(\beta)$ of the stationary solution of (3.83) of the following form $\Pi^0 = v_0^f = v_0^s = 0$, $\rho_0^f > 0, \rho_0^s > 0$, and ρ_0^f, ρ_0^s as well as $\frac{\partial}{\partial x} u_0^s$ are constant. Material parameters E, K, n_E do not have to be constant.

On the other hand in the case $\tau = \beta$ initial data (3.82) are a perturbation of the stationary solution of (3.83) of the form $\Pi^0 = 0, v_0^f = v_0^s$.

In the work [36] we prove for different cases the propositions of the following general form

Proposition 1 *Let $\tau = \beta^2, \Pi^0 = v_0^f = v_0^s = 0$, and $\rho_0^f > 0, \rho_0^s > 0, \frac{\partial}{\partial x} u_0^s$ are constant and $n_E(\rho^f, \rho^s) \neq \text{const}$. Then asymptotic solutions of the system (3.81) exist on a finite time interval with any accuracy with respect to β . They possess the following properties: they are smooth approximations with respect to β of order $O(\beta)$ of strong discontinuities either of v_f, ρ_f or of $v_s, \rho_s, \varepsilon_s$ as well as of the infinitely thin soliton function of order $O(\beta)$ for Π along a small perturbation of characteristics of the linearized problem. They are stable in the sense that any small perturbation of either of the characteristic speeds of the front $\dot{x}_E = \pm\sqrt{K}, \dot{x}_E = \pm\sqrt{\frac{E(n_0)}{\rho_0^s}}$ tends to zero. $\widehat{\rho}_0^s$ denotes the value of ρ_0^s on the front of the singularity*

Proposition 2 *Let $\tau = \beta, \Pi^0 = 0, v_0^f = v_0^s$, and $\rho_0^f > 0, \rho_0^s > 0, \frac{\partial}{\partial x} u_0^s$ are constant and $n_E(\rho^f, \rho^s) = n_0 = \text{const}$. Then asymptotic solutions of the system (3.81) exist on a finite time interval with any accuracy and they possess the following properties: they are smooth approximations with respect to β of order $O(1)$ of weak discontinuities either of v_f, ρ_f or of $v_s, \rho_s, \varepsilon_s$ as well as a smooth approximation of order $O(\beta)$ of a strong discontinuity of Π along a small perturbation of characteristics of the linearized problem*

Roughly speaking these two propositions indicate the character of solutions of 1D Riemann problems of the model. We concentrate on the first case. Then the proposition shows that there exists a kink solution for the partial velocity in solely one of the two components, and the propagation of this kink which is almost characteristic is accompanied by a soliton-like solution for dynamical changes of porosity. This is also an indication of the property of the model, at least in the above formulated simple version, that we do not need boundary conditions for the porosity.

In the paper [36] we prove the existence of a weak solution of the above Cauchy problem such that

$$\begin{aligned}\rho_f &= \rho_{as}^f + w_f, & \rho_s &= \rho_{as}^s + w_s, & \Delta &= \Delta_{as} + w^p, \\ v_f &= v_{as}^f + \omega_f, & v_s &= v_{as}^s + \omega_s, & \varepsilon_s &= \varepsilon_{as}^s + \varphi_s,\end{aligned}\quad (3.84)$$

and

$$\begin{aligned}\|w_f\|_{L_2(Q_T)} + \|w_s\|_{L_2(Q_T)} + \|\omega_f\|_{L_2(Q_T)} + \|\omega_s\|_{L_2(Q_T)} + \\ + \|\varphi_s\|_{L_2(Q_T)} &= O\left(\beta^{3/2}\right), \\ \|w^p\|_{L_2(Q_T)} &= O\left(\beta^{5/2}\right),\end{aligned}\quad (3.85)$$

where $\rho_{as}^f, \rho_{as}^s, v_{as}^f, v_{as}^s$, and Δ_{as} are asymptotic solutions of the above set of equations.

The construction of asymptotic solutions is based on the following ansatz. The solution consists of two parts. The first part is constructed in a neighborhood Ω_Γ of the front $\Gamma_T := \bigcup_{0 \leq t \leq T} \Gamma_t$, $\Gamma_t := \{x \in \Omega, x = x(t)\}$, where $x(t)$ denotes the current position of the front. For instance, it has the following form for the changes of porosity

$$\Pi_{as} = \Upsilon_0(t) + \Pi_0(\sigma, t) + \sum_{j=1}^N \beta^j \left(\Upsilon_1(x, t) + Y_j^p(\sigma, x, t) \right), \quad (3.86)$$

where

$$Y_j^p = H_1^p(x, t) z_0(\sigma, t) + \Pi_j(\sigma, t), \quad j \geq 1, \quad \sigma := \frac{x - x(t)}{\beta^2}. \quad (3.87)$$

$\Pi_j(\sigma, t)$ are smooth, bounded soliton-like functions, and $z_0(\sigma, t)$ is a kink-like function. Both of them are *stabilized in the infinity*, i.e.

$$\sigma^k \frac{d^j}{d\sigma^j} \frac{d^l}{dt^l} \frac{d^i}{dx^i} (y - y^\pm) = 0, \quad \forall k, j, i \geq 0 \quad \text{if } \sigma \rightarrow \infty, \quad (3.88)$$

for $y = z_0$ and $y = \Pi_j$, where

$$z_0^-(t) = \lim_{\sigma \rightarrow -\infty} z_0 = 0, \quad z_0^+(t) = \lim_{\sigma \rightarrow +\infty} z_0 = 1, \quad (3.89)$$

or *vice versa*.

The ansatz for smooth approximations of the strong discontinuity of velocity v_f and of the density ρ_f has the similar form

$$\begin{aligned}v_{as}^f &= \sum_{j=1}^N \beta^j \left(V_1^f(x, t) + Y_j^f(\sigma, x, t) \right), \\ \rho_{as}^f &= \rho_0^f(x, t) + \sum_{j=1}^N \beta^j \left(\rho_1^f(x, t) + Y_j^{\rho f}(\sigma, x, t) \right).\end{aligned}\quad (3.90)$$

In these relations

$$\begin{aligned} Y_j^f &= H_j^f(x, t) z_0(\sigma, t) + \mathcal{A}_j^f(\sigma, t), \\ Y_j^{\rho f} &= H_j^{\rho f}(x, t) z_0(\sigma, t) + \mathcal{A}_j^{\rho f}(\sigma, t), \end{aligned} \quad (3.91)$$

and

$$\widehat{H}_j^f, \widehat{H}_j^{\rho f} \neq 0, \quad j \geq 1. \quad (3.92)$$

We use here the notation $(\widehat{\dots})$ for the truncation of the function in the brackets to the front Γ_T .

The functions $H_j^f, H_j^{\rho f}$ are smooth and bounded while the functions $\mathcal{A}_j^f, \mathcal{A}_j^{\rho f}$ are smooth, bounded soliton-like functions satisfying condition (3.88).

The second part of solutions describes the behavior in the exterior, i.e. far from the front of discontinuity. We do not show it here.

In addition we assume that the position of the front is described by the following relation for the speed of propagation

$$\dot{x}(t) = \dot{x}_E(t) + \beta \dot{x}_1(t) + O(\beta^2). \quad (3.93)$$

As usual the form of contributions to the last relation follows from solvability conditions.

The main result can be formulated as follows.

Theorem 1 *Let $\tau = \beta^2$, and the following inequalities be satisfied*

$$1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s \right) \frac{K'(n_0)}{K(n_0)} > 0, \quad \varphi(n_0) - \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s \right) > 0. \quad (3.94)$$

Let us suppose that the speed of propagation of the front Γ_T is defined by the equation

$$\dot{x}_E^2 = K(n_0), \quad (3.95)$$

and that the first correction to this speed is described by the equation

$$\dot{x}_1 = \widehat{V}_1^f + \frac{1}{2} \left(1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s \right) \frac{K'(n_0)}{K(n_0)} \right) \widehat{H}_1^f. \quad (3.96)$$

Then asymptotic solution (3.86) and similar solutions for all other fields exist with any accuracy on the short time interval $(0, T)$. The leading part of this asymptotics

$$\begin{aligned}
\rho_{as}^f &= \rho_0^f(x) + \beta \left(\rho_1^f(x, t) + H_1^{\rho f}(x, t) z_0(\sigma, t) \right) + O(\beta^2), \\
\rho_{as}^s &= \rho_0^s(x) + \beta \left(\rho_1^s(x, t) + H_1^{\rho s}(x, t) z_0(\sigma, t) \right) + O(\beta^2), \\
v_{as}^f &= \beta \left(V_1^f(x, t) + H_1^f(x, t) z_0(\sigma, t) \right) + O(\beta^2), \\
v_{as}^s &= \beta \left(V_1^s(x, t) + H_1^s(x, t) z_0(\sigma, t) \right) + O(\beta^2), \\
\varepsilon_{as}^s &= \beta \left(\Phi_1^s(x, t) + H_1^{\Phi s}(x, t) z_0(\sigma, t) \right) + O(\beta^2), \\
\Pi_{as} &= \Upsilon_0(t) + \Pi_0(\sigma, t) + O(\beta^2),
\end{aligned} \tag{3.97}$$

satisfies the system (3.81) up to the order $O(\beta)$.

Background functions $V_1^f, V_1^s, \rho_1^f, \rho_1^s$ and Φ_1^s are the solution of the following Cauchy problem in the strip Q_T :

$$\begin{aligned}
\rho_0^f \frac{\partial V_1^f}{\partial t} + K(n_0) \frac{\partial \rho_1^f}{\partial x} + \pi \left(V_1^f - V_1^s \right) &= 0, \\
\rho_0^s \frac{\partial V_1^s}{\partial t} - E(n_0) \frac{\partial \Phi_1^s}{\partial x} - \pi \left(V_1^f - V_1^s \right) &= 0, \\
\frac{\partial \rho_1^f}{\partial t} + \rho_0^f \frac{\partial V_1^f}{\partial x} &= 0, \quad \frac{\partial \rho_1^s}{\partial t} + \rho_0^s \frac{\partial V_1^s}{\partial x} = 0, \\
\frac{\partial \Phi_1^s}{\partial t} - \frac{\partial V_1^s}{\partial x} &= 0,
\end{aligned} \tag{3.98}$$

with the initial data

$$\begin{aligned}
V_1^f|_{t=0} &= V_f^1(x), \quad V_1^s|_{t=0} = V_f^1(x), \quad \Phi_1^s|_{t=0} = \Phi_s^1(x), \\
\rho_1^f|_{t=0} &= \rho_f^1(x), \quad \rho_1^s|_{t=0} = \rho_s^1(x).
\end{aligned} \tag{3.99}$$

On the other hand **amplitudes of discontinuity** $H_1^f, H_1^s, H_1^{\rho f}, H_1^{\rho s}$ and $H_1^{\Phi s}$ satisfy the following Cauchy problem

$$\begin{aligned}
\rho_0^f \frac{\partial H_1^f}{\partial t} + K(n_0) \frac{\partial H_1^{\rho f}}{\partial x} + \pi \left(H_1^f - H_1^s \right) &= 0, \\
\rho_0^s \frac{\partial H_1^s}{\partial t} - E(n_0) \frac{\partial H_1^{\Phi s}}{\partial x} - \pi \left(H_1^f - H_1^s \right) &= 0, \\
\frac{\partial H_1^{\rho f}}{\partial t} + \rho_0^f \frac{\partial H_1^f}{\partial x} &= 0, \quad \frac{\partial H_1^{\rho s}}{\partial t} + \rho_0^s \frac{\partial H_1^s}{\partial x} = 0, \\
\frac{\partial H_1^{\Phi s}}{\partial t} - \frac{\partial H_1^s}{\partial x} &= 0,
\end{aligned} \tag{3.100}$$

with the following Cauchy data on the front Γ_T

$$H_1^f|_{\Gamma_T} = H(x), \quad H_1^{\rho f}|_{\Gamma_T} = \frac{\widehat{\rho}_0^f}{\dot{x}_E} H(x), \quad H_1^s|_{\Gamma_T} = H_1^{\Phi s}|_{\Gamma_T} = H_1^{\rho s}|_{\Gamma_T} = 0. \quad (3.101)$$

Simultaneously we have

$$\Upsilon_0(t) = \Upsilon_0^0 \exp\left(-\frac{t}{\beta^2}\right), \quad \Upsilon_0^0 = \text{const}. \quad (3.102)$$

The fast part of the porosity Π_0 is the soliton-like function

$$\Pi_0 = \widehat{\rho}_0^f \left(1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right) \frac{K'(n_0)}{K(n_0)}\right) \left(\widehat{H}_1^f\right)^2 z_0 (1 - z_0),$$

where the kink-like function z_0 is a strictly monotonic solution of the equation

$$\begin{aligned} & \left(\varphi(n_0) - \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right) - \dot{x}_E \widehat{\rho}_0^f \widehat{H}_1^f \left(1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right) \frac{K'(n_0)}{K(n_0)}\right) (1 - 2z_0) \right) z_0' = \\ & = -\widehat{\rho}_0^f \left(1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right) \frac{K'(n_0)}{K(n_0)}\right) \widehat{H}_1^f z_0 (1 - z_0) \end{aligned} \quad (3.103)$$

Primes in the above relations denote the ordinary differentiation with respect to an appropriate argument.

In addition to the above Theorem one can prove an important property of solutions of equation (3.103). Namely

Theorem 2 (entropy condition for the amplitude of discontinuity of v_f)

The function z_0 increases if

$$\widehat{H}_1^f < 0, \quad t \in [0, T], \quad (3.104)$$

and it decreases if

$$\widehat{H}_1^f > 0, \quad t \in [0, T]. \quad (3.105)$$

These relations correspond to the entropy condition for the problem of propagation.⁴

The strictly monotonic bounded solution of equation (3.103) exists if the following condition is satisfied

$$\left| \widehat{H}_1^f \right| < \frac{\varphi(n_0) - \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right)}{\widehat{\rho}_0^f \sqrt{K(n_0)} \left(1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right) \frac{K'(n_0)}{K(n_0)}\right)}. \quad (3.106)$$

⁴Note that the product $\widehat{H}_1^f z_0$ is always decreasing. For this reason we can consider this result to be an analogy of the classical *entropy condition*.

Amplitude \widehat{H}_1^f of the jump of strong discontinuity of v_f is defined uniquely as a smooth solution of the following differential equation

$$2\widehat{\rho}_0^f \frac{d\widehat{H}_1^f}{dt} + \pi \widehat{H}_1^f = 0, \quad \widehat{H}_1^f \Big|_{t=0} = H_1^f(0), \quad (3.107)$$

i. e.

$$\widehat{H}_1^f = H_1^f \exp\left(-\frac{t}{2\rho_0^f}\right) \quad (3.108)$$

By the same technique we derive Rankine-Hugoniot conditions which we shall not quote in these notes.

Analogous results were obtained in [36] for the second characteristic $\dot{x}_E = \sqrt{\frac{E(n_0)}{\rho_0^s}}$.

Bearing the above results in mind we can distinguish the following four cases for the propagation of discontinuities initiated by a jump of the fluid velocity v_f .

1. If $\dot{x}_E = \sqrt{K(n_0)}, \widehat{H}_1^f > 0$ and equation (3.96) for \dot{x}_1 is satisfied then function Π_0 is positive and the profile of velocity v_f determined by Y_1^f becomes sharper on the front than that of z_0 because it is the product of z_0 and H_1^f . Moreover if the background function $V_1^f = 0$ then the direction of the correction \dot{x}_1 coincides with the direction of propagation. Hence the front is accelerating. This corresponds to a physically plausible situation when an increment of velocity v_f behind the front yields an increment of porosity in Ω_T . It means that an increment of velocity of the front Γ_T results in local opening of pores. This corresponds to the classical condition of fluid mechanics called the stable displacement (**push**), i.e. a denser fluid pushes the one which is less dense

$$\left(\widehat{Y}_1^{\rho f}\right)^- > \left(\widehat{Y}_1^{\rho f}\right)^+ \quad \text{on } \Gamma_T. \quad (3.109)$$

2. If $\dot{x}_E = \sqrt{K(n_0)}, \widehat{H}_1^f < 0$ and equation (3.96) for \dot{x}_1 is satisfied then function Π_0 is positive and the profile of velocity v_f determined by Y_1^f becomes flatter on the front than that of z_0 . This is an analogy of the rarefaction wave and it is physically not plausible because a local increment of porosity is coupled with the decrement of velocity v_f ahead of the front Γ_T .
3. Let $\dot{x}_E = \sqrt{K(n_0)}, \widehat{H}_1^f < 0$ and in equation (3.96) for \dot{x}_1 we have the minus sign instead of the plus, i.e.

$$\dot{x}_1 = \widehat{V}_1^f - \frac{1}{2} \left(1 + \widehat{\rho}_0^f \frac{\partial n_E}{\partial \rho_f} \left(\widehat{\rho}_0^f, \widehat{\rho}_0^s\right) \frac{K'(n_0)}{K(n_0)}\right) \widehat{H}_1^f. \quad (3.110)$$

This corresponds to negative solitons of porosity. Again we have here a physically plausible situation in which local closing of pores decelerates the front Γ_T . Consequently a less dense fluid pushes a denser one. The following condition holds in this case

$$\left(\widehat{Y_1^{\rho f}}\right)^- > \left(\widehat{Y_1^{\rho f}}\right)^+ \quad \text{on } \Gamma_T. \quad (3.111)$$

This case is called the **Saffman-Taylor instability**.

4. If $\dot{x}_E = \sqrt{K(n_0)}, \widehat{H_1^f} < 0$ and equation (3.110) for \dot{x}_1 holds we deal again with a nonphysical rarefaction wave.

Summing up the soliton-like solution for porosity can be interpreted as a "filter". It opens in the case of denser fluid pushing the less dense one, and it closes in the opposite case. Consequently for sufficiently small perturbations of \dot{x}_E , i.e. under the existence condition of the kink-like solution (3.106) the amplitude $\widehat{H_1^f}$ of discontinuity of v_f decreases in time according to the relation (3.108) ($|\dot{x}_1| \rightarrow 0$) and the system is stable.

We do not consider here any further possibilities connected with the propagation of soliton-like disturbances of porosity. We refer for details to the paper [36]. The technique of asymptotic analysis is similar to this presented above.

Let us make two final remarks connected with the above presented Riemann problem. First of all we have shown that the character of solution depends on an entropy condition which, within the scope of this paper, was formulated directly. In a general case it should be a consequence of thermodynamical considerations which are still missing. Secondly the 1D case considered in these notes hides certain essential properties of soliton-like solutions which can be observed solely in a higher dimension. In particular this concerns an influence of the curvature of the front on stability properties. Preliminary numerical analysis performed by O. A. VASSILIEVA (Moscow State University) for 2D case indicates that there appear instabilities leading to the loss of the symmetry of initial data and a creation of mushy regions.

4 Mass exchange

4.1 Adsorption for large channel diameter

Mass exchange between components of porous, and granular materials belongs to the most important problems of practical bearing within theories of such materials. This is connected primarily with a very large internal surface per unit volume on which the exchange takes place. For instance in sandstones it reaches the value of $1.5 \times 10^5 \frac{m^2}{m^3}$ in comparison with $6 \frac{m^2}{m^3}$ for the external surface. This property is used in many technological processes. For instance in the growth of SiC crystals by

sublimation the vapour of silicium flows through a porous graphite wall in which it forms various carbide connections. A charcoal granular material is also used in gas masks. Lungs, many filters and chemical reactors are made of porous materials for the same reason. Transport of pollutants by ground water in soils is an important example of such processes appearing spontaneously in nature.

Classically processes of mass exchange in porous materials were divided into two classes of *chemical and physical adsorption*. This classification seems to be not very sharp but it is still useful by construction of various macroscopic models. In the case of chemical adsorption we deal with exchange of mass between skeleton and one or more fluid components in which particles build chemical bindings. In contrast to such processes in a physical adsorption particles of skeleton and of a fluid component form weak van der Waals bindings. The first type of exchange is connected with essential thermal effects connected with a release or absorption of energy, i.e. with the latent heat of reaction. On the other hand the heat of reaction by physical adsorption is very small and we use the approximation of isothermal processes. We discuss such a model in this Section.

The physical adsorption model of a mass exchange between a fluid component, and a solid in porous, and granular materials is based on the classical work of Langmuir (see: [7,37] for references). In the original works of Langmuir the theory of adsorption was limited to flat solid surfaces interacting with a gas. However for porous materials whose pores are large - their diameter is greater than app. $500 = 50nm$ - one can still rely on the assumption that the influence of the curvature of the surface is small. We discuss the influence of the curvature in the next Section. This is why we say that we describe materials with a *large diameter of channels*.

We assume as well that the fluid component consists of two phases. The bulk phase does not exchange mass, and it is solely a carrier of the adsorbate. The second phase - the adsorbate - moves with the velocity of the bulk fluid and its concentration is small

$$c := \frac{\rho_t^A}{\rho_t^F + \rho_t^A} \ll 1, \quad (4.1)$$

where ρ_t^A, ρ_t^F denote current partial mass densities of the adsorbate, and of the fluid carrier, respectively.

According to Langmuir a transfer of particles of adsorbate from the fluid phase to the surface of the solid depends on the number of available *bare sites* on this surface. The notion of bare sites is introduced in connection with the landscape of interaction energy between adsorbate and the solid skeleton. This landscape depends on a crystallographic or structural symmetry of the skeleton and it has the form of a regular pattern with distributed spots of a maximum interaction energy. On these spots one can expect particles of adsorbate to settle down most frequently. This process is controlled in the first place by the partial pressure of adsorbate in the fluid phase. Certainly there are also particles which desorbate again if they overcome an energy barrier through thermal fluctuations. In the equilibrium one expects these two processes to have the same rate.

Such an adsorption process is called the *monolayer adsorption*. If in a certain domain all available bare sites on the surface of the skeleton are already occupied the adsorption may take place due to interaction of adsorbate particles in their fluid phase and those which are already bound to the skeleton. We deal then with the *multilayer adsorption*. In the case of a small concentration of the fluid adsorbate such exchange of mass between the two phases may be assumed to be less important. We limit attention in these notes solely to the monolayer adsorption model.

In order to construct a continuous model in the spirit of this work we have to construct the mass sources in partial mass balance equations. On the semimacroscopic level (i.e. in the *representative elementary volume* (REV) of a porous or granular material) the normalized fraction of bare sites per unit volume is denoted by $1 - x$, i.e. x is the fraction of *occupied sites*. If the area of the internal surface contained in the representative elementary volume is denoted by f_{int} , and the mass of adsorbate per unit area of the internal surface by m^A then the amount of mass which is already adsorbed in the representative elementary volume is in average equal to the product $m^A x f_{int}$.

Let us denote by V the volume of the representative elementary volume. Then the amount of mass of adsorbate transferred in unit time from the fluid phase to the solid skeleton is given by the macroscopic balance relation

$$\hat{\rho}_t^A = -m^A \frac{d(xy)}{dt}, \quad y := \frac{f_{int}}{V}, \quad (4.2)$$

where $\hat{\rho}_t^A$ denotes the current intensity of mass source per unit time, and unit macroscopic volume.

In order to construct the model we have to specify the rates in this relation. The quantities x, y play in the model the role of additional *microstructural variables*. Field equations for these quantities are assumed to have a form of evolution equations which are specified by relations for the above mentioned rates.

For the rate $\frac{dx}{dt}$ we assume that changes of the fraction x are described by the Langmuir relation

$$\frac{dx}{dt} = a(1 - x)p^A - bx e^{-\frac{E_b}{kT}}, \quad (4.3)$$

where p^A denotes the partial pressure of the adsorbate in the fluid phase, E_b is the energy barrier for particles adsorbed on the solid surface due to the van der Waals interaction forces, and it is assumed to be constant, a , and b are material parameters which within the present model may depend solely on the temperature, k is the Boltzmann constant, and T is the absolute temperature. In the case of full phase equilibrium in which the adsorption rate (the first contribution with coefficient a), and the desorption rate (the second contribution with coefficient b) are equal but of opposite sign we obtain from the equation (4.3) the following relation for the equilibrium fraction of occupied sites

$$x = x_L := \frac{\frac{p^A}{p_0}}{1 + \frac{p^A}{p_0}}, \quad p_0 := \frac{b}{a} e^{-\frac{E_b}{kT}}. \quad (4.4)$$

This relation describes the so-called the *Langmuir isotherm*. It begins in the origin $\frac{p^A}{p_0} = 0$ with the zero value of occupied sites and saturates at the value 1 for $\frac{p^A}{p_0} \rightarrow \infty$. At any given partial pressure p^A the fraction x is uniquely determined and it may change its value if we vary the pressure. This corresponds to a slow transition from one thermodynamical equilibrium to another one. In reality such processes are conducted through nonequilibrium states which are described by the rate equation (4.3) and are connected with the dissipation.

In the mass source (4.2) we have also another contribution connected with the change of the internal surface. Consequently we must formulate a relation for the rate $\frac{dy}{dt}$. We make the assumption that changes of the internal surface are coupled with dissipative changes of the porosity n which in turn describe relaxation processes of semimacroscopic changes of volume of the skeleton.

First of all let us notice that for sufficiently smooth internal surfaces of porous, and granular materials with a random geometry of pore spaces a change of an average characteristic linear dimension of the internal surface, and this of pores in the elementary representative volume can be assumed to be proportional: $\delta f_{int}^{\frac{1}{2}} \sim \delta(nV)^{\frac{1}{3}}$. Simultaneously dissipative changes of the porosity are given by a source \hat{n} which describes the intensity of these changes per unit time and volume of the porous material. Bearing the above assumption in mind we obtain immediately

$$\frac{1}{y} \frac{dy}{dt} = \zeta \frac{\hat{n}}{n}, \quad (4.5)$$

where the proportionality factor ζ is assumed to be constant for the purpose of this work.

Obviously in a thermodynamical phase equilibrium $\hat{n} \equiv 0$, and the equilibrium fraction x is connected with the partial pressure p^A through the relation (4.4). Then the mass source (4.2) vanishes identically.

The behavior of the continuous model based on the above assumptions has been checked on a simple bench-mark homogeneous problem [6]. It was found that results are indeed qualitatively in agreement with observations.

We present here the set of field equations which covers a much more extensive class of problems. In particular we can describe couplings of adsorption and diffusion as well as we can incorporate boundary conditions on permeable boundaries which are characteristic for the majority of practical problems. According to the above remarks we have to determine the following fields

$$\left\{ \rho_t^S, \rho_t^L, c, \mathbf{v}^S, \mathbf{e}^S, \mathbf{v}^F, n, x, y \right\}, \quad \rho_t^L := \rho_t^F + \rho_t^A, \quad (4.6)$$

where the concentration c is defined by (4.1). The velocity of the third component does not appear because the adsorbate in the fluid phase moves with the same velocity as the other fluid component. Therefore we use only two momentum balance equations, for the skeleton and for both fluid components together.

Field equations follow from three mass balance equations, two momentum balance equations, the balance equation of porosity, integrability condition for the deformation of the skeleton, and two evolution equations for two additional microstructural variables. In the case of small deformations of the skeleton, small changes of the fluid mass density, small concentrations of adsorbate, and small changes of porosity, i.e.

$$\|\mathbf{e}^S\| \ll 1, \quad \left| \frac{\rho_t^L - \rho_0^L}{\rho_0^L} \right| \ll 1, \quad 0 < c \ll 1, \quad |\Delta| \ll 1, \quad (4.7)$$

they have the form

- mass balance

$$\begin{aligned} \frac{\partial \rho_t^S}{\partial t} + \operatorname{div}(\rho_t^S \mathbf{v}^S) &= -\rho_t^L \hat{c}, & \frac{\partial \rho_t^L}{\partial t} + \operatorname{div}(\rho_t^L \mathbf{v}^F) &= \rho_t^L \hat{c}, \\ \frac{\partial c}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} c &= (1-c) \hat{c}, & \hat{c} := \frac{\hat{\rho}_t^A}{\rho_t^L} &= -\frac{m^A}{\rho_0^L} \frac{d(xy)}{dt}, \end{aligned} \quad (4.8)$$

- momentum balance

$$\begin{aligned} \frac{\partial \rho_t^L \mathbf{v}^F}{\partial t} + \operatorname{div}(\rho_t^L \mathbf{v}^F \otimes \mathbf{v}^F + p^L \mathbf{1}) + \pi(\mathbf{v}^F - \mathbf{v}^S) &= 0 \\ \rho_t^S \frac{\partial \mathbf{v}^S}{\partial t} &= \operatorname{div} \mathbf{T}^S + \pi(\mathbf{v}^F - \mathbf{v}^S), \end{aligned} \quad (4.9)$$

- porosity balance

$$\frac{\partial \Delta}{\partial t} + \varphi(n_E) \operatorname{div}(\mathbf{v}^F - \mathbf{v}^S) = -\frac{\Delta}{\tau},$$

where

$$\begin{aligned} \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu^S \mathbf{e}^S + \beta \Delta \mathbf{1}, \\ p^L &= p_0^L + \kappa(\rho_t^L - \rho_0^L) + \beta \Delta, \quad p^F = (1-c)p^L, \quad p^A = cp^L, \end{aligned} \quad (4.10)$$

with material parameters $\lambda^S, \mu^S, \kappa, \beta, \pi$ being constant. They depend parametrically on the constant initial porosity n_E . In addition we have

- integrability condition

$$\frac{\partial \mathbf{e}^S}{\partial t} = \text{symgrad} \mathbf{v}^S, \quad (4.11)$$

- evolution equations for microstructural variables

$$\begin{aligned} \frac{d \ln \frac{y}{y_0}}{dt} &= -\zeta \frac{\Delta}{n_E}, & y(t=0) &= y_0 \equiv \frac{f_{int}(t=0)}{V}, \\ \frac{dx}{dt} &= \frac{1}{\tau_{ad}} \left[(1-x) \frac{cp^L}{p_0} - x \right], & x(t=0) &= \frac{\frac{c_0 p_0^L}{p_0}}{1 + \frac{c_0 p_0^L}{p_0}}, \\ \tau_{ad} &:= \frac{1}{b} e^{\frac{E_b}{kT}}, & c_0 &:= c(t=0), \end{aligned} \quad (4.12)$$

Again the material parameters ζ, p_0, τ_{ad} are assumed to be constant.

General results for this system of equations have not been obtained as yet. However some important particular problems have been solved under the assumptions of negligible accelerations, and a negligible explicit time dependence of porosity. Their discussion can be found in the Ph-D Thesis [7]. The most important results have been published [38,39]. We quote here solely the most important conclusions of these works. The latter assumption yields the following relation for changes of porosity, and the constitutive relations for partial stresses

$$\begin{aligned} \Delta &= -\tau \varphi \text{div} (\mathbf{v}^F - \mathbf{v}^S), \\ \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S \text{tr} \mathbf{e}^S \mathbf{1} + 2\mu^S \mathbf{e}^S - \beta \tau \varphi \text{div} (\mathbf{v}^F - \mathbf{v}^S) \mathbf{1}, \\ p^L &= p_0^L + \kappa (\rho_t^L - \rho_0^L) - \beta \tau \varphi \text{div} (\mathbf{v}^F - \mathbf{v}^S). \end{aligned} \quad (4.13)$$

Hence changes of porosity appear in equations of motion in a similar way as contributions of bulk viscosities in mechanics of fluids. Relation (4.13)₁ allows to eliminate dynamical changes of porosity from field equations.

Let us note that the initial value problem has been already formulated in the above relations. It remains to add boundary conditions. In the two-component case it has the following form

$$\begin{aligned} (\mathbf{T}^S - p^L \mathbf{1}) \mathbf{n} \Big|_{\partial \mathcal{B}_t} &= \mathbf{t}_{ext}, \\ \rho^L (\mathbf{v}^F - \mathbf{v}^S) \cdot \mathbf{n} \Big|_{\partial \mathcal{B}_t} &= \alpha (p^L - n p_{ext}) \Big|_{\partial \mathcal{B}_t}, \end{aligned} \quad (4.14)$$

$$(\mathbf{v}^F - \mathbf{v}^S) - (\mathbf{v}^F - \mathbf{v}^S) \cdot \mathbf{n} \mathbf{n} \Big|_{\partial \mathcal{B}_t} = 0.$$

It is easy to check that these conditions are of the third art (see: a simple example in [27], the second Note).

Such a problem has been analyzed for a 1-D case in the above quoted works [7,38,39]. Most important results for applications of the model are connected with couplings

of adsorption and diffusion. For instance, it has been shown that the mass source as a function of relative velocity of components possesses a maximum. It is zero for the relative velocity equal to zero and it tends again to zero when this velocity goes to infinity. In the first case the system is in equilibrium and the number of bare sites remains constant and determined by the partial pressure p^A . The internal surface is also constant because the porosity does not change. In the second extreme case relaxation times for internal variables: $\tau_{ad}, \frac{1}{\zeta}$ are too long for adsorbate to settle down on the skeleton. This indicates the way for an optimal design of filters. Particularly useful is a dependence of the relative velocity from the surface permeability coefficient α because this parameter can be controlled.

4.2 Adsorption for small channel diameter; capillarity

As we have already mentioned processes in materials with the so-called mesopores, i.e. in materials in which the diameter of channels lies in the range 20 – 500 are connected with the appearance of hysteresis loops in adsorption isotherms. Measurements of their size are used in practical application to find microgeometrical properties of such materials. The reason for the appearance of loops is a capillary condensation. In contrast to the previous case a macroscopic multicomponent model for adsorption with hysteresis has not been constructed as yet.

We proceed to explain the notion of capillary condensation in terms of semimacroscopic description by means of the Kelvin equation.

We begin with mechanical and thermodynamical equilibrium conditions on an interface between two different phases of a liquid. These conditions determine properties of an interface between a fluid and its vapour in the analysis of condensation.

Let us consider a surface in a three-dimensional Euclidean space described by the position vector \mathbf{r}

$$\mathbf{r} = \mathbf{r}(\xi^1, \xi^2), \quad (4.15)$$

where (ξ^1, ξ^2) are surface coordinates. Base vectors, a normal vector, and the first and second metric tensors of this surface are defined by the relations

$$\begin{aligned} \mathbf{g}_\Delta & : = \frac{\partial \mathbf{r}}{\partial \xi^\Delta}, & \mathbf{n} & := \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|}, \\ a_{\Delta\Lambda} & : = \mathbf{g}_\Delta \cdot \mathbf{g}_\Lambda, & b_{\Delta\Lambda} & := \frac{\partial^2 \mathbf{r}}{\partial \xi^\Delta \partial \xi^\Lambda} \cdot \mathbf{n}. \end{aligned} \quad (4.16)$$

It can be shown [5] that the normal component of the mechanical equilibrium condition on such a surface has the following general form

$$[[\mathbf{T}\mathbf{n}]] \cdot \mathbf{n} = -S^{\Delta\Lambda} b_{\Delta\Lambda}, \quad (4.17)$$

where \mathbf{T} denotes the Cauchy stress tensor in bulk substances on both sides of the surface, and $S^{\Delta\Lambda}$ are components of the surface stress tensor, both on the level of semimacroscopic description.

In the case of ideal fluids on both sides of the interface which is itself a membrane we have $\mathbf{T} = -p\mathbf{1}$, and $S^{\Delta\Lambda} = \sigma a^{\Delta\Lambda}$, where σ is a *surface tension*. Then relation (4.17) reduces to the following one

$$p^G - p^F = \sigma \sum_{\alpha=1}^2 K^{(\alpha)} \beta^{(\alpha)} \cdot \beta^{(\alpha)}, \quad (4.18)$$

where $K^{(\alpha)}$ are eigenvalues, and $\beta^{(\alpha)}$ eigenvectors of the second metric tensor

$$(b_{\Delta\Lambda} - K a_{\Delta\Lambda}) \beta^\Lambda = 0. \quad (4.19)$$

On the left hand side of (4.18) we have the difference of pressures between the gaseous phase, and the fluid phase, respectively.

In addition to the mechanical equilibrium condition (4.18) an equilibrium on the interface is determined by the thermodynamical condition of equal chemical potentials (Gibbs free energies) of both phases

$$\mu^G(T^G, p^G) = \mu^F(T^F, p^F), \quad (4.20)$$

where p^G, p^F are pressures in the gaseous phase, and in the fluid phase, respectively. We assume the temperatures to be equal and constant: $T^G = T^F = T = \text{const}$.

We apply the above conditions to a small change of the equilibrium. We have then

$$d\mu^G \equiv V^G dp^G = d\mu^F \equiv V^F dp^F, \quad (4.21)$$

and, hence, by substitution in (4.18)

$$dp^G - dp^F = \left(1 - \frac{V^G}{V^F}\right) dp^G \approx -\frac{V^G}{V^F} dp^G = \sigma d\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \quad (4.22)$$

where $\frac{1}{r_1}, \frac{1}{r_2}$ denote the main curvatures. It was assumed that the volume of fluid phase is much smaller than this of the gaseous phase: $V^F \ll V^G$ and that the surface tension σ is constant. For ideal gases we have

$$\frac{RT}{V^F} d \ln p^G = -\sigma d\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \quad (4.23)$$

where R is the gas constant.

Integration of this relation yields the *Kelvin relation*

$$\ln \frac{p}{p_0} = -\frac{\sigma V^F}{RT} \left(\frac{1}{r_1} + \frac{1}{r_2}\right), \quad (4.24)$$

in which p_0 is the integration constant which we identify with the saturation pressure.

We demonstrate the properties described by relation (4.24) on a simple example of a cylinder open on both ends.

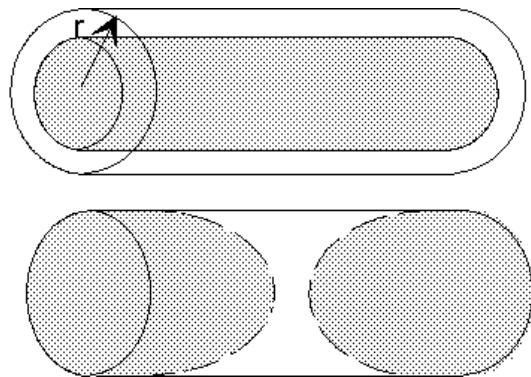


Figure 2: *Scheme of capillary condensation and evaporation in a cylinder*

Imagine a carrier of adsorbate filling the cylinder in the initial state with the partial pressure p of the adsorbate given by (4.24) for $r_1 = r, r_2 = \infty$, where r denotes the radius of the cylinder (Figure 2). The process of condensation may occur below the saturation pressure p_0 if there is a thin film of adsorbate on the internal wall of the cylinder. This yields the existence of a surface tension and, according to Kelvin formula, the partial pressure p at which condensation occurs is lower than p_0 . Once the nucleation on this thin meniscus begins the partial pressure of adsorbate decreases and this yields a decrement of the radius of the cylindrical space still filled with the gas (upper part of Fig.2). This process is, of course, unstable and it means that it terminates when the cylinder contains solely a condensed fluid form of the adsorbate. In Figure 3 it corresponds to the lower curve of the hysteresis loop reflecting the process of condensation. On the other hand the evaporation does not require a nucleation. It proceeds from the heads of the cylinder by moving hemispherical meniscus from both ends of the cylinder to its interior (lower part of Fig.2). In this process $r_1 = r_2 = r$, which means that the partial pressure is lower than in the case of condensation. Consequently we observe a hysteresis shown on the example of Figure 3.

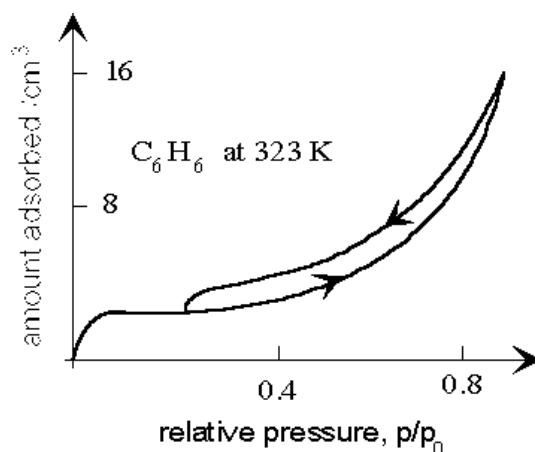


Figure 3: *Condensation and evaporation on a sodium-rich montmorillonite [37]*

Processes in channels of real materials with mesopores are similar to this described in the above example. However the appearance of hysteresis loops is connected with various geometrical properties and their macroscopic description usually requires essential modifications of the model for particular materials.

Mathematical methods leading to hysteresis effects are well known [40,41] but they were not yet applied to a construction of a multicomponent model of porous materials.

5 Perspectives

Continuum multicomponent theories of porous materials which we presented in these notes are still in a rather early stage of development. They require essential physical modifications as well as investigations of their mathematical properties. Let us list a few most important questions which should be answered in near future.

1. *Nonisothermal processes.*

We have mentioned already problems arising in connection with a macroscopic definition of thermodynamical temperature or multiple temperatures. There may be a way out of this problem if we work with some alternative notion such as an internal partial energy itself. One of the problems which can be solved easily in such a formulation is a transfer of energy by convection (relative motion) with negligibly small contributions of heat conduction. Such a problem may clear some couplings of diffusion and of a heterogeneous temperature distribution. But the main question of measurable thermal quantities cannot be answered in this way.

2. *Entropy condition and entropy flux.*

These problems arise in general thermodynamical considerations as well as in connection with the selection criterion for weak solutions, satisfying Rankine-Hugoniot conditions. An example of such a problem has been shown in the section on Riemann problems.

All thermodynamical results which were presented in these notes and which led to essential constitutive restrictions were based on the assumptions that the partial heat fluxes and entropy fluxes are proportional to each other with the inverse of the absolute temperature as the proportionality factor, and that processes are isothermal. Such assumptions are too restrictive for both thermodynamical purposes and for a theory of shock waves in porous materials.

3. *Nonelastic and nonlinear elastic behavior of the skeleton.*

These problems are primarily connected with practical applications of theories of porous materials in soil mechanics of clays, thermomechanics of sponges and other filters undergoing large deformations, etc. The way seems to be straightforward because there exist already one-component models for such a behavior, and methods of extended thermodynamics seem to be appropriate to incorporate those results in the scheme of multicomponent models.

4. *Extensions of microstructure and multiscaling problems.*

Some of these problems are being approached already. This concerns particularly systems with a double porosity, materials with a heterogeneous initial porosity, but also problems connected with mass exchange which require additional microstructural variables. A big open question is a transition between various levels of observation and averaging techniques in the derivation of macrosopic constitutive laws from microscopic and semimacroscopic relations.

5. *Chemical reactions and phase transitions. Capillarity and hysteresis.*

The first group, chemical reactions, can be approached in a way suggested by classical continuum theories of mixtures. The condition for a successful construction of such models is an answer on the first question listed above.

On the other hand, problems connected with the description of phase transitions, such as condensation and evaporation, capillarity, melting etc. follow primarily from the lack of transitions between various scales of description. A commonly used mathematical model of capillarity in porous materials which follows from the diffusion-reaction equations is physically very doubtful due to physical notions appearing in macroscopic description (such as pressure, saturation pressure, temperature) which are taken over from semimacroscopic models without any justification by, for example, averaging procedures.

6. *Nonlinear waves in porous materials.*

We have shown in these notes an example of a nonlinear wave asymptotic analysis following from some weak nonlinearities of the model. There exists no systematic research of this question. The reason is that nonlinear multicomponent models are not sufficiently developed as yet. Apart from nonlinearities connected with constitutive laws for the skeleton which we have mentioned above, laws for fluid components should go beyond simple linear compressible and ideal fluids. Such problems arise, for instance, in modelling of fluids containing bubbles of vapour. Analysis should concern not only strong discontinuity waves but also, due to their practical importance, surface waves. This research has been already initiated.

7. *Weak solutions, numerical methods.*

There is a considerable progress in recent years within the second part of this problem. Some multicomponent models of porous materials, particularly these based on the assumption of incompressibility and commonly used in soil mechanics, have an extensive numerical literature. We shall not quote it here because it concerns a different class of models. For nonlinear models of compressible components as presented in these notes, there exist only few numerical works.

>From the point of view of analytical properties of these models first steps have been made and some of the results were indicated in this work. From the purely mathematical standpoint properties of weak solutions for the model presented in these notes are not available as yet.

8. *Granular materials, unilateral constraints.*

We have mentioned that certain classes of processes in granular materials can be described by continuous models identical with those for porous materials. However a natural condition which should be incorporated into constitutive models of granular materials not carrying tension has never been investigated with a tacit assumption that solutions should not be extended so far. A proper constitutive model should rely on additional mechanical unilateral constraints.

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