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A Riemann Problem for Poroelastic Materials with the Balance Equation for Porosity
Part I

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Abstract

The paper is devoted to the asymptotic analysis of the multicomponent model of poroelastic materials in which the porosity is described by its own field equation. The model is weakly nonlinear due to the kinematic contributions and a nonlinear dependence of material parameters on an equilibrium porosity. It is shown that the model contains two small parameters. The first one describes the coupling of the skeleton (solid component) with the fluid and its contribution through partial stresses is similar to a dynamical pressure of extended thermodynamics. Mathematically it leads to a dispertion effect similar to this appearing in the Korteweg - de Vries equation. The second small parameter describes a relaxation of porosity. We consider two cases. In the first one the order of magnitude of both small parameters is the same. This seems to correspond to usual porous materials. In the second case the dimensionless relaxation time is propotional to the square of the other parameter. We call such materials granular-like porous. They seem to correspond to compact granular materials with hard and smooth particles.

We prove the existence of soliton-like solutions for the porosity and kink-like solutions for the partial velocities under a natural entropy-like selection condition which is also presented in the paper. The proof is based on the asymptotic analysis in which two steps of approximations were investigated. We show that the diffusive interaction force of components - a kind of an internal friction - yields decaying amplitudes of discontinuities. We show as well that in one class of Riemann problems a Saffman-Taylor instability appears.

The paper is divided into two parts solely for technical reasons. Therefore the references appear after the second part. In the Appendix we show a few examples of a numerical simulation of a two-dimensional Riemann problem. These were obtained by dr. O. A. Vassilieva (Moscow University). The full numerical analysis shall be presented separately.

1 Physical preliminaries

Continuous multicomponent models of porous materials with diffusion differ from the classical theory of mixtures due to the fact that one of the components is a solid. This yields a microstructure whose description requires additional fields. In the simplest case of porous materials it is a single scalar field of porosity.

In a few recent works (e.g. [1, 2, 3]) a new continuous model of such porous materials has been introduced. For the sake of brevity we call it the W-model. There are two main features which distinguish this model from previous multicomponent models. First of all on the basis of geometrical arguments it is argued that the porosity satisfies a balance equation of its own. Secondly there is an additional coupling of field equations which is due to changes of porosity. These two features motivate the analysis presented in this work.

A mathematical asymptotic analysis of this work is based on the assumption that the coupling coefficient is small. In this section we present the physical justification of this assumption.

Let us begin with the presentation of the linearized version of the model for poroelastic materials saturated with an ideal fluid. In such a case the fields of a continuous model reduce to the partial mass densities of components ρ_s, ρ_f, the displacement of the skeleton $\mathbf{u}_s$, the velocity of the fluid $\mathbf{v}_f$, and the porosity n. Instead of the latter we use the deviation of porosity $\Delta := n - n_E$ from the equilibrium value n_E. The equilibrium porosity is either assumed to be constant or to be given by a constitutive relation in which solely a dependence on partial mass densities may appear. Further in this paper we use dimensionless quantities which shall be defined further in this Section. In such a dimensionless form the relations for equilibrium porosity are either

$$n_E(\varrho_f, \varrho_s) \equiv \text{const},$$

or, in a weaker form, solely

$$n_E(\varrho_0^f, \varrho_0^s) = n_0 = \text{const}, \tag{1.1}$$

and

$$\partial_{\varrho_f} n_E(\varrho_0^f, \varrho_0^s) = \gamma_0^f = \text{const} > 0, \quad \partial_{\varrho_s} n_E(\varrho_0^f, \varrho_0^s) = \gamma_0^s = \text{const} < 0,$$

where ϱ_0^f, ϱ_0^s are reference equilibrium values of the dimensionless partial mass densities of fluid and skeleton, respectively.

Physical justification of this extension of the original W-model shall be discussed elsewhere. All above fields are defined on the domain $Q_t^3 := B_t \times (0, T)$, $B_t \subset \Re^3$. Within the W-model they satisfy the following set of local field equations

∘ mass balance equations (without mass sources!)

$$\partial_t \rho_f + \operatorname{div}(\rho_f \mathbf{v}_f) = 0, \quad \partial_t \rho_s + \operatorname{div}(\rho_s \mathbf{v}_s) = 0, \quad \mathbf{v}_s := \partial_t \mathbf{u}_s; \tag{1.2}$$

∘ momentum balance equations

$$\rho_f \left(\partial_t + (\mathbf{v}_f, grad)\right) \mathbf{v}_f = \operatorname{div} \mathbf{T}_f - \pi \left(\mathbf{v}_f - \mathbf{v}_s\right), \tag{1.3}$$

$$\rho_s \left(\partial_t + (\mathbf{v}_s, grad)\right) \mathbf{v}_s = \operatorname{div} \mathbf{T}_s + \pi \left(\mathbf{v}_f - \mathbf{v}_s\right),$$

where $(\mathbf{v}_f, grad)$, and $(\mathbf{v}_s, grad)$ are scalar products, $\mathbf{T}_f, \mathbf{T}_s$ denote the partial Cauchy stress tensors given by the following constitutive relations

$$\mathbf{T}_f = -\left[p_f^0 + \kappa(n_E)\left(\rho_f - \rho_f^0\right) + \beta\Delta\right]\mathbf{1},$$

$$\mathbf{T}_s = \lambda_s(n_E)\operatorname{tr}\mathbf{e}_s\mathbf{1} + 2\mu_s(n_E)\mathbf{e}_s + \beta\Delta\mathbf{1}, \quad \mathbf{e}_s := \operatorname{sym} grad\mathbf{u}_s,$$

and p_f^0 is a reference value of the partial fluid pressure, $\kappa(n_E)$ denotes a coefficient of compressibility of the fluid component, $\lambda_s(n_E), \mu_s(n_E)$ are effective Lamé coefficients of the skeleton. β denotes the above mentioned coupling coefficient, and π is the permeability coefficient;

- balance equation of porosity

$$\partial_t (\Delta + n_E) + (\mathbf{v}_s, grad)(\Delta + n_E) + \mathrm{div}\,(\varphi(n_E)(\mathbf{v}_f - \mathbf{v}_s)) = -\frac{\Delta}{\tau}, \qquad (1.4)$$

where τ is a constant relaxation time of porosity. Functions λ_s, μ_s, κ and φ depend smoothly on n_E. The structure of the flux in equation (1.4), in particular the form of the coefficient φ, is explaind in [1].

All material parameters are chosen in such a way that they satisfy thermodynamic conditions (the second law of thermodynamics and the stability of the thermodynamic equilibrium) if they are *positive*. In particular the dissipation in the system is described by the following local dissipation function

$$d := \pi (\mathbf{v}_f - \mathbf{v}_s) \cdot (\mathbf{v}_f - \mathbf{v}_s) + \frac{\Delta^2}{\tau} \geq 0. \qquad (1.5)$$

The W-model follows from the assumption that deviations of processes from the thermodynamic equilibrium: $d = 0$ are small, i.e. the model is linear with respect to the diffusion velocity $\mathbf{v}_f - \mathbf{v}_s$, as well as with respect to the deviation of porosity Δ.

The above described model has been analysed for various steady-state processes - also for large deformations of the skeleton, and for dynamic processes of propagation of acoustic waves (e.g. see: $[4-9]$). Results have been compared with experimental data quoted, for instance in the book [10]. We proceed to present those results obtained in these papers which are essential for our present work.

First of all in Table 1 we show typical orders of magnitude of various material parameters appearing in the W-model.

Table 1

reference porosity n_E	0.3
mass density of the fluid $\frac{\rho_f}{n}$	$10^3\ \frac{kg}{m^3}$
mass density of the skeleton $\frac{\rho_s}{1-n}$	$3 \times 10^3\ \frac{kg}{m^3}$
effective Lamé coefficient λ_s	$40\ GPa$
coefficient of compressibility κ	$10^6\ \frac{m^2}{s^2}$
coefficient of permeability π	$10^8\ \frac{kg}{m^3 s}$
relaxation time τ	$10^{-6}\ s$
coupling coefficient β	$10^2\ MPa$

These values yield the following characteristic orders of magnitude of the fields.

Table 2

speeds of longitudinal acoustic waves c_s, c_f	$3\frac{km}{s}$ and 1 $\frac{km}{s}$
changes of the fluid mass density $\left|\rho_f - \rho_f^0\right|$	0.1 $\frac{kg}{m^3}$
diffusion velocity $\left|\mathbf{v}_f - \mathbf{v}_s\right|$	0.1 $\frac{m}{s}$
partial pressure in the fluid	0.1 MPa
partial stresses in the skeleton	200 MPa
changes of porosity $\left|\Delta\right|$	10^{-7}

We use these estimates to write the field equations in the dimensionless form. Variables and fields shall be replaced by dimensionless quantities according to the following scheme.

$$\begin{aligned} t &\to \frac{t}{t_0}, \quad x_k \to \frac{x_k}{L}, \\ \mathbf{v}_f &\to \frac{\mathbf{v}_f}{v_0}, \quad \mathbf{v}_s \to \frac{\mathbf{v}_s}{v_0}, \quad v_0 := \frac{L}{t_0} \\ \rho_f &\to \frac{\rho_f}{\rho_f^0}, \quad \rho_s \to \frac{\rho_s}{\rho_f^0}, \\ p_f &\to \frac{p_f}{p_f^0}, \quad \mathbf{T}_s \to \frac{\mathbf{T}_s}{p_f^0}. \end{aligned} \tag{1.6}$$

This scheme is chosen in order to preserve the structure of field equations. For instance we normalize both partial mass densities with the same reference mass density ρ_f^0, and both velocities with the same reference value v_0 in order to keep the same form of the diffusion term: $\pi\left(\mathbf{v}_f - \mathbf{v}_s\right)$ in both momentum balance equations. Let us mention that initial values of p_f, ϱ_f are frequently heterogeneous. Then the parameters used in normalization should be understood as (for instance) maximum equilibrium values.

Certainly the number of independent reference parameters can be reduced according to methods of dimensional analysis. For instance we can choose according to the momentum balance equation for the fluid

$$p_f^0 = \frac{\rho_f^0 L}{t_0^2}, \quad t_0 := \frac{\rho_f^0}{\pi} \sim 10^{-5}\, s, \quad \Longrightarrow L := \sqrt{\frac{p_f^0 \rho_f^0}{\pi^2}} \sim 10^{-4}\, m, \tag{1.7}$$

where we accounted for data of Table 1 and chosen $\rho_f^0 \sim 10^3 \frac{kg}{m^3}$, $p_f^0 \sim 0.1 MPa$.

Now we have to compare the contributions to the partial pressure in the fluid $\kappa\left(\rho_f - \rho_f^0\right)$ and $\beta\Delta$. We have

$$\kappa\left(\rho_f - \rho_f^0\right) = p_f^0 \left[\frac{\kappa \rho_f^0}{p_f^0}\right]\left(\frac{\rho_f}{\rho_f^0} - 1\right). \tag{1.8}$$

The quantity in the square barckets defines the dimensionless coefficient of compressibility. For the above quoted data it has the value: $\frac{\kappa\rho_f^0}{p_f^0} = 10^4$. Simultaneously the changes in the round brackets are of the order (see: Table 2) 10^{-4}. On the other hand the second contribution has the form

$$\beta\Delta = p_f^0 \left[\frac{\beta\Delta_0}{p_f^0}\right] \frac{\Delta}{\Delta_0}. \tag{1.9}$$

Δ_0 is a parameter which normalizes Δ to the same order as changes of mass density in (1.8), i.e. it must be of the order 10^{-3}. Consequently the normalized coefficient $\frac{\beta\Delta_0}{p_f^0}$ appearing in the square brackets has the order of magnitude 1. This should be compared with the value 10^4 estimated above for the dimensionless compressibility coefficient $\frac{\kappa\rho_f^0}{p_f^0}$. Hence it is clear that the coupling term is a small correction and we can rescale the problem in such a way that these contributions appear with a small parameter in the field equations. Simultaneously the above scaling yields the conclusion that dimensionless quantities $\frac{\beta\Delta_0}{p_f^0}$ and $\frac{\tau}{t_0}$ (compare: τ and the value t_0 in (1.7)) differ on one order of magnitude. This difference may be much bigger for systems in which the porosity relaxation time is much shorter. For instance, this seems to be the case in granular materials made of very rigid and smooth particles. This observation shall be usefull in further considerations. In the sequel we denote dimensionless quantities by the same symbols as in the case of quantities possessing physical dimensions.

2 Motivation of the mathematical problem - dispersion in the W-model of poroelastic materials

Bearing the above dimensional analysis in mind we consider for simplicity a 1-D problem in the strip $Q_T = \{x \in R^1, t \in [0,T]\}$. However all results can be obtained analogously for 2-D and 3-D cases. We reduce the system (1.1)-(1.4) to the equivalent system of hyperbolic equations of the first order in the dimensionless form. We have

$$\partial_t (\Delta + n_E) + v_s \partial_x (\Delta + n_E) + \partial_x (\varphi(n_E)(v_f - v_s)) = -\frac{\Delta}{\tau}, \tag{2.1}$$

and

$$\begin{gathered}
\partial_t \rho_f + \partial_x (\rho_f v_f) = 0, \\
\partial_t \rho_s + \partial_x (\rho_s v_s) = 0, \\
\rho_f(\partial_t v_f + v_f \partial_x v_f) + \partial_x \left(\kappa(n_E)(\rho_f - \varrho_0^f)\right) + \beta\partial_x\Delta + \pi (v_f - v_x) = 0, \\
\rho_s(\partial_t v_s + v_s \partial_x v_s) - \partial_x (E(n_E)\Phi_s) - \beta\partial_x\Delta - \pi (v_f - v_x) = 0, \\
\partial_t \Phi_s - \partial_x v_s = 0
\end{gathered} \tag{2.2}$$

with the initial conditions

$$
\begin{aligned}
v_s|_{t=0} &= v_0^s(x) + \mathcal{V}_0^s(x,\beta,\tau), \qquad v_f|_{t=0} = v_0^f(x) + \mathcal{V}_0^f(x,\beta,\tau), \\
\rho_s|_{t=0} &= \rho_0^s(x) + \mathcal{R}_0^s(x,\beta,\tau), \qquad \rho_f|_{t=0} = \rho_0^f(x) + \mathcal{R}_0^f(x,\beta,\tau), \\
\Delta|_{t=0} &= \Delta^0(x) + \Delta_0(x,\beta,\tau) \quad \Phi_s|_{t=0} = \partial_x u_s^0 + \mathcal{F}_0^s(x,\beta,\tau).
\end{aligned}
\tag{2.3}
$$

The functions E, κ, φ of n_E are smooth. All quantities are dimensionless and the motion of both components is assumed to appear only in the x-direction. As usual we denote by E an elastic coefficient which is a dimensionless combination of $\lambda_s + 2\mu_s$ divided by the normalization parameter.

Equilibrium values of velocities and mass densities appearing in (2.3) are solutions of the stationary limit problem of (1.2-3) as $\beta = 0$:

$$
\varrho_s^0 v_s^0 \partial_x v_s^0 - E(n_0)\partial_x \left(\partial_x u_s^0\right) - \pi(v_f^0 - v_s^0) = 0, \tag{2.4}
$$

$$
\varrho_f^0 v_f^0 \partial_x v_f^0 + \pi(v_f^0 - v_s^0) = 0,
$$

and

$$
v_0^s = \text{const}, \quad \varrho_s^0 v_s^0 = const, \quad \varrho_f^0 v_f^0 = const, \quad \varphi(n_E(\varrho_0^f, \varrho_0^s)) = \text{const}, \quad \Delta^0 = 0.
$$

Whence one obtains that v_0^s, ϱ_0^s are constant. Equilibrium values $v_0^f, v_0^s, \varrho_0^s, \Phi_0^s, \varrho_0^f$ are either any stationary solution of (2.4) in the case of $n_E \equiv$ const, or a constant stationary solution of (2.4) in the case of the state equation (1.1).

As already mentioned in Sec.1 an estimation of characteristic values of positive parameters β and τ as well as a preliminary asymptotic analysis of a possible structure of solutions yields the smallness of the parameter β and

$$
\tau = \tau_0 \beta^\gamma, \quad \gamma = 1, 2, \quad \tau_0 = O(1). \tag{2.5}
$$

Let for simplicity $\tau_0 = 1$. In the case $\gamma = 2$ we shall speak about granular-like porous materials (compare the remark at the end of Section 1). It corresponds to the class of materials which are made of hard and smooth grains remaining in contact with each other in the range of processes under considerations.

Taking into account (2.5) the transformation

$$
\Delta = \beta \Pi \tag{2.6}
$$

yields the following form of the system $(2.1-2)$ with the small parameter β:

$$
\beta \partial_t \Pi + \beta v_s \partial_x \Pi + \partial_x \left(\varphi(n_E)\left(v_f - v_s\right)\right) = -\frac{\Pi}{\beta^{\gamma-1}} - \partial_t n_E, \tag{2.7}
$$

$$
\begin{gathered}
\partial_t \varrho_f + \partial_x (\varrho_f v_f) = 0, \\
\partial_t \varrho_s + \partial_x (\varrho_s v_s) = 0, \\
\varrho_f(\partial_t v_f + v_f \partial_x v_f) + \partial_x \left(\kappa(n_E)(\varrho_f - \varrho_0^f)\right) + \beta^2 \partial_x \Pi + \pi (v_f - v_s) = 0, \\
\varrho_s(\partial_t v_s + v_f \partial_x v_f) - \partial_x (E(n_E)\Phi_s) - \beta^2 \partial_x \Pi - \pi (v_f - v_s) = 0, \\
\partial_t \Phi_s - \partial_x v_s = 0,
\end{gathered}
\tag{2.8}
$$

Firstly in the case $\tau = \beta^2$, initial data (2.3) are a small perturbation of order $O(\beta)$ of a the stationary solution of (2.4) such that $\Delta^0 = v_s^0 = v_f^0 = 0$ and $\varrho_0^f > 0, \varrho_0^s > 0$, Φ_0^s are constant. The functions E, κ, n_E are not necesserly constant.

Secondly in the case $\tau = \beta$, initial data (2.3) are a perturbation of stationary solution of (2.4) such that $\Delta_0 = 0, v_f^0 = v_s^0$ and $\varrho_0^f, \varrho_0^s, v_0^f, v_0^s, \Phi_0^s$ are constant. In this case without loss of genarality we assume that κ, E, n_E are constants. Then $\varphi(n_E)$ is constant too.

Our goal is an investigation of the evolution of solutions of system (1.2-3), when either the functions $\mathcal{V}_1^f, \mathcal{V}_1^s, \mathcal{R}_1^f, \mathcal{R}_1^s, \mathcal{F}_1^s$ of initial data (2.3) (in the first case) or $\mathcal{V}_0^f, \mathcal{V}_0^s, \mathcal{R}_0^f, \mathcal{R}_0^s, \mathcal{F}_0^s$ of initial data (2.3) (the second case) are smooth approximations of discontinuities (strong and weak) with respect to τ, β.

Recently grows an interest of physicists, mathematicians and mechanicians for the problem of description of wave processes in media with *dispersion*. The simplest model of this type with the dispersion β^2 is described by the following Korteweg - de Vries equation

$$
\partial_t u + 3\partial_x u^2 + \beta^2 \partial_x^3 u = 0, \quad x \in \Re^1, \ t \in [0, T]. \tag{2.9}
$$

If $\beta = 0$ this equation, similar to the Burgers equation, is quasi-linear hyperbolic. However limit solutions as $\beta \to 0$ of the Korteweg - de Vries equation are principally different from limit solutions of the Burgers equation. For example consider the following particular solution of the equation (2.9):

$$
u(x, t, \beta) = u_0 + A \cosh^{-2} (\gamma(x - x(t))/\beta), \tag{2.10}
$$

where $A = a^2$, $\gamma = a/2$, $x(t) = (a^2 + 6u_0)\,t$; $a > 0$, u_0 is some constant. Indeed the limit in $\mathcal{D}'$ [1] as $\beta \to 0$ of the function $(u(x,t,\beta) - u_0)/\beta$ is equal to $2a\delta(x - x(t))$ [2]. At the same time the pointwise limit as $\beta \to 0$ of the function $u(t,x,\beta) - u_0$ for $x \neq x(t)$ is equal to 0. However the limit as $\beta \to 0$ of the maximum of the function $u(t,x,\beta) - u_0$ is equal to A. As indicated in [11-14, 17] these solutions are infinitely thin soliton-like functions.

[1] i.e. $H(x,t)$ is called a weak limit of the function $H(x,t,\beta)$ in $\mathcal{D}'$ as $\beta \to 0$ if for any $t \in [0,T]$ the following relation holds: $(H(x,t), \psi(x)) = \int_{-\infty}^{\infty} H(x,t)\psi(x)dx = \lim_{\beta\to 0} \int_{-\infty}^{\infty} H(x,t,\beta)\psi(x)dx$, $\forall \psi \in C_0^\infty$.

[2] $\delta(x - \varphi)$ is the Dirac function: $(\delta(x - \varphi), \psi(x)) = \psi(\varphi)$.

Let us analyse the equation for porosity (2.7). Considering smooth approximations of strong discontinuities for the porosity and the velocities v_s and v_f in the form of kink functions one gets soliton-like functions in all terms of the left-hand side of this equation and a kink function in the right-hand side. This contradiction means that a solution for Δ should be a soliton-like function. On the other hand if we consider smooth approximations of weak discontonuities for v_s and v_f the same analysis shows that a smooth approximation of the strong discontinuity for porosity in the form of a kink function may exist. In the sequel we construct for W-model the solution using two-scales asymptotic approach [11-14] which allows us to describe singularities analogous to those which appear for the Korteweg - de Vries equation.

We show that the term $\beta\partial_x\Delta$ (see (2.2)) plays the role of dispersion with respect to porosity.

Proposition 2.1 *Let $\tau=\beta^2$, $\Delta_0=v_0^f=v_0^s=0$, $\varrho_0^f>0, \varrho_0^s>0$, Φ_0^s are constant, satisfying (1.1), with $n_E\neq$ const. Then asymptotic solutions of any accuracy with respect to β of system (2.7-8) exist on a finite time interval, and they possess the following properties: they are smooth approximations with respect to β of order $O(\beta)$ of strong discontinuities either for v_f, ϱ_f or for v_s, ϱ_s, Φ_s and the infinitely thin soliton function [11] of order $O(\beta)$ for Δ along a small perturbation of characteristics of the linearized problem to (1.2-3). These solutions have properties of solutions of the model for a pendulum around the stable equilibrium point, i.e. any small perturbations of the following two equlibrium velocities of the front $\overset{\circ}{x}_E=\pm\sqrt{\kappa(n_0)}$, $\overset{\circ}{x}_E=\pm\sqrt{E(n_0)/\widehat{\varrho_0^s}}$ tend to zero. $\widehat{\rho_0^s}$ denotes the value of ρ_0^s on the front.*

Proposition 2.2 *Let $\tau=\beta$, $\Delta_0=0, \varrho_0^f=\varrho_0^s$, $\varrho_0^f>0, \varrho_0^s>0, v_0^f, v_0^s, \Phi_0^s$ are constant. Then there exist asymptotic solutions of any accuracy of system (2.7-8) with the following properties: they are smooth approximations with respect to β of order $O(1)$ of weak discontinuities either for v_f, ϱ_f or v_s, ϱ_s, Φ_s and a smooth approximation of order $O(\beta)$ of a strong discontinuity for Δ along a small perturbation of characteristics of linearized problem to (1.2-3).*

Next we prove the existence of a weak solution for the Cauchy problem (1.1-3), (2.3) such that

$$v_s=v_{as}^s+\omega_s,\ \ v_f=v_{as}^f+\omega_f, \Phi_s=\Phi_{as}^s+\varphi_s,$$
$$\varrho_s=\varrho_{as}^s+w_s,\ \ \varrho_f=\varrho_{as}^f+w_f,\ \ \Delta=\Delta_{as}+w^p,$$
$$\omega_s,\omega_f,w_s,w_f,w_\pi\in W_p^1(Q_T),\ \forall p\geq 2,$$

and

$$||\omega_s,L_2(Q_T)||+||\omega_f,L_2(Q_T)||+||\varphi_s,L_2(Q_T)||$$
$$+||w_s,L_2(Q_T)||+||w_f,L_2(Q_T)||=O(\beta^{3/2}),$$
$$||w^p,L_2(Q_T)||=O(\beta^{5/2}),$$

where $v_{as}^f, v_{as}^s, \varrho_{as}^f, \varrho_{as}^s$ and Δ_{as} are asymptotic solutions of (2.1-2).

3 Construction of the asymptotic solution for the propagation of the front

We consider two cases. In the first one the functions v^f_{as} and ϱ^f_{as} are smooth approximations of discontinuous functions with respect to the small parameter β, simultaneously the functions v^s_{as}, Φ^s_{as} and ϱ^s_{as} are smooth approximations of continuous functions. In the other case the functions $v^s_{as}, \varrho^s_{as}, \Phi^s_{as}$ are smooth approximations of discontinuous functions and the functions v_f, ϱ_f are smooth approximations of continuous functions.

In the paper we investigate in details the first case, i.e. when the functions v^f_{as} and ϱ^f_{as} are smooth approximations of discontinuous functions with respect to the small parameter β. For the second case we present results.

3.1 Strong discontinuity of v_f

We proceed to investigate the first case, i.e. a smooth approximation of strong discontinuities of v_f and ϱ_f. We begin with the derivation of equations specifying the leading part of asymptotics for v_f and Π. These should be equations having kink-like and soliton-like solutions, respectively.

Setting $\tau = \beta^2$ we investigate the system (2.7), (2.8).

The ansatz for the asymptotic expansion of an unknown function Π in a neighborhood Ω_Γ of the propagation front Γ_T has the form:

$$\Pi_{as} = \Upsilon_0(t) + \Pi_0(\sigma, t) + \sum_{j=1}^{N} \beta^j (\Upsilon_1(x,t) + Y^p_j(\sigma, x, t)), \tag{3.1}$$

where

$$Y^p_j = \Pi_j(\sigma, t)) + H^p_1(x,t) z_0(\sigma, t)), \; j \geq 1,$$

and

$$\sigma = (x - x(t))/\beta^2,$$

is a fast variable. We assume that Υ_0 depends solely on t and $\Upsilon_j(x,t), \Pi_j, H^p_j, \; z_0, j \geq 1$ are smooth bounded functions. The functions Π_j, $z_0(\sigma, t)$ are smooth, bounded soliton-like and kink-like functions, respectively, stabilized in the infinity:

$$\sigma^k \frac{d^j}{d\sigma^j} \frac{d^l}{d\,t^l} \frac{d^i}{d\,x^i} (y - y^{\pm}) = 0, \quad \forall k, j, i \geq 0 \;\; \text{if} \;\; \sigma \to \pm\infty \tag{3.2}$$

for $y = z_0$ and $y = \Pi_j$, where

$$z^-_0(t) = \lim_{\sigma \to -\infty} z_0 = 0, \;\; z^+_0(t) = \lim_{\sigma \to +\infty} z_0 = 1,$$

or

$$z^-_0(t) = 1, \quad z^+_0(t) = 0. \tag{3.3}$$

The ansatz for the smooth approximation of the strong discontinuity of velocity v_f and density ϱ_f has the form

$$v^f_{as} = \sum_{j=1}^{N} \beta^j (V^f_1(x,t) + Y^f_j(\sigma,x,t)),\;\; \varrho^f_{as} = \varrho^f_0(x,t) + \sum_{j=1}^{N} \beta^j (\varrho^f_j(x,t) + Y^{\varrho,f}_j(\sigma,x,t)). \tag{3.4}$$

Here

$$Y^f_j = H^f_j(x,t) z_0(\sigma,t) + \mathcal{A}^f_j(\sigma,t), \quad Y^{\varrho,f}_j = H^{\varrho,f}_j(x,t) z_0(\sigma,t) + \mathcal{A}^{\varrho,f}_j(\sigma,t), \tag{3.5}$$

and

$$\widehat{H^f_j},\; \widehat{H^{\varrho,f}_j} \neq 0,\, j \geq 1. \tag{3.6}$$

The current position of the front is described by the following asymptotic formula

$$\overset{\circ}{x}(t) = \overset{\circ}{x}_E(t) + \beta\, \overset{\circ}{x}_1(t) + O(\beta^2).$$

We show further that the zero order approximation $\overset{\circ}{x}_E$ is identical with either of the two zero order approximations of two eigenvalues of field equations which are given by elastic properties of both components.

On the other hand the first order approximation $\overset{\circ}{x}_1$ follows from the solvability condition of the equation for the kink-like and soliton-like solutions (compare formula (3.30)).

Notation

Further on we use the following notation

$$\widehat{G}(t,\sigma) = G(x,t,\sigma)\Big|_{x=x(t)},$$

for any smooth function $G(x,t,\sigma)$. This is the cutting of the function G to the following front

$$\Gamma_T = \cup_{0\leq t\leq T}\Gamma_t, \quad \Gamma_t = \{x \in \Omega,\, x = x(t)\}.$$

Conditions (3.5), (3.6) mean that we are smoothing the strong discontinuities. In relations (3.5) $\mathcal{A}^f_j$ and $\mathcal{A}^{\varrho,f}_j, j \geq 1$ are smooth, bounded soliton-like functions. It means that in condition (3.2) $y^{\pm}$ equals zero.

The functions $V^f_j + H^f_j z_0$, $\varrho^f_j + H^{\varrho,f}_j z_0$ are smooth approximations of strong discontinuities.

The anzatz of asymptotic expansions for v_s, Φ_s and ϱ_s is of the form

$$v^s_{as} = \sum_{j=1}^{N} \beta^j (V^s_1(x,t) + Y^s_j(\sigma,x,t), \quad \varrho^s_{as} = \varrho^s_0 + \sum_{j=1}^{N} \beta^j (\rho^s_j(x,t) + Y^{\varrho,s}_j(\sigma,x,t)), \tag{3.7}$$

and

$$\Phi^s_{as} = \sum_{j=1}^{N} \beta^j (\Phi^s_j(x,t) + Y^{\Phi,s}_j(\sigma, x, t)),$$

where

$$Y^s_j = H^s_j(x,t) z_0 + \mathcal{A}^s_j, \quad Y^{\varrho,s}_j = H^{\varrho,s}_j(x,t) z_0 + A^{\varrho,f}_j, \quad Y^{\Phi,s}_j = \Phi^s_j(x,t) z_0 + \mathcal{A}^{\Phi,s}_j,$$

and

$$\widehat{H^s_1} = \widehat{H^{\varrho,s}_1} = \widehat{H^{\Phi,s}_1} = 0, \quad \widehat{\partial_x H^f_1} \neq 0, \quad \text{and} \quad \widehat{\partial_x H^{\varrho,f}_j} \neq 0, \quad j \geq 1.$$

Then the functions $v^s_j + H^s_j z_0$, $\varrho^s_j + H^{\varrho,s}_j z_0$, $\Phi^s_j + H^{\Phi,s}_j z_0$ are smooth approximations of weak discontinuities.

The functions

$$V^f_j, V^s_j, \Phi^s_j, \varrho^s_j, \varrho^f_j,$$

$$H^f_j, H^s_j, H^{\varrho,s}_j, H^{\Phi,s}_j, H^{\varrho,f}_j, \ j \geq 1,$$

are smooth and bounded, and, on the other hand, the functions

$$\mathcal{A}^f_j, \mathcal{A}^{\varrho,f}_j \mathcal{A}^s_j, \mathcal{A}^{\varrho,s}_j, \mathcal{A}^{\varrho,s}_j, \quad j \geq 1,$$

are as before smooth, bounded soliton-like functions, satisfying condition (3.2).

Corresponding to (3.3) the outer asymptotic expansion has the following form:

$$V^{as}_s = \begin{cases} \beta V^s_1(x,t) + O(\beta^2) & \text{when} \quad x < x(t), \\ \beta(V^s_1(x,t) + H^s_1(x,t)) + O(\beta^2) & \text{when} \quad x > x(t) \end{cases} \tag{3.8}$$

$$\Phi^{as}_s = \begin{cases} \beta \Phi^s_1(x,t) + O(\beta^2) & \text{when} \quad x < x(t), \\ \beta(\Phi^s_1(x,t) + H^{\Phi,s}_1(x,t)) + O(\beta^2) & \text{when} \quad x > x(t) \end{cases}$$

$$V^{as}_f = \begin{cases} \beta V^f_1(x,t) + O(\beta^2) & \text{when} \quad x < x(t), \\ \beta(V^f_1(x,t) + H^f_1(x,t)) + O(\beta^2) & \text{when} \quad x > x(t) \end{cases}$$

$$\varrho^{as}_s = \begin{cases} \varrho^s_0 + \beta \varrho^s_1(x,t) + O(\beta^2) & \text{when} \quad x < x(t), \\ \varrho^s_0 + \beta(\varrho^s_1(x,t) + H^{\varrho,s}_1(x,t)) + O(\beta^2) & \text{when} \quad x > x(t) \end{cases}$$

$$\varrho^{as}_f = \begin{cases} \varrho^f_0 + \beta \varrho^f_1(x,t) + O(\beta^2) & \text{when} \quad x < x(t), \\ \varrho^f_0 + \beta(\varrho_1(x,t) + H^{\varrho,f}_1(x,t)) + O(\beta^2) & \text{when} \quad x > x(t) \end{cases}$$

$$P_{as} = \Upsilon_0(t, \beta) + O(\beta),$$

or

$$V_s^{as} = \begin{cases} \beta V_1^s(x,t) + O(\beta^2) & \text{when} \quad x > x(t), \\ \beta(V_1^s(x,t) + H_1^s(x,t)) + O(\beta^2) & \text{when} \quad x < x(t) \end{cases} \tag{3.8$_{\text{cont.}}$}$$

$$\Phi_s^{as} = \begin{cases} \beta \Phi_1^s(x,t) + O(\beta^2) & \text{when} \quad x > x(t), \\ \beta(\Phi_1^s(x,t) + H_1^{\Phi,s}(x,t)) + O(\beta^2) & \text{when} \quad x < x(t) \end{cases},$$

$$V_f^{as} = \begin{cases} \beta V_1^f(x,t) + O(\beta^2) & \text{when} \quad x > x(t), \\ \beta(V_1^f(x,t) + H_1^f(x,t)) + O(\beta^2) & \text{when} \quad x < x(t) \end{cases},$$

$$\varrho_s^{as} = \begin{cases} \varrho_0^s + \beta \varrho_1^s(x,t) + O(\beta^2) & \text{when} \quad x > x(t), \\ \varrho_0^s + \beta(\varrho_1^s(x,t) + H_1^{\varrho,s}(x,t)) + O(\beta^2) & \text{when} \quad x < x(t) \end{cases},$$

$$\varrho_f^{as} = \begin{cases} \varrho_0^f + \beta \varrho_1^f(x,t) + O(\beta^2) & \text{when} \quad x > x(t), \\ \varrho_0^f + \beta(\varrho_1(x,t) + H_1^{\varrho,f}(x,t)) + O(\beta^2) & \text{when} \quad x < x(t) \end{cases},$$

$$\Upsilon_{as} = \Upsilon_0(t,\beta) + O(\beta).$$

For the smooth part of the asymptotic expansions we obtain the following Cauchy problem in the strip Q_T:

$$\varrho_0^f \partial_t V_1^f + \kappa(n_0)\partial_x \varrho_1^f + \pi(V_1^f - V_1^s) = 0, \tag{3.9}$$

$$\partial_t \varrho_1^f + \varrho_0^f \partial_x V_1^f = 0,$$

$$\varrho_0^s \partial_t V_1^s - E(n_0)\partial_x \Phi_1^s - \pi(V_1^f - V_1^s) = 0,$$

$$\partial_t \Phi_1^s - \partial_x V_1^s = 0,$$

$$\partial_t \varrho_1^s + \varrho_0^s \partial_x V_1^s = 0,$$

with initial data:

$$V_1^f|_{t=0} = V_f^1(x), \; V_1^s|_{t=0} = V_s^1(x), \; \Phi_1^s|_{t=0} = \Phi_s^1(x),$$

$$\varrho_1^f|_{t=0} = \varrho_f^1(x), \quad \varrho_1^s|_{t=0} = \varrho_s^1(x).$$

On the other hand the amplitudes of discontinuities should satisfy the following characteristic Cauchy problem in a sufficiently small neighborhood Ω_Γ of the front Γ_T:

$$\varrho_0^f \partial_t H_1^f + \kappa(n_0)\partial_x H_1^{\varrho,f} + \pi(H_1^f - H_1^s) = 0, \tag{3.10}$$

$$\partial_t H_1^{\varrho,f} + \varrho_0^f \partial_x H_1^f = 0,$$

$$\varrho_0^s \partial_t H_1^s - E(n_0)\partial_x H_1^{\Phi,s} - \pi(H_1^f - H_1^s) = 0,$$

$$\partial_t H_1^{\Phi,s} - \partial_x H_1^s = 0,$$

$$\partial_t H_1^{\varrho,s} + \varrho_0^s \partial_x H_1^s = 0,$$

with the Cauchy data on the front Γ_T:

$$H_1^f|_{\Gamma_T} = H(x),\ H_1^{\varrho,f}|_{\Gamma_T} = \frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E} H(x),\ H_1^s|_{\Gamma_T} = H_1^{\Phi,s}|_{\Gamma_T} = H_1^{\varrho,s}|_{\Gamma_T} = 0,$$

which corresponds to the linear Rankine-Hugoniot conditions.

This solution can be continued as a solution of the system (3.9) either in the domain $Q_T^+ = \{(x,t) \in Q_T, x > x(t), t \in (0,T)\}$, if $z_0^+ = 1$ (see (3.2)), or in the domain $Q_T^- = \{(x,t) \in Q_T, x < x(t), t \in (0,T)\}$, if $z_0^+ = 0$.

3.2 Propagation equation for the front

Now let us analyse nonlinear terms in (2.7), (2.8). One has

$$n_E = n_0 + \beta\Big(\frac{\partial n_E}{\partial \varrho_f}(\varrho_0^f, \varrho_0^s)(\varrho_1^f + Y_1^{\varrho,f}) + \frac{\partial n_E}{\partial \varrho_s}(\varrho_0^f, \varrho_0^s)(\varrho_1^s + Y_1^{\varrho,s})\Big)$$

$$+\frac{1}{2}\beta^2\Big\{\frac{\partial^2 n_E}{\partial \varrho_f^2}(\varrho_0^f, \varrho_0^s)(\varrho_1^f + Y_1^{\varrho,f})^2 + 2\frac{\partial^2 n_E}{\partial \varrho_f \partial \varrho_s}(\varrho_0^f, \varrho_0^s)(\varrho_1^f + Y_1^{\varrho,f})(\varrho_1^s + Y_1^{\varrho,s}) +$$

$$\frac{\partial^2 n_E}{\partial \varrho_f^2}(\varrho_0^f, \varrho_0^s)(\varrho_1^s + Y_1^{\varrho,s})^2\Big\} + O(\beta^3).$$

Also we have

$$\partial_x\Big(\kappa(n_E)(\varrho_{as}^f - \varrho_f^0)\Big) = \beta\kappa(n_0)\partial_x(\varrho_1^f + Y_1^{\varrho,f}) + \beta^2\Big\{\kappa(n_0)\partial_x(\varrho_2^f + Y_2^{\varrho,f}) +$$

$$+\kappa'(n_0)(\varrho_1^f + Y_1^{\varrho,f})\partial_x\Big(\frac{\partial n_E}{\partial \varrho_f}(\varrho_0^f, \varrho_0^s)(\varrho_1^f + Y_1^{\varrho,f}) + \frac{\partial n_E}{\partial \varrho_s}(\varrho_0^f, \varrho_0^s)(\varrho_1^s + Y_1^{\varrho,s})\Big)\Big\} + O(\beta^3).$$

Using equations for mass densities in (2.8), one obtains

$$\partial_t n_E = \mathcal{N}_E - \gamma_0^s \varrho_0^s \partial_x v_s - \gamma_0^f \varrho_0^f \partial_x v_f,$$

where

$$\mathcal{N}_E = -\partial_{\varrho_f} n_E \partial_x\Big((\varrho_f - \varrho_0^f)v_f\Big) - \partial_{\varrho_s} n_E \partial_x\Big((\varrho_s - \varrho_0^s)v_s\Big) - \partial_{\varrho_f} n_E\, v_f \partial_x \varrho_0^f - \partial_{\varrho_s} n_E\, v_s \partial_x \varrho_0^s -$$

$$(\partial_{\varrho_f} n_E - \gamma_0^f)\varrho_0^f \partial_x v_f - (\partial_{\varrho_s} n_E - \gamma_0^s)\varrho_0^s \partial_x v_s.$$

Then it is not difficult to show that $\mathcal{N}_E = O(\beta)$.

Substituting the expansions Π_{as}, v_{as}^f and ϱ_{as}^f into (2.7), (2.8) one gets from the lowest approximation (of the order $O(1/\beta)$) :

Lemma 3.1 *Strong discontinuities cannot exist simultaneously for v_s and v_f. A strong discontinuity of v_s yields a weak discontinuity of v_f and vice versa.*

Propagation velocities of the fronts of strong discontinuity of v_s and v_f satisfy the equation

$$\overset{\circ}{x}{}_E^2 = \frac{E(n_0)}{\widehat{\varrho_0^s}}, \tag{3.11}$$

or

$$\overset{\circ}{x}{}_E^2 = \kappa(n_0), \tag{3.12}$$

respectively.

Comment 1 *In what follows we consider in detail the case of a strong discontinuity for v_f. For the case of a strong discontinuity for v_s we solely quote general results.*

Let us denote

$$\Upsilon^N = \sum_{j=0}^{N} \beta^j \Upsilon_j, \quad \mathcal{P}_N^p = \sum_{j=1} \beta^j \Pi_j, \quad x_N^\beta(t) = \sum_{j=0}^{N} \beta^j x_j(t),$$

$$\mathcal{V}_N^s = \sum_{j=0}^{N} \beta^j V_j^s, \;\; \mathcal{V}_N^f = \sum_{j=0}^{N} \beta^j V_j^f, \;\; \mathcal{Y}_N^s = \sum_{j=1} \beta^j Y_j^s, \;\; \mathcal{Y}_N^f = \sum_{j=1} \beta^j Y_j^f,$$

and use the fact that in Ω_Γ any smooth function $g(\sigma, x, t)$ admits an expansion

$$g(\sigma, x, t) = \sum_{j=0}^{N} \frac{1}{j!} \frac{\partial^j}{\partial x^j} g\Big|_\Gamma \sigma^j (\beta^{1+\alpha})^j + O(|x - x(t)|^{N+1}).$$

We need the lemma following from the stabilization condition (3.2):

Lemma 3.2 *Let the function $y(\sigma)$ satisfy stabilization condition (3.2) and $\sigma = (x - x(t))/\beta^k$, $k \geq 1$. Then for any smooth function G one has*

$$G(x,t) y'((x - x(t))/\beta^k) = \widehat{G}(x,t) y'((x - x(t))/\beta^k)$$
$$+ \beta^k \frac{\partial}{\partial t} \widehat{G}(x,t) \sigma y'((x - x(t))/\beta^k) + O(\beta^{2k}).$$

Proof. Let us expand the function G in the Taylor series on the front Γ_T. One obtains:

$$G(x,t) = \widehat{G}(t) + \widehat{\partial_x G}(t)(x - x(t)) + O((x - x(t))^2) =$$
$$= \widehat{G}(t) + \beta^k \widehat{\partial_x G} \sigma + O(\beta^{2k} \sigma^2),$$

and

$$G(x,t) y' = \widehat{G}(x,t) y' + \beta^k \widehat{\partial_x G}(t) \sigma \, y' + O(\beta^{2k}).$$

Due to (3.2) $\sigma^2|y'| < C_2$ and, consequently, $O((x-x(t))^2)y'((x-x(t)/\beta^k) = O(\beta^{2k})$.

Now let us substitute (3.1), (3.4), (3.6), (3.7) in (2.7). Using a standard procedure one obtains the following relation for the functions Υ^N and $\mathcal{P}_N^p$:

$$\beta\partial_t(\Upsilon_N + \Pi_N) + \frac{\Upsilon_N}{\beta} + \partial_x(\varphi(n_E)(V_N^f + Y_N^f - V_N^s - Y_N^s)) + \frac{1}{\beta}\Big(- \overset{\circ}{x}_N \,\Pi_N' + \Pi_N$$

$$+(V_N^s + Y_N^s)\Pi_N' + \frac{d}{d\sigma}(\varphi(n_E)(V_N^f + Y_N^f - V_N^s - Y_N^s) = \beta^{1+N} f_N^p(\sigma, x, t), \qquad (3.13)$$

where $Y' = \frac{d}{d\sigma}Y$, $f_N^p \in C^\infty(\Omega_\Gamma \times R^1 \times [0,T])$ is some function bounded in the norm $C(\Omega_\Gamma)$. It should be noted that in (3.13) the slow variables x and t and the fast variable σ are assumed to be independent.

The leading part of the asymptotic expansion of (2.7) has the form:

$$\beta\partial_t\Upsilon_0 + \frac{\Upsilon_0}{\beta} - \overset{\circ}{x}_0 \,\Pi_0' + \Pi_0 + + \frac{d}{d\sigma}(\varphi(n_0)(Y_0^f - Y_0^s) = O(1).$$

Whence the function $\Upsilon_0(t,\beta)$ defines a boudary layer with respect to initial data

$$\beta\partial_t\Upsilon_0 + \frac{\Upsilon_0}{\beta} = 0 \;\rightarrow\; \Upsilon_0(t,\beta) = \Upsilon_0^0 \exp(-t/(\beta^2)), \;\; \Upsilon_0^0 = \text{const},$$

and one gets the following equation for the function $\widehat{\Pi}_0$

$$- \overset{\circ}{x}_E \,\widehat{\Pi}_0' + \widehat{\Pi}_0 + \Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\Big)(Y_1^s)' = 0. \qquad (3.14)$$

Condition 3.1 *Let us assume that the initial data ρ_0^f and ρ_0^s are such that*

$$\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) > 0 \quad on \quad \Gamma_T.$$

Upon substitution of (3.4) into (2.8) one obtains:

$$- \overset{\circ}{x}_E \,(\widehat{Y_1^{\varrho,f}})' + \widehat{\varrho}_0^s(\widehat{Y_1^f})' = O(1).$$

Obviously due to (3.2) the following equation holds true

$$\overset{\circ}{x}_E \,\widehat{Y_1^{\varrho,f}} = \widehat{\varrho_0^s}\widehat{Y_1^f}. \qquad (3.15)$$

Analogously, from equations (2.10) we obtain:

$$\overset{\circ}{x}_E \,\widehat{\varrho_0^f}(\widehat{Y_1^1})' + \kappa(n_E)(\widehat{Y_1^{\varrho,f}})' = 0.$$

This equation has a nontrivial solution

$$\widehat{Y_1^{\varrho,f}} = -\frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}\widehat{Y_1^f}, \tag{3.16}$$

if

$$(\overset{\circ}{x}_E)^2 = \kappa(n_0). \tag{3.17}$$

Thus from (3.17) it follows that discontinuities of v_s and v_f cannot exist simultaneously. Indeed, the front of the smooth approximation to the strong discontinuity of v_f moves with velocity $\pm\sqrt{\kappa(n_0)}$ but the front of the smooth approximation to the strong discontinuity of v_s moves with velocity $\pm\sqrt{E(n_0)/\widehat{\varrho_0^s}}$. However, due to the physical interpretation (characteristic speeds of the so-called $P1$ - and $P2$ - waves)

$$\sqrt{E(n_0)/\widehat{\varrho_0^s}} \neq \sqrt{\kappa(n_0)}.$$

Let us define the functions

$$\widehat{Y_1^f} = \widehat{H_1^f}(t)z_0(\sigma,t) \quad \widehat{Y_1^{\varrho,f}} = \widehat{H_1^{\varrho,f}}(t)z_0,$$

where

$$\widehat{H_1^{\varrho,f}} = \frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}\widehat{H_1^f}. \tag{3.18}$$

The kink-function z_0 will be defined below.

Let us note that the equation (3.18) defines the relation between the jumps of the velocity v_f and the density ϱ_f (the so-called Rankine- Hugoniot condition on the front of propagation). Equation (3.17) is the equation for the propagation velocity $\overset{\circ}{x}_E$ of the front Γ_T.

Lemma 3.2 is proved.

Let us note that functions V_1^f, H_1^f, z_0, Π_0 are still unknown.

3.3 General result

Now we can formulate the general result. We show that profiles of solutions depend on the initial propagation velocity $\overset{\circ}{x}\,|_{t=0}$ of the front Γ_T, the initial jump $H_1^f|_{t=0}$ of the velocity v_f and the initial correction $\overset{\circ}{x}_1\,|_{t=0}$ to the propagation velocity of the front.

The results of the previous sections can be summarized in the form of the following

Theorem 3.1 *Let*

$$\tau = \beta^2,$$

and the following inequalities hold:

$$1+\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}>0, \quad \textit{and} \quad \varphi(n_0)-\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})>0.$$

Let us suppose that the propagation velocity of the front Γ_T *is defined by the equation*

$$\overset{\circ}{x}{}_E^2=\kappa(n_0),$$

and that the first correction to the propagation velocity of the front is described by the equation:

$$\overset{\circ}{x}_1=\widehat{V_1^f}+\frac{1}{2}\Big(1+\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}. \tag{3.19}$$

Then asymptotic solutions (3.1), (3.4), (3.7) exist with any accuracy on the short time interval $(0,T)$. *The leading part of this asymptotics*

$$\begin{aligned}
v^s_{as} &= \beta(V_1^s(x,t)+H_1^s(x,t)z_0(\sigma,t))+O(\beta^2),\\
\Phi^s_{as} &= \beta(\Phi_1^s+H_1^{\Phi,s}z_0(\sigma,t)+O(\beta^2),\\
v^f_{as} &= \beta(U_1^f(x,t)+H_1^f(x,t)z_0(\sigma,t))+O(\beta^2),\\
\varrho^s_{as} &= \varrho_0^s(x)+\beta(\varrho_1^s(x,t)+H_1^{\varrho,s}z_0)+O(\beta^2),\\
\varrho^f_{as} &= \varrho_0^f(x)+\beta(\varrho_1^f(x,t)+H_1^{\varrho,f}(x,t)z_0)+O(\beta^2),\\
\Pi_{as} &= \Upsilon_0(t,\beta)+\Pi_0(\sigma,t)+O(\beta),
\end{aligned}$$

satisfies system (2.7), (2.8) up to the order $O(\beta)$.

The background functions $V_1^f, V_1^s, \varrho_1^f, \varrho_1^s$ *and* Φ_1^s *are the solution of the Cauchy problem (3.9) in the strip* Q_T. *The functions* $H_1^f, V_1^s, \varrho_1^f, \varrho_1^s$ *and* Φ_1^s *solve the characteristic Cauchy problem (3.10).*

Also one has

$$\Upsilon_0(t,\beta)=\Upsilon_0^0\exp(-\frac{t}{\beta^2}),\ \Upsilon_0^0=\text{const}\,.$$

Π_0 *is the soliton-like function*

$$\Pi_0=\widehat{\varrho_0^f}\Big(1+\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(\widehat{H_1^f})^2\, z_0(1-z_0), \tag{3.20}$$

where the kink-like function z_0 *satisfying (3.2) is a strictly monotonic solution of the nonlinear equation*

$$\Big(\varphi(n_0)-\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})-\overset{\circ}{x}_E\ \widehat{\varrho_0^f}\widehat{H_1^f}\Big(1+\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(1-2z_0)\Big)z_0'= \tag{3.21}$$

$$=-\widehat{\varrho_0^f}\Big(1+\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f},\widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}\, z_0(1-z_0).$$

Moreover the function z_0 increases if

$$\widehat{H_1^f} < 0 \quad t \in [0, T], \tag{3.22}$$

and it decreases if

$$\widehat{H_1^f} > 0 \quad t \in [0, T].$$

The strictly monotonic bounded solution of nonlinear equation (3.10) exists if the following condition is satisfied:

$$\left|\widehat{H_1^f}\right| < \frac{\left(\varphi(n_0) - \widehat{\varrho_0^f} \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\right)}{\widehat{\varrho_0^f} \sqrt{\kappa(n_0)} \left(1 + \widehat{\varrho_0^f} \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) \frac{\kappa'(n_0)}{\kappa(n_0)}\right)}. \tag{3.23}$$

Amplitude $\widehat{H_1^f}$ of the jump of strong discontinuity for v_f is defined uniquely as the smooth solution of the following ordinary differential equation

$$2\widehat{\varrho_0^f} \frac{d}{dt} \widehat{H_1^f} + \pi \, \widehat{H_1^f} = 0, \quad \widehat{H_0^F}|_{t=0} = H_1^f(0), \tag{3.24}$$

i.e.

$$\widehat{H_1^f}(t) = H_1^f(0) \exp(-\frac{1}{2\varrho_0^f} \, t),$$

which is, of course, the decreasing function.

Comment 2 *Inequalities (3.22) define the time of existence of asymptotic solutions. They form an entropy-like condition for the jump of discontinuity of v_f.*

3.4 Soliton-like solution for porosity

Now we find the functions z_0 and Π_0.

Lemma 3.3 *Let the correction $\overset{\circ}{x}_1$ of the propagation velocity $\overset{\circ}{x}_E$ of the front Γ_T satisfy the equation:*

$$\overset{\circ}{x}_1 = \widehat{V_1^f} + \frac{1}{2}\left(1 + \widehat{\varrho_0^f} \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) \frac{\kappa'(n_0)}{\kappa(n_0)}\right) \widehat{H_1^f}.$$

Then the soliton-like function Π_0 has the form:

$$\Pi_0 = \widehat{\varrho_0^f}\left(1 + \widehat{\varrho_0^f} \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) \frac{\kappa'(n_0)}{\kappa(n_0)}\right) (\widehat{H_1^f})^2 \, z_0(1 - z_0). \tag{3.25}$$

The kink function z_0 is strictly monotonic solution of the nonlinear equation

$$\left(\varphi(n_0) - \widehat{\varrho_0^f} \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) - \overset{\circ}{x}_E \, \widehat{\varrho_0^f} \widehat{H_1^f} \left(1 + \widehat{\varrho_0^f} \frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) \; frac \kappa'(n_0) \kappa(n_0)\right)(1 - 2z_0)\right) z_0' =$$

$$= -\widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}\, z_0(1 - z_0), \tag{3.26}$$

satisfying the Cauchy data (3.2). Here the variable t is a parameter.

The function z_0 increases if

$$\widehat{H_1^f} < 0 \quad t \in [0, T], \tag{3.27}$$

and it decreases if

$$\widehat{H_1^f} > 0 \quad t \in [0, T].$$

The solution of (3.26) exists on the time interval $(0, T)$ in which the following inequalities hold:

$$1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)} > 0, \quad \varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) > 0,$$

and

$$\left|\widehat{H_1^f}\right| < \frac{\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})}{\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{\varrho_0^f}\sqrt{\kappa(n_0)}}. \tag{3.28}$$

The first correction to the equations (2.8) yields:

$$- \overset{\circ}{x}_E\ \widehat{\varrho_0^f}(\widehat{Y_2^f})' + \kappa(n_0)(\widehat{Y_2^{\varrho,f}})' = \Big(\ \overset{\circ}{x}_1\ \widehat{\varrho_0^s} + \overset{\circ}{x}_E\ \varrho_1^f - \widehat{\varrho_0^f}\widehat{V_1^f})(\widehat{Y_1^f})'$$

$$-\Pi_0' - 2\widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\kappa'(n_0)\frac{\widehat{\varrho_0^f}}{\kappa(n_0)}Y_1^f(Y_1^f)', \tag{3.29}$$

$$- \overset{\circ}{x}_E\ (\widehat{Y_2^{\varrho,f}})' + \widehat{\varrho_0^f}(\widehat{Y_2^f})' = (\overset{\circ}{x}_1 - \widehat{\varrho_1^f} - \widehat{V_1^f}\frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E})(Y_1^f)' - 2\frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}\widehat{Y_1^f}(\widehat{Y_1^f})'.$$

The system (3.29) has the solution

$$\overset{\circ}{x}_E\ \widehat{Y_2^{\varrho,f}} = \widehat{\varrho_0^f}\widehat{Y_2^f} + \Big(\ \overset{\circ}{x}_1 - \widehat{\varrho_1^f} - \widehat{V_0^f}\frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}\Big)Y_1^f - \frac{\widehat{\varrho_0^f}}{\overset{\circ}{x}_E}\Big(Y_1^f\Big)^2,$$

if the following compatibility condition is satisfied

$$2\widehat{\varrho_0^f}\Big(\ \overset{\circ}{x}_1 - \widehat{V_1^f}\Big)(\widehat{Y_1^f})' - 2\widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)Y_1^f(Y_1^f)' = \Pi_0'. \tag{3.30}$$

Integrating (3.30) one gets

$$\Pi_0 = 2\widehat{\varrho_0^f}\Big(\ \overset{\circ}{x}_1 - \widehat{V_1^f}\Big)\widehat{Y_1^f} - \widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(Y_1^f)^2. \tag{3.31}$$

From (3.30) one can define the first correction $\overset{\circ}{x}_1$ for the velocity $\overset{\circ}{x}_E$ of the front Γ_T:

$$2\widehat{\varrho_0^f}\Big(\overset{\circ}{x}_1 - \widehat{V_1^f}\Big) = \widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}.$$

Then the expression (3.31) for Π_0 can be simplified

$$\Pi_0 = \widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(\widehat{H_1^f})^2\, z_0(1 - z_0). \tag{3.32}$$

Comment 3 *Let us assume that*

$$1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)} > 0, \quad \text{and} \quad \varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) > 0. \tag{3.33}$$

The first one follows from the physical assumption (1.1) while the second one limits the flux function φ.

Substituting (3.32) into (3.14) one gets an equation for the kink function z_0:

$$\Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) - \overset{\circ}{x}_E\, \widehat{\varrho_0^f}\widehat{H_1^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(1 - 2z_0)\Big) z_0' =$$

$$= -\widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}\, z_0(1 - z_0). \tag{3.34}$$

Condition 3.2 *The amplitude $\widehat{H_1^f}$ should be bounded*

$$\Big|\widehat{H_1^f}\Big| < \frac{\Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\Big)}{\widehat{\varrho_0^f}\sqrt{\kappa(n_0)}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)}, \quad t \in [0, T], \tag{3.35}$$

for the finite time of existence of the soliton-like solution.

Obviously under condition (3.35) equation (3.34) has the unique strictly monotonic solution z_0 satisfying (3.2). The function z_0 increases if

$$\widehat{H_1^f} < 0 \quad t \in [0, T], \tag{3.36}$$

and it decreases if

$$\widehat{H_1^f} > 0 \quad t \in [0, T].$$

Simultaneously

$$\lim_{\sigma \to -\infty} z_0 = 0, \quad \lim_{\sigma \to \infty} z_0 = 1,$$

or

$$\lim_{\sigma \to -\infty} z_0 = 0, \quad \lim_{\sigma \to \infty} z_0 = -1.$$

Comment 4 *The product $(\widehat{H_1^f}\, z_0)$ is always decreasing. Hence the inequality (3.36) is the entropy-like condition for the jump of discontinuity of v_f.*

Lemma 3.3 is proved.

3.5 Negative soliton-like solution

Let us formulate the similar result for the case of the first correction $\overset{\circ}{x}_1$ of the propagation velocity of the front $\overset{\circ}{x}_E$ corresponding to the negative soliton-like solution of the system (3.14), (3.30).

Theorem 3.2 *Let*

$$\tau = \beta^2$$

and the following inequalities hold:

$$1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0}{\kappa(n_0)} > 0, \quad and \quad \varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) > 0.$$

Let us suppose that the propagation velocity of the front Γ_T is defined by the equation

$$\overset{\circ}{x}{}_E^2 = \kappa(n_0)$$

and the first correction to the propagation velocity of the front is described by the equation:

$$\overset{\circ}{x}_1 = \widehat{V_1^f} - \frac{1}{2}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f}. \tag{3.37}$$

Then the asymptotic solution of (3.1), (3.4), (3.7) exists with an arbitrary accuracy on the short time interval $(0, T)$. The leading part of this asymptotics

$$\begin{aligned}
v_{as}^s &= \beta(V_1^s(x,t) + H_1^s(x,t)z_0(\sigma,t)) + O(\beta^2),\\
\Phi_{as}^s &= \beta(\Phi_1^s + H_1^{\Phi,s}z_0(\sigma,t) + O(\beta^2),\\
v_{as}^f &= \beta(U_1^f(x,t) + H_1^f(x,t)z_0(\sigma,t)) + O(\beta^2),\\
\varrho_{as}^s &= \varrho_0^s(x) + \beta(\varrho_1^s(x,t) + H_1^{\varrho,s}z_0) + O(\beta^2),\\
\varrho_{as}^f &= \varrho_0^f(x) + \beta(\varrho_1^f(x,t) + H_1^{\varrho,f}(x,t)z_0) + O(\beta^2),\\
\Pi_{as} &= \Upsilon_0(t,\beta) + \Pi_0(\sigma,t) + O(\beta),
\end{aligned}$$

satisfies system (2.7)-(2.12) up to the order $O(\beta)$.

The background functions $V_1^f, V_1^s, \varrho_1^f, \varrho_1^s$ and Φ_1^s are the solution of the Cauchy problem (3.9) in the strip Q_T. The functions $H_1^f, V_1^s, \varrho_1^f, \varrho_1^s$ and Φ_1^s satisfy the characteristic Cauchy problem (3.10).

Also one has

$$\Upsilon_0(t,\beta) = \Upsilon_0^0\exp(-\frac{t}{\beta^2}),\ \Upsilon_0^0 = \text{const}.$$

Π_0 *is the soliton-like function*

$$\Pi_0 = -\widehat{\varrho_0^f}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)(\widehat{H_1^f})^2\, z_0(1+z_0). \tag{3.38}$$

The kink-like function z_0 is strictly monotonic solution of the nonlinear equation

$$\Big(\varphi(n_0) - \widehat{\varrho_0^f}\frac{\partial n}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s}) + \overset{\circ}{x}_E\, \widehat{\varrho_0^f}\widehat{H_1^f}\big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\big)(1+2z_0)\Big)z_0' = \tag{3.39}$$

$$= \widehat{\varrho_0^f}\big(1 + \widehat{\varrho_0^f}\frac{\partial n}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\big)\widehat{H_1^f}\, z_0(1+z_0),$$

satisfying the following conditions in infinity

$$\lim_{\sigma\to-\infty} z_0 = -1, \quad \lim_{\sigma\to-\infty} z_0 = 0,$$

or

$$\lim_{\sigma\to-\infty} z_0 = 0, \quad \lim_{\sigma\to-\infty} z_0 = -1.$$

Hence the function z_0 increases if

$$\widehat{H_1^f} > 0 \quad t \in [0, T], \tag{3.40}$$

and it decreases if

$$\widehat{H_1^f} < 0 \quad t \in [0, T].$$

The strictly monotonic bounded solution of nonlinear equation (3.39) exists if the condition (3.23) holds.

The amplitude $\widehat{H_1^f}$ of jump of strong discontinuity of v_f is defined uniquely as a smooth solution of the ordinary differential equation (3.24), so that

$$\widehat{H_1^f}(t) = H_1^f(0)\exp(-\frac{1}{2\varrho_0^f}\, t),$$

is the decreasing function.

3.6 Comments.

We have shown above that the asymptotic solution has a strongly defined structure of the leading part. Namely:

1) the monotonic increment of the leading part of asymptotic solution to v_f

$$\widehat{H_1^f}(t)(z_0^+ - z_0^-) > 0, \quad t \in (0, T),$$

corresponds to the negative leading part of the asymptotic solution for Π and vice-versa. Hence one has always

$$\Big((\widehat{Y_1^f})^+ - (\widehat{Y_1^f})^-\Big)\,\mathrm{sign}\,\Pi_0 < 0 \quad t \in (0, T). \tag{3.41}$$

2) the following inequality, which is the consequence of the relation (3.18)

$$\overset{\circ}{x}_E \left((\widehat{Y_1^f})^+ - (\widehat{Y_1^f})^-\right)\left((\widehat{Y_1^{\varrho,f}})^+ - (\widehat{Y_1^{\varrho,f}})^-\right) =$$

$$= \overset{\circ}{x}_E \widehat{H_1^f}\widehat{H_1^{\varrho,f}}(z_0^+ - z_0^-)^2 > 0 \quad t \in (0,T), \tag{3.42}$$

holds true, i.e. the sign of the product $\left((\widehat{Y_1^f})^+ - (\widehat{Y_1^f})^-\right)\left((\widehat{Y_1^{\varrho,f}})^+ - (\widehat{Y_1^{\varrho,f}})^-\right)$ corresponds to the sign of the propagation velocity of the front.

Let us investigate the influence of the behavior of the leading part of the amplitude H_1^f on the profile of the function Y_1^f in a neighborhood Ω_Γ of Γ_T.

Due to equation (3.10) there exists a potential function $Z(x,t)$ such that

$$\partial_t Z = \varrho_0^f H_1^f, \quad \partial_x Z = -H_1^{\varrho,f} \text{ in } \Omega_\Gamma,$$

because, according to the Cauchy data, the functions $H_1^s, H_1^{\varrho,s}$ may be zero in a neighborhood $\Omega'_\Gamma \in \Omega_\Gamma$. Then condition (3.18) on Γ_T can be rewritten as

$$\overset{\circ}{x}_E \partial_x Z + \partial_t Z = \frac{d}{dt} Z = 0 \quad t \in (0,T).$$

Hence we can suppose that on the propagation front

$$Z|_{\Gamma_T} = 0. \tag{3.43}$$

The system of the first two equations (3.10) can be reduced to the equation

$$\partial_t^2 Z - \kappa(n_0)\partial_x^2 Z + \frac{\pi}{\varrho_0^f}\partial_t Z = 0 \text{ in } \Omega_T. \tag{3.44}$$

After the transformation

$$\xi = \frac{1}{2}(t - \frac{x}{\overset{\circ}{x}_E}), \quad \eta = \frac{1}{2}(t + \frac{x}{\overset{\circ}{x}_E}),$$

(3.44) has the form

$$\partial_\xi \partial_\eta Z + \frac{\pi}{2\varrho_0^f}(\partial_\xi + \partial_\eta) Z = 0, \tag{3.45}$$

in a neighborhood of the line $\xi = 0$ if $\overset{\circ}{x}_E = \sqrt{\kappa(n_0)}$ or $\eta = 0$ if $\overset{\circ}{x}_E = -\sqrt{\kappa(n_0)}$ such that $Z|_{\xi=0} = 0$ or $Z|_{\eta=0} = 0$, respectively.

A solution of the problem (3.45) is sought in the form:

$$Z = \sum_{j\geq 1} a_j(\eta)\frac{\xi^j}{j!} \quad \text{or} \quad Z = \sum_{j\geq 1} a_j(\xi)\frac{\eta^j}{j!},$$

depending on whether $\xi = 0$ or $\eta = 0$. We show that it corresponds to the following coefficients

$$a_j(\eta) = P_j(\eta)\exp(-\frac{\pi}{2\varrho_0^f}\eta), \quad \text{or} \quad a_j(\xi) = P_j(\xi)\exp(-\frac{\pi}{2\varrho_0^f}\xi),$$

where P_j are polynomials of degree $j-1$, $j \geq 1$.

To be specific let us consider the case when $\overset{\circ}{x}_E = \sqrt{\kappa(n_0)}$. Then for example the coefficient a_1 satisfies the following equation

$$\frac{d}{d\eta}a_1 + \frac{\pi}{2\varrho_0^f}a_1 = 0 \implies a_1 = a_1(0)\exp(-\frac{\pi}{2\varrho_0^f}\eta).$$

Hence the solution has the form:

$$H_1^f(x,t) = \frac{1}{\varrho_0^f}\partial_t\Big(\sum_{j\geq 1} P_j(\frac{1}{2}(t+\frac{x}{\overset{\circ}{x}_E}))\exp(-\frac{\pi}{4\varrho_0^f}(t+\frac{x}{\overset{\circ}{x}_E}))\frac{(t-\frac{x}{\overset{\circ}{x}_E})^j}{2^j j!}\Big),$$

or

$$H_1^f(x,t) = \frac{1}{\varrho_0^f}\partial_t\Big(\sum_{j\geq 1} P_j(\frac{1}{2}(t-\frac{x}{\overset{\circ}{x}_E}))\exp(-\frac{\pi}{4\varrho_0^f}(t-\frac{x}{\overset{\circ}{x}_E}))\frac{(t+\frac{x}{\overset{\circ}{x}_E})^j}{2^j j!}\Big),$$

and

$$H_1^{\varrho,f}(x,t) = -\partial_x\Big(\sum_{j\geq 1} P_j(\frac{1}{2}(t+\frac{x}{\overset{\circ}{x}_E}))\exp(-\frac{\pi}{4\varrho_0^f}(t+\frac{x}{\overset{\circ}{x}_E}))\frac{(t-\frac{x}{\overset{\circ}{x}_E})^j}{2^j j!}\Big),$$

or

$$H_1^{\varrho,f}(x,t) = -\partial_x\Big(\sum_{j\geq 1} P_j(\frac{1}{2}(t-\frac{x}{\overset{\circ}{x}_E}))\exp(-\frac{\pi}{4\varrho_0^f}(t-\frac{x}{\overset{\circ}{x}_E}))\frac{(t+\frac{x}{\overset{\circ}{x}_E})^j}{2^j j!}\Big),$$

respectively.

Therefore for both cases of $\overset{\circ}{x}_E = \pm\sqrt{\kappa(n_0)}$ the function

$$H(t) = H_1^f\Big|_{\Gamma_T} = a_1(0)\exp(-\frac{\pi}{2\varrho_0^f}t),$$

is decreasing in Ω_T and obviously the second condition of (3.18) is fulfilled. Hence we have four different profiles of the function Y_1^f in Ω_T:

1) if $\overset{\circ}{x}_E = \sqrt{\kappa(n_0)}$, $\widehat{H_1^f} > 0$, and condition (3.19) is satisfied, then the function Π_0 is positive and the function Y_1^f being a product of z_0 and $H_1^f, \widehat{H_1^f} > 0$, becomes faster decreasing than z_0 ahead of Γ_T and sharper than z_0 behind Γ_T. It means that the function Y_1^f has a sharper profile. Moreover the first correction to the propagation velocity of the front has the form

$$\overset{\circ}{x}_1 = V_1^f + \frac{1}{2}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f},$$

so that if the background function $V_1^f = 0$ then the direction of $\overset{\circ}{x}_1$ coincides with the direction of propagation velocity of the front. The latter means that the front Γ_T is accelerating. This case correcponds to the physically plausible situation when an increment of velocity v_f behind the front is coupled to an increment of porosity in Ω_T and vice-versa. An increment of the velocity of the front Γ_T results in the local opening of pores. This correcponds to **the classical condition of the stable displacement (push)** [11]. Namely a denser liquid pushes the one which is less dense:

$$(\widehat{Y_1^{\varrho,f}})^- > (\widehat{Y_1^{\varrho,f}})^+ \quad \text{on} \quad \Gamma_T.$$

Local opening of pores in a neighborhood of Γ_T induces a wave in the liquid which accelerates Γ_T.

2) opposite to the case 1); let $\overset{\circ}{x}_E = \sqrt{\kappa(n_E)}$, $\widehat{H_1^f} < 0$, and condition (3.19) is satisfied. Then the function Π_0 is positive, the function Y_1^f, being a product of z_0 and H_1^f, becomes flater than z_0 in Ω_T. This case is an analogy of a rarefaction wave. It does not correspond to any physically reasonable situation, since an increment of porosity is connected with a decrement of velosity v_f ahead of the front Γ_T.

3) if $\overset{\circ}{x}_E = \sqrt{\kappa(n_0)}$, $\widehat{H_1^f} > 0$, and condition (3.37) holds true , then the function Π_0 is negative and the function Y_1^f being a product of z_0 and $H_1^f, \widehat{H_1^f} > 0$, becomes faster increasing than z_0 ahead of Γ_T and sharper then z_0 behind Γ_T. It means that the function Y_1^f has a sharper profile. The first correction to the propagation velocity of the front has the form

$$\overset{\circ}{x}_1 = V_1^f - \frac{1}{2}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n_E}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_0)}{\kappa(n_0)}\Big)\widehat{H_1^f},$$

so that if the background function $V_1^f = 0$ then the direction of $\overset{\circ}{x}_1$ is opposite to the direction of the propagation velocity of the front. This leads to the deceleration of the front Γ_T. This case correcponds to the physically plausible situation when locally in Ω_T a decrement of porosity Π results in a decrement of velocity v_f ahead of Γ_T. A decrement of velocity of the front is connected with the effect of the local closing of pores. In this case, opposite to the case 1), the condition of the so-called Saffman-Taylor instability

$$(\widehat{Y_1^{\varrho,f}})^- < (\widehat{Y_1^{\varrho,f}})^+ \quad \text{on} \quad \Gamma_T.$$

holds true. This correspond to the case when a less dense liquid pushes a denser one. Then the direction of the first correction $\overset{\circ}{x}_1$ is opposite to the direction of velocity $\overset{\circ}{x}_E$ of Γ_T. The local closing of pores results in a weakening of the Saffman-Taylor instability. The local closing of pores in a neighborhood of Γ_T yields a reflection wave in the liquid which delays the front Γ_T.

4) opposite to the case 3); let $\overset{\circ}{x}_E = \sqrt{\kappa(n_0)}$, $\widehat{H_1^f} < 0$, and condition (3.50) holds true. Then the function Π_0 is negative and the function Y_1^f, being a product of z_0 and H_1^f, becomes flater than z_0 in Ω_T. This is also an analogy of a rarefaction wave. This case is physically meaningless , since there is a decrease of porosity whereas the velosity v_f increases behind Γ_T.

In the case of the negative velocity $\overset{\circ}{x}_E = -\sqrt{\kappa(n_0)}$ we follow the same way of argument with an appropriate change of the sign. It is easy to see that in this case two waves are physically meaningful.

5) for $\widehat{H_1^f} < 0$ and under the condition (3.19) we obtain the case 1) considered before with z_0 replaced by $-z_0$.

6) for $\widehat{H_1^f} < 0$ and under the condition (3.37) we obtain the case 3) corresponding to the Saffman-Taylor instability.

Comment 5 *In the case when $\tau = \beta^2$ we have four different waves: each propagation velocity $\overset{\circ}{x}_E$ of the front defines two waves: one is characterized by the closing of pores and another one is characterized by the opening of pores. The analysis of properties of these waves, i.e. an analysis of structure of the profiles of Y_1^f and Π_0 allows one to choose proper initial data. This choice is an analogy of the entropy-like conditions which define necessary and sufficient conditions of the stable solution and necessary and sufficient conditions for the uniqueness theorem.*

Comment 6 *We illustrate the case 3) (Saffman-Taylor instability) with a numerical example. We present a sequence of 10 graphs for Π and v_f as the functions of x.*

They were obtained for the following conditions on the front Γ_T

$$v_f\big|_{\sigma=0} = \beta y\left(\frac{x}{\beta^2}\right), \quad \rho_f\big|_{\sigma=0} = \rho_0^f + \beta\frac{\rho_0^f}{\sqrt{\kappa}}y\left(\frac{x}{\beta^2}\right), \; v_s\big|_{\sigma=0} = \Phi_s\big|_{\sigma=0} \equiv 0,$$

$$\rho_s\big|_{\sigma=0} = \rho_0^s, \quad \Pi\big|_{\sigma=0} = 0, \tag{3.46}$$

where

$$y = -\frac{e^{-\sigma}}{1+e^{-\sigma}} \tag{3.47}$$

and the parameters were chosen to be $\beta = 0.1$, $\rho_0^s = 1$, $\rho_0^f = \frac{1}{7}$, $\kappa = 2.25$, $\pi = 3$. The value of the small parameter β is for real materials much smaller. However the above choice speeds up the computer calculations without changing a qualitative behavior of solutions. The curves presented in Figure 1 correspond to the following dimensionless time instances $\{0.075, 0150, 0225, 0.300, 0.375, 0.450, 0.525, 0.600, 0.675, 0.750\}$. The choice (3.47) of the function y follows as a solution of equation (3.21) in the case of $\varphi = 1$, $n_E \equiv \mathrm{const}$. In such a case the profile of v_f forms the main part of the solution. Evolution of this solution yields the correction of the solution in porosity and, simultaneously, it is smoothing the front of the velocity v_f. The coefficient β and its powers create a series of filters which result in solutions of the order β, β^2, β^3, and so on. Asymptotic analysis shous that the modes β^3 and higher lead to stagnation waves. This is visible in the Figure. First two modes propagate in a positive direction of x - axis and remaining modes stay behind the point indicationed by the thin line in the Figure.

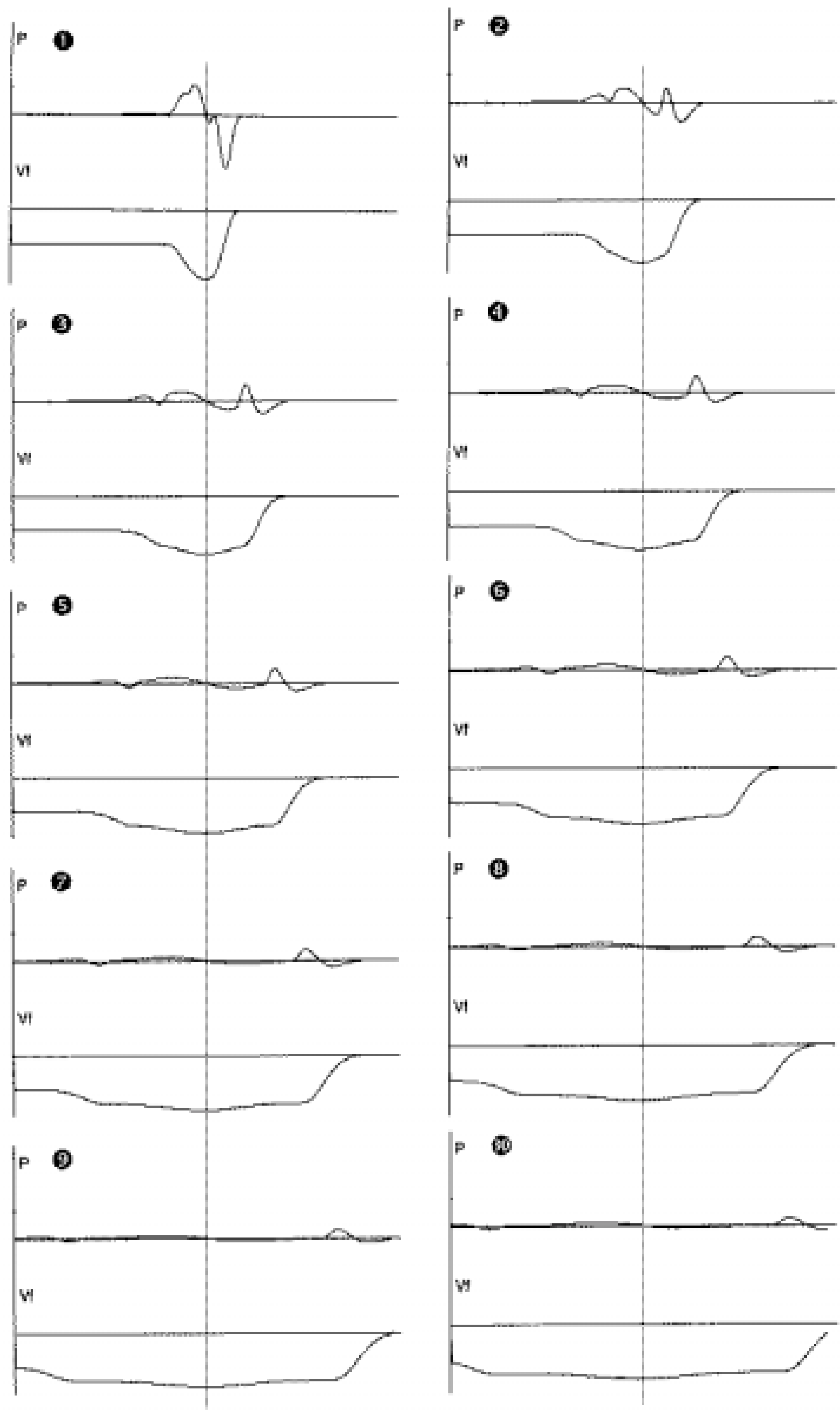

Figure 1: Propagation of the soliton of porosity and of the kink of velocity v_f for the case 3

Comment 7 (Contact problem) *Asymptotic solutions constructed above allow one to investigate the contact problem when the propagation velocity*

$$\overset{\circ}{x}_E(t,\beta) = \overset{\circ}{x}_E + \beta\,\overset{\circ}{x}_1 + \ldots$$

of the front is known and has the form $\widehat{g} = (g_0 + \beta\, g_1(x) + \ldots)|_{\Gamma_T}$*, where* g_0 *is constant. In this case if* κ *is not constant we can define uniquely the initial values* ϱ_0^f, ϱ_0^s *and the jump* H_1^f *on the front. Namely for the above conditions 1), 3) with the amplitude* $\widehat{H_1^f}(t) > 0$*, the following equation:*

$$\overset{\circ}{x}_E = \pm\sqrt{\kappa(n_0)} = g_0, \tag{3.46}$$

in which we choose the sign according to the sign of g_0*, together with the equation*

$$\widehat{g_1} = V_1^f \pm \frac{1}{2}\Big(1 + \widehat{\varrho_0^f}\frac{\partial n}{\partial \varrho_f}(\widehat{\varrho_0^f}, \widehat{\varrho_0^s})\frac{\kappa'(n_E)}{\kappa(n_E)}\Big)\widehat{H_1^f}, \tag{3.47}$$

determine these quantities provided we make use of the state equation

$$n_E(\varrho_0^f, \varrho_0^s) = n_0.$$

In equations (3.47) the sign is also chosen according to the sign of $g(x)_1$*. Consequently boundary conditions for the functions* $v_f, v_s, \varrho_s, \Phi_s$ *and* ϱ_f *on* Γ_T *can be defined from the relation (3.10). The latter means that we define the leading part of the asymptotic solution of contact problem. Following the same procedure one can define next corrections* ϱ_j^f, ϱ_j^s*,* $j \geq 1$ *to* ϱ_0^f, ϱ_0^s *and* H_j^f*,* $j \geq 2$ *to* H_1^f*, using corresponding equations for next corrections* $\overset{\circ}{x}_j$ *of the velosity* $\overset{\circ}{x}_E$ *of the front.*

Let $\widehat{g}$ *be a known velocity of* Γ_T *and* $\overset{\circ}{x}_E$ *the velocity of the longitudinal wave of the second kind (P2) corresponding to the initial density* ϱ_0^f*. Then it is not difficult to show that the difference* $\widehat{g} - \overset{\circ}{x}_E$ *defines the sign of changes of the porosity* $n - n_E$ *and the amplitude of the leading part of asymptotic solution to* $n - n_E$*. It means that the acceleration of the front* Γ_T *induces a local increament of porosity and, vice-versa, a local increament of porosity accelerates the front* Γ_T*. Therefore the soliton-like solution for porosity can be interpreted as a „filter". It opens in the case when a denser liquid pushes a less dense one, and it closes in the opposite case. Consequently, in the case of the above considered Cauchy problem, the filter prevents the Saffman-Taylor instability. For sufficiently small perturbations of* $\overset{\circ}{x}_E$*, i.e. under the existence condition (3.23) of the kink-like solution due to the solvability condition (3.24) for a solution of the characteristic Cauchy problem (3.10) the amplitude* $\widehat{H_1^f}$ *of discontinuity of* v_f *decreases in time according to (3.24) and* $|\overset{\circ}{x}_1| \to 0$*. Hence the filter is an analogy of a pendulum in the vicinity of the stable equilibrium point.*