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Exponentially sensitive internal layer solutions of one-side and their asymptotic expansions

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Abstract

We consider a singularly perturbed boundary value problem with Dirichlet conditions and study the sensitivity of the internal layers solutions with respect to small changes in the boundary data. Our approach exploits the existence of smooth invariant manifolds and their asymptotic expansions in the small parameter of perturbation. We show that the phenomenon is extremely sensitive since the shock layers are only obtained by exponentially small perturbations of the boundary data.

1 Introduction

We investigate the sensitivity of the internal layer solutions of singularly perturbed boundary value problem of the form

$$P_\varepsilon(A,B)\ \begin{cases} \varepsilon x'' = g(x)x' + r(t) & 0 < t < 1, \\ x(0) = A & x(1) = B, \end{cases} \tag{1}$$

for a small positive parameter ε and smooth functions g and r under the assumptions

a) For a given A there exists B^* and two smooth functions u_L and u_R satisfying the reduced problem $g(u)u' + r(t) = 0$ with $u_L(0) = A$ and $u_R(1) = B^*$ respectively and such that $\int_{u_L(t)}^{u_R(t)} g(s)ds \equiv 0$ in $[0,1]$.

b) $\int_{u_L(t)}^{x} g(s)ds > 0$ for all $0 \le t \le 1$ and $u_L(t) < x < u_R(t)$ (we assume that $u_L(t) < u_R(t)$.)

c) For all $0 \le t \le 1$, $g(x) < 0$ if $x \ge u_R(t)$ and $g(x) > 0$ if $x \le u_L(t)$.

Examples of such boundary value problems include some viscous shock problems in the homogeneous case $r = 0$ [7] and some stiff ordinary differential equations where interior shock layers can occur in the inhomogeneous case [4].

The phenomenon we consider arises when studying the location t_0 of the jumps of the solutions of $P_\varepsilon(A,B)$ as a function of the boundary values we introduce small changes in the boundary data (A, B^*). These boundary values have been selected in such a way that the location t_0^* of the jump that connects the limiting solutions $u_L(t)$ and $u_R(t)$ cannot be determined by the classical Rankine-Hugoniot condition. As a consequence of these small perturbations, the asymptotic behavior of the solution with $B = B^*$ and with $B(\varepsilon) \to B^*$ as $\varepsilon \to 0^+$ is the same but the shock location t_0 of the jump changes of $O(1)$. More precisely, for each $t_0 \in (0,1)$ there exists B close to B^* such that $P_\varepsilon(A,B)$ has an internal layer solution which is closely approximated by $u_L(t)$ and $u_R(t)$ throughout most the interval $[0,1]$ with the exception of the shock layer region. Near t_0 the solution changes rapidly in order to transfer from one solution of the reduced problem to the other. Outside the shock layer region every internal layer solution has the same leading-order term in the asymptotic expansion.

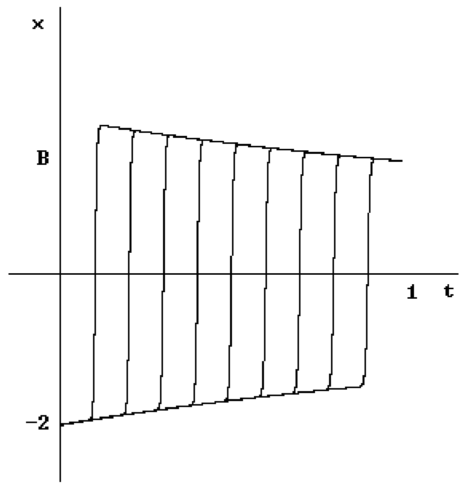

Fig. 1. *The shock layer solution of* $\varepsilon x'' = -2xx' + 2(t-2)$, $x(0) = -2$ *and* $x(1) = B$ *is plotted for several values of* B *near* 1. *Changes of order* $O(exp(-1/\varepsilon))$ *in* B *causes the shock layer location to move by* $O(1)$.

This internal layer behavior was first studied in [1] for a certain class of autonomous equations and later investigated in [2] in a more general context including non autonomous equations. Necessary and sufficient conditions implying the existence of the phenomenon and a result about the monotonicity of the shock location were given in [2]. By assuming uniqueness of solutions of the boundary value problems on each subinterval of $[0,1]$, it was proved the existence of a small interval $I = [B_1, B_2]$, with $B_1(\varepsilon) < B^* < B_2(\varepsilon)$ such that the shock location $t_0(B)$ is a continuous decreasing function of B in $[B_1, B_2]$. The values $B_i,\ i = 1,2$ are such that the corresponding problem $P_\varepsilon(A, B_i)$ has a boundary layer at one or at the other side of $[0,1]$, more precisely, $t_0(B) \to 0$ as $B \to B_2$ (and towards 1 as $B \to B_1$). Then, the sensitivity of the solution of $P_\varepsilon(A, B)$ to small perturbations of the boundary values follows as a consequence of the existence of the small interval I and it is characterized by the set ξ of values $B \in I$ for which there are only internal layers.

The determination of ξ for autonomous equations of the form $\varepsilon x'' = g(x)F(x')$ has been obtained in [1]. These results reveal the extremely sensitivity of the phenomenon when F is a linear function since the shock layer positions can be perturbed significantly by introducing exponentially small changes in the boundary values.

This exponentially sensitive phenomenon has been the object of much recent work for boundary value problems for both ordinary and partial differential equations (cf. [6,7,10]). Most of these works have focused then their attention on deriving equations for the location of the jump and some methods have been successfully applied (cf. [3,7,11]). These methods typically present two steps in their approach. The first one consists on determining the location of the jump for the unperturbed problem. In the second step the sensitivity of the solutions to slight changes of the boundary values for the unperturbed problem is then investigated.

It is interesting to remark that in most of these works the boundary values for the unperturbed problem $P_\varepsilon(A, B^*)$ lead to an internal layer solution. Then, the other internal layers positions may be obtained by positive or negative exponentially small of perturbations of B^*. That is, the sensitivity is "centered" at B^*. However, this is not always the case even if the shock condition for (A, B^*) is satisfied exactly. The difficulty with the class of problems $P_\varepsilon(A, B)$ is that in some cases the solution with $B = B^*$ exhibits a jump located near (at a distance of $O(\varepsilon)$ to) one of the endpoints of the interval instead of an internal layer. Therefore the internal layers arise for $B \in I$ but only at *one-side* of B^* and perturbations of this value may not exhibit the family of internal layers solutions. Such new behavior is caused by the term $r(t)$ and to the best of our knowledge, it has not been studied before.

The first goal in this work is to extend the results of [1] to the quasi-linear problem $P_\varepsilon(A, B)$. We prove the existence of a critical value of $B \in \xi$ labeled by B_c, around which the sensitivity is centered and give the corresponding shock location. We prove that B is in ξ if only if $\mid B - B_c \mid = \exp(-b/\varepsilon)$ with some $b > 0$, we determine the shock location as a function of the parameter b and we also prove that the internal layer solutions are exponentially close in the regions where they are close to the same reduced solution.

In addition, we show that if $r(t) \equiv 0$ then the sensitivity is centered at $B_c = B^*$. On the contrary if $r(t)$ is such that $u'_L(0) \neq u'_R(1)$ then we prove that $P_\varepsilon(A, B^*)$ has a boundary layer behavior which leads to a one side sensitivity. In that case, the internal solutions arise around some B_c with $B_c < B^*$ or $B_c > B^*$ depending on the $sign(u'_L(0) - u'_R(1))$.

The major obstacle in capturing the internal layer solutions for these problems is the determination of some $B \in \xi$. The second goal in this work is to prove the existence of a unique series $\hat{B}(\varepsilon) = \sum_{i=0}^{\infty} b_i \varepsilon^i$ with $b_0 = B^*$ such that any $B \in \xi$ has $\hat{B}$ as its asymptotic expansion and to give recurrence formulae to compute the coefficients b_i. For singularly perturbed problems it is usual to perform asymptotic expansions of the solutions rather of the boundary values. However, such expansion $\hat{B}$ turns to be a main tool in these problems. By using a least term cut-off process (see [9]) that optimally truncates the expansion in order to achieve an exponentially small error, we give a good approximation of some value B that provides an internal layer solution. The numerical results of B are then compared with corresponding asymptotic results in an example.

2 Main results

In virtue of Lorenz's results [8], the solutions of the boundary value problems associated with $\varepsilon x'' = g(x)x' + r(t)$ on each subinterval of $[0, 1]$ are unique. Thus, the problem $P_\varepsilon(A, B)$ considered under the assumptions a)b)c), is a sensitive B.V.P. with respect to (A, B^*) in a small interval $I = [B_1, B_2]$ (see [2]) and the shock location is a decreasing function of B in I. The solutions $x_B(t)$ of $P_\varepsilon(A, B)$ with A fixed and B varying in I only intersect themselves at the initial time $t = 0$.

To characterize the set ξ we study the difference of two internal layer solutions x_B and $x_{\tilde{B}}$

with B and $\tilde{B} \in \xi$. We first remark that the boundary value problem (1) can be reduced to the one-parameter family of fast-slow equations

$$\varepsilon x' = \int_{u_L(t)}^{x} g(u)du + C_B, \tag{2}$$

where $C_B = \varepsilon x'_B(0) \sim 0$ for all $B \in \xi$.

Condition a) and the fact that C_B is asymptotically small imply that the slow curve Γ_0 of the associated system of (2) in $\mathbb{R}^2$ has two branches $x = u_L(t)$ and $x = u_R(t)$. Condition b) implies that $u_L(t)$ is attractive while $u_R(t)$ is repulsive.

We now derive an estimate of the difference of $\mid C_B - C_{\tilde{B}} \mid$ by using the results of [5].

The differential equation of (1) viewed as a fast-slow system in $\mathbb{R}^3$, has a slow surface S given by $g(x)v + r(t) = 0$. Under the hypothesis c), S is the graph of a function $Z_0 : \mathcal{U} \to \mathbb{R}$ in those ouverts $\mathcal{U} \subset \mathbb{R}^2$ where $g(x) \neq 0$. Here $Z_0 = -\frac{r(t)}{g(x)}$.

It was proved in [5] that $\forall$ open set $\mathcal{V}, \overline{\mathcal{V}} \subset \mathcal{U}$ there exists a slow invariant manifold $\mathcal{M}$, graph of $Z(t,x) : \mathcal{V} \to \mathbb{R}$, where $Z(t,x) \sim Z_0(t,x)$ in $\mathcal{V}$, with the following property:

Let be $\gamma(t) = (t, x(t), v(t))$ a slow trajectory for $t \in J$ such that $(t, x(t)) \in \mathcal{V}$, $\gamma(t) \notin \mathcal{M}$ and $x(t) \sim u(t)$ in J where $(t, u(t))$ satisfies the slow dynamics $\dot{u} = Z_0(t, u(t))$. Then for each τ and $t \in J$ the vertical distance $d(t) := |v(t) - Z(t, x(t))|$ from the slow trajectory γ to the invariant manifold $\mathcal{M}$ at the point $(t, x(t))$ satisfies:

$$d(t) = d(\tau) \exp(\frac{1 + \delta(\tau,t)}{\varepsilon} \int_{\tau}^{t} g(u(s))ds) \tag{3}$$

where $\delta(\tau,t) \sim 0$ when $\tau - t$ is not asymptotically small (see [5, lemma 3.12]. Since γ is a slow trajectory it is always possible to find $\tau \in J$ where $\tau < t$ if $g(u) < 0$ or $\tau > t$ if $g(u) > 0$ and such that $\tau - t = O(1)$. Thus, (3) applied to such a τ shows that the vertical distance $d(t)$ is exponentially small. In particular this holds for two slow trajectories evaluated at the same point $(t, x(t))$.

Lemma 1 *If B and $\tilde{B}$ are in I such that $m_0 = \min\{t_0(B), t_0(\tilde{B})\} \gg 0$ then there exist $k_0 > 0$ such that $\mid C_B - C_{\tilde{B}} \mid = \varepsilon \exp(-k_0/\varepsilon)$ where $k_0 \sim \int_0^{m_0} g(u_L(s))ds$.*

Proof: Suppose $B > \tilde{B}$ then $t_0(B) < t_0(\tilde{B})$ because the shock location is a decreasing function of B. Since $m_0 \gg 0$ the corresponding trajectories of (2) are slow in some interval $[0, t_1]$ where $0 \ll t_1 < t_0(B)$ and both $x_B(t)$ and $x_{\tilde{B}}(t)$ are close to $u_L(t)$ in that interval. Thus both $(t, x_B(t))$ and $(t, x_{\tilde{B}}(t))$ are contained in the same $\mathcal{V}$ where $g(x) > 0$. Let be $\mathcal{M}$ the slow invariant manifold defined on $\mathcal{V}$ and containing the slow trajectory $\gamma_{\tilde{B}}$. Then, the vertical distance between these two slow trajectories at the initial point $t = 0$, $x = A$ satisfies (3) for any $\tau \in [0, t_1]$. Actually (3) holds up to some $\tau_0 \sim t_0(B)$ such that $d(\tau_0) = O(1)$ which yields

$$x'_B(0) - x'_{\tilde{B}}(0) = d(\tau_0)\exp(\frac{(1+\delta(\tau_0,0))}{\varepsilon}\int_{\tau_0}^{0} g(u_L(s))ds) \tag{4}$$

where $\delta(\tau_0, 0) \sim 0$.

It follows then the existence of a value $k_0 > 0$ such $x'_B(0) - x'_{\tilde{B}}(0) = \exp(-k_0/\varepsilon)$ with $k_0 = (1+\delta(\tau_0,0))\int_0^{\tau_0} g(u_L(s))ds - \varepsilon \ln d(\tau_0) \sim \int_0^{t_0(B)} g(u_L(s))ds > 0$.

Remark: Note that the above estimate does not hold if the value of $\tilde{B}$ is such that the solution of $P_\varepsilon(A, \tilde{B})$ has a boundary layer at 0. The value of k_0 is determined by the smallest shock location m_0 and $k_0 \to 0$ as $m_0 \to 0$.

The next step is to derive an estimate of $B - \tilde{B}$ for two values in ξ by studying the difference $x_B(t) - x_{\tilde{B}}(t)$.

Theorem 1 *Assume hypotheses a),b),c) for the boundary value problem* $P_\varepsilon(A, B)$.
Let be B *and* $\tilde{B} \in \xi$, $m_0 = \min\{t_0(B), t_0(\tilde{B})\}$ *and* $M_0 = max\{t_0(B), t_0(\tilde{B})\}$. *Then there exist* $b \sim \min\{\int_0^{m_0} g(u_L(s))ds, \int_1^{M_0} g(u_R(s))ds\}$ *such that* $\mid \tilde{B} - B \mid = exp(-b/\varepsilon)$.

Proof: Let be $B > \tilde{B}$ and both in ξ (the case $B < \tilde{B}$ follows analogously), then $m_0 = t_0(B)$ and $M_0 = t_0(\tilde{B})$. For all $t_0(\tilde{B}) < t \leq 1$ with t not close to $t_0(\tilde{B})$ the solutions $x_B(t)$ and $x_{\tilde{B}}(t)$ are slow and close to $u_R(t)$, $E(t) := x_B(t) - x_{\tilde{B}}(t) \sim 0$ and satisfies

$$\varepsilon E'(t) = H(t)E(t) + (C_B - C_{\tilde{B}}) \tag{5}$$

where $H(t) \sim g(u_R(t))$ and $E(t)$ and $C_B - C_{\tilde{B}} = \varepsilon E'(0)$ are positive because of the uniqueness of solutions of the boundary value problem . By performing the change of variable $W = E^{[\varepsilon]} = E| E |^{\varepsilon-1}$ and using lemma 1 we deduce the existence of k_0 such that the equation (5) becomes

$$\frac{W'}{W} = H(t) + \varepsilon(\frac{exp(-k_0)}{W})^{[1/\varepsilon]}, \tag{6}$$

If $exp(-k_0) \gg W > 0$ the composant in W is large and the trajectories are nearly vertical.

If $exp(-k_0) \ll W$ or $W \sim exp(-k_0)$ such that $\varepsilon(\frac{exp(-k_0)}{W})^{[1/\varepsilon]} \sim 0$ the equation (6) is a regular perturbation of $\frac{W'}{W} = g(u_R(t))$.

For all $t_0(\tilde{B}) < t \leq 1$ the image of the trajectory of (5) with $E(t) \sim 0$ appears, in the (t, W) plane , contained in the region where $exp(-k_0) \ll W < 1$ or $W \sim exp(-k_0)$ while changes of $E(t)$ that are either small (but not exponentially small in ε) or of order $O(1)$ arise for $W \sim 1$.

Then, in virtue of the behavior of $E(t)$ and of the trajectories of (6), there exists $\tilde{t} \sim t_0(\tilde{B})$ such that $E(\tilde{t}) = O(1)$ and $W(\tilde{t}) \sim 1$ and, in the new variable, $E(t)$ satisfies

$$W(t) = W(\tilde{t})exp((1+\eta(\tilde{t},t))\int_{\tilde{t}}^{t} g(u_R(s))ds), \tag{7}$$

as long as $t \geq \tilde{t}$ is such that $F(t) = \varepsilon(\frac{exp(-k_0)}{W})^{[1/\varepsilon]} \sim 0$. The value $\eta \sim 0$ if $t - \tilde{t}$ is not small.

For $t \gg \tilde{t}$, $\int_{\tilde{t}}^{t} g(u_R(s))ds) = O(1)$ while $\int_{\tilde{t}}^{t_0(\tilde{B})} g(u_R(s))ds) \sim 0$ then we can write (7) as

$$W(t) = W(\tilde{t})exp(-(1 + \eta_1(\tilde{t}, t))\int_{t}^{t_0(\tilde{B})} g(u_R(s))ds), \tag{8}$$

where $\eta_1(\tilde{t}, t)) \sim 0$ which yields

$$E(t) = E(\tilde{t})exp(-\frac{(1 + \eta_1(\tilde{t}, t))}{\varepsilon}\int_{t}^{t_0(\tilde{B})} g(u_R(s))ds). \tag{9}$$

Let be $K_0 = (1 + \eta_1(\tilde{t}, 1))\int_{1}^{t_0(\tilde{B})} g(u_R(s))ds \sim \int_{1}^{t_0(\tilde{B})} g(u_R(s))ds > 0$ then

$$W(1) = W(\tilde{t})exp(-K_0), \;\; F(1) = \frac{\varepsilon}{E(\tilde{t})}exp((K_0 - k_0)/\varepsilon). \tag{10}$$

If $K_0 \leq k_0$ or $K_0 > k_0$ with $K_0 - k_0 = O(\varepsilon)$ obviously $F(1) \sim 0$ and (9) holds up to $t = 1$ which yields $E(1) = B - \tilde{B} = E(\tilde{t})exp(-K_0/\varepsilon) = exp(-b/\varepsilon)$ where $b = K_0 - \varepsilon \ln E(\tilde{t}) \sim \int_{1}^{t_0(\tilde{B})} g(u_R(s))ds.$

If $K_0 - k_0 \gg O(\varepsilon)$, $W(1) \ll exp(-k_0)$ and $F(1)$ is large. Thus, (9) is satisfied up to some $\tau \ll 1$ such that $W(\tau) \sim exp(-k_0)$ and $F(\tau) \sim 0$ and $E(t)$ must spend the left over time $1 - \tau$ in the region where $W \sim exp(-k_0)$ and $F(t) = O(1)$. Thus, necessarily at $t = 1$, $\varepsilon\frac{exp(-k_0/\varepsilon)}{E(1)} = O(1)$, which implies the existence of $b = k_0 - \varepsilon \ln(\varepsilon\alpha) \sim \int_{0}^{m_0} g(u_L(s))ds$ for some $\alpha = O(1)$ such that $E(1) = exp(-b/\varepsilon)$.

Remark: By the same time the estimate (9) shows that two internal solutions are exponentially close when they are close to the same reduced solution outside the internal layer region. An estimate of $E(t)$ near the repulsive branch $u_R(t)$ is obtained in a similar way.

Theorem 1 gives both a necessary condition for a value B to be in ξ and an estimate of the parameter b that makes the shock location $t_0(\tilde{B})$ for some $\tilde{B} \in \xi$ moves to $t_0(B)$. We now give a sufficient condition that characterizes completely ξ and we derive an equation for the shock location.

We will see that the behavior of

$$I(t) = \int_{0}^{t} g(u_L(s))ds - \int_{1}^{t} g(u_R(s))ds \tag{11}$$

for $t \in [0, 1]$ plays an important role on determining the set ξ.

Theorem 2 *Assume hypotheses a), b), c) for the boundary value problem $P_\varepsilon(A, B)$. There exist a unique internal layer location t_c that satisfies* $\int_{0}^{t_c} g(u_L(s))ds = \int_{1}^{t_c} g(u_R(s))ds$

and a unique $B_c \in \xi$ *with* $t_0(B_c) = t_c$ *such that if* $B = B_c + \alpha \exp(-b/\varepsilon)$ *with* $\alpha = \pm 1$ *and* $0 \ll b \leq b_c$ *where* $b_c = \int_0^{t_0(B_c)} g(u_L(s))ds$ *then* $B \in \xi$ *and the shock location* $t_0(B)$ *satisfies asymptotically* $b \sim \int_0^{t_0(B)} g(u_L(s))ds$ *if* $\alpha = 1$ *or* $b \sim \int_1^{t_0(B)} g(u_R(s))ds$ *if* $\alpha = -1$.

Proof: The existence and the uniqueness of t_c follows from the fact that $I(t)$ defined in (11), is a continuous and strictly monotonic function with $signI(0) \neq signI(1)$ because of assumption c). The existence and the uniqueness of B_c follows from using that $t_0(B)$ is a continuous and strictly monotonic function of B.

Let be $B = B_c + exp(-b/\varepsilon)$ with $0 \ll b \leq b_c$ (we only consider the case of positive perturbations, the case of negative perturbations is handled in the similar way). Then $0 \leq t_0(B) < t_0(B_c) < 1$ and a boundary layer at $t = 1$ is precluded. Suppose that $x_B(t)$ has a boundary layer at 0. Let be $t_0(\hat{B})$ any internal layer located before $t_0(B_c)$ then $B_c < \hat{B} < B$. By theorem 1 there exist $\hat{b} > 0$ such that $\hat{B} - B_c = \exp(-\hat{b}/\varepsilon)$ where $\hat{b} \sim \min\{\int_0^{t_0(\hat{B})} g(u_L(s))ds, \int_1^{t_0(B_c)} g(u_R(s))ds\}$. Since $\hat{B} - B > 0$ then necessarily $0 \ll b < \hat{b}$.

Then, for any $0 \ll t_0(\hat{B}) < t_0(B_c)$, $0 \ll b \lesssim K$ where $K = \int_0^{t_0(\hat{B})} g(u_L(s))ds \to 0$ as $t_0(\hat{B}) \to 0$ then b must be ~ 0 which is absurd. Thus, $B \in \xi$ and it follows from theorem 1 that $b \sim \min\{\int_0^{t_0(B)} g(u_L(s))ds, \int_1^{t_0(B_c)} g(u_R(s))ds\}$. The fact that $b \leq b_c$ and $I(t_0(B_c)) = 0$ leads to the announced estimate.

Note that if $b \to b_c$ then $t_0(B) \to t_0(B_c)$ while for positive (negative) perturbations $t_0(B) \to 0$ (or $t_0(B) \to 1$) as $b \to 0$.

To summarize, the *main result* of this section is *the existence of a critical internal layer position* given by

$$\int_0^{t_0(B_c)} g(u_L(s))ds = \int_1^{t_0(B_c)} g(u_R(s))ds \tag{12}$$

and *a critical value* B_c such that $B \in \xi \Leftrightarrow |B - B_c| = \exp(-b/\varepsilon)$ with $b \sim \int_0^{t_0(B)} g(u_L(s))ds$ if $B > B_c$ or $b \sim \int_1^{t_0(B)} g(u_R(s))ds$ if $B < B_c$.

In this case we say that the sensitivity is *centered* at B_c since any internal layer may be captured by introducing exponentially small perturbations of B_c.

It is interesting to note that, usually, in a singularly perturbed boundary value problem with internal layer behavior, one of the main focus is to determine the shock location as a function of the given boundary conditions. Here, for this class of sensitive boundary value problems, the critical internal layer position $t_0(B_c)$ is determined by (12), on the contrary, the value of B_c may be not equal to B^* and so, in that case, it is not known.

In the next subsections we analyze the influence of the term $r(t)$ on the value of B_c and show when this class of equations may exhibit either a centered sensitivity at B^* or at one-side of B^*.

2.1 The case of the sensitivity centered at B^* $(r(t) \equiv 0)$

When $r(t) \equiv 0$ the reduced solutions of (1) are the constants $u_L(t) \equiv A$ and $u_R(t) \equiv B^*$ where $\int_{B^*}^{A} g(s)ds = 0$. In this case it follows from (12) that the critical shock layer location corresponding to B_c is given by $g(A)t_0(B_c) = g(B^*)(t_0(B_c) - 1)$ and thus

$$t_0(B_c) = \frac{g(B^*)}{g(B^*) - g(A)}. \tag{13}$$

On the other hand, it was proved in [1] that, when $r(t) \equiv 0$, $B^* \in \xi$ and that the shock location $t_0(B^*)$ is given by (13) . Therefore, in this case $B_c = B^*$, the internal layers may be obtained by exponentially small changes of B^* and from theorem 2 their location satisfy

$$t_0(B) \sim \begin{cases} \frac{b}{g(A)} & \text{if } B \geq B^* \\ 1 + \frac{b}{g(B^*)} & \text{if } B < B^* \end{cases} \tag{14}$$

for $B - B^* = \pm e^{-b/\varepsilon}$ and $0 < b \leq b_c = g(A)t_0(B_c) = g(B^*)(t_0(B_c) - 1)$. This shock location was also derived by using different methods, see [1],[3], [6], [7], [10], [11].

Two examples of centered sensitivity at B^* for (1) are $g(x) = -2x$ and with $B^* = 1 = -A$ which yields the two-point problem for the well-known Burgers equation and $g(x) = (1-x^{-2})$ with $AB^* = 1$ which arises in modelling compressible flows in a straight duct. For Burgers equation the critical $t_0(B^*) = 1/2$ while for the second case $t_0(B^*) = A/(A+B^*)$.

2.2 The case of the sensitivity at one-side of B^*

Let us now study the effect of the term $r(t) \neq 0$ in the behavior of the sensitive internal layers solution of (1) with respect to B^* .

We first remark that for this form of boundary value problems, the condition

$$\int_{u_L(t)}^{u_R(t)} g(s)ds \equiv 0 \quad \text{in} \quad [0, 1] \tag{15}$$

which is necessary to have sensitivity yields

$$g(u_L(t))u'_L(t) \equiv g(u_R(t))u'_R(t) \equiv -r(t) \quad \text{in} \quad [0, 1]. \tag{16}$$

Since we assume in c) that $sign(g(u_L(t))) \neq sign(g(u_R(t)))$ then $u'_L(t)$ and $u'_R(t)$ are either zero at the same time t or satisfy $sign(u'_L(t)) \neq sign(u'_R(t))$.

The second remark concerns the solution $x_{B^*}(t)$. When $B = B^*$ the initial and final values of the first derivatives are balanced so we have $x'_{B^*}(0) = x'_{B^*}(1)$.

Using this and an analysis of the trajectories in the phase space we derive the following result for $P_\varepsilon(A, B^*)$.

Theorem 3 *If under the hypotheses a),b),c) the reduced solutions $u_L(t)$ and $u_R(t)$ are such that $u'_L(0)) \neq u'_R(1)$ then $B^* \notin \xi$. Moreover,*

i)If $u'_L(0) < u'_R(1)$ the solution $x_{B^*}(t)$ has a boundary layer at $t = 0$,

ii) If $u'_L(0) > u'_R(1)$ the solution $x_{B^*}(t)$ has a boundary layer at $t = 1$.

Proof: Let us suppose that $x_{B^*}(t)$ has an internal layer at some $t_0(B^*)$.

Then $x_{B^*}(t) \sim u_L(t)$ for $0 \leq t \ll t_0(B^*)$ while $x_{B^*}(t) \sim u_R(t)$ for $t_0(B^*) \ll t \leq 1$ and for those t, $x'_{B^*}(t) \sim -\frac{r(t)}{g(x_{B^*}(t))}$. Thus, $x'_{B^*}(t)$ at $t = 0$ and $t = 1$ has the asymptotic limits $u'_L(0)$ and $u'_R(1)$ respectively as $\varepsilon \to 0$ where $u'_L(0)) \neq u'_R(1)$ which is absurd since $x'_{B^*}(0) = x'_{B^*}(1)$.

Let us now consider the case i) $u'_L(0) < u'_R(1)$ and suppose that $x_{B^*}(t)$ has a boundary layer at $t = 1$. Since $x_{B^*}(t)$ must start at $t = 0$ with $x'_{B^*}(0) \sim u'_L(0)$ and reach $t = 1$ with $x'_{B^*}(1) = x'_{B^*}(0)$ there exist $\tau_1 < 1$ and $\tau_1 \sim 1$ such that $x'_{B^*}(\tau_1) = O(1/\varepsilon)$. In addition, since $u'_L(0) < u'_R(1)$ there exist $\tau_1 < \tau_2 < 1$ such that $x'_{B^*}(\tau_2) = u'_R(1)$ and $x_{B^*}(\tau_2) \sim u_R(\tau_2) \sim B^*$. Thus, $x'_{B^*}(\tau_2) = u'_R(1) = -\frac{r(1)}{g(u_R(1))} \sim -\frac{r(\tau_2)}{g(x_{B^*}(\tau_2))}$ where $sign(g(x_{B^*}(\tau_2))) = sign(g(u_R(\tau_2))) < 0$. This implies that the trajectory $\gamma(t) = (t, x_{B^*}(t), x'_{B^*}(t))$ reaches at time τ_2, a point near the attracting part of the slow surface. Then, in $[\tau_2, 1]$, we must necessarily have $x_{B^*}(t) \sim u_R(t)$ and $x'_{B^*}(t) \sim u'_R(t)$ and so, at $t = 1$, $x'_{B^*}(1) \sim u'_R(1) > u'_L(0) \sim x'_{B^*}(0)$ which is not possible.

Corollary 1 *The boundary value problem (1) exhibits an exponential sensitivity either on the right side of B^* if $u'_L(0)) > u'_R(1)$ or on the left side of B^* if $u'_L(0)) < u'_R(1)$.*

Remark: An important consequence of the above results in the case of *one-side sensitivity* is that the value $B^* - B_c$ is small but *not exponentially small.* Therefore, exponentially small changes of B^* will never capture the internal layers and they will only exhibit a boundary layer.

We close the section with an example that illustrates theorem 3.

Example 2.3.1:

$$\begin{cases} \varepsilon x'' = -2xx' + \exp(t) & 0 < t < 1, \\ x(0) = A & x(1) = B, \end{cases} \tag{17}$$

For the boundary values $A = -1$ and $B^* = \sqrt{e}$, the two reduced solutions $u_L(t) = -\exp(t/2)$ and $u_R(t) = \exp(t/2)$ satisfy the assumptions a)b)c) and thus, (17) is a supersensitive boundary value problem with respect to (A, B^*) in a small interval I of B^*. Since $u'_L(0) = -1/2 < u'_R(1) = \sqrt{e}/2$, it follows from Theorem 3 that the solution of (17) for $A = -1$ and $B^* = \sqrt{e}$ has a boundary layer at $t = 0$ instead of an internal layer and it exhibits an exponential sensitivity on the left side of B^*. So the internal layer transition arises for values of B in I with $B < \sqrt{e}$ and it may be obtained by positive and negative exponentially small changes of some $B_c < \sqrt{e}$. From (12), the corresponding critical internal layer $t_0(B_c)$ for B_c is given by

$$t_0(B_c) = 2\ln(\frac{1+\sqrt{e}}{2}). \tag{18}$$

The other internal layers can be seen by adding an exponentially small term of the form $\pm\exp(-b/\varepsilon)$ with $0 < b \leq b_c = 2(\sqrt{e}-1)$ into B_c. The inclusion of these perturbations moves the shock location away from $t_0(B_c)$ and from theorem 3 the internal layer is now located at

$$t_0(B) \sim \begin{cases} 2\ln(1+\frac{b}{4}) & \text{for } B > B_c \\ 2\ln(\sqrt{e}-\frac{b}{4}) & \text{for } B < B_c \end{cases} \tag{19}$$

3 Asymptotic expansions for the boundary values B in ξ

As we have already mentioned, the difficulty with these problems in the case of one-side sensitivity is that we do not have the explicit value of B_c or at least of one value of B providing an internal layer solution.

In this section we now prove the existence of a unique asymptotic expansion in powers of ε for any $B \in \xi$ and show how it is possible to characterize the B's by means of the expansion.

The result is derived by using the asymptotic expansions for the first derivatives $x'_B(0)$ and $x'_B(1)$. In [5] it was proved that any slow invariant manifold $Z(t,x)$ has an expansion on $\mathcal{V} \subset \mathcal{U}$ of the form $\sum_{i=0}^{\infty} \varepsilon^i Z_i(t,x))$ where the family of functions $(Z_i)_{i\in\mathbb{N}}, Z_i : \mathcal{U} \to \mathbb{R}$ of class C^∞ are such that for any n: $(Z(t,x) - \sum_{i=0}^{n} \varepsilon^i z_i(t,x))/_{\varepsilon^n} \to 0$ as $\varepsilon \to 0$. Moreover, the slow invariant manifolds defined on a subset of the same $\mathcal{U}$ have the same expansion. Since the vertical distance from a slow trajectory to an invariant manifold is exponentially small, then for an internal layer solution, the first derivative at the initial (at the final) time and the corresponding invariant manifold $Z(t,x)$ evaluated at the point $(0,A)$ (at $(1,B)$) have the same expansion on $\mathcal{U}$.

Let us first derive the functions $Z_i(t,x))$ for the boundary value problem (1). We seek for a slow solution of the equation of (1) such that

$$x'(t) = Z_0(t,x(t)) + \varepsilon Z_1(t,x(t)) + \varepsilon^2 Z_2(t,x(t)) + \cdots \tag{20}$$

By introducing this formal expansion in the differential equation and equating coefficients of powers of ε we obtain a sequence of algebraic equations for the coefficients $Z_i(t,x)$ which finally gives:

$$\begin{cases} Z_0(t,x) &= -\frac{r(t)}{g(x)} \\ Z_i(t,x) &= \frac{1}{g(x)}[\frac{\partial Z_{i-1}}{\partial t} + \sum_{j=0}^{i-1} \frac{\partial Z_j}{\partial x} Z_{i-j-1}], \quad i \geq 1. \end{cases} \tag{21}$$

where the slow manifold S defined by $f(t,x,v)=0$ with $f(t,x,v)=g(x)v+r(t)$ is the graph of $Z_0(t,x)$. The repulsive part of S where $\partial f/\partial v = g(x)$ is positive is denoted S_R while the attracting part where $g(x)$ is negative is denoted S_A. These two surfaces are the graphs of $Z_0(t,x)$ defined in some open sets $\mathcal{U_R}$ and $\mathcal{U_A}$ of $\mathbb{R}^2$ respectively. Condition c) implies that for all $t \in [0,1]$, $(t,u_L(t)) \in \mathcal{U_R}$ and $(t,u_R(t)) \in \mathcal{U_A}$, in particular, the points $(1,B) \in \mathcal{U_A}$ for all $B \sim B^*$.

Theorem 4 *Assume hypotheses a)-c) for the boundary value problem (1). If g and r are of class C^∞ then there is a unique formal series $\hat{B} = \sum_{i=0}^{\infty} b_i \varepsilon^i$ such that any $B \in \xi$ has $\hat{B}$ as its asymptotic expansion. The coefficients b_i are given by the recursive formulas*

$$\begin{cases} b_0 &= B^* \\ b_1 &= \frac{1}{g(B^*)}(Z_0(1,B^*) - Z_0(0,A)) \\ b_2 &= \frac{1}{g(B^*)}[Z_1(1,B^*) - Z_1(0,A) + \frac{\partial Z_0}{\partial x}(1,B^*)b_1 - \frac{g'(B^*)}{2} b_1^2], \\ b_{i+1} &= \frac{1}{g(B^*)}[Z_{i-1}(1,B^*) - Z_{i-1}(0,A) + \sum_{l=0}^{i-1}(\frac{\partial Z_l}{\partial x}(1,B^*)b_{i-l} - \frac{g^{l+1}(B^*)}{(l+2)!}\alpha_{i-l}(l+2)) + \\ & \sum_{l=0}^{i-2}\sum_{j=2}^{i-l} \frac{1}{j!}\frac{\partial^j Z_l}{\partial x^j}(1,B^*)\alpha_{i+1-l-j}(j)], \quad i \geq 2. \end{cases} \tag{22}$$

where $Z_i(t,x)$ is given by (21) and $\alpha_k(j)$ is given, for $k \geq 1$, by the recursive formulas:

$$\alpha_k(j) = \begin{cases} \sum_{n=1}^{k} b_{k+1-n} b_n & \text{for } j = 2 \\ \sum_{n=1}^{k} \alpha_{k+1-n}(j-1) b_n & \text{for } j \geq 3. \end{cases} \tag{23}$$

Proof: For all $B \in \xi$, $\gamma(t, x_B(t), x'_B(t))$ is a slow trajectory in $[0,1]$ except $t \sim t_0(B)$ which lies near S_R for $t \in [0,t_1]$ and near S_A for $t \in [t_2,1]$ with $t_1 < t_0(B) < t_2$. Let us consider $\mathcal{V_R}$ and $\mathcal{V_A}$ two open sets of $\mathbb{R}^2$ containing $(t, x_B(t))$ for $t \in (-\delta, t_1)$ and for $t \in (t_2, 1+\delta)$ for some $\delta > 0$ respectively and such that $\overline{\mathcal{V_R}} \subset \mathcal{U_R}$ and $\overline{\mathcal{V_A}} \subset \mathcal{U_A}$. According to [5], condition c) implies the existence of both two slow invariant surfaces $Z^R(t,x)$ and $Z^A(t,x)$ defined on $\mathcal{V_R}$ and $\mathcal{V_A}$ respectively and of their asymptotic expansions such that

$$x'_B(0) = Z^R(0,A) + \exp(-k_0/\varepsilon), \quad x'_B(1) = Z^A(1,B) + \exp(-k_1/\varepsilon) \tag{24}$$

where the constants $k_i, i = 0,1$ in the exponentially small corrections are of $O(1)$.

Moreover, it follows from [5] that any slow invariant manifold defined on a subset of the same $\mathcal{U}$ has the same asymptotic expansion of the form $\sum_{i=0}^{\infty} \varepsilon^i Z_i(t,x))$ where $Z_i(t,x)$ are defined on $\mathcal{U}$. Here, the functions $Z_i(t,x)$ are given by (21) and, in particular, at the points $(0,A)$ and $(1,B)$ we have

$$Z^R(0,A) \sim \sum_{i=0}^{\infty} \varepsilon^i Z_i(0,A)\,; \quad Z^A(1,B) \sim \sum_{i=0}^{\infty} \varepsilon^i Z_i(1,B) \tag{25}$$

On the other hand, from (2)

$$\varepsilon(x'_B(1) - x'_B(0)) = \int_{B^*}^{B} g(u)du \text{ for all } \; B \in \xi. \tag{26}$$

By formally introducing the asymptotic expansions (25) and the Taylor expansion for the term on the right side of (26) we obtain

$$\sum_{i=0}^{\infty} \varepsilon^{i+1}(Z_i(1,B) - Z_i(0,A) - \frac{g^{i+1}(B^*)}{(i+2)!}\frac{(B-B^*)^{i+2}}{\varepsilon^{i+1}}) = g(B^*)(B-B^*). \tag{27}$$

Next, by expanding each $Z_i(1,B)$ around B^* in (27) we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} \varepsilon^{i+1}[Z_i(1,B^*) - Z_i(0,A) \;&+\; \sum_{j=1}^{\infty} \frac{1}{j!}\frac{\partial^j Z_i}{\partial x^j}(1,B^*)(B-B^*)^j - \frac{g^{i+1}(B^*)}{(i+2)!}\frac{(B-B^*)^{i+2}}{\varepsilon^{i+1}}] \\ &-\; g(B^*)(B-B^*) = 0.\end{aligned} \tag{28}$$

We now seek for a formal expansion $\sum\limits_{k=0}^{\infty} b_k \varepsilon^k$ for $B \in \xi$ where, obviously, the zeroth-order term is $b_0 = B^*$.

We introduce this series in (28) and write each $(B - B^*)^j = \sum\limits_{k=0}^{\infty} \alpha_k(j)\varepsilon^{k+j-1}$ for $j \geq 2$ where $\alpha_k(j)$ is given by (23). Finally we separate the sums having the same power of ε which gives

$$\begin{aligned}&\sum_{i=0}^{\infty}(Z_i(1,B^*) - Z_i(0,A) - g(B^*)b_{i+1})\varepsilon^{i+1} + \\ &\quad \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}[\frac{\partial Z_l}{\partial x}(1,B^*)b_k - \frac{g^{(l+1)}(B^*)}{(l+2)!}\alpha_k(l+2)]\varepsilon^{k+l+1} + \\ &\quad \sum_{l=0}^{\infty}\sum_{j=2}^{\infty}\frac{1}{j!}\frac{\partial^j Z_l}{\partial x^j}(1,B^*)(\sum_{k=1}^{\infty}\alpha_k(j)\varepsilon^{k+l+j}) = 0.\end{aligned} \tag{29}$$

It follows from (29) that only the term obtained for $i = 0$ in the first series, contributes to the coefficient of ε. This yields

$$b_1 = (Z_0(1,B^*) - Z_0(0,A))/g(B^*).$$

The contributions to the coefficient of ε^2 are obtained from the first serie for $i = 1$ and from the second one for those values of k and l such that $k + l = 1$ with $k \geq 1, l \geq 0$ while the third series provides higher powers of ε. Thus,

$$b_2 = (Z_1(1, B^*) - Z_1(0, A) + \frac{\partial Z_0}{\partial x}(1, B^*)b_1 - \frac{g'(B^*)}{2}b_1^2)/g(B^*).$$

Finally, for $i \geq 2$, the coefficient b_{i+1} of ε^{i+1} in the expansion for B is formed from collecting $S_1 = Z_i(1, B^*) - Z_i(0, A)$, the contributions from the second sum for those k, l such that $k + l = i$ with $1 \leq k \leq i$ and $0 \leq l \leq i - 1$, i.e.

$$S_2 = \sum_{l=0}^{i-1}(\frac{\partial Z_l}{\partial x}(1, B^*)b_k i - l - \frac{g^{(l+1)}(B^*)}{(l+2)!}\alpha_{i-l}(l+2)),$$

and finally those terms of the third sum obtained for those k, l, j such that $k+j+l = i+1$ where $0 \leq l \leq i-2$ and $2 \leq j \leq i-l$, i.e.

$$S_3 = \sum_{l=0}^{i-2}\sum_{j=2}^{i-l}\frac{1}{j!}\frac{\partial^j Z_l}{\partial x^j}(1, B^*)\alpha_{i+1-l-j}(j).$$

Therefore, for $i \geq 2$,

$$b_{i+1} = (S_1 + S_2 + S_3)/g(B^*)$$

which yields the announced formula for the b_i's.

Remark: This result gives not only a theoretical proof of the existence of the asymptotic expansion $\hat{B}$ but also a practical way to compute its coefficients. In fact, using the recurrence formulae (21),(22) and (23) and a formal computation software like MAPLE it is possible to compute high-order terms to obtain, in some cases, good estimates of the "internal layers boundary values" B.

On the other hand, we note that the coefficient b_1 is nothing more that $b_1 = (u'_R(1) - u'_L(0))/g(B^*)$ and that from theorem 3, b_1 must be zero if $B^* \in \xi$. Now, by means of the asymptotic expansion for B we can give a strong necessary condition for B^* to be in ξ.

Corollary 2 *If $B^* \in \xi$ then the coefficients of the asymptotic expansion $\hat{B} = \sum_{i=0}^{\infty} b_i\varepsilon^i$ satisfy $b_i \equiv 0$ for $i \geq 1$.*

It follows easily from (22) and (23) that, if $b_i \equiv 0$ for $i \geq 1$ then

$$Z_i(0, A) = Z_i(1, B^*) \text{ for all } \quad i \geq 0. \tag{30}$$

3.1 The asymptotic expansion $\hat{B}$ in case of sensitivity centered at B^*

Obviously, the result of the corollary 2 agrees with the result obtained in [1] for the homogeneous case $r(t) \equiv 0$. In fact, in such case $B^* \in \xi$ and, on the other hand, the slow surface S given by the horizontal plane $v = 0$ is also a slow invariant manifold so its expansion gives $Z_i(t, x(t)) \equiv 0$ for all $i \geq 0$. Thus for the viscous shock problem

$$P_\varepsilon(A,B) \left\{ \begin{array}{ll} \varepsilon x'' = g(x)x' & 0 < t < 1, \\ x(0) = A & x(1) = B, \end{array} \right. \tag{31}$$

the asymptotic expansion $\hat{B}$ is just

$$B^* + 0\varepsilon + 0\varepsilon^2 + \cdots. \tag{32}$$

Remark: The condition $b_i \equiv 0$ for all $i \geq 1$ is not sufficient for B^* to be in ξ. The following s-family of perturbed viscous shock problems

$$P_s \left\{ \begin{array}{ll} \varepsilon x'' = g(x)x' + e^{-b/\varepsilon^s} & 0 < s \leq 1, b > 0 \\ x(0) = A & x(1) = B^*, \end{array} \right. \tag{33}$$

provides a good counterexample.

Results of [1], [3], [6] and [10] show that (31) is sensitive to exponentially small changes not only in the boundary values but also in the coefficients of the differential operator. More precisely, it was proved that a small perturbation of order $e^{-b/\varepsilon}$ moves the internal layer location $t_0(B^*)$, given by (13) for $B = B^*$, to $t_0 \sim 1 - \frac{b}{|g(B^*)|}$. Since t_0 moves toward the right endpoint of $[0,1]$ as b tends to 0, any perturbation of the form e^{-b/ε^s} with $0 < s < 1$ locates the jump close to $t_0 \sim 1$. Therefore the solution of P_s with $0 < s < 1$ exhibits, for the boundary values A and B^*, a boundary layer instead of an internal layer, so $B^* \notin \xi$. However, the inclusion of the exponentially small term $r(t) \equiv e^{-b/\varepsilon^s}$ leads to the trivial asymptotic expansion (32) for any $0 < s \leq 1$.

3.2 Gevrey expansions and least term summation in case of sensitivity of one side

In contrast with the example (33) the inclusion of a term $r(t) \neq 0$ or $r(t,\varepsilon)$ small but not exponentially small usually provides a divergent series $\hat{B}$. Nevertheless, in some cases it is possible, to obtain good approximations of the values of B that are in ξ by means of the partial sums. In this section, we determine, under some hypotheses, a value of B for which there is an internal layer solution. The asymptotic results are then compared by the numerical computations on an example.

We first note that since the formal series $\hat{B}$ satisfies

$$\forall n \geq 0 \quad \lim_{\varepsilon \to +0} \frac{B(\varepsilon) - \sum_{i=0}^{n} b_i \varepsilon^i}{\varepsilon^n} = 0, \tag{34}$$

the error $R_n(\varepsilon)$ after n terms is of order $O(\mid b_n \mid \varepsilon^n)$. So the optimal value of n such that the remainder $R_n(\varepsilon)$ is as small as possible would be given by the index where the sequence $v_n = \mid b_n \mid \varepsilon^n$ takes its minimum value.

The results of [9] show that if the series is Gevrey of order $1/k$ and of type $\mathcal{A}$, i.e. if there are constants $K > 0$ and $\beta \geq 0$ such that

$$\forall n,\ n > 0,\ \mid b_n \mid \leq K n^{\beta} (\frac{\mathcal{A}}{k})^{\frac{n}{k}} (n!)^{\frac{1}{k}}, \tag{35}$$

such index exists. More precisely, these results, which require the analycity of $B(\varepsilon)$ in ε on a sector of the complex plane, show that there exists an index $N_s(\varepsilon) = [\frac{k}{\mathcal{A}\varepsilon^k}]$ such that

$$\mid B(\varepsilon) - \sum_{n=0}^{N_s} b_n \varepsilon^n \mid \leq O(e^{-1/\mathcal{A}\varepsilon^k}). \tag{36}$$

In other words, if the expansion is Gevrey of order $1/k$ the values of B could be optimally approximated by truncating the series at the least term N_s since the error will be exponentially small.

The existence of $N_s(\varepsilon)$ follows from the fact that if the expansion satisfies (35), then for a given ε, n can be chosen such that the upper bound $a_n = K n^{\beta} (\frac{\mathcal{A}}{k})^{\frac{n}{k}} (n!)^{\frac{1}{k}}$ is optimal with respect to ε.

In fact, using Stirling's formula we have $\ln((n!)^{\frac{1}{k}} \varepsilon^n) \sim \frac{n}{k} \ln(n \varepsilon^k e^{-1})$ for large n. This yields

$$\ln(a_n \varepsilon^n) \sim \ln(K n^{\beta}) + \frac{n}{k} \ln(\frac{n}{k} \mathcal{A} \varepsilon^k e^{-1}), \tag{37}$$

for large n, where the expression on the right of (37) has a minimum with respect to n at a value close to $N = \frac{k}{\mathcal{A}\varepsilon^k}$. Thus, the minimum of $a_n \varepsilon^n$ is of order $e^{-1/\mathcal{A}\varepsilon^k}$.

These results lead us to the following theorem.

Theorem 5 *If $B(\varepsilon)$ is analytic on an open sector $\mathcal{S}$ and $\hat{B}(\varepsilon)$ is Gevrey of order* 1 *and of type $\mathcal{A}$ then there exist $N_s(\varepsilon) = [\frac{k}{\mathcal{A}\varepsilon^k}]$ such that*

$$B_s(\varepsilon) := \sum_{n=0}^{N_s} b_n \varepsilon^n \in \xi. \tag{38}$$

Proof: The existence of $N_s(\varepsilon) = O(1/\varepsilon)$ such that $\forall B \in \xi$, $\mid B(\varepsilon) - B_s(\varepsilon) \mid \leq O(e^{-1/\mathcal{A}\varepsilon})$ follows from [9, th. 6.9]. The desired result is then obtained by application of theorem 2.

3.2.1. An example

$$\begin{cases} \varepsilon x'' = -2xx' + 2(t-2) & 0 < t < 1, \\ x(0) = A & x(1) = B, \end{cases} \tag{39}$$

The phenomenon of super-sensitivity of (39) arises for $B \sim 1$ when the boundary values are $A = -2$ and $B^* = 1$. The reduced solutions given by $u_L(t) = -u_R(t) = (t-2)$ satisfy $u_L'(0) > u_R'(1)$. It follows from theorem 3 that, for this choice of boundary values, the solution of (39) has a boundary layer at $t = 1$ and thus, the problem exhibits a sensitivity on the right-side of $B^* = 1$. Therefore the shock-type transition layer solutions will appear by exponentially small perturbations of some critical value B_c which is here greater than $B^* = 1$.

The equation (12) provides the following equation for the critical location $t_0(B_c)$:

$$t_0{}^2(B_c) - 4t_0(B_c) + 3/2 = 0 \tag{40}$$

which gives $t_0(B_c) \sim 0.418$ while $b_c = 3/2$.

Letting B varies like $B = B_c \pm \exp(-b/\varepsilon)$ with $0 < b \leq 3/2$ the shock location is found to satisfy either $t_0{}^2 - 4t_0 + b \sim 0$ for positive perturbations or $t_0{}^2 - 4t_0 + 3 - b \sim 0$ for negative perturbations and finally,

$$t_0(B) \sim \begin{cases} 2 - \sqrt{4-b} & \text{for } B > B_c \\ 2 - \sqrt{1+b} & \text{for } B < B_c. \end{cases} \tag{41}$$

Let us now analyze the asymptotic expansion $\hat{B}(\varepsilon)$ for this problem. We first remark that the two reduced solutions $u_L(t) = -u_R(t) = (t-2)$ satisfy the full equation of (39), so they are actually two slow solutions. Since $u_L'(t) \equiv 1$ and $u_R'(t) \equiv -1$, the associated expansion $u'(t) = Z_0(t, u(t)) + \varepsilon Z_1(t, u(t)) + \varepsilon^2 Z_2(t, u(t)) + \cdots$ for these slow solutions (see (20)), is such that their coefficients, given by (21), satisfy at $(0, A) = (0, -2)$

$$\begin{cases} Z_0(0, -2) = 1 \\ Z_n(0, -2) = 0 \quad \text{for all} \quad n \geq 1. \end{cases} \tag{42}$$

while for those evaluated at $(1, B^*) = (1, 1)$ we have

$$\begin{cases} Z_0(1, 1) = -1 \\ Z_n(1, 1) = 0 \quad \text{for all} \quad n \geq 1. \end{cases} \tag{43}$$

In addition, since any internal layer solution $x_B(t)$ of (39) is a slow solution that is close to $u_L(t)$ for $t < t_0(B)$, then $x'_B(0)$ has the same asymptotic (and convergent) expansion as $u'_L(0)$ with an exponentially small correction term (see (24)). More precisely,

$$x'_B(0) = 1 + \exp(-b/\varepsilon), \ \ b > 0. \tag{44}$$

This estimate of the first derivative will be very useful to compute the solutions of (39) numerically.

Formulas (42) and 43) simplify the calcul of the b_n, $n \geq 2$ in (22) and provides the following expansion

$$B = 1 + \varepsilon - \varepsilon^2 + \frac{5}{2}\varepsilon^3 - \frac{37}{4}\varepsilon^4 + \frac{353}{8}\varepsilon^5 + \cdots \tag{45}$$

for $B \in \xi$.

Using MAPLE we have calculated more terms of the asymptotic expansion. We observe that the coefficients are of the form

$$b_n = (-1)^{n-1}\frac{a_n}{2^{n-2}} \ \textit{for}\ n \geq 0 \tag{46}$$

where $a_0 = -1/4$, $a_1 = 1/2$; $a_2 = 1$; $a_3 = 5$, $a_4 = 37$, etc. and that $\mid b_n \mid$ increases very fast.

In table 1 we display the values of b_n for $0 \leq n \leq 32$ in a decimal floating-point form.

Table 1.
Coefficients of the asymptotic expansion for Example 3.2.1 with $B \sim 1$

n	b_n
0	1
1	1
2	-1
3	2.5
4	-9.25
5	44.125
6	-255.0625
7	1725.15625
8	-13346.82812
9	116219.00781
10	-1125073.12890
11	11990066.25976
12	-139533491.75488
13	1761075373.29736
14	-23964453644.47290
15	349807707386.89514
16	-5452782881870.17315
17	90409633553220.17190
18	-1588873312064303.24708
19	29503471702357018.18363
20	-577211508360208755.57509
21	11867572245692828645.23748
22	-255825207532718634125.86235
23	5769764361963295673110.67756
24	-135882330167800664588505.92165
25	3335674694299457524244579.27030
26	-85213146545883984655481980.67496
27	2261893281890785361852776703.19528
28	-62297230480972443151737767963.75131
29	1777985066966228736180519624398.19752
30	-52519497829293954909816472745105.34103
31	1603804317704747941436304776925733.8289
32	-50577507144194022234506563696140944.822

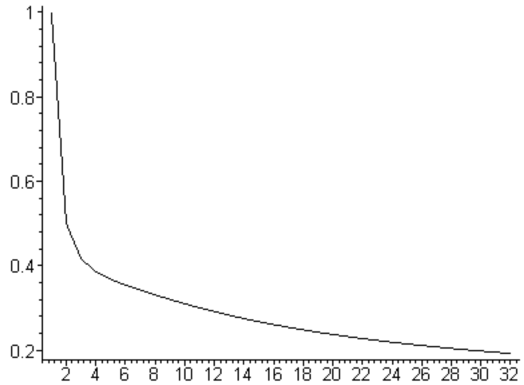

Fig. 2. *The sequence $c_n = b_n/n!$ for the example 3.2.1.*

In order to investigate a possible Gevrey behavior of the expansion, we compare $\mid b_n \mid$ with $n!$. Figure 2 illustrates the behavior of the sequence $c_n := \frac{|b_n|}{n!}$ as a function of n.

Note that c_n decreases as n increases and $c_n \leq 1$ for $0 \leq n \leq 32$. These results suggest that (45) would be a Gevrey expansion of order 1 and of type $\mathcal{A} = 1$ with $\beta = 0$ and $K = 1$.

According to this and in order to obtain one value of $B \sim 1$ that provides an internal layer solution, we employ the strategy of the summation at the least term. In tables 2, 3, 4 and 5 we display both the values of $\mid b_n \varepsilon^n \mid$ and of the partial sums $S_n = \sum_{i=0}^{n} b_i \varepsilon^i$ with $n = 0, 32$ for $\varepsilon = 1/10$, $\varepsilon = 1/15$, $\varepsilon = 1/20$ and $\varepsilon = 1/25$ respectively. To obtain the optimal truncation of the expansion, we look for the index $N_s(\varepsilon)$ that gives the minimum value of $\mid b_n \mid \varepsilon^n$. Note that in each case $N_s(\varepsilon) = 1/\varepsilon$. Then, $B_s(\varepsilon) = S_{N_s(\varepsilon)}$ should be a boundary value for which the solution of (39) has an internal layer. The following table summarizes the values $S_{N_s(\varepsilon)}$ of the optimal truncation of the expansion (45) for different values of ε.

Optimal truncation

ε	$B_s(\varepsilon) = S_{N_s(\varepsilon)}$
1/10	1.0918039465387
1/15	1.0628231164415
1/20	1.0477654694129
1/25	1.0385400109915

Table 2.
Partials sums for Example 3.2.1 with $B \sim 1$ and $\varepsilon = 1/10$

n	$\lvert b_n \varepsilon^n \rvert$	S_n
0	1	1
1	0.1	1.1
2	0.01	1.09
3	2.5E-3	1.0925
4	9.25E-4	1.091575
5	4.4125E-4	1.09201625
6	2.550625E-4	1.0917611875
7	1.72515625E-4	1.091933703125
8	1.3346828125E-4	1.0918002348438
9	1.162190078125E-4	1.0919164538516
10	**1.1250731289063E-4**	**1.0918039465387**
11	1.1990066259766E-4	1.0919238472013
12	1.3953349175488E-4	1.0917843137095
13	1.7610753732974E-4	1.0919604212468
14	2.3964453644473E-4	1.0917207767104
15	3.4980770738689E-4	1.0920705844178
16	5.4527828818702E-4	1.0915253061296
17	9.0409633553220E-4	1.0924294024651
18	15.8887331120643E-4	1.090840529153
19	29.503471702375E-4	1.0937908763232
20	57.721150836021E-4	1.0880187612396
21	118.6757225E-4	1.0998863334854
22	255.8252075E-4	1.0743038127321
23	576.9764362E-4	1.1320014563517
24	0.1358823302	0.99611912618395
25	0.3335674694	1.3296865956139
26	0.8521314655	0.47755513015506
27	2.261893282	2.73944841204584
28	6.229723048	-3.4902746360513
29	17.77985067	14.2895760336108
30	52.51949783	-38.229921795683
31	160.3804318	122.150509974791
32	505.7750714	-383.62456146714

Table 3.
Partials sums for Example 3.2.1 with $B \sim 1$ and $\varepsilon = 1/15$

n	$\lvert b_n \varepsilon^n \rvert$	S_n
0	1	1
1	6.6666667E-2	1.0666666666666
2	4.4444444E-3	1.0622222222222
3	7.40740740E-4	1.0629629629629
4	1.8271604938272E-4	1.0627802469136
5	5.8106995884772E-5	1.0628383539095
6	2.2392318244169E-5	1.0628159615913
7	1.0096936442615E-5	1.0628260585277
8	5.2077244322511E-6	1.0628208508033
9	3.0231231011533E-6	1.0628238739264
10	1.9510489322428E-6	1.0628219228775
11	1.3861739515771E-6	1.0628233090515
12	1.0754329873458E-6	1.0628222336185
13	9.0488122281704E-7	1.0628231384997
14	8.2089933786122E-7	1.0628223176004
15	**7.9884126056811E-7**	**1.0628231164415**
16	8.3015284473412E-7	1.0628222862886
17	9.1762091775335E-7	1.0628232039095
18	1.0750944189681E-6	1.0628221288151
19	1.3308809257707E-6	1.0628234596960
20	1.7358404334284E-6	1.0628217238556
21	2.3792793978564E-6	1.0628241031350
22	3.4192876359967E-6	1.0628206838474
23	5.1411363083269E-6	1.0628258249837
24	8.0718441570004E-6	1.0628177531395
25	1.3209981635132E-5	1.0628309631212
26	2.2497479602686E-5	1.0628084656416
27	3.9811461052269E-5	1.0628482771026
28	7.3099345131622E-5	1.0627751777575
29	1.3908541684822E-4	1.0629142631743
30	2.7389417320249E-4	1.0626403690011
31	5.5759946398589E-4	1.0631979684651
32	1.1722955875193E-3	1.0620256728776

Table 4.
Partials sums for Example 3.2.1 with $B \sim 1$ and $\varepsilon = 1/20$

n	$\lvert b_n \varepsilon^n \rvert$	S_n
0	1	1
1	5E-2	1.05
2	2.5E-3	1.0475
3	3.125E-4	1.0478125
4	5.78125E-5	1.0477546875
5	1.37890625E-5	1.0477684765625
6	3.9853515625E-6	1.0477644912109
7	1.3477783203125E-6	1.0477658389892
8	5.2136047363281E-7	1.0477653176287
9	2.2699024963379E-7	1.0477655446189
10	1.0987042274476E-7	1.0477654347485
11	5.8545245409014E-8	1.0477654932937
12	3.4065793885469E-8	1.0477654592279
13	2.1497502115447E-8	1.0477654807254
14	1.4626741726363E-8	1.0477654660987
15	1.0675284038907E-8	1.0477654767741
16	8.3202863798068E-9	1.0477654684537
17	6.8977076380325E-9	1.0477654753514
18	6.0610706789562E-9	1.0477654692903
19	5.6273406414713E-9	1.0477654749176
20	**5.5047179065724E-9**	**1.0477654694129**
21	5.658899425E-9	1.0477654694130
22	6.099348248E-9	1.0477654689726
23	6.878095108E-9	1.0477654758507
24	8.099218022E-9	1.0477654677515
25	9.941085263E-9	1.0477654776925
26	12.69774832E-9	1.0477654649948
27	16.85241820E-9	1.0477654818472
28	23.20752683E-9	1.0477654586397
29	33.11755260E-9	1.0477654917572
30	48.91259393E-9	1.0477654428446
31	74.68295832E-9	1.0477655175276
32	117.7599355E-9	1.0477653997677

Table 5.
Partials sums for Example 3.2.1 with $B \sim 1$ and $\varepsilon = 1/25$

n	$\lvert b_n \varepsilon^n \rvert$	S_n
0	1	1
1	4E-2	1.04
2	1.6E-3	1.0384
3	1.6E-4	1.03856
4	2.368E-5	1.03853632
5	4.5184E-6	1.0385408384
6	1.044736E-6	1.038539793664
7	2.826496E-7	1.0385400763136
8	8.74697728E-8	1.0385399888438272
9	3.0466115584E-8	1.038540019309942784
10	1.179724681216E-8	1.0385400075126959718
11	5.02899828736E-9	1.0385400125416942592
12	2.340983530405888E-9	1.0385400102007107288
13	1.18183767720361984E-9	1.0385400113825484060
14	6.432909041844944896E-10	1.0385400107392575018
15	3.7560316577886306304E-10	1.0385400111148606676
16	2.3419524149821025026E-10	1.0385400108806654261
17	1.5532256774176996553E-10	1.0385400110359879938
18	1.0918654260485415512E-10	1.0385400109268014512
19	8.1098525491254297097E-11	1.0385400110078999767
20	6.3465076512817336139E-11	1.0385400109444349002
21	5.2194134710444007314E-11	1.0385400109966290349
22	4.5005246457669197150E-11	1.0385400109516237884
23	4.0601107235239790622E-11	1.0385400109922248956
24	3.8247475719371361266E-11	1.0385400109539774199
25	**3.7556358275690575160E-11**	**1.0385400109915337782**
26	3.8376589503111058162E-11	1.0385400109531571887
27	4.0746646965900676895E-11	1.0385400109939038357
28	4.4889885436851424045E-11	1.0385400109490139503
29	5.1246930464380250325E-11	1.0385400110002608808
30	6.0550858458545620191E-11	1.0385400109397100223
31	7.3962419482524624995E-11	1.0385400110136724418
32	9.3299033017516358794E-11	1.0385400109203734088

Table 6.
Comparison of the numerical value for B with
the asymptotic value $B_s(\varepsilon) = 1.0918039465387$ for $\varepsilon = 1/10$

b	$x'(0)$	$t_0(B)$	B
0.25	1.082084998	0.132	1.1276035178
0.5	1.0067379470	0.265	1.0921584115
0.8	1.000335462628	0.352	1.0918754775
1	1.000045399929	0.418	1.0918623507
1.2	1.0000061442124	0.475	1.0918583354
1.3	1.0000022603293	0.508	1.0918536249
1.4	1.0000008315287	0.55	1.0918411828
1.5	1.0000003059023	0.57	1.0918078332
1.6	1.0000001125352	0.612	1.0917099123
1.7	1.0000000413994	0.645	1.0914261442
1.8	1.0000000152299	0.688	1.0906381144
2	1.0000000020612	0.768	1.0814066899

Now we are going to compare these asymptotic results with those obtained numerically. Note that we want to compute solutions of the boundary value problem (39) exhibiting an internal layer for $B \sim 1$, where these values are, a priori, unknown. Our approach to capture such solutions is to solve an initial value problem with initial data at $t = 0$, $x(0) = A = -2$ and $x'(0)$ satisfying (44). That is, the values of $x'(0)$ that we considered, are exponentially small perturbations of $u_L'(0) = 1$.

In tables 6, 7, 8 and 9 we display the results of the numerical experiences for $\varepsilon = 1/10$, $\varepsilon = 1/15$, $\varepsilon = 1/20$ and $\varepsilon = 1/25$ respectively.

For each fixed ε, we have computed several solutions with $x'(0) = 1 + \exp(-b/\varepsilon)$ as b is varied in a range $0.25 \leq b \leq 2$. For each solution we have determined the location t_0 of the shock (which is indicated in the third column) and we have found the value of B by evaluating the solution at time $t = 1$ (see the fourth column).

From these tables we observe that the first four, five or six decimals (depending on ε) in the numerical values of B, do not change for those B such that the corresponding solution exhibits an internal layer. We also remark that the asymptotic value of B that occurs from truncating the expansion to the least term and the numerical results agree to several decimal places of accuracy. Finally we note that the agreement between the asymptotic and the numerical values of B increases as $\varepsilon \to 0$.

Our conclusion is that the optimal truncation of the expansion, provided it is Gevrey of order 1, provides an accurate value of the B's in ξ to these one-side sensitive boundary value problems.

Table 7.
Comparison of the numerical value for B with the asymptotic value $B_s(\varepsilon) = 1.0628231164415$ for $\varepsilon = 1/15$.

b	$x'(0)$	$t_0(B)$	B
0.25	1.023517746	0.159	1.063541018
0.5	1.000553084	0.227	1.062839629
0.8	1.000006144	0.312	1.062822917
1	1.000000306	0.373	1.062822738
1.2	1.000000015	0.437	1.062822728
1.3	1.00000000339	0.471	1.062822724
1.4	1.00000000075826	0.496	1.062822717
1.5	1.00000000016919	0.577	1.062822399
1.6	1.0000000000377513	0.622	1.062819658
1.7	1.0000000000084235	0.639	1.062818266

Table 8.
Comparison of the numerical value for B with the asymptotic value $B_s(\varepsilon) = 1.0477654694129$ for $\varepsilon = 1/20$.

b	$x'(0)$	$t_0(B)$	B
0.25	1.006737947	0.138	1.047922947608043
0.5	1.000045400	0.206	1.047766548350165
0.8	1.000000113	0.293	1.047765489483575
1	1.0000000020611536	0.352	1.047765485310023
1.1	1.0000000002789468093	0.768	1.047707954952546

Table 9.
Comparison of the numerical value for B with the asymptotic value $B_s(\varepsilon) = 1.0385400109915$ for $\varepsilon = 1/25$.

b	$x'(0)$	$t_0(B)$	B
0.25	1.001930454	0.125	1.038576570451286
0.5	1.000003727	0.193	1.038540097072739
0.75	1.0000000071941330303	0.265	1.038540049234752
0.8	1.000000002	0.278	1.038540025574494
0.85	1.0000000005	0.288	1.038540025500596
0.9	1.0000000001691897923	0.557	1.038540021204803

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