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A Riemann Problem for Poroelastic Materials with the Balance Equation for Porosity
Part II

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Abstract

In the first part of this work we considered the properties of the first order approximation with respect to the small parameter β. In this part we present the second order approximation which describes the evolution of amplitudes and, consequently, establishes conditions of stability.

3.7 Equation for the amplitude of discontinuity

In the Part I we have exploited the conditions following from the first order approximation. I order to find the evolution of amplitudes we proceed to investigation of the second order approximation.

We begin with the second approximation to equations (2.8). We obtain the following relations:

$$
\begin{aligned}
&- \overset{\circ}{x}_E \, \widehat{\varrho_0^f} (\widehat{Y_3^f})' + \kappa(n_0)(\widehat{Y_3^{\varrho,f}})' = (\widehat{\varrho_0^f} \, \overset{\circ}{x}_1 + \widehat{\varrho_1^f} \, \overset{\circ}{x}_E - \widehat{\varrho_0^f}\widehat{V_1^f})(\widehat{Y_2^f})' + \Pi_1' \qquad (3.48)\\
&-2(\widehat{\varrho_0^f})^2 \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) \frac{\kappa'(n_0)}{\kappa(n_0)} (\widehat{Y_1^f}\widehat{Y_2^f})' + (\widehat{\varrho_0^f} \, \overset{\circ}{x}_2 + \widehat{\varrho_2^f} \, \overset{\circ}{x}_E - \widehat{\varrho_0^f}\widehat{V_2^f})(\widehat{Y_1^f})' + \mathcal{F}_1\\
&- \varrho_0^f \partial_t (V_1^f + H_1^f z_0) - \kappa(n_0)\partial_x(\varrho_1^f + H_1^{\varrho,f} z_0) - \pi(V_1^f - V_1^s + (H_1^f - H_1^s) z_0) = O(\beta),
\end{aligned}
$$

and

$$
\begin{aligned}
&- \overset{\circ}{x}_E \, (\widehat{Y_3^{\varrho,f}})' + \widehat{\varrho_0^f}(\widehat{Y_3^f})' = \qquad (3.49)\\
&= \frac{1}{\overset{\circ}{x}_E}\Big(\overset{\circ}{x}_E \overset{\circ}{x}_1 - \widehat{\varrho_1^f} \, \overset{\circ}{x}_E - \widehat{\varrho_0^f}\widehat{V_1^f}\Big)(\widehat{Y_2^f})' + 2\frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}(\widehat{Y_1^f}\widehat{Y_2^f})'\\
&- \frac{1}{\overset{\circ}{x}_E}(\widehat{\varrho_0^f}\widehat{V_2^f} + \widehat{\varrho_2^f} \, \overset{\circ}{x}_E)(\widehat{Y_1^f})' + \mathcal{F}_2 - \partial_t(\varrho_1^f + H_1^{\varrho,f} z_0) - \varrho_0^f \partial_x (V_1^f + H_1^f z_0) + O(\beta),
\end{aligned}
$$

where the functions $\mathcal{F}_1, \mathcal{F}_2$ depend on the functions $z_0, \widehat{H_1^f}, V_1^f, \varrho_1^f$ and V_1^s.

We define the background functions $V_1^f, \varrho_1^f, V_1^s, \Phi_1^s$ and ϱ_1^s as a solution of the Cauchy problem in Q_T:

$$
\varrho_0^f \partial_t V_1^f - \kappa(n_0)\partial_x \varrho_1^f + \pi(V_1^f - V_1^s) = 0, \qquad (3.50)
$$

$$
\partial_t \varrho_1^f + \varrho_0^f \partial_x V_1^f = 0,
$$

$$\varrho_0^s \partial_t V_1^s - E(n_0)\partial_x \Phi_1^s - \pi(V_1^f - V_1^s) = 0,$$

$$\partial_t \Phi_1^s - \partial_x V_1^s = 0,$$

and

$$\partial_t \varrho_1^s + \varrho_0^s \partial_x V_1^s = 0,$$

with the initial data

$$V_1^f|_{t=0} = V_f^1(x), \quad V_1^s|_{t=0} = V_f^s(x), \quad \Phi_1^s|_{t=0} = \Phi_f^1(x),$$

$$\varrho_1^f|_{t=0} = \varrho_f^1(x), \quad \varrho_1^s|_{t=0} = \varrho_s^1(x).$$

The functions $H_1^f, H_1^s, H_1^{\Phi,s}, H_1^{\varrho,f}$ and $H_1^{\varrho,s}$ are a solution of the characteristic Cauchy problem in Ω_Γ:

$$\partial_t H_1^{\varrho,f} + \varrho_0^f \partial_x H_1^f = 0, \tag{3.51}$$

$$\varrho_0^f \partial_t H_1^f - \kappa(n_0)\partial_x H_1^{\varrho,f} + \pi(H_1^f - H_1^s) = 0,$$

$$\varrho_0^s \partial_t H_1^s - E(n_0)\partial_x H_1^{\Phi,s} - \pi(H_1^f - H_1^s) = 0,$$

$$\partial_t H_1^{\Phi,s} - \partial_x H_1^s = 0,$$

and

$$\partial_t H_1^{\varrho,s} + \varrho_0^s \partial_x H_1^s = 0,$$

with the initial data on the front Γ_T:

$$H_1^f|_{t=0} = H(x), \quad H_1^{\varrho,f}|_{t=0} = \frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E} H(x), \quad H_1^s = H_1^{\Phi,s}|_{t=0} = H_1^{\varrho,s} = 0,$$

$$\varrho_1^f|_{t=0} = \varrho_f^1(x), \quad \varrho_1^s|_{t=0} = \varrho_s^1(x).$$

This solution can be continued as a solution of the system (3.10) either in the domain $Q_T^+ = \{(x,t) \in Q_T, x > x(t), t \in (0,T)\}$, if $z_0^+ = 1$ (see (3.3)), or in the domain $Q_T^- = \{(x,t) \in Q_T, x < x(t), t \in (0,T)\}$, if $z_0^+ = 0$. The same result follows in the case when (3.40) holds. We have commented already on this solution in the Part I (p. 25-26).

The amplitude $\widehat{H_1^f}$ of jump for v_f follows from the condition of existence of a solution of the characteristic Cauchy problem (3.51). We obtain the following equation

$$2\widehat{\varrho_0^f}\frac{d}{dt}\widehat{H_1^f} + \pi\,\widehat{H_1^f} = 0, \quad \widehat{H_0^F}|_{t=0} = H_1^f(0),$$

which has the solution
$$\widehat{H_1^f}(t) = H_1^f(0)\exp(-\frac{1}{2\varrho_0^f}\,t).$$

Then due to (3.48), (3.49) one obtains for the fast corrections

$$-\overset{\circ}{x}_E\ \widehat{\varrho_0^f}(\widehat{Y_3^f})' + \kappa(n_0)(\widehat{Y_3^{\varrho,f}})' = (\widehat{\varrho_0^f}\ \overset{\circ}{x}_1 + \widehat{\varrho_1^f}\ \overset{\circ}{x}_E - \widehat{\varrho_0^f}\widehat{V_1^f})(\widehat{Y_2^f})' + (\Pi_1)' \tag{3.52}$$
$$-2(\widehat{\varrho_0^f})^2\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}(\widehat{Y_1^f}\widehat{Y_2^f})' + (\widehat{\varrho_0^f}\ \overset{\circ}{x}_2 + \widehat{\varrho_2^f}\ \overset{\circ}{x}_E - \widehat{\varrho_0^f}\widehat{V_2^f})(\widehat{Y_1^f})'$$
$$+\mathcal{F}_1 - \varrho_0^f H_1^f\partial_t z_0 + O(\beta),$$

and

$$-\overset{\circ}{x}_E\ (\widehat{Y_3^{\varrho,f}})' + \widehat{\varrho_0^f}(\widehat{Y_3^f})' = \frac{1}{\overset{\circ}{x}_E}\Big(\ \overset{\circ}{x}_E\overset{\circ}{x}_1 - \widehat{\varrho_1^f}\ \overset{\circ}{x}_E - \widehat{\varrho_0^f}\widehat{V_1^f}\Big)(\widehat{Y_2^f})' \tag{3.53}$$
$$+2\frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}(\widehat{Y_1^f}\widehat{Y_2^f})' - \frac{1}{\overset{\circ}{x}_E}(\widehat{\varrho_0^f}\widehat{V_2^f} - \overset{\circ}{x}_2\ \widehat{\varrho_0^f} + \widehat{\varrho_2^f}\ \overset{\circ}{x}_E)(\widehat{Y_1^f})' + \mathcal{F}_2 - H_1^{\varrho,f}\partial_t z_0 + O(\beta).$$

3.8 Stability of the structure of soliton-kink solutions

The system (3.42), (3.43) has the nontrivial kink solutions $\widehat{Y_3^f}, \widehat{Y_3^{\varrho,f}}$ if the following compatibility condition holds

$$2\Big(\widehat{\varrho_0^f}\ \overset{\circ}{x}_1 - \widehat{\varrho_0^f}\widehat{V_1^f}\Big)(\widehat{Y_2^f})' + \Pi_1'$$
$$-2\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}(\widehat{Y_1^f}\widehat{Y_2^f})' + 2(\widehat{\varrho_0^f}\ \overset{\circ}{x}_2 - \widehat{\varrho_0^f}\widehat{V_2^f})(\widehat{Y_1^f})'$$
$$+\mathcal{F}_1 + \overset{\circ}{x}_E\ \mathcal{F}_2 - 2\varrho_0^f H_1^f\partial_t z_0.$$

Due to (3.19) we can simplify this equation

$$-\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\widehat{H_1^f}(1-2z_0)(\widehat{Y_2^f})' +$$
$$-2\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\widehat{H_1^f}z_0'\widehat{Y_2^f} + \Pi_1' = \mathcal{G}_1, \tag{3.54}$$

where
$$\mathcal{G}_1 = -2(\widehat{\varrho_0^f}\ \overset{\circ}{x}_2 - \widehat{\varrho_0^f}\widehat{V_2^f})(\widehat{Y_1^f})' - \mathcal{F}_1 - \overset{\circ}{x}_E\ \mathcal{F}_2 + 2\varrho_0^f H_1^f\partial_t z_0.$$

Simultaneously, the first approximation to the equation (2.7) yields

$$-\overset{\circ}{x}_E\ \Pi' + (\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s}))(\widehat{Y_2^f})' + \Pi_1 = \mathcal{F}_3, \tag{3.55}$$

where the function $\mathcal{F}_3$ depends on the functions $z_0, \widehat{H_1^f}, V_1^f, \varrho_1^f$ and V_1^s. Let $\widehat{Y_2^f} = H_2^f z_0 + \mathcal{A}_2^f$, i.e. we separate a kink part from the soliton part of the second order approximation. Then we can rewrite the system (3.54), (3.55) in the form

$$\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\widehat{H_1^f}(1-2z_0)(\mathcal{A}_2^f)' + \tag{3.56}$$

$$-2\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\widehat{H_1^f} z_0' \mathcal{A}_2^f + \Pi_1' = \mathcal{K}_1 - 2(\widehat{\varrho_0^f}\ \overset{\circ}{x}_2 - \widehat{\varrho_0^f}\widehat{V_2^f})(\widehat{Y_1^f})',$$

and

$$-\ \overset{\circ}{x}_E\ \Pi' + (\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}))(\mathcal{A}_2^f)' + \Pi_1 = \mathcal{K}_2, \tag{3.57}$$

where the soliton-like functions $\mathcal{K}_1, \mathcal{K}_2$ depend on $V_1^f, \varrho_0^f\ \varrho_1^f, z_0, \widehat{H_1^f}, \widehat{H_2^f}$ and $\overset{\circ}{x}_1, \overset{\circ}{x}_E$. Integrating equation (3.45) one obtains

$$\Pi_1 = \Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\widehat{H_1^f}\mathcal{A}_2^f - \tag{3.58}$$

$$-2\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\widehat{H_1^f}\ z_0\mathcal{A}_2^f + \int_{-\infty}^{\sigma}\mathcal{K}_1 d\sigma_1 - 2(\widehat{\varrho_0^f}\ \overset{\circ}{x}_2 - \widehat{\varrho_0^f}\widehat{V_2^f})\widehat{H_1^f} z_0$$

i.e. the form of equation (3.56) leads to the product of the kink z_0 and the soliton $\mathcal{A}_2^f$ which is again a solition-like structure. This means that the first order approximation of porosity Π_1 is defined by a soliton-like contribution $\mathcal{A}_2^f$ of the second approximation of the velocity v_f.

Since $\Pi_1, \mathcal{A}_2^f$ should be soliton-like functions we get

$$\widehat{\varrho_0^f}\ \overset{\circ}{x}_2 = \widehat{\varrho_0^f}\widehat{V_2^f} + \frac{1}{2}\int_{-\infty}^{\infty}\mathcal{K}_1 d\sigma_1. \tag{3.59}$$

Substituting (3.58) in (3.57) we obtain the so-called variational equation for (3.21). We remind that the first approximation for the propagation velocity of the front had a form of a pendulum. On the other hand the above second approximation is unique. It means that the solution is **stable**. Namely

$$\Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) -\ \overset{\circ}{x}_E\ \widehat{\varrho_0^f}\widehat{H_1^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(1-2z_0)\Big)(\mathcal{A}_2^f)'$$

$$+2\ \overset{\circ}{x}_E\ \widehat{\varrho_0^f}\widehat{H_1^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\mathcal{A}_2^f\ z_0'$$

$$+\widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}(1-2z_0)\mathcal{A}_2^f = \mathcal{K}_3, \tag{3.60}$$

where due to (3.59) $\mathcal{K}_3$ is a soliton-like function. Certainly, the unique solution of the homogeniuos equation (3.60) is z_0'. Therefore the solution of the equation (3.55) can be sought in the form

$$\mathcal{A}_2^f = C(t,\sigma)z_0' + C_0 z_0',$$

where C is an unknown kink function and C_0 is constant. Then for the function C one has the equation:

$$\Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s}) - \overset{\circ}{x}_E\, \widehat{\varrho_0^f}\widehat{H_1^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_E)}{\kappa(n_E)}\Big)(1-2z_0)\Big)\, C' = \frac{\mathcal{K}_3}{z_0'}.$$

Whence

$$C(\sigma,t) = \int_0^\sigma \Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s}) -$$
$$- \overset{\circ}{x}_E\, \widehat{\varrho_0^f}\widehat{H_1^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(1-2z_0)\Big)^{-1}\frac{\mathcal{K}_3}{z_0'}d\sigma_1.$$

It is not difficulty to see that

$$C(\sigma,t) = O\big(|\sigma|\big) \quad |\sigma| \gg 1.$$

Comment 8 *The solution is constructed in an analogous manner for each step of asymptotics. Consequently, we can construct the solution with an arbitrary accuracy. Hence we have proved the Theorem 3.1.*

3.9 Strong discontinuity of v_s

Analogously to section 3.3 we can obtain the results for the case when v_{as}^s, ϱ_{as}^s are the smooth approximations of discontinuous functions and functions v_s, ϱ_s are smooth approximations of some continuous functions. As above we have

$$\tau = \beta^2.$$

Then

Theorem 3.3 *Let the following condition hold:*

$$E'(n_E) \neq 0.$$

Then the propagation velocity $\overset{\circ}{x}_E$ of the front Γ_T and its first correction $\overset{\circ}{x}_1$ are defined by the relations

$$\overset{\circ}{x}{}_E^2 = \frac{E(n_0)}{\widehat{\varrho_0^s}},$$

$$\overset{\circ}{x}_1 = \widehat{V_1^f} + (\widehat{\varrho_0^s})^2\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{E'(n_0)}{E(n_0)}\widehat{H_1^f}.$$

The asymptotic solution (3.1), (3.4), (3.7) exists on the short time interval $(0,T)$ *with any accuracy. The leading part of this asymptotics*

$$v^s_{as} = \beta(V^s_1(x,t) + H^s_1(x,t)z_0(\sigma,t)) + O(\beta^2),$$
$$\Phi^s_{as} = \beta(\Phi^s_1 + H^{\Phi,s}_1 z_0(\sigma,t) + O(\beta^2),$$
$$v^f_{as} = \beta(U^f_1(x,t) + H^f_1(x,t)z_0(\sigma,t)) + O(\beta^2),$$
$$\varrho^s_{as} = \varrho^s_0(x) + \beta(\varrho^s_1(x,t) + H^{\varrho,s}_1 z_0) + O(\beta^2),$$
$$\varrho^f_{as} = \varrho^f_0(x) + \beta(\varrho^f_1(x,t) + H^{\varrho,f}_1(x,t)z_0) + O(\beta^2),$$
$$\Pi_{as} = \Upsilon_0(t,\beta) + \mathcal{A}^p_0(\sigma,t) + O(\beta),$$

satisfies system (2.7)-(2.12) up to the order $O(\beta)$. *Corresponding to (3.2) the leading part of the outer expansion has the form (3.8). The background functions* $V^f_1, V^s_1, \varrho^f_1, \varrho^s_1$ *and* Φ^s_1 *satisfy the Cauchy problem (3.9) in the strip* Q_T.

The functions $H^f_1, H^s_1, H^{\Phi,s}_1, H^{\varrho,f}_1$ *and* $H^{\varrho,s}_1$ *are the solution of the characteristic Cauchy problem (3.10) in a sufficiently small neighborhood* Ω_T *of the front* Γ_T *with the Cauchy data on the front* Γ_T:

$$H^s_1|_{\Gamma_T} = H(x), \quad H^{\Phi,s}|_{\Gamma_T} = -\frac{1}{\overset{\circ}{x}_E}H(x), \quad H^{\varrho,s} = \frac{\widehat{\varrho^s_0}}{\overset{\circ}{x}_E}H(x),$$

$$H^f_1|_{\Gamma_T} = H^{\varrho,f}_1|_{\Gamma_T} = 0.$$

This solution can be continued as a solution of the system (3.9) either in the domain $Q^+_T = \{(x,t) \in Q_T, x > x(t), t \in (0,T)\}$, *if* $z^+_0 = 1$*(see (3.2)), or in the domain* $Q^-_T = \{(x,t) \in Q_T, x < x(t), t \in (0,T)\}$, *if* $z^+_0 = 0$.

Also

$$\Upsilon_0(t,\beta) = \Upsilon^0_0 \exp(-\frac{t}{\beta^2}), \; \Upsilon^0_0 = \mathrm{const}\,.$$

The function

$$\Pi_0 = -(\widehat{\varrho^s_0})^2 \frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) \frac{E'(n_0)}{E(n_0)} (\widehat{H^f_1})^2\, z_0(1-z_0),$$

is soliton-like. The kink-like function z_0 *satisfying (3.2) is a strictly monotonic solution of the nonlinear equation*

$$\Big(\varphi(n_0) + \widehat{\varrho^s_0}\frac{\partial n}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) + (\overset{\circ}{x}_E\; \widehat{\varrho^s_0})^2 \frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0})\widehat{H^s_1}\frac{E'(n_0)}{E(n_0)}(1-2z_0)\Big) z'_0 =$$

$$= (\widehat{\varrho^s_0})^2 \frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) \frac{E'(n_0)}{E(n_0)} \widehat{H^f_1}\; z_0(1-z_0). \tag{3.61}$$

Moreover the function z_0 *increases if*

$$\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) E'(n_0) \widehat{H^f_1} > 0, \quad t \in [0,T],$$

and it decreases if

$$\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})E'(n_0)\widehat{H_1^f} < 0, \quad t \in [0, T].$$

The solution of (3.61) exists if

$$\left|\widehat{H_1^s}\right| < \frac{\left(\varphi(n_0) + \widehat{\varrho_0^s}\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\right)\sqrt{E(n_0)}}{(\widehat{\varrho_0^s})^{3/2}\left|\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})E'(n_0)\right|}. \tag{3.62}$$

The amplitude $\widehat{H_1^s}$ of the strong discontinuity of v_s satisfies the following equation

$$2\widehat{\varrho_0^s}\frac{d}{dt}\widehat{H_1^s} + \pi\,\widehat{H_1^s} = 0, \quad \widehat{H_0^s}|_{t=0} = H_1^s(0), \tag{3.63}$$

so that

$$\widehat{H_1^s}(t) = H_1^s(0)\exp\Big(-\frac{\pi}{2\widehat{\varrho_0^s}}\,t\Big).$$

Remark 3.1 *Let us note that due to the physical condition (1.1) the inequality*

$$\varphi(n_0) - \widehat{\varrho_0^s}\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) > 0,$$

is identically satisfied.

In the case of the opposite sign of the first correction to $\overset{\circ}{x}_E$ of the propagation velocity of the front a Theorem, analogous to the Theorem 3.3 holds with an appropriate change of signs. For completeness we quote it here in the full form.

Theorem 3.4 *Let the following condition hold:*

$$E'(n_0) \neq 0.$$

Then the propagation velocity $\overset{\circ}{x}_E$ of the front Γ_T and its first correction $\overset{\circ}{x}_1$ are defined by the relations

$$\overset{\circ}{x}{}_E^2 = \frac{E(n_0)}{\widehat{\varrho_0^s}},$$

$$\overset{\circ}{x}_1 = \widehat{V_1^f} - (\widehat{\varrho_0^s})^2\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{E'(n_0)}{E(n_0)}\widehat{H_1^f}.$$

The asymptotic solution (3.1), (3.4), (3.7) exists on the short time interval $(0, T)$ with any accuracy. The leading part of this asymptotics

$$v_{as}^s = \beta(V_1^s(x,t) + H_1^s(x,t)z_0(\sigma,t)) + O(\beta^2),$$

$$\Phi^s_{as} = \beta(\Phi^s_1 + H^{\Phi,s}_1 z_0(\sigma,t) + O(\beta^2),$$
$$v^f_{as} = \beta(U^f_1(x,t) + H^f_1(x,t)z_0(\sigma,t)) + O(\beta^2),$$
$$\varrho^s_{as} = \varrho^s_0(x) + \beta(\varrho^s_1(x,t) + H^{\varrho,s}_1 z_0) + O(\beta^2),$$
$$\varrho^f_{as} = \varrho^f_0(x) + \beta(\varrho^f_1(x,t) + H^{\varrho,f}_1(x,t)z_0) + O(\beta^2),$$
$$\Pi_{as} = \Upsilon_0(t,\beta) + \mathcal{A}^p_0(\sigma,t) + O(\beta),$$

satisfies system (2.7)-(2.12) up to the order $O(\beta)$. In accordance to (3.2) the leading part of the outer expansion has the form (3.8). The background functions $V^f_1, V^s_1, \varrho^f_1, \varrho^s_1$ and Φ^s_1 satisfy the Cauchy problem (3.9) in the strip Q_T.

The functions $H^f_1, H^s_1, H^{\Phi,s}_1, H^{\varrho,f}_1$ and $H^{\varrho,s}_1$ are the solution of characteristic Cauchy problem (3.10) in sufficiently small neighborhood Ω_T of the front Γ_T with the Cauchy data on Γ_T:

$$H^s_1|_{\Gamma_T} = H(x), \quad H^{\Phi,s}|_{\Gamma_T} = -\frac{1}{\overset{\circ}{x}_E}H(x), \quad H^{\varrho,s} = \frac{\widehat{\varrho^s_0}}{\overset{\circ}{x}_E}H(x),$$

$$H^f_1|_{\Gamma_T} = H^{\varrho,f}_1|_{\Gamma_T} = 0.$$

This solution can be continued as a solution of the system (3.9) either in the domain $Q^+_T = \{(x,t) \in Q_T, x > x(t), t \in (0,T)\}$, if $z^+_0 = 1$(see (3.2)), or in the domain $Q^-_T = \{(x,t) \in Q_T, x < x(t), t \in (0,T)\}$, if $z^+_0 = 0$.

Also

$$\Upsilon_0(t,\beta) = \Upsilon^0_0 \exp(-\frac{t}{\beta^2}), \ \Upsilon^0_0 = \mathrm{const}\,.$$

The function

$$\Pi_0 = (\widehat{\varrho^s_0})^2 \frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) \frac{E'(n_0)}{E(n_0)} (\widehat{H^f_1})^2 \, z_0(1+z_0),$$

is soliton-like. The kink-like function z_0 is a strictly monotonic solution of the nonlinear equation

$$\Big(\varphi(n_0) + \widehat{\varrho^s_0}\frac{\partial n_e}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) - \overset{\circ}{x}_E \, (\widehat{\varrho^s_0})^2 \frac{\partial n}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0})\widehat{H^s_1}\frac{E'(n_0)}{E(n_0)}(1+2z_0)\Big) z'_0 =$$
$$= -(\widehat{\varrho^s_0})^2 \frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) \frac{E'(n_0)}{E(n_0)} \widehat{H^f_1} \, z_0(1+z_0), \tag{3.64}$$

satisfying the following conditions in infinity

$$\lim_{\sigma\to-\infty} z_0 = -1, \quad \lim_{\sigma\to-\infty} z_0 = 0,$$

or

$$\lim_{\sigma\to-\infty} z_0 = 0, \quad \lim_{\sigma\to-\infty} z_0 = -1,$$

The function z_0 increases if

$$\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho^f_0}, \widehat{\varrho^s_0}) E'(n_0)\widehat{H^f_1} < 0 \quad t \in [0,T],$$

and it decreases if

$$\frac{\partial n_E}{\partial \varrho_s}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})E'(n_0)\widehat{H_1^f} > 0 \quad t \in [0, T].$$

A solution of (3.64) exists if relation (3.62) holds.

The amplitude $\widehat{H_1^s}$ of the strong discontinuity for v_s satisfies equation (3.63), so that

$$\widehat{H_1^s}(t) = H_1^s(0)\exp\Big(-\frac{\pi}{2\varrho_0^s}t\Big).$$

4 Strong discontinuity of porosity.

Now let us investigate the case when Π_{as} is a smooth approximation of the strong discontinuity of porosity. Simultaneously the functions $v_{as}^s, \Phi_{as}^s, \varrho_{as}^s, v_{as}^f$ and ϱ_{as}^f are smooth approximations of weak discontinuous functions.

We consider two cases. In the first one the functions v_{as}^f and ϱ_{as}^f are smooth approximations of weakly discontinuous functions with respect to the small parameter β, and, simultaneously, the functions v_{as}^s, Φ_{as}^s and ϱ_{as}^s are smooth approximations of smooth functions. In the other case the functions $v_{as}^s, \varrho_{as}^s, \Phi_{as}^s$ are smooth approximations of weakly discontinuous functions and the functions v_f, ϱ_f are smooth approximations of smooth functions.

Further we investigate again in details the first case, i.e. when the functions v_{as}^f and ϱ_{as}^f are smooth approximations of weakly discontinuous functions with respect to the small parameter β. For the second case we present results.

4.1 Weak discontinuity of v_f

We proceed to investigate the first case, i.e. a smooth approximation of a strong discontinuity of porosity and weak discontinuities of v_f and ϱ_f. We begin with the derivation of equations specifying the leading part of asymptotics for v_f and Π. These should be equations having weakly discontinuous and kink-like solutions, respectively.

Setting $\tau = \beta$ we investigate the system (2.7), (2.8).

The ansatz for the asymptotic expansion of an unknown function Π in a neighborhood $\Omega_\Gamma = \{x \in R^1, |x - x(t)| < \delta, t \in [0, T]\}$, δ being sufficiently small, of the front $\Gamma_T = \{x \in R^1, x = x(t), t \in [0, T]\}$ has the form:

$$\Pi_{as} = \sum_{j=0}^{N} \beta^j(\Upsilon_j(x, t) + Y_j^p(\sigma, x, t)), \tag{4.1}$$

where

$$Y_j^p = \Pi_j(\sigma, t)) + H_1^p(x, t)z_0(\sigma, t)),\ j \geq 1,$$

and

$$\sigma = (x - x(t))/\beta,$$

is the fast variable. The function $z_0(\sigma,t)$ is a smooth bounded kink-like function and the functions $\Pi_j(\sigma,t), j \geq 1$, are smooth bounded soliton-like functions. The functions Π_j, z_0 are stabilized in the infinity:

$$\sigma^k \frac{d^j}{d\sigma^j}\frac{d^l}{dt^l}\frac{d^k}{dx^k}(y - y^{\pm}) \ \to 0, \ \forall k,j,l \geq 0 \quad \text{if } \sigma \to \pm\infty, \quad \lim_{\sigma\to-\infty} y = 0, \tag{4.2}$$

for $y = z_0$ and $y = \Pi_j$, where

$$z_0^+ = \lim_{\sigma\to\infty} z_0 = 1, \quad \Pi_j^{\pm} = \lim_{\sigma\to\pm\infty} \Pi_j = z_0^- = \lim_{\sigma\to-\infty} = 0. \tag{4.3}$$

Here $y^+(x,t) = \lim_{\sigma\to+\infty} y$.

In order to get a smooth approximation of the discountinuous function Π, we assume that $\Upsilon_j(x,t), H_j^p(x,t)\ H_j^p(x,t), j \geq 1$ are smooth bounded functions and

$$H_1^p(x,t)\Big|_{\Gamma_T} \neq 0.$$

The ansatz for the smooth approximation of weak discontinuities of the velocity v_f and the density ϱ_f has the form:

$$v_{as}^f = V_0^f(x,t) + H_0^f(x,t)z_0 + \sum_{j=1}^{N} \beta^j (V_j^f(x,t) + Y_j^f), \tag{4.4}$$

and

$$\varrho_{as}^f = \varrho_0^f(x,t) + H_0^{\varrho,f}(x,t)z_0 + \sum_{j=1}^{N} \beta^j (\varrho_j^f(x,t) + Y_j^{\varrho,f}),$$

where

$$Y_j^f = H_j^f(x,t)z_0(\sigma,t) + \mathcal{A}_j^f(\sigma,t), \quad Y_j^{\varrho,f} = H_j^{\varrho,f} z_0(\sigma,t) + \mathcal{A}_j^{\varrho,f}(\sigma,t).$$

The functions V_j^f, ϱ_j^f and $H_j^f, H_j^{\varrho,f}, j \geq 1$ are smooth in Ω_Γ. The functions $\mathcal{A}_j^f, \mathcal{A}_j^{\varrho,f}$ are smooth bounded soliton-like functions.

To define the smooth approximation of weak discontinuities one has to require

$$\widehat{H_0^{\varrho,f}} = \widehat{H_0^f} = 0,$$

and

$$\widehat{\partial_x H_0^{\varrho,f}}, \quad \widehat{\partial_x H_0^f} \neq 0.$$

The ansatz for the smooth approximation of the velocity v_s, the deformation Φ_s and the density ϱ_s has the form:

$$v^s_{as} = V^s_0(x,t) + H^s_0(x,t)z_0 + \sum_{j=1}^{N} \beta^j (V^s_j(x,t) + Y^s_j(\sigma,x,t)), \tag{4.5}$$

$$\Phi_{as} = \Phi^s_0(x,t) + H^{\Phi,s}_0(x,t)z_0 + \sum_{j=1}^{N} \beta^j (\Phi^s_j(x,t) + Y^{\Phi,s}_j(\sigma,x,t)),$$

$$\varrho^s_{as} = \varrho^s_0(x,t) + H^{\varrho,s}_0(x,t)z_0 + \sum_{j=1}^{N} \beta^j (\varrho^s_j(x,t) + Y^{\varrho,s}_j),$$

where

$$Y^s_j = H^s_j(x,t)z_0(\sigma,t) + \mathcal{A}^s_j(\sigma,t), \quad Y^{\Phi,s}_j = H^{\Phi,s}_j(x,t)z_0(\sigma,t) + \mathcal{A}^{\Phi,s}_j(\sigma,t),$$
$$Y^{\varrho,s}_j = H^{\varrho,s}_j z_0(\sigma,t) + \mathcal{A}^{\varrho,s}_j(\sigma,t).$$

In this case

$$\widehat{H^s_0} = \widehat{H^{\Phi,s}_0} = \widehat{H^{\varrho,s}_0} = \partial_x \widehat{H^{\varrho,f}_0} = \partial_x \widehat{H^{\Phi,s}_0} = \partial_x \widehat{H^f_0} = 0,$$

and

$$\partial^2_x \widehat{H^s_0}, \ \partial^2_x \widehat{H^{\Phi,s}_0}, \ \partial^2_x \widehat{H^{\varrho,s}_0}, \neq 0.$$

The functions $V^s_j, \Phi^s_j, V^f_j, \varrho^s_j, \varrho^f_j, H^s_j, H^{\Phi,s}_j, H^f_j, H^{\varrho,s}_j$ and $H^{\varrho,f}_j, j \geq 1$ are smooth in Ω_Γ. The functions $\mathcal{A}^s_j, \mathcal{A}^{\Phi,s}_j$ are smooth bounded soliton-like functions.

Then the functions

$$V^s_0 + H^s_0 z_0, \ \Phi^s_0 + H^{\Phi,s}_0 z_0, \ \varrho^s_0 + H^{\varrho,s}_0 z_0, \ V^f_0 + H^f_0 z_0, \ \varrho^f_0 + H^{\varrho,f}_0 z_0,$$

are smooth approximations of C^1-functions.

For the smooth part of asymptotic expansions we obtain the following Cauchy problem in the strip Q_T:

$$\varrho^s_0(\partial_t V^s_0 + V^s_0 \partial_x V^s_0) - E\partial_x \Phi^s_0 - \pi(V^f_0 - V^s_0)) = 0,$$
$$\partial_t \Phi^s_0 - \partial_x V^s_0 = 0,$$
$$\varrho^f_0(\partial_t V^f_0 + V^f_0 \partial_x V^f_0) + \kappa \partial_x \varrho^f_0 + \pi(V^f_0 - V^s_0) = 0,$$

$$\partial_t \varrho^f_0 + \partial_x (V^f_0 \varrho^f_0) = 0,$$

$$\partial_t \varrho^s_0 + \partial_x (V^s_0 \varrho^s_0) = 0, \tag{4.6}$$

with the initial data

$$V^s_0|_{t=0} = V^0_s(x), \ V^f_0|_{t=0} = V^0_f(x), \ \Phi^s_0|_{t=0} = \Phi^0_s(x),$$

$$\varrho_0^s|_{t=0} = \varrho_s^0(x), \quad \varrho_0^f|_{t=0} = \varrho_f^0(x). \tag{4.7}$$

The functions $H_0^s, H_0^f, H_0^{\Phi,s}, H_0^{\varrho,s}$ and $H_0^{\varrho,f}$ can be defined as the solutions of the Cauchy problem in a sufficiently small neighborhood Ω_Γ of Γ_T:

$$\varrho_0^s(\partial_t H_0^s + H_0^s \partial_x H_0^s + \partial_x(V_0^s H_0^s)) - E\partial_x H_0^{\Phi,s} - \pi(H_0^f - H_0^s))$$
$$+H_0^{\varrho,s}(\partial_t(V_0^s + H_0^s) + (V_0^s + H_0^s)\partial_x(V_0^s + H_0^s) = 0,$$
$$\partial_t H_0^{\Phi,s} - \partial_x H_0^s = 0,$$
$$\partial_t \varrho_0^s + \partial_x(H_0^s H_0^{\varrho,s}) + \partial_x(H_0^s \varrho_0^s + V_0^s H_0^{\varrho,s}) = 0, \tag{4.8}$$

and

$$\varrho_0^f(\partial_t H_0^f + H_0^f \partial_x H_0^f + \partial_x(V_0^f H_0^f)) + \kappa \partial_x H_0^{\varrho,f} + \pi(H_0^f - H_0^s)$$
$$+H_0^{\varrho,f}(\partial_t(V_0^f + H_0^f) + (V_0^f + H_0^f)\partial_x(H_0^f + V_0^f)) = 0,$$

$$\partial_t H_0^{\varrho,f} + \partial_x(H_0^f H_0^{\varrho,f}) + \partial_x(H_0^f \varrho_0^f + V_0^f H_0^{\varrho,f}) = 0, \tag{4.9}$$

with the initial data on Γ_T

$$\widehat{H_0^s} = \widehat{H_0^f} = \widehat{H_0^{\Phi,s}} = \widehat{H_0^{\varrho,s}} = \widehat{H_0^{\varrho,f}} = 0. \tag{4.10}$$

Theorem 4.1 *Let*

$$\tau = \beta.$$

Consider a strictly decreasing function $z_0(\sigma)$, satisfying (4.2), (4.3) such that

$$\lim_{\sigma \to \pm\infty} \frac{z_0'}{\exp(-(\pm\, l^\pm)\sigma)} = l^\pm, \quad \int_{-\infty}^{\infty} \sigma\, z_0' d\sigma = 0, \tag{4.11}$$

where $z_0' = \frac{d}{d\sigma} z_0$, the positive constants $l^\pm$ fulfil

$$\max(l^+, l^-) < \frac{1}{\sqrt{\kappa}} \quad \text{if} \quad \overset{\circ}{x}{}_E^2 = \kappa, \tag{4.12}$$

and

$$\overset{\circ}{x} = \frac{d}{dt} x(t) = \overset{\circ}{x}_E + O(\beta).$$

Then an asymptotic solution (4.1), (4.4), (4.5) of (2.7), (2.8) exists on the short time interval $(0, T)$ with any accuracy.

The leading part of this asymptotics

$$\Pi_{as} = \Upsilon_0(x,t) + H_0^p(x,t) z_0 + \mathcal{A}_0^p(\sigma, x, t)) + O(\beta), \tag{4.13}$$

$$v_{as}^s = V_0^s(x,t) + H_0^s(x,t) z_0 + \beta(V_1^s + Y_1^s) + O(\beta^2),$$
$$\Phi_{as} = \Phi_0^s(x,t) + H_0^{\Phi,s}(x,t) z_0 + \beta(\Phi_1^s + Y_1^{\Phi,s}) + O(\beta^2),$$

$$
\begin{aligned}
v^f_{as} &= V^f_0(x,t) + H^f_0(x,t)z_0 + \beta(V^f_1 + Y^f_1) + O(\beta^2),\\
\varrho^s_{as} &= \varrho^s_0(x,t) + H^{\varrho,s}_0(x,t)z_0 + \beta(\varrho^s_1 + Y^{\varrho,s}_1) + O(\beta^2),\\
\varrho^f_{as} &= \varrho^f_0(x,t) + H^{\varrho,f}_0(x,t)z_0 + \beta(\varrho^f_1 + Y^{\varrho,f}_1) + O(\beta^2),
\end{aligned}
$$

satisfies the system (2.7), (2.8) up to the order $O(\beta)$.

Here

$$
\Upsilon_0(x,t) = -\partial_x(V^f_0 - V^s_0), \quad H^p_0(x,t) = -\partial_x(H^f_0 - H^s_0). \tag{4.14}
$$

The equilibrium velocity $\overset{\circ}{x}_E$ of the front Γ_T satisfies the following equation:

$$
(\widehat{V^f_0} - \overset{\circ}{x}_E)^2 = \kappa \quad \textit{and} \quad \widehat{V^s_0} - \overset{\circ}{x}_E \neq 0. \tag{4.15}
$$

The functions

$$
\mathcal{A}^p_0 = \exp\Big(-\frac{\sigma}{\widehat{V^s_0} - \overset{\circ}{x}_E}\Big)\int_{-\infty}^{\sigma} \exp\Big(\frac{\sigma_1}{\widehat{V^s_0} - \overset{\circ}{x}_E}\Big) F^p_0 d\sigma_1 \quad \textit{if } \widehat{V^s_0} - \overset{\circ}{x} > 0,
$$

and

$$
\mathcal{A}^p_0 = -\exp\Big(-\frac{\sigma}{\hat{V}^s_0 - \overset{\circ}{x}_E}\Big)\int_{\sigma}^{\infty} \exp\Big(\frac{\sigma_1}{\widehat{V^s_0} - \overset{\circ}{x}_E}\Big) F^p_0 d\sigma_1 \quad \textit{if } \widehat{V^s_0} - \overset{\circ}{x} < 0,
$$

are soliton-like functions, where

$$
F^p_0 = -\widehat{H^p_0}\, z_0' + \frac{\widehat{\partial_x H^f_0}}{\overset{\circ}{x}_E - \widehat{V^s_0}} \sigma\, z_0'.
$$

The background functions $V^f_0, V^s_0, \Phi_0, \varrho^s_0$ and ϱ^f_0 are the solutions of the Cauchy problem (4.6), (4.7) in the strip Q_T. The functions $H^s_0, H^f_0, H^{\Phi,s}_0, H^{\varrho,s}_0$ and $H^{\varrho,f}_0$ are the solutions of characteristic Cauchy problem (4.8)-(4.10) in sufficiently small neighborhood $\Omega_\Gamma = \{(x,t), |x - x(t)| < \delta, t \in [0,T]\}$ of the front Γ_T.

The amplitude of jump $\widehat{\partial_x H^f_0}(t)$ of a weak discontinuity of v_f is the solution of the nonlinear problem:

$$
\frac{d}{dt}\widehat{\partial_x H^f_0} + (\widehat{\partial_x H^f_0})^2 + \Big[\frac{\pi}{2\widehat{\varrho^f_0}} + \widehat{\partial_x V^f_0} - \tag{4.16}
$$

$$
-\frac{1}{2}(\widehat{V^f_0} - \overset{\circ}{x}_E)\frac{\widehat{\partial_x \varrho^f_0}}{\widehat{\varrho^f_0}} - \frac{\pi(\widehat{V^f_0} - \widehat{V^s_0})}{2\widehat{\varrho^f_0}(\widehat{V^f_0} - \overset{\circ}{x}_E)}\Big]\widehat{\partial_x H^f_0} = 0.
$$

The function $\widehat{H^f_1}(t)$ is defined by the nonhomogeneuos linear equation on Γ_T:

$$
\frac{d}{dt}\widehat{H^f_1} + \Big\{\frac{\pi}{2\widehat{\varrho^f_0}} + q_f(\widehat{V^f_0}, \widehat{\varrho^f_0}, \widehat{\partial_x H^f_0})\Big\}\widehat{H^f_1} = g_f(\widehat{\varrho^f_0}, \widehat{V^f_0}, \widehat{\partial_x V^f_0}\widehat{\varrho^f_1}, \widehat{V^f_1}), \tag{4.17}
$$

where the function q_f depends on the functions $V^f_0, \partial_x H^f_0, \varrho^f_0$, while the function g_f depends on the functions $\varrho^f_0, V^f_0, \varrho^f_1, V^f_1$ and their first derivatives.

Remark 4.1 *Let us note that the solution of (4.9)- (4.11) exists in* Ω_Γ *if* $\delta < \delta_0$ *is sufficiently small. This solution can be extended to the region* $Q_T^- = \{(x,t), x < x(t), t \in (0,T)\}$ *such that it remains to be a soliton for (4.9), (4.10).*

As in the preceding cases the proof of the theorem 4.1 is devided into several steps. First we obtain the equation for the propagation velocity of the front Γ_T.

4.2 Propagation velocity of the front.

Lemma 4.1 *Weak discontinuities of* v_f *and* v_s *cannot exist simultaneously. The propagation velocity of the front of the weak discontinuity of* v_f *satisfies the equation*

$$(\widehat{V_0^f} - \overset{\circ}{x}_E)^2 = \kappa. \tag{4.18}$$

and the propagation velocity of the front of the weak discontinuity of v_s *satisfies the equation*

$$(\widehat{V_0^s} - \overset{\circ}{x})^2 = \frac{E}{\widehat{\varrho_0^s}}.$$

Let us denote

$$\Upsilon^N = \beta^\alpha \sum_{j=1}^{N} \beta^{j-1}\Upsilon_j, \quad \mathcal{Y}_N^p = \beta^\alpha \sum_{j=1} \beta^{j-1} Y_j^p, \quad x_N^\beta(t) = \sum_{j=0}^{N} \beta^j x_j(t). \tag{4.19}$$

$$\mathcal{V}_N^s = \sum_{j=0}^{N} \beta^j V_j^s, \;\; \mathcal{V}_N^f = \sum_{j=0}^{N} \beta^j V_j^f, \;\; \mathcal{Y}_N^s = \sum_{j=1} \beta^j Y_j^s, \;\; \mathcal{Y}_N^f = \sum_{j=1} \beta^j Y_j^f$$

Now let us substitute (4.1), (4.4), (4.5) in the first equation of (2.7). Using standard procedure one gets the following relation specifying the functions Υ^N and $\mathcal{Y}_N^p$:

$$-x_\beta^N \frac{\partial}{\partial\sigma}\mathcal{Y}_N^p + \mathcal{Y}_N^p + (\mathcal{V}_N^s + \mathcal{Y}_N^s)\frac{\partial}{\partial\sigma}\mathcal{Y}_N^p + \frac{\partial}{\partial\sigma}(\mathcal{Y}_N^f - \mathcal{Y}_N^s) + \partial_t(\Upsilon^N + \mathcal{Y}_N^p)$$

$$+(\mathcal{V}_N^s + \mathcal{Y}_N^s)\partial_x \mathcal{Y}_N^p + \partial_x(\mathcal{V}_N^f - \mathcal{V}_N^s + \mathcal{Y}_N^f - \mathcal{Y}_N^s) + \Upsilon^N = \beta^{1+N} f_N^p(\sigma, x, t), \tag{4.20}$$

where $f_N^p \in C^\infty(\Omega_\Gamma \times R^1 \times [0,T])$ is some function bounded in the norm $C(\Omega_\Gamma)$. It should be noted that in (4.20) the slow variables x and t and the fast variable σ are assumed, as before, to be independent.

Setting coefficients of β^j equal to zero one gets the system of recurrent equations for the definition of the functions $\Upsilon_j(x,t), H_j^p(x,t)$ and $\mathcal{A}_j^p(\sigma,t)$. The lowest approximation leads to the following relation:

$$(\widehat{V_0^s} - \overset{\circ}{x}_E)(\mathcal{A}_0^p)' + \mathcal{A}_0^p = -(\widehat{V_0^s} - \overset{\circ}{x}_E)\widehat{H_0^p} z_0' - \Upsilon_1 - H_0^p z_0 \tag{4.21}$$

$$-\partial_x(V_0^f - V_0^s) - \partial_x(H_0^f - H_0^s)z_0 - (\widehat{\partial_x H_0^f} - \widehat{\partial_x H_0^s})\sigma\, z_0' + O(\beta).$$

This relation is fulfilled if

$$\Upsilon_0 = -\partial_x (V_0^f - V_0^s), \quad H_0^p = -\partial_x (H_0^f - H_0^s),$$

and

$$(\hat{V}_0^s - \overset{\circ}{x}_E)(\mathcal{A}_0^p)' + \mathcal{A}_0^p = -(\hat{V}_0^s - \overset{\circ}{x}_E)\hat{H}_0^p z_0' - (\partial_x \hat{H}_0^f - \partial_x \hat{H}_0^s)\sigma\, z_0' = F_0^p, \tag{4.22}$$

where

$$F_0^p = -(\hat{V}_0^s - \overset{\circ}{x}_E)\hat{H}_0^p z_0' - (\partial_x \hat{H}_0^f - \partial_x \hat{H}_0^s)\sigma\, z_0'.$$

Then the soliton-like solution $\mathcal{A}_0^p$ of (4.22) has the form:

$$\mathcal{A}_0^p = \exp(-\frac{\sigma}{\hat{V}_0^s - \overset{\circ}{x}_E}) \int_{-\infty}^{\sigma} \exp(\frac{\sigma_1}{\hat{V}_0^s - \overset{\circ}{x}_E}) \frac{F_0^p}{\hat{V}_0^s - \overset{\circ}{x}_E} d\sigma_1 \quad \text{if } \hat{V}_0^s - \overset{\circ}{x} > 0,$$

or

$$\mathcal{A}_0^p = -\exp(-\frac{\sigma}{\hat{V}_0^s - \overset{\circ}{x}_E}) \int_{\sigma}^{\infty} \exp(\frac{\sigma_1}{\hat{V}_0^s - \overset{\circ}{x}_E}) \frac{F_0^p}{\hat{V}_0^s - \overset{\circ}{x}_E} d\sigma_1 \quad \text{if } \hat{V}_0^s - \overset{\circ}{x} < 0.$$

Remark 4.2 *Under conditions (4.11), (4.12) the following estimate holds true:*

$$\frac{\mathcal{A}_0^p}{z_0'} = O(|\sigma|) \quad |\sigma| \geq 1.$$

>From the subsequent equations of (2.7) one obtains in the lowest approximation:

$$\begin{gathered}
\widehat{\varrho_0^s}(\widehat{V_0^s} - \overset{\circ}{x}_E)(\widehat{Y_1^s})' - E(\widehat{Y_1^{\Phi,s}})' = \\
-\Big(\widehat{\varrho_0^s}(\widehat{V_0^s} - \overset{\circ}{x}_E)\partial_x \widehat{H_0^s} - E\partial_x \widehat{H_0^{\Phi,s}}\Big)\sigma\, z_0' \\
-(\varrho_0^s + H_0^{\varrho,s} z_0)\Big((\partial_t (V_0^s + H_0^s z_0) + (V_0^s + H_0^s z_0)\partial_x (V_0^s + H_0^s z_0)\Big) \\
-E\partial_x (\Phi_0 + H_0^{\Phi,s} z_0) - \pi(V_0^f - V_0^s + (H_0^f - H_0^s) z_0) + O(\beta),
\end{gathered} \tag{4.23}$$

$$\begin{gathered}
-\overset{\circ}{x}_E\, (\widehat{Y_1^{\Phi,s}})' - (\widehat{Y_1^s})' = \\
(\overset{\circ}{x}_E\, \partial_x \widehat{H_0^{\Phi,s}} + \partial_x \widehat{H_0^s})\sigma z_0' - \partial_t (\Phi_0 + H_0^{\Phi,s} z_0) - \partial_x (V_0^s + H_0^s z_0) + O(\beta),
\end{gathered} \tag{4.24}$$

$$\begin{gathered}
(\widehat{V_0^s} - \overset{\circ}{x}_E)(\widehat{Y_1^{\varrho,s}})' + (\widehat{\varrho_0^s}\widehat{Y_1^s})' = \\
-\Big((\widehat{V_0^s} - \overset{\circ}{x}_E)\partial_x \hat{H}_0^{\varrho,s} + \partial_x (\widehat{\varrho_0^s H_0^s})\Big)\sigma z_0' \\
-\partial_t (\varrho_0^s + H_0^{\varrho,s} z_0) - \partial_x ((\varrho_0^s + H_0^{\varrho,s})(V_0^s + H_0^s z_0)) + O(\beta),
\end{gathered} \tag{4.25}$$

and

$$\widehat{\varrho_0^f}(\widehat{V_0^f} - \overset{\circ}{x}_E)(\widehat{Y_1^f})' + \kappa(\widehat{Y_1^{\varrho,s}})' =$$
$$-\Big(\widehat{\varrho_0^f}(\widehat{V_0^f} - \overset{\circ}{x}_E)\partial_x\widehat{H_0^f} - \kappa\partial_x\widehat{H_0^{\varrho,f}}\Big)\sigma\, z_0'$$
$$-(\varrho_0^f + H_0^{\varrho,f}z_0)\Big((\partial_t(V_0^f + H_0^f z_0) + (V_0^f + H_0^f z_0)\partial_x(V_0^f + H_0^f z_0)\Big)$$
$$-\kappa\partial_x(\varrho_0^f + H_0^{\varrho,f}z_0) + \pi(V_0^f - V_0^s + (H_0^f - H_0^s)z_0) + O(\beta), \tag{4.26}$$

$$(\widehat{V_0^f} - \overset{\circ}{x}_E)(\widehat{Y_1^{\varrho,f}})' - \widehat{\varrho_0^f}(\widehat{Y_1^f})' =$$
$$-\Big((\hat{V}_0^f - \overset{\circ}{x}_E)\partial_x\widehat{H_0^{\varrho,f}} + \partial_x(\widehat{\varrho_0^f H_0^f})\Big)\sigma z_0'$$
$$-\partial_t(\varrho_0^f + H_0^{\varrho,f}z_0) + \partial_x((\varrho_0^f + H_0^{\varrho,f})(V_0^f + H_0^f z_0)) + O(\beta). \tag{4.27}$$

First we define the smooth background functions $V_0^f, V_0^s, \Phi_0, \varrho_0^s$ and ϱ_0^f as the solution of the Cauchy problem in the strip Q_T:

$$\varrho_0^s(\partial_t V_0^s + V_0^s\partial_x V_0^s) - E\partial_x\Phi_0^s - \pi(V_0^f - V_0^s)) = 0,$$
$$\partial_t\Phi_0^s - \partial_x V_0^s = 0,$$
$$\varrho_0^f(\partial_t V_0^f + V_0^f\partial_x V_0^f) + \kappa\partial_x\varrho_0^f + \pi(V_0^f - V_0^s) = 0,$$

$$\partial_t\varrho_0^f + \partial_x(V_0^f\varrho_0^f) = 0,$$

$$\partial_t\varrho_0^s + \partial_x(V_0^s\varrho_0^s) = 0, \tag{4.28}$$

with the initial data

$$V_0^s|_{t=0} = V_s^0(x),\ V_0^f|_{t=0} = V_f^0(x),\ \Phi_0^s|_{t=0} = \Phi_s^0(x),$$
$$\varrho_0^s|_{t=0} = \varrho_s^0(x),\ \ \varrho_0^f|_{t=0} = \varrho_f^0(x). \tag{4.29}$$

Then the functions $H_0^s, H_0^f, H_0^{\Phi,s}, H_0^{\varrho,s}$ and $H_0^{\varrho,f}$ can be defined as the solutions of the Cauchy problem in a sufficiently small neighborhood Ω_Γ of Γ_T:

$$\varrho_0^s(\partial_t H_0^s + H_0^s\partial_x H_0^s + \partial_x(V_0^s H_0^s)) - E\partial_x H_0^{\Phi,s} - \pi(H_0^f - H_0^s))$$

$$+H_0^{\varrho,s}(\partial_t(V_0^s + H_0^s) + (V_0^s + H_0^s)\partial_x(V_0^s + H_0^s) = 0,$$

$$\partial_t H_0^{\Phi,s} - \partial_x H_0^s = 0,$$
$$\partial_t H_0^{\varrho,s} + \partial_x(H_0^s\varrho_0^s) + \partial_x(H_0^s H_0^{\varrho,s} + V_0^s H_0^{\varrho,s}) = 0, \tag{4.30}$$

and

$$\varrho_0^f(\partial_t H_0^f + H_0^f\partial_x H_0^f + \partial_x(V_0^f H_0^f)) + \kappa\partial_x H_0^{\varrho,f} + \pi(H_0^f - H_0^s)$$
$$+H_0^{\varrho,f}(\partial_t(V_0^f + H_0^f) + (V_0^f + H_0^f)\partial_x(H_0^f + V_0^f)) = 0,$$

$$\partial_t H_0^{\varrho,f} + \partial_x(H_0^f\varrho_0^f) + \partial_x(H_0^f H_0^{\varrho,f} + V_0^f H_0^{\varrho,f}) = 0, \tag{4.31}$$

with the initial data on Γ_T

$$\hat{H}_0^s = \hat{H}_0^f = \hat{H}_0^{\Phi,s} = \hat{H}_0^{\varrho,s} = \hat{H}_0^{\varrho,f} = 0. \tag{4.32}$$

As it is well-known (4.30)-(4.32) has a nontrivial solution if this problem is characteristic, i.e. either

$$(\hat{V}_0^f - \overset{\circ}{x}_E)^2 = \kappa. \tag{4.33}$$

if

$$\widehat{\partial_x H_0^f},\ \widehat{\partial_x H_0^{\varrho,f}} \neq 0.$$

or

$$(\hat{V}_0^s - \overset{\circ}{x}_E)^2 = E/\hat{\varrho}_0^s \tag{4.34}$$

if

$$\widehat{\partial_x H_0^s},\ \widehat{\partial_x H_0^{\varrho,s}},\ \widehat{\partial_x H_0^{\Phi,s}} \neq 0.$$

Due to the physical interpretation of speeds of P1- and P2- waves the relations (4.33), (4.34) cannot hold simultaneuosly.

The solution of (4.30)-(4.32) can be extended to the region Q_T^+ so that it remains a solution of the above equations for a sufficiently small $t \leq T_0 \leq T$. Any solution of (4.30)-(4.32) has the following properties:

Lemma 4.2 *For the characteristic Cauchy problem (4.30)-(4.32) on the front the amplitude of the jumps $\widehat{\partial_x H_0^f}$ of the weak discontinuity of v_f satisfies the following nonlinear equation:*

$$\frac{d}{dt}\widehat{\partial_x H_0^f} + (\widehat{\partial_x H_0^f})^2 + \Big[\frac{\pi}{2\widehat{\varrho_0^f}} + \widehat{\partial_x V_0^f} -$$

$$-\frac{1}{2}(\widehat{V_0^f} - \overset{\circ}{x}_E)\frac{\widehat{\partial_x \varrho_0^f}}{\widehat{\varrho_0^f}} - \frac{\pi(\widehat{V_0^f} - \widehat{V_0^s})}{2\widehat{\varrho_0^f}(\widehat{V_0^f} - \overset{\circ}{x}_E)}\Big]\widehat{\partial_x H_0^f} = 0, \quad \widehat{\partial_x H_0^f}|_{t=0} = \partial_x H_f^0(0) \tag{4.35}$$

and the following relations on Γ_T hold:

$$\widehat{\partial_x H_0^{\varrho,f}} = -\frac{\widehat{\varrho_0^f}}{\widehat{V_0^f} - \overset{\circ}{x}_E}\widehat{\partial_x H_0^f}, \quad \widehat{\partial_x H_0^s} = \widehat{\partial_x H_0^{\Phi,s}} = \widehat{\partial_x H_0^{\varrho,s}} = 0,$$

$$\widehat{H_1^{\varrho,f}} = -\frac{\widehat{\varrho_0^f}}{\widehat{V_0^f} - \overset{\circ}{x}_E}\widehat{H_1^f} \neq 0, \quad \widehat{H_1^s} = \widehat{H_1^{\Phi,s}} = \widehat{H_1^{\varrho,s}} = 0,$$

where the function $\widehat{H_1^f}$ *satisfies the following equation*

$$\frac{d}{dt}\widehat{H_1^f} + \Big\{\frac{\pi}{2\widehat{\varrho_0^f}} + q_f(\widehat{V_0^f}, \widehat{\varrho_0^f}, \widehat{\partial_x H_0^f})\Big\}\widehat{H_1^f} = g_f(\widehat{\varrho_0^f}, \widehat{V_0^f}, \widehat{\partial_x V_0^f}, \widehat{\varrho_1^f}, \widehat{V_1^f}), \tag{4.36}$$

with the function q_f *dependent on the functions* $V_0^f, \partial_x H_0^f, \varrho_0^f$, *and the function* g_f *dependent on the functions* $\varrho_0^f, V_0^f, \varrho_1^f, V_1^f$ *and their first derivatives.*

Proof Due to (4.32) it follows from (4.30), (4.31) on Γ_T that

$$\widehat{\varrho_0^f}\widehat{\partial_x H_0^f} + (\widehat{V_0^f} - \mathring{x}_E)\widehat{\partial_x H_0^{\varrho,f}} = 0, \tag{4.37}$$

and

$$\widehat{\partial_x H_0^s} = \widehat{\partial_x H_0^{\Phi,s}} = \widehat{\partial_x H_0^{\varrho,s}} = 0.$$

Differentiating (4.30), (4.31) with respect to x one gets the following relations on Γ_T:

$$\widehat{\varrho_0^f}\frac{d}{dt}\widehat{\partial_x H_0^f} + (\widehat{V_0^f} - \mathring{x}_E)\Big[\widehat{\varrho_0^f}\widehat{\partial_x\partial_x H_0^f} + (\widehat{V_0^f} - \mathring{x}_E)\widehat{\partial_x\partial_x H_0^{\varrho,f}}\Big]$$
$$+\Big\{\pi + (\widehat{V_0^f} - \mathring{x}_E)\widehat{\partial_x\varrho_0^f} - \kappa\frac{\widehat{\partial_x\varrho_0^f}}{\widehat{\varrho_0^f}} - \pi\frac{\widehat{V_0^f} - \widehat{V_0^s}}{\widehat{\varrho_0^f}}\Big\}\widehat{\partial_x H_0^f} = 0,$$

and

$$\frac{d}{dt}\widehat{\partial_x H_0^{\varrho,f}} + \widehat{\varrho_0^f}\widehat{\partial_x\partial_x H_0^f} + (\widehat{V_0^f} - \mathring{x}_E)\widehat{\partial_x\partial_x H_0^{\varrho,f}}$$
$$-\frac{2\widehat{\varrho_0^f}}{\widehat{V_0^f} - \mathring{x}_E}\Big(\widehat{\partial_x H_0^f}\Big)^2 + 2\Big(\widehat{\partial_x\varrho_0^f} - \frac{\widehat{\varrho_0^f}}{\widehat{V_0^f} - \mathring{x}_E}\widehat{\partial_x V_0^f}\Big)\widehat{\partial_x H_0^f} = 0.$$

Whence

$$\widehat{\varrho_0^f}\frac{d}{dt}\widehat{\partial_x H_0^f} - (\widehat{V_0^f} - \mathring{x}_E)\frac{d}{dt}\widehat{\partial_x H_0^{\varrho,f}} + 2\widehat{\varrho_0^f}\Big(\widehat{\partial_x H_0^f}\Big)^2 \tag{4.38}$$
$$+\Big\{\pi - (\widehat{V_0^f} - \mathring{x}_E)\widehat{\partial_x\varrho_0^f} + 2\widehat{\varrho_0^f}\widehat{\partial_x V_0^f} - \kappa\frac{\widehat{\partial_x\varrho_0^f}}{\widehat{\varrho_0^f}} - \frac{\pi(\widehat{V_0^f} - \widehat{V_0^s})}{\widehat{\varrho_0^f}}\Big\}\widehat{\partial_x H_0^f} = 0.$$

>From (4.37) one has

$$-(\widehat{V_0^f} - \mathring{x}_E)\frac{d}{dt}\widehat{\partial_x H_0^{\varrho,f}} = \widehat{\varrho_0^f}\frac{d}{dt}\widehat{\partial_x H_0^f} + \widehat{\partial_x\varrho_0^f}\widehat{\partial_x H_0^f}.$$

The statement of Lemma 4.2 follows from (4.38).

Taking into account (4.28), (4.30), (4.32) and Lemma 3.1, one gets the following estimates:

$$\partial_t(\varrho_0^f + H_0^{\varrho,f}z_0) + \partial_x((\varrho_0^f + H_0^{\varrho,f})(V_0^f + H_0^f z_0)) = O(\beta),$$

$$(\varrho_0^f + H_0^f z_0)\Big(\partial_t(V_0^f + H_0^f z_0) + (V_0^f + H_0^f z_0)\partial_x(V_0^f + H_0^f z_0)\Big)$$
$$+\kappa(n_0)\partial_x(\varrho_0^f + H_0^f z_0) + \pi(V_0^f - V_0^s + (H_0^f - H_0^s)z_0) = O(\beta),$$

$$(\varrho_0^s + H_0^{\varrho,s} z_0)\Big((\partial_t(V_0^s + H_0^s z_0) + (V_0^s + H_0^s z_0)\partial_x(V_0^s + H_0^s z_0)\Big)$$
$$-E\partial_x(\Phi_0 + H_0^{\Phi,s} z_0) - \pi(V_0^f - V_0^s + (H_0^f - H_0^s)z_0) = O(\beta),$$

$$\partial_t(\Phi_0 + H_0^{\Phi,s} z_0) - \partial_x(V_0^s + H_0^s z_0) = O(\beta),$$

$$\partial_t(\varrho_0^s + H_0^{\varrho,s} z_0) + \partial_x((\varrho_0^s + H_0^{\varrho,s})(V_0^s + H_0^s z_0)) = O(\beta).$$

Thus relations (4.23)-(4.27) hold if

$$\widehat{\varrho_0^s}(\widehat{V_0^s} - \overset{\circ}{x}_E)(\mathcal{A}_1^s)' - E(\mathcal{A}_1^{\Phi,s})' =$$
$$-\Big(\widehat{\varrho_0^s}(\widehat{V_0^s} - \overset{\circ}{x}_E)\widehat{H_1^s} - E\widehat{H_1^{\Phi,s}}\Big)z_0', \tag{4.39}$$

$$-\overset{\circ}{x}_E\,(\mathcal{A}_1^{\Phi,s})' - (\mathcal{A}_1^s)' = -\Big((\overset{\circ}{x}_E\,\widehat{H_1^{\Phi,s}} - \widehat{H_1^f}\Big)z_0',$$

$$(\widehat{V_0^s} - \overset{\circ}{x}_E)(\mathcal{A}_1^{\varrho,s})' + \widehat{\varrho_0^s}(\mathcal{A}_1^s)' = -\Big((\widehat{V_0^s} - \overset{\circ}{x}_E)\widehat{H_0^{\varrho,s}} - \widehat{\varrho_0^s}\widehat{H_1^s}\Big)z_0',$$

and

$$\widehat{\varrho_0^f}(\widehat{V_0^f} - \overset{\circ}{x}_E)(\mathcal{A}_1^f)' + \kappa(\mathcal{A}_1^{\varrho,s})' = -\Big(\widehat{\varrho_0^f}(\widehat{V_0^f} - \overset{\circ}{x}_E)\hat{H}_1^f + \kappa\widehat{H_1^{\varrho,f}}\Big)z_0',$$

$$(\widehat{V_0^f} - \overset{\circ}{x}_E)(\mathcal{A}_1^{\varrho,f})' - \widehat{\varrho_0^f}(\mathcal{A}_1^f)' = -\Big((\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{H_1^{\varrho,f}} + \widehat{\varrho_0^f}\widehat{H_1^f})\Big)z_0'.$$

Due to (4.2) the solvability condition of the system (4.39) in the class of soliton-like functions is equivalent to the existence of a nontrivial solution $\widehat{H_1^f}, \widehat{H_1^{\varrho,f}}, \widehat{H_1^s}, \widehat{H_1^{\varrho,s}}, \widehat{H_1^{\Phi,s}}$ for the following system

$$-\widehat{\varrho_0^s}(\widehat{V_0^s} - \overset{\circ}{x}_E)\widehat{H_1^s} + E\widehat{H_1^{\Phi,s}} = 0, \tag{4.40}$$

$$\overset{\circ}{x}_E\,\widehat{H_1^{\Phi,s}} - \widehat{H_1^f} = 0,$$

$$(\widehat{V_0^s} - \overset{\circ}{x}_E)\widehat{H_0^{\varrho,s}} - \widehat{\varrho_0^s}\widehat{H_1^s} = 0,$$

and

$$\widehat{\varrho_0^f}(\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{H_1^f} + \kappa\widehat{H_1^{\varrho,f}} = 0, \tag{4.41}$$

$$(\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{H_1^{\varrho,f}} + \widehat{\varrho_0^f}\widehat{H_1^f} = 0.$$

Condition 4.1 *Let*

$$\widehat{V_0^s} - \overset{\circ}{x}_E \neq 0 \quad \textit{if} \quad (\widehat{V_0^f} - \overset{\circ}{x}_E)^2 = \kappa.$$

Then the algebraic noncharacteristic system (4.40) has only the trivial solution

$$\widehat{H_1^s} = \widehat{H_1^{\Phi,s}} = \widehat{H_1^{\varrho,s}} = 0.$$

The characteristic algebraic system (4.41) results in the relation:

$$\widehat{H_1^{\varrho,f}} = -\frac{\widehat{\varrho_0^f}}{\widehat{V_0^f} - \overset{\circ}{x}_E}\widehat{H_1^f}.$$

Whence we get the nontrivial soliton-like solutions of (4.39)

$$\mathcal{A}_1^{\varrho,f} = -\frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}\mathcal{A}_1^f,\ \mathcal{A}_1^s = \mathcal{A}_1^{\Phi,s} = \mathcal{A}_1^{\varrho,s} = 0.$$

Comment 9 *For the velocity v_f the smooth approximation of weak discontinuities of order $O(1)$ genarates a smooth approximation of the strong discontinuity of order $O(\beta)$. For v_s the smooth approximation of the weak discontinuity of order $O(\beta)$ may or may not appear.*

4.3 Correction of the propagation velocity of the front

Now we are in the position to show which functions can be specified by the first approximation.

Lemma 4.3 *The first correction $\overset{\circ}{x}_1$ to the velocity $\overset{\circ}{x}_E$ of the front Γ_T satisfies the following equation:*

$$\overset{\circ}{x}_1 = \widehat{V_1^f} + \frac{\widehat{H_0^p}}{2\widehat{\varrho_0^f}\widehat{H_1^f}}.$$

Consequently either

$$\widehat{V_1^f} - \overset{\circ}{x}_1 > 0 \quad \textit{if} \quad \frac{\widehat{\partial_x H_0^f}}{\widehat{H_1^f}} > 0,$$

or

$$\widehat{V_1^f} - \overset{\circ}{x}_1 < 0 \quad if \quad \frac{\widehat{\partial_x H_0^f}}{\widehat{H_1^f}} < 0.$$

It means that, when the background function $V_1^f = 0$, the propagation front decelerates if the amplitude of weak jump H_0^f of velocity v_f increases and $\widehat{H_1^f} > 0$. The latter corresponds to the local increment of porosity $\mathcal{A}_0^p$ and the global decrement of porosity Y_0^p behind the front. Since due to (4.37)

$$\partial_x \widehat{H_0^{\varrho,f}} = \frac{\hat{\varrho}_0^f}{\overset{\circ}{x}_E - \hat{V}_0^f} \partial_x \hat{H}_0^f,$$

this case corresponds to the Saffman-Taylor instability.

>From the second approximation of the equations (2.8) one has:

$$\hat{\varrho}_0^f(\widehat{V_0^f} - \overset{\circ}{x}_E)(\hat{Y}_2^f)' + \kappa(\widehat{Y_2^{\varrho,f}})' = -\Big((\widehat{V_1^f} - \overset{\circ}{x}_1)\widehat{\varrho_0^f} + (\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{\varrho_1^f}\Big)(\widehat{Y_1^f})' - (\mathcal{A}_0^p)'$$

$$-H_0^p z_0' - \varrho_0^f\Big(\partial_t V_1^f + \partial_t(H_1^f)z_0 + (V_0^f + H_0^f z_0)\partial_x(V_1^f + H_1^f z_0)\Big) - \kappa\partial_x(\varrho_1^f + H_1^{\varrho,f} z_0)$$

$$-(\varrho_1^f + H_1^{\varrho,f} z_0)\Big(\partial_t V_0^f + \partial_t(H_0^f)z_0 + (V_0^f + H_0^f z_0)\partial_x(V_0^f + H_0^f z_0)\Big)$$

$$-\pi(V_1^f - V_1^s + (H_1^f - H_1^s)z_0) - \partial_x(\widehat{\varrho_0^f V_0^f})\sigma(\hat{Y}_1^f)', \tag{4.42}$$

and

$$(\widehat{V_0^f} - \overset{\circ}{x}_E)(\widehat{Y_2^{\varrho,f}})' + \widehat{\varrho_0^f}(\widehat{Y_2^f})' = \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}(\widehat{V_1^f} - \overset{\circ}{x}_1)(\widehat{Y_1^f})'$$

$$-\hat{\varrho}_1^f(\widehat{Y_1^f})' + \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}((\widehat{Y_1^f})^2)' - \partial_t\varrho_1^f - \partial_t(H_1^{\varrho,f})z_0 - \partial_x\Big((\varrho_1^f + H_1^{\varrho,f} z_0)(V_0^f + H_0^f z_0)$$

$$+(\varrho_0^f + H_0^{\varrho,f} z_0)(V_1^f + H_1^f z_0)\Big) - \Big(\partial_x \hat{\varrho}_0^f - \partial_x \hat{V}_0^f \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}\Big)\sigma(\widehat{Y_1^f})'. \tag{4.43}$$

The smooth background functions $\varrho_1^f, \varrho_1^s, V_1^f, V_1^s$ and Φ_1^s are defined as a solution of the following Cauchy problem in Q_T:

$$\varrho_0^f(\partial_t V_1^f + \partial_x(V_0^f V_1^f)) + \kappa\partial_x\varrho_1^f + \varrho_1^f(\partial_t V_0^f + V_0^f\partial_x V_0^f) + \pi(V_1^f - V_1^s) = 0, \tag{4.44}$$

$$\partial_t\varrho_1^f + \partial_x(\varrho_0^f V_1^f + \varrho_1^f V_0^f) = 0,$$

$$\varrho_0^s(\partial_t V_s^f + \partial_x(V_0^f V_1^s)) - E\partial_x\Phi_1^s + \varrho_1^s(\partial_t V_0^s + V_0^s\partial_x V_0^s) - \pi(V_1^f - V_1^s) = 0,$$

$$\partial_t\Phi_1^s - \partial_x V_1^s = 0,$$

$$\partial_t\varrho_1^f + \partial_x(\varrho_0^f V_1^f + \varrho_1^f V_0^f) = 0,$$

with the initial data

$$V_1^s|_{t=0} = V_s^1(x), V_1^f|_{t=0} = V_f^1(x), \Phi_1^s|_{t=0} = \Phi_s^1(x), \varrho_1^s|_{t=0} = \varrho_s^1(x), \varrho_1^f|_{t=0} = \varrho_f^1(x).$$

Then the functions $H_1^f, H_1^s, H_1^{\Phi,s}, H_1^{\varrho,s}$ and $H_1^{\varrho,f}$ can be obtained as a solution of the Cauchy problem in Ω_Γ:

$$\begin{aligned}\varrho_0^f(\partial_t H_1^f + \partial_x(V_0^f H_1^f + H_0^f V_1^f)) + \kappa\partial_x H_1^f \\ +H_1^{\varrho,f}(\partial_t(V_0^f + H_0^f) + (V_0^f + H_0^f)\partial_x(V_0^f + H_0^f)) \\ +\varrho_1^f(\partial_t H_0^f + H_0^f\partial_x H_0^f + \partial_x(V_0^f H_0^f)) + \pi(H_1^f - H_1^s) = 0\end{aligned} \tag{4.45}$$

$$\partial_t H_1^{\varrho,f} + \partial_x(H_0^{\varrho,f} H_1^f + H_1^{\varrho,f} H_0^f) + \partial_x(\varrho_1^f H_0^f + \varrho_0^f H_1^f + H_1^{\varrho,f} V_0^f + H_0^{\varrho,f} V_1^f) = 0,$$

$$\begin{aligned}\varrho_0^s(\partial_t H_1^s + \partial_x(H_0^s V_1^s + H_1^s V_0^s + H_0^s H_1^s)) - E\partial_x H_1^{\Phi,s} \\ +H_1^{\varrho,s}(\partial_t(V_0^s + H_0^s) + (V_0^s + H_0^s)\partial_x(V_0^s + H_0^s)) \\ +\varrho_1^s(\partial_t H_0^s + H_0^s\partial_x H_0^s + \partial_x(V_0^s H_0^s)) - \pi((v_1^f - V_1^s) = 0,\end{aligned}$$

$$\partial_t H_1^{\Phi,s} - \partial_x H_1^s = 0,$$

$$\partial_t H_1^{\varrho,s} + \partial_x(H_0^{\varrho,s} H_1^s + H_1^{\varrho,s} H_0^s) + \partial_x(\varrho_1^s H_0^s + \varrho_0^s H_1^s + H_1^{\varrho,s} V_0^s + H_0^{\varrho,s} V_1^s) = 0,$$

with the initial data on Γ_T:

$$H_1^{\varrho,f} = \frac{\varrho_0^f}{\overset{\circ}{x}_E - \widehat{V_0^f}} H_1^f, \quad H_1^{\Phi,s} = H_1^s = H_1^{\varrho,s} = 0.$$

Equations (4.42), (4.43) can be simplified to:

$$\begin{aligned}\hat{\varrho}_0^f(\widehat{V_0^f} - \overset{\circ}{x}_E)(\widehat{Y_2^f})' + \kappa(\widehat{Y_2^{\varrho,f}})' = -\widehat{H_0^p} z_0' - \Big((\widehat{V_1^f} - \overset{\circ}{x}_1)\widehat{\varrho_0^f} \\ +(\widehat{V_0^f} - \overset{\circ}{x}_E)\hat{\varrho}_1^f\Big)(\widehat{Y_1^f})' - (\mathcal{A}_0^p)' - \partial_x(\widehat{\varrho_0^f V_0^f})\sigma(\hat{Y}_1^f)',\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}(\widehat{V_0^f} - \overset{\circ}{x}_E)(\widehat{Y_2^{\varrho,f}})' + \hat{\varrho}_0^f(\widehat{Y_2^f})' = -\Big(\widehat{\varrho_1^f} - \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}(\widehat{V_1^f} - \overset{\circ}{x}_1)\Big)(\widehat{Y_1^f})' \\ +\Big(- \widehat{\partial_x \varrho_0^f} + \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}\widehat{\partial_x V_0^f}\Big)\sigma\, z_0' + \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}\Big((\widehat{Y_1^f})^2\Big)' = 0.\end{aligned} \tag{4.47}$$

Analogously to (4.34) we can derive the equation for the amplitude $\hat{H}_1^f$ of the strong discontinuity of v_f from the characteristic problem (4.45):

$$\frac{d}{dt}\widehat{H_1^f} + \Big\{\frac{\pi}{2\hat{\varrho}_0^f} + q_f(\widehat{V_0^f}, \hat{\varrho}_0^f, \widehat{\partial_x H_0^f})\Big\}\widehat{H_1^f} = g_f(\varrho_0^f, V_0^f, \varrho_1^f, V_1^f).$$

Here the function q_f depends on the functions $\widehat{V_0^f}, \widehat{\partial_x H_0^f}, \hat{\varrho}_0^f$, the function g_f depends on $\varrho_0^f, V_0^f, \varrho_1^f, V_1^f$ and their first derivatives.

System (4.46), (4.47) has a nontrivial solution

$$(\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{Y_2^{\varrho,f}} = -\widehat{\varrho_0^f}\widehat{Y_2^f} - \Big(\widehat{\varrho_1^f} - \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}(\widehat{V_1^f} - \overset{\circ}{x}_1)\Big)\widehat{Y_1^f}$$

$$+\Big(-\widehat{\partial_x \varrho_0^f} + \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}\widehat{\partial_x V_0^f}\Big)\int_{-\infty}^{\sigma} \sigma_1\, z_0' d\sigma_1 + \frac{\hat{\varrho}_0^f}{\widehat{V_0^f} - \overset{\circ}{x}_E}(\widehat{Y_1^f})^2,$$

if the following solvability condition holds:

$$2\hat{\varrho}_0^f(\widehat{V_1^f} - \overset{\circ}{x}_1)(\mathcal{A}_1^f)' = -(\mathcal{A}_0^p)' - \widehat{H_0^p} z_0'$$

$$-2\hat{\varrho}_0^f(\widehat{V_1^f} - \overset{\circ}{x}_1)\widehat{H_1^f} z_0' + \Big(-\partial_x(\widehat{\varrho_0^f V_0^f}) + 2(\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{\partial_x \varrho_0^f}\Big)\sigma\, z_0'.$$

Here we used the fact that, due to conditions of the Theorem 4.1, $\partial_t z_0 \equiv 0$. Integrating this equation with respect to σ and taking into account (4.6) we get the equation for the correction $\overset{\circ}{x}_1$ of the velocity of the front Γ_T:

$$\overset{\circ}{x}_1 = \widehat{V_1^f} + \frac{\widehat{H_0^p}}{2\hat{\varrho}_0^f \widehat{H_1^f}}, \tag{4.48}$$

and we define the function $\mathcal{A}_1^f$:

$$\mathcal{A}_1^f = -\frac{1}{2\hat{\varrho}_0^f(\widehat{V_1^f} - \overset{\circ}{x}_1)}\Big(\mathcal{A}_0^p + \Big(-\partial_x(\widehat{\varrho_0^f V_0^f}) + 2(\widehat{V_0^f} - \overset{\circ}{x}_E)\widehat{\partial_x \varrho_0^f}\Big)\int_{-\infty}^{\sigma} \sigma_1 z_0' d\sigma_1\Big),$$

where

$$\hat{V}_1^f - \overset{\circ}{x}_1 \neq 0 \quad \text{since} \quad \frac{\widehat{H_0^p}}{\widehat{H_1^f}} = -\frac{\widehat{\partial_x H_0^f}}{\widehat{H_1^f}} \neq 0.$$

Conclusion 4.1 *At any step of asymptotic construction the solution is derived analogously to the previous case. Thus this asymptotic solution can be constructed with any accuracy.*

4.4 Comments

Comment 10 *For simplicity let us consider the case when the background functions* $V_0^f = V_1^f = V_0^s = \varrho_1^f = 0$. *Then the equation (4.16) has the form*

$$\frac{d}{dt}\widehat{\partial_x H_0^f} + (\widehat{\partial_x H_0^f})^2 + \frac{\pi}{2\varrho_0^f}\widehat{\partial_x H_0^f} = 0. \tag{4.49}$$

Solving this nonlinear equation one obtains

$$\widehat{\partial_x H_0^f} = \frac{\widehat{\partial_x H_0^f}(0)\exp(-\frac{\pi}{2\varrho_0^f}t)}{\frac{\pi}{2\varrho_0^f} + \widehat{\partial_x H_0^f}(0)\big(1 - \exp(-\frac{\pi}{2\varrho_0^f}t)\big)}$$

This solution decreases and it is bounded if

$$\widehat{\partial_x H_0^f}(0) > -\frac{\pi}{2\varrho_0^f}. \tag{4.50}$$

For the same case the functions $\hat{H}_1^f, H_1^{\varrho,f}$ *satisfy the following system on the front*

$$\varrho_0^f \partial_t H_1^f + \kappa \partial_x H_1^{\varrho,f} + \pi\, H_1^f = 0,$$

$$\partial_t H_1^{\varrho,f} + \varrho_0^f \partial_x H_1^f + 2\frac{\varrho_0^f}{\overset{\circ}{x}_E}\widehat{\partial_x H_0^f} H_1^f = 0.$$

As above we reduce this system to the equation

$$\frac{d}{dt}\widehat{H_1^f} + \Big(\frac{\pi}{2\varrho_0^f} + \widehat{\partial_x H_0^f}\Big)\widehat{H_1^f} = 0. \tag{4.51}$$

Due to the equation (4.49) the equation (4.51) has the unique solution

$$\widehat{H_1^f}(t) = \frac{\widehat{H_1^f}(0)}{\widehat{\partial_x H_0^f}(0)}\widehat{\partial_x H_0^f}(t).$$

Therefore under the assumption (4.50) the functions $\widehat{\partial_x H_0^f}(t), \widehat{H_1^f}(t)$ *decrease.*

Comment 11 *The conditions of the decrement of amplitudes* $\widehat{\partial_x H_0^f}, \widehat{H_1^f}$ *allow one to separate two asymptotic solutions for each direction of the equilibrium velocity* $\overset{\circ}{x}_E$, *which correspond to physically plausible situations.*

The first asymptotic solution describes the process being an analogy of the classical Saffman-Taylor instability with the global closing of pores behind the front Γ_T *and the second one is a non-classical case in which the Saffman-Taylor instability interacts with the global opening of pores ahead of the front (see: Fig 2 and 3 in which the position of the front is indicated by the vertical broken line). Namely*

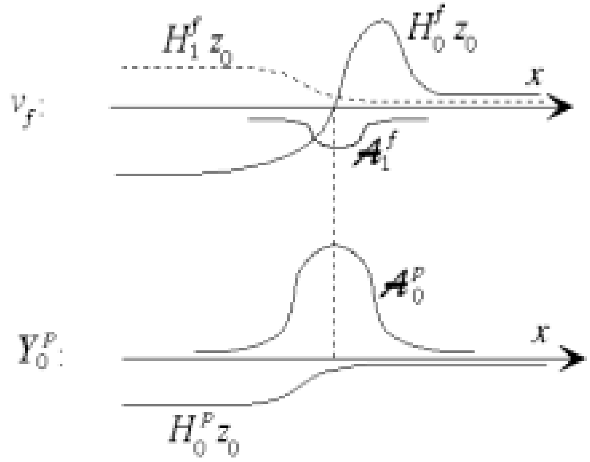

Figure 2: A schematic x-distribution illustrating the appearance of the Saffman-Taylor instability with the closing of pores behind the front Γ_T (the case 1 in Comment 11)

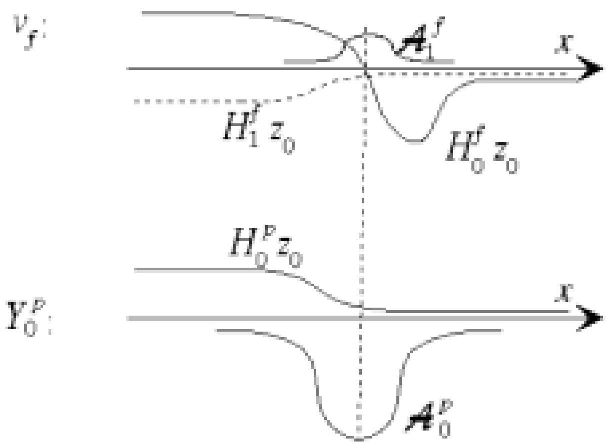

Figure 3: A schematic x-distribution illustrating the appearance of the global opening of pores behind the front Γ_T (the case 2 in Comment 11)

1) the first case corresponds to the choice $\overset{\circ}{x}_E > 0$, *the background functions* $v_0^f = V_1^f = V_0^s = \varrho_1^f = 0$ *and* $\partial_x \hat{H}_0^f(0) > 0$ *and* $\widehat{H_1^f}(0) > 0$. *Then due to (4.48)* $\overset{\circ}{x}_1 < 0$. *The leading fast part of the asymptotic solution of porosity* $Y_0^p = H_0^p z_0 + \mathcal{A}_0^p$ *yields a local opening of pores (the soliton-like term* $\mathcal{A}_0^p$ *is positive) in a small neighborhood of the front, similar to the push displacement considered in the Section 3.6, and a global closing of pores behind the front* Γ_T. *A global closing of pores dominates and this yields the deceleration of the front since (4.51)* $\overset{\circ}{x}_1 = -\partial_x \widehat{H_0^f}(0)/\widehat{H_1^f}(0) < 0$. *One obtains as well negative values of the fluid velocity* $v_{as}^f = H_0^f z_0 + \beta(H_1^f z_0 + \mathcal{A}_1^f) + O(\beta^2)$ *behind the front. We deal in this case with an interaction of a wave of a local opening* *of pores and a reflection wave following from a global closing of pores behind of the front.*

2) The second case corresponds to the following choice: $\overset{\circ}{x}_E > 0$*, the background functions* $V_0^f = V_1^f = V_0^s = \varrho_1^f = 0$*, and* $\widehat{\partial_x H_0^f}(0) < 0$*,* $\widehat{H_1^f}(0) > 0$*.*

The leading fast part of the asymptotic solution of porosity $Y_0^p = H_0^p z_0 + \mathcal{A}_0^p$ *defines the local closing of pores (the soliton-like term* $\mathcal{A}_0^p$ *is negative) in a small neighborhood of the front, considered in the Section 3.6, and the global opening of pores behind the front* Γ_T*. The global opening of pores dominates and this yields the acceleration of the front since* $\overset{\circ}{x}_1 > 0$*.*

The kink-like member $H_0^p z_0$ *drives the global increment of porosity behind the front, which yields a push displasement in the fluid. We deal in this case with an interaction of a stable displacement and a reflection wave ahead of the front following from a local closing of pores.*

The condition (4.50) bounding the amplitude of the jump from below eliminates the breaking (blow up) of the push displacement.

As in the subsection 3.6 we have for these two cases an analogy of the entropy-like conditions.

The proof of existence of the special exact solution of the considered problem, having the same initial data as the asymptotic solution shows that the entropy-like conditions yield the condition of stability of the asymptotic solution and the uniqueness condition of the special solution.

Comment 12 *Let us remind that due to (4.49), (4.51) the functions* H_0^f, H_1^f *decrease. On times* $t = O(\ln(1/\beta))$ *one obtains that* $H_0^f = O(\beta)$*,* $H_1^f = O(\beta)$*.*

For such asymptotic solutions these equations are discoupled and functions $\widehat{\partial_x H_0^f}$*,* $\widehat{H_1^f}$ *satisfy the equation of the form:*

$$\frac{d}{dt}H + \frac{\pi}{2\varrho_0^f}H = 0.$$

Then for the first correction one obtains

$$\widehat{H_1^f}\, \overset{\circ}{x}_1 = \widehat{\partial_x H_0^f} \quad on \quad \Gamma_T.$$

Hence due to the appearence of the term from the second order approximation we may solely have the following solution

$$\widehat{\partial_x H_0^f} = 0, \quad \overset{\circ}{x}_1 = 0 \quad and \quad \widehat{H_1^f} \neq 0,$$

which was impossible in the previous case. This yields a bifurcation of $\overset{\circ}{x}_1$*.*

To prove this statement exactly it is necessary to pass to large times by means of the following transformation

$$t = \eta/\beta, \quad x - x_E(t) = y/\beta, \quad \eta > \eta_* \beta^\delta, \; 0 < \delta < 1, \, \eta_* = \text{const} > 0.$$

Then the system (2.8) for $\gamma = 1$ ie reduced to the system (2.8) with $\gamma = 2$, corresponding to the case $\tau = \beta^2$ with π replaced by π/β, as considered in Section 3. Consequently, we can consider the present case as a pendulum similar to this considered in Section 3.

For large times $t = O(\ln(1/\beta))$ the soliton-like solution for porosity can be again interpreted as a filter"which is an analogy of a pendulum in the vicinity of the stable equilibrium point.

4.5 Weak discintinuity of the velocity v_s

Theorem 4.2 *Let*

$$\tau = \beta.$$

Consider strictly decreasing function $z_0(\sigma)$, satisfying (4.2), (4.3) such that

$$\lim_{\sigma \to \pm\infty} \frac{z_0'}{\exp(-(\pm\, l^\pm)\sigma)} = l^\pm, \quad \int_{-\infty}^{\infty} \sigma\, z_0' d\sigma = 0,$$

$$(\widehat{V_0^s} - \overset{\circ}{x}_E)^2 = \frac{E}{\hat{\varrho}_0^s}, \tag{4.52}$$

where the positive constans $l^\pm$ are as follows

$$\max(l^+, l^-) < \sqrt{\frac{\varrho_0^s}{E}}.$$

Then asymptotic solution (4.1),(4.4) and (4.5) of the system (2.7)-(2.12), exists on the short time interval $(0, T)$ with any accuracy.

Leading part of this asymptotics

$$\Pi_{as} = \Upsilon_0(x,t) + H_0^p(x,t) z_0 + \mathcal{A}_0^p(\sigma, x, t)) + O(\beta),$$

$$\begin{aligned}
v_{as}^s &= V_0^s(x,t) + H_0^s(x,t) z_0 + \beta(V_1^s + Y_1^s) + O(\beta^2), \\
\Phi_{as} &= \Phi_0^s(x,t) + H_0^{\Phi,s}(x,t) z_0 + \beta(\Phi_1^s + Y_1^{\Phi,s}) + O(\beta^2), \\
v_{as}^f &= V_0^f(x,t) + H_0^f(x,t) z_0 + \beta(V_1^f + Y_1^f) + O(\beta^2), \\
\varrho_{as}^s &= \varrho_0^s(x,t) + H_0^{\varrho,s}(x,t) z_0 + \beta(\varrho_1^s + Y_1^{\varrho,s}) + O(\beta^2), \\
\varrho_{as}^f &= \varrho_0^f(x,t) + H_0^{\varrho,f}(x,t) z_0 + \beta(\varrho_1^f + Y_1^{\varrho,f}) + O(\beta^2),
\end{aligned}$$

satisfies the system (2.7)-(2.12) up to order $O(\beta)$.

Here

$$\Upsilon_0(x,t) = -\partial_x(V_0^f - V_0^s), \quad H_0^p(x,t) = -\partial_x(H_0^f - H_0^s),$$

and the equilibrium velocity $\overset{\circ}{x}_E$ *of the front* Γ_T *satisfies the following equation:*

$$(\widehat{V_0^s} - \overset{\circ}{x}_E)^2 = \frac{E}{\hat{\varrho}_0^s}. \tag{4.53}$$

The functions

$$\mathcal{A}_0^p = \exp\Big(-\frac{\sigma}{\hat{V}_0^s - \overset{\circ}{x}_E}\Big)\int_{-\infty}^{\sigma}\exp\Big(\frac{\sigma_1}{\hat{V}_0^s - \overset{\circ}{x}_E}\Big)\frac{F_0^p}{\hat{V}_0^s - \overset{\circ}{x}_E}d\sigma_1 \quad \text{if } \hat{V}_0^s - \overset{\circ}{x}_E > 0,$$

and

$$\mathcal{A}_0^p = -\exp\Big(-\frac{\sigma}{\hat{V}_0^s - \overset{\circ}{x}_E}\Big)\int_{\infty}^{\sigma}\exp\Big(\frac{\sigma_1}{\hat{V}_0^s - \overset{\circ}{x}_E}\Big)\frac{F_0^p}{\hat{V}_0^s - \overset{\circ}{x}_E}d\sigma_1 \quad \text{if } \hat{V}_0^s - \overset{\circ}{x}_E < 0.$$

are soliton-like , where

$$F_0^p = (\overset{\circ}{x}_E - \hat{V}_0^s)\hat{H}_0^p\, z_0' - \partial_x \hat{H}_0^f \sigma\, z_0'.$$

The background functions $V_0^f, V_0^s, \Phi_0, \varrho_0^s$ *and* ϱ_0^f *are the solutions of the Cauchy problem (4.21), (4.29) in the strip* Q_T. *The functions* $H_0^s, H_0^f, H_0^{\Phi,s}, H_0^{\varrho,s}$ *and* $H_0^{\varrho,f}$ *are the solutions of characteristic Cauchy problem (4.30), (4.31) in a sufficiently small neighborhood* $\Omega_\Gamma = \{(x,t), |x - x(t)| < \delta, t \in [0,T]\}$ *of the front* Γ_T.

The amplitude of jump $\widehat{\partial_x H_0^s}(t)$ *of weak discontinuity of* v_s *is defined by the nonlinear problem:*

$$\frac{d}{dt}\widehat{\partial_x H_0^s} + (\widehat{\partial_x H_0^s})^2 + \Big[\frac{\pi}{2\hat{\varrho}_0^s} + \frac{1}{2}\frac{d}{dt}\hat{\varrho}_0^s + \partial_x \widehat{V_0^f}$$

$$-(\widehat{V_0^s} - \overset{\circ}{x}_E)\frac{\widehat{\partial_x \varrho_0^f}}{\hat{\varrho}_0^s} + \frac{\pi(\widehat{V_0^f} - \widehat{V_0^s})}{2\hat{\varrho}_0^f(\widehat{V_0^s} - \overset{\circ}{x}_E)} + \frac{E\widehat{\partial_x \Phi_0^s}}{2\hat{\varrho}_0^s(\widehat{V_0^s} - \overset{\circ}{x}_E)}\Big]\widehat{\partial_x H_0^s} = 0. \tag{4.54}$$

The fuction $\widehat{\partial_x H_0^s}$ *is defined by the equation*

$$-(\widehat{V_0^s} - \overset{\circ}{x}_E)\frac{d}{dt}\partial_x\widehat{H_0^{\varrho,s}} = \varrho_0^s\frac{d}{dt}\widehat{\partial_x H_0^s} + \widehat{\partial_x \varrho_0^s}\widehat{\partial_x H_0^s} + \frac{1}{2}\frac{d}{dt}\widehat{\varrho_0^f}.$$

Also the function $\widehat{H_1^s}(t)$ *is defined by the nonhomogeniuos linear equation on* Γ_T:

$$\frac{d}{dt}\widehat{H_1^s} + \Big\{\frac{\pi}{2\hat{\varrho}_0^s} + q_s(\widehat{V_0^s}, \hat{\varrho}_0^s, \widehat{\partial_x H_0^s})\Big\}\widehat{H_1^s} = g_s(\varrho_0^s, V_0^s, \Phi_0^s, \hat{\varrho}_1^s, \widehat{V_1^s}, \widehat{\Phi_1^s}), \tag{4.55}$$

where the function q_s *depends on the functions* $V_0^s, \partial_x H_0^s, \varrho_0^s$, *the function* g_s *depends on* $\varrho_0^s, V_0^s, \varrho_1^s, V_1^s$ *and their first derivatives.*

Remark 4.3 *Let us note that the solution of (4.30)- (4.32) exists in* Ω_Γ *if* $\delta < \delta_0$ *is sufficiently small. This solution can be extended to the region* Q_T^- *so that it remains a solition-like solution of (4.30), (4.32).*

5 APPENDIX

In the paper we presented main results for a 1-D case of the Riemann problem. As already mentioned most of them can be also proved for 2-D and 3-D cases.

However there are additional effects accompanying multidimensional cases which cannot appear in the 1-D case. The most important effect of this type is an instability appearing due to the curvature of an initial profile of the velocities. The evolution of this curvature yields instabilities connected with a local concaving of the front. They appear, for instance, in the form of loss of symmetry of the initial data.

We demonstrate these effects with a simple 2-D example in which the initial value problem is chosen to be either

1)

$$v_{fx}(t=0,x,y)=0,\ v_{fy}(t=0,x,y)=\beta\, z_0(\sigma),$$
$$\sigma := \frac{y-\sin(x)}{\beta^2},\quad 0<x<\pi,\ -\infty<y<\infty,$$

where $z_0(\sigma)$ has the form (3.47) and we use periodic boundary conditions in x-direction,

or

2)

$$\sigma = (y+\sin(x))/\beta^2$$

with other conditions being the same as above.

In Figures 4 and 5 we show the behavior of components of the fluid velocity v_{fx}, v_{fy} and of the changes of porosity Π for a chosen instant of time.

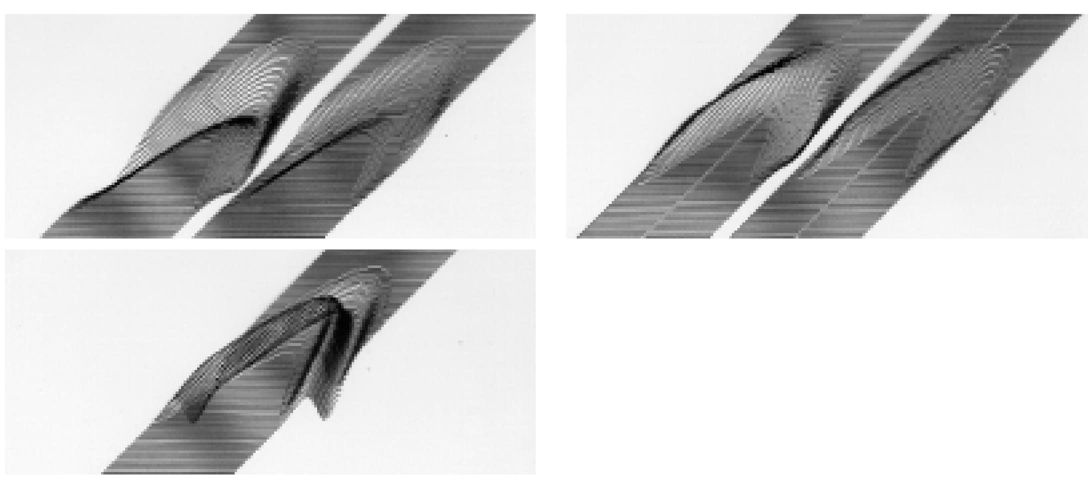

Figure 4: Distribution of velocity components v_{fy} and v_{sy} (left upper part), v_{fx}, v_{sx} (right upper part), and changes of the porosity (lower part) for the dimensionless time instant 0.3125 in the first case of initial conditions

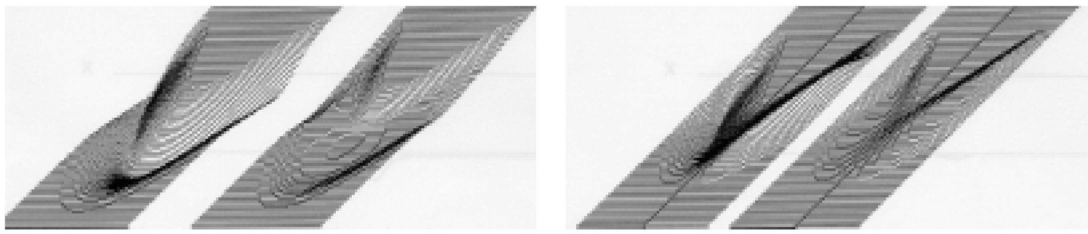

Figure 5: Distribution of velocity components v_{fy} and v_{sy} (left part), v_{fx}, v_{sx} (right part) for the dimensionless time instant 0.46875 in the second case of initial conditions

It is clear that, in contrast to a stable behavior of the case 1) in Fig. 4, in the case 2) (Fig. 5) the velocity profile is loosing its symmetry. As in the 1-D case its disturbance hase the character of a soliton wave (in the normal sections) but the instability of the 2-D front yields a creation of a mushy region (compare the middle part of regions shown in the Figure).

These problems will be investigated in details in a forthcoming paper.

6 FINAL REMARKS

We have shown that the balance equation of porosity leads to a definite structure of solutions of Riemann problems for partial velocities v_f and v_s. Kink-like solutions for velocities propagate with characteristic velocities modified in the second order terms by contributions whose sign depends on initial amplitudes.

This is conneced with the existence of two effects. In the case of a positive second order correction of the propagation velocity there appears a Saffman-Taylor instability. On the other hand in the case of a negative second order correction we deal with the so-called stable push displacement.

Simultaneously we have proved that disturbances of porosity propagate as soliton-like waves accompanying the disturbances of velocity. This proves as well that the model of porous materials used in this paper does not require additional boundary conditions which could arise due to the additional balance equation.

We have shown as well that the model is stable in large times which is the consequence of the attenuation arising due to diffusive forces.

In addition we have indicated that the existence of kink-like and soliton-like solutions requires entropy-like conditions. These were formulated in the paper but their physical interpretation in terms of thermodynamics is still missing.

It has been indicated as well that the behavior of the model depends on the choice of relation between two small parameters of the model: the relaxation time of porosity τ and the coupling parameter β. In the case of the same order of magnitude of these parameters, i.e. $O(\tau/\beta) = 1$, we deal with classical porous materials. In the second case, when the relaxation time τ is much smaller than β, c.g. $O(\tau/\beta) = O(\beta)$, we deal with granular-like maaterials. Both cases were investigated in the paper. They are quite similar in the first order effects but they do differ considerably in the behavior of amplitudes of the waves. This is the second order effect.

Finally let us mention that the 1-D character of the model hides certain essential properties of waves connected with the curvature of the front. This was illustrated by a numerical example in Appendix and it should be investigated in the forthcoming paper.

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