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Unidirectional transport in stochastic ratchets

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Abstract. Constructive conditions for existence of the unidirectional transport are given for systems with state-dependent noise and for forced thermal ratchets. Using them, domains of parameters corresponding to the unidirectional transport are indicated. Some results of numerical experiments are presented.

1. Introduction

Stochastic ratchets are defined as systems which are able to produce a directed current through the rectification of noise although on average no macroscopic force is acting. A lot of recent research has been devoted to these systems (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein). Much interest in these simple nonequilibrium models is stimulated by their potential relevance with respect to protein motors, transport in noncentrosymmetric materials, and novel particle pumps and separation techniques.

Analytical and numerical studies of noise-induced directed transport mainly deal with evaluating the mean velocity. Having values of the mean velocity only, it is impossible to describe the noise-induced transport in detail. For example, it may occur so that the mean velocity is small but the trajectories walk far in both positive and negative directions. Large mean velocity may be in the case of the unidirectional transport, when there is practically no movement in one of the directions, or may be in the case when the trajectories walk in both directions far. In the paper we are interested in the noise-induced unidirectional transport. One of motivations for consideration of the unidirectional transport may be possible biological applications of the Brownian ratchets to modeling molecular motors [3, 5, 7, 10, 11, 12]. As is known (see, e.g., [7, 12]), molecular motors are microscopic objects that unidirectionally move along one-dimensional periodic structures and the problem of explaining this unidirectionality belongs to a larger class of such problems involving rectifying processes at a small scale.

In section 2, we study detailed structure of the transport in systems with state-dependent noise [1, 2] and get an analytical condition for the unidirectional transport. We consider the probability q that the trajectory $X_x(t)$, $X(0) = x$, reaches first the right (or left) end of the interval $(x - L,\ x + L)$, L is a period of the ratchet potential. This probability is found analytically by solving the corresponding boundary value problem for a second-order ordinary differential equation. The condition for the unidirectional transport in positive direction consists in closeness of the probability q to 1.

In section 3, we consider forced thermal ratchets [3, 5, 6, 7, 8, 13, 14] (see (3.1) below). Here, to propose a condition for the unidirectional transport in, e.g., positive direction, we have to consider two probabilities: the probability $Q^{<}_{0,x}$ that the trajectory $X_{0,x}(t)$, $X(0) = x$, escapes from $(x - L,\ x + L)$ through the right end during the first half-period $[0, T/2)$ of the periodic forcing with a period T, and the probability $P^{>}_{T/2,x}$ that $X_{T/2,x}(t)$, $X(T/2) = x$, does not escape from $(x-L,\ x+L)$ through the left end during the second half-period $[T/2, T)$. For large T, we get a qualitative condition of unidirectionality attracting results of section 2. But to obtain a general condition, we should evaluate the probabilities $Q^{<}_{0,x}$, $P^{>}_{T/2,x}$ by solving two boundary value problems for parabolic equations.

The condition for the unidirectional transport in positive direction consists in closeness of the product $Q^{<}_{0,x} \cdot P^{>}_{T/2,x}$ to 1. As a result, we propose an effective (numerical) tool for indicating domains of parameters, where the noise-induced unidirectional transport is realized. The technique is universal and can be applied to various systems with the noise-induced transport. Some results of numerical experiments are presented.

2. Systems with state-dependent diffusion

It was shown in [1, 2] (see also [6, 14]) that state-dependent diffusion can induce transport in a system which is at equilibrium in the presence of thermal noise only. Here we are interested in detailed structure of the transport in such systems. We consider the stochastic differential equation (SDE) in the sense of Ito:

$$dX = f(X)dt + \sigma(X)dw(t), \tag{2.1}$$

where $f(x)$ and $\sigma(x)$ are L-periodic functions and $w(t)$ is a standard Wiener process.

Introduce the process $\Phi(t) = X(t) \pmod L$ on a circle of radius $L/2\pi$. The process $\Phi(t)$ is continuous on the circle. Due to the periodicity of f and σ, we can write (2.1) in the form

$$dX = f(\Phi)dt + \sigma(\Phi)dw(t). \tag{2.2}$$

Under a sufficiently wide assumptions (e.g., $\sigma(x) \neq 0$, $x \in R$), $\Phi(t)$ is an ergodic process (see, e.g., [15]). Its invariant density $p(\varphi)$, $0 \leq \varphi \leq L$, is L-periodic and it satisfies the stationary Fokker-Planck equation

$$\frac{1}{2}\frac{\partial^2}{\partial\varphi^2}(\sigma^2 p) - \frac{\partial}{\partial\varphi}(fp) = 0,\ p(0) = p(L),\ \int_0^L p(\varphi)d\varphi = 1.$$

Solving this problem, we get

$$p(\varphi) = \frac{Cr(\varphi)}{\sigma^2(\varphi)}[r(L)\int_\varphi^L r^{-1}(\vartheta)d\vartheta + \int_0^\varphi r^{-1}(\vartheta)d\vartheta], \tag{2.3}$$

where

$$r(\varphi) = \exp(\rho(\varphi)),\ \rho(\varphi) = 2\int\limits_0^\varphi \frac{f(\xi)}{\sigma^2(\xi)}d\xi,$$

and $C > 0$ is found from the condition of normalization.

Let $EX(0) < \infty$. Due to the ergodicity of $\Phi(t)$, we have for the mean velocity $\bar{v}$ of $X(t)$:

$$\bar{v} := \lim_{t\to\infty}\frac{EX(t)}{t} = \lim_{t\to\infty}\frac{EX(0)}{t} + \lim_{t\to\infty}\frac{1}{t}\int_0^t Ef(\Phi(s))ds \tag{2.4}$$

$$= \int_0^L f(\varphi)p(\varphi)d\varphi = \frac{LC}{2}[e^\rho - 1],\quad \rho := \rho(L) = 2\int_0^L \frac{f(\xi)}{\sigma^2(\xi)}d\xi.$$

The sign of $\bar{v}$ depends on the sign of $e^\rho - 1$ only. Evidently, the necessary and sufficient condition for zero mean velocity consists in the equality $\rho = \rho(L) = 0$ (cf. [1, 2, 14]).

For instance, if $\sigma \equiv$const and the potential

$$F(x) = -\int f(x)dx$$

is an L-periodic function (e.g., a ratchet potential), we get the well-known fact of thermodynamics [16] that $\bar{v} = 0$. At the same time, for an L-periodic potential $F(x)$, one can find an L-periodic state-dependent $\sigma(x)$ such that $\rho \neq 0$, i.e., $\bar{v} \neq 0$, and therefore there is a noise-induced transport [1, 2]. Our urgent aim is to find sufficient conditions for unidirectional transport.

Let $X_{s,x}(t)$ be the solution of (2.2) which starts from the point x at the moment s. If $s = 0$, we write $X_x(t)$ instead of $X_{0,x}(t)$. Consider an interval $(x - mL, x + nL)$, where m and n are positive integers. The trajectory $X_x(t)$ reaches one of the points $x-mL,\ x+nL$ for a finite (random) time τ with probability 1. Here τ is the first exit time of $X_x(t)$ from the interval $(x - mL, x + nL)$. Denote

$$p_{m,n} := P(X_x(\tau) = x - mL),\ q_{m,n} := P(X_x(\tau) = x + nL).$$

Theorem 1. *The following expressions can be established*

$$p_{m,n} = \frac{e^{n\rho} - 1}{e^{(m+n)\rho} - 1},\ q_{m,n} = \frac{e^{n\rho}(e^{m\rho} - 1)}{e^{(m+n)\rho} - 1}. \tag{2.5}$$

In particular,

$$p := p_{1,1} = \frac{1}{1 + e^{\rho}}\ ,\ q := q_{1,1} = \frac{1}{1 + e^{-\rho}}\ .$$

Proof. Let τ_y, $x - mL < y < x + nL$, be the first exit time of trajectory $X_y(t)$ from the interval $(x - mL, x + nL)$. Clearly $\tau = \tau_x$. Let φ be a function defined on the set consisting of two points: $x-mL$ and $x+nL$. It is well known [17, 18, 19] that the function

$$u(y) := E\varphi(X_y(\tau_y))$$

satisfies the following boundary value problem

$$\frac{1}{2}\sigma^2(y)\frac{d^2u}{dy^2} + f(y)\frac{du}{dy} = 0,\ x - mL < y < x + nL, \tag{2.6}$$

$$u(x - mL) = \varphi(x - mL),\ u(x + nL) = \varphi(x + nL). \tag{2.7}$$

Clearly, $u(y)$ depends on x as on a parameter, i.e., $u(y) = u(y; x)$. If $\varphi(x - mL) = 1$, $\varphi(x + nL) = 0$, then $u(y; x) = P(X_y(\tau_y) = x - mL)$ and $p_{m,n} = u(x; x)$. The function $u(y; x)$ can easily be found from (2.6)-(2.7):

$$u(y;x) = \frac{\int\limits_x^{x+nL} \exp(-2\int\limits_x^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi - \int\limits_x^{y} \exp(-2\int\limits_x^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi}{\int\limits_{x-mL}^{x+nL} \exp(-2\int\limits_x^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi}\ .$$

Consequently,

$$
p_{m,n} = u(x;x) = \frac{\int\limits_{x}^{x+nL} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi}{\int\limits_{x-mL}^{x+nL} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi} \ . \tag{2.8}
$$

We have for $k = 1, ..., m$:

$$
\int\limits_{x-kL}^{x-(k-1)L} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi = \int\limits_{x}^{x+L} \exp(-2\int\limits_{x}^{\xi-kL} \frac{f(s)}{\sigma^2(s)}ds)d\xi \ . \tag{2.9}
$$

Further, due to the periodicity of the function $\dfrac{f(s)}{\sigma^2(s)}$, we get

$$
2\int\limits_{x}^{\xi-kL} \frac{f(s)}{\sigma^2(s)}ds = 2\int\limits_{x}^{x-kL} \frac{f(s)}{\sigma^2(s)}ds + 2\int\limits_{x-kL}^{\xi-kL} \frac{f(s)}{\sigma^2(s)}ds = 2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds - k\rho \ .
$$

Therefore, from (2.9), we obtain for $k = 1, ..., m$:

$$
\int\limits_{x-kL}^{x-(k-1)L} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi = e^{k\rho}\int\limits_{x}^{x+L} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi \ . \tag{2.10}
$$

Analogously, we have for $k = 1, ..., n$:

$$
\int\limits_{x+(k-1)L}^{x+kL} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi = e^{-(k-1)\rho}\int\limits_{x}^{x+L} \exp(-2\int\limits_{x}^{\xi} \frac{f(s)}{\sigma^2(s)}ds)d\xi \ . \tag{2.11}
$$

Substituting (2.10) and (2.11) in (2.8), we obtain the first formula of (2.5). The formula for $q_{m,n}$ is a consequence of the equality $q_{m,n} = 1 - p_{m,n}$. Theorem 1 is proved.

Remark 1. The formulas (2.5) remain true with the same ρ if we consider a SDE in the sense of Stratonovich:

$$
dX = f(X)dt + \sigma(X) * dw(t).
$$

It is equivalent to the Ito equation

$$
dX = f(X)dt + \frac{1}{2}\sigma(X)\frac{d\sigma}{dx}(X)dt + \sigma(X)dw(t).
$$

We have

$$
\rho_{str}(\varphi) = 2\int\limits_{0}^{\varphi} \frac{1}{\sigma^2(\xi)}(f(\xi) + \frac{1}{2}\sigma(\xi)\frac{d\sigma}{dx}(\xi))d\xi = \rho_{ito}(\varphi) + \ln\frac{\sigma(\varphi)}{\sigma(0)} \ .
$$

Due to the periodicity of $\sigma(\varphi)$, we obtain $\rho_{str} = \rho_{str}(L) = \rho_{ito}(L) = \rho$.

We mark that the probabilities $p_{m,n}$ and $q_{m,n}$ do not depend on x. Let $\tau_0 = \tau$ be the first exit time of $X_{x_0}(t)$ from the interval $(x_0 - L, x_0 + L)$ and $x_1 := X_{x_0}(\tau_0)$. Then, $x_1 = x_0 - L$ with the probability p and $x_1 = x_0 + L$ with the probability $q = 1 - p$. Let τ_1 be the first exit time of $X_{x_1}(t)$ from the interval $(x_1 - L, x_1 + L)$. Clearly, the conditional probabilities $P(X_{x_0}(\tau_0 + \tau_1) = X_{x_1}(\tau_1) = x_1 \mp L \mid X_{x_0}(\tau_0) = x_1)$ are equal to p and q correspondingly. If we continue, we obtain the random sequences $\tau_0, \tau_1, ..., \tau_k, ...$ and $x_0 = 0,\ x_1 = X_{x_0}(\tau_0),\ ...,\ x_k = X_{x_0}(\tau_0 + ... + \tau_{k-1}),\ ...$.

The sequence τ_k. The sequence consists of independent identically distributed (i.i.d.) random variables with distribution of τ. We observe that all the basic probabilistic characteristics of the random variable τ can be found by solving deterministic differential equations. For instance, the probability $P(\tau < s,\ X_x(\tau) = x - L)$ can be evaluated by solving a mixed problem for a backward Kolmogorov equation and the characteristic function or Laplace transform for τ can be evaluated by solving a boundary value problem for an ordinary differential equation (see, e.g., [17]). It turns out that such important characteristics as $E\tau$ and $D\tau$ can be found by quadratures. Namely $E\tau = u(0)$, where $u(x)$ is a solution to the following boundary value problem [17, 18]

$$\frac{1}{2}\sigma^2(x)u'' + f(x)u' = -1,\ u(-L) = u(L) = 0. \tag{2.12}$$

We get

$$u(x) = -G(x) + C_1 \int_0^x \exp(-2\int_0^\xi \frac{f(s)}{\sigma^2(s)}ds)d\xi + C_2,$$

where

$$G(x) = \int_0^x \left[\frac{2}{\sigma^2(\xi)} \int_\xi^x \exp(2\int_\eta^\xi \frac{f(s)}{\sigma^2(s)}ds)d\eta\right] d\xi,$$

and the constants $C_1,\ C_2$ have to be found from (2.12). Thus

$$E\tau = u(0) = C_2 = \frac{e^\rho G(L) + G(-L)}{1 + e^\rho}. \tag{2.13}$$

The second moment $E\tau^2$ can also be expressed in quadratures: $E\tau^2 = v(0)$, where $v(x)$ is a solution to the boundary value problem [17]

$$\frac{1}{2}\sigma^2(x)v'' + f(x)v' = -2u(x),\ v(-L) = v(L) = 0.$$

We do not write the corresponding explicit expression for $v(x)$ because of its bulky form.

Clearly, behavior of the sum $\tau_0 + ... + \tau_N$ as $N \to \infty$ is governed by the law of large numbers and by the central limit theorem.

The sequence x_k. It is not difficult to see that $x_{k+1} = x_k + \xi_k$, where ξ_k are i.i.d. random variables. Any ξ is a Bernoulli random variable: it takes two values $-L$ and L with the probabilities $P(\xi = -L) = p = 1/(1 + e^\rho),\ P(\xi = L) = q = 1/(1 + e^{-\rho})$. The sequence x_k

can be considered as a trajectory of a random walk. The theory of such a random walk is well developed (see, e.g., [20]). In particular, the behavior of x_N for large N is governed by the central limit theorem. We note that if p is small and Np is not sufficiently large (for example, $Np < 10$) then the Poisson law for x_N is most preferable [20].

Unidirectional transport. Let $\rho > 0$. Then the transport is positive. If the probability p is small, then the retrograde steps are infrequent and in such a case it is natural to consider the transport as unidirectional one. An acceptable condition of the noise-induced unidirectional transport to the right is closeness of the probability p to zero. The probability p is smaller, when ρ is larger. So, we have got the very simple characteristic ρ of transport unidirectionality for the model (2.1). Analogously, for $\rho < 0$ the transport is negative and an acceptable condition of the noise-induced unidirectional transport to the left is closeness of the probability q to zero.

Note that there is no definite relation between the mean velocity $\bar{v}$ (see formula (2.4)) and the characteristic ρ of unidirectionality because for large (small) ρ the constant C in (2.4) can be small (large). The following relation between ρ, $\bar{v}$, and $E\tau$ holds:

$$\bar{v} = \frac{L}{E\tau} \cdot \frac{e^\rho - 1}{e^\rho + 1}\,. \tag{2.14}$$

A heuristic proof of (2.14) is as follows. The mean number of jumps to the right in the sequence $x_0, x_1, ..., x_N$ is equal to $N/(1+e^{-\rho})$, and to the left is equal to $N/(1+e^{\rho})$. Consequently, the mean advance (for the mean time $NE\tau$) is equal to $NL(e^\rho - 1)/(1+e^\rho)$. From here formula (2.14) follows. The strong proof consists in direct checking formula (2.14). It is possible due to the known expressions for $E\tau$ and the constant C.

Remark 2. Consider the piece $X_{x_k}(t)$, $\tau_{k-1} \le t \le \tau_k$, of the trajectory $X(t)$. Let $x_{k+1} = X_{x_k}(\tau_k) = x_k + L$, i.e., the considered trajectory shifts to the right at the $(k+1)$-th step. It is not to be supposed that the trajectory could not step back the distance L or more during the time (τ_{k-1}, τ_k). Indeed, the trajectory may come up close to $x_k + L$, then turn back and come up to $x_k - L$, and finally reach $x_k + L$. In such a situation, we can assert only that the trajectory does not step back the distance $2L$.

Example 1. To illustrate the results of this section, we take the coefficients of (2.1) in the form [1]:

$$f(x) = f_0 \sin(2\pi x),\ \sigma(x) = \frac{\sigma_0}{\sqrt{1 - \alpha\cos(2\pi x + \phi)}} \tag{2.15}$$

with $f_0, \sigma_0 > 0$, $0 < \alpha < 1$. In this case $\rho = \dfrac{\alpha f_0}{\sigma_0^2}\sin\phi$. Note that if $\phi = k\pi$ with integer k, ρ is equal to 0 and there is no transport.

Figure 1 gives typical trajectories of the solution $X(t)$ to (2.1) with the coefficients of (2.15). Figure 1 (left) corresponds to the regime of the unidirectional transport: there is practically no movement to the left ($p = 0.0095$). If we increase the noise intensity, the mean velocity of the transport increases but the transport becomes non-unidirectional (see figure 1 (right), the corresponding $p = 0.475$).

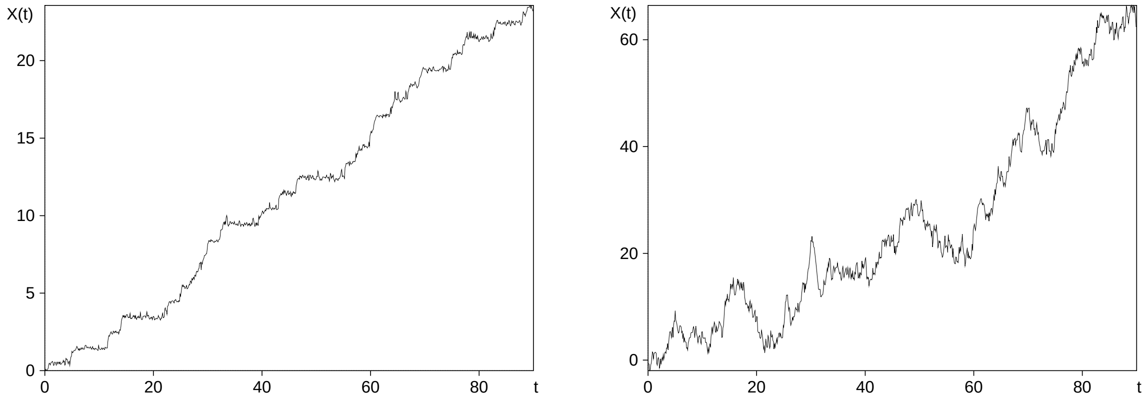

FIGURE 1. Systems with state-dependent noise. Sample trajectories of the solution to (2.1) with the coefficients of (2.15) for $f_0 = 1$, $\alpha = 0.9$, $\phi = \pi/2$, and $\sigma_0 = 0.44$ (left) and $\sigma_0 = 3$ (right).

3. Forced thermal ratchets

We consider a periodically forced thermal ratchet of the form [3] (see also [5, 6, 7, 8, 13, 14]):

$$dX = f(X)dt + A\chi(t;T)dt + \sigma dw(t), \tag{3.1}$$

where $F(x) = -\displaystyle\int f(x)dx$ is an L-periodic ratchet potential $F(x) = F(x+L)$, $x \in R$, possessing no reflection symmetry (there is no ϕ such that $F(x+\phi) = F(-x+\phi)$ for all $x \in (0, L/2)$), A, T, and σ are some positive constants,

$$\chi(t;T) = \begin{cases} \;\;\,1, & 0 \le t < T/2, \\ -1, & T/2 \le t < T, \end{cases}$$

and $\chi(t;T)$ is T-periodical.

As is known [3] (see also [5, 6, 7, 8, 13, 14]), forced thermal ratchets exhibit the noise-induced transport. Here we study conditions when the transport is unidirectional.

A qualitative condition of unidirectionality. In connection with (3.1), consider two SDEs

$$dX^+ = f(X^+)dt + Adt + \sigma dw(t), \tag{3.2}$$

$$dX^- = f(X^-)dt - Adt + \sigma dw(t). \tag{3.3}$$

Just as we get (2.4), it is possible to find expressions for the mean velocities (see, e.g., [14]): $\bar{v}^\pm = \lim_{t\to\infty} EX^\pm(t)/t$. The asymmetry of the ratchet potential $F(x)$ can result in $\bar{v}^+ \neq -\bar{v}^-$. If the period T of $\chi(t;T)$ is sufficiently large, the mean velocity $\bar{v}$ of $X(t)$ can be approximately evaluated by $\bar{v} \doteq (\bar{v}^+ + \bar{v}^-)/2$ (see [3] and also [5, 6, 7, 8, 13, 14]).

Consider the probabilities p^+ and q^+ (p^- and q^-) and the random time τ^+ (τ^-) for the process $X^+(t)$ ($X^-(t)$) introduced analogously to p, q, and τ of the previous section, namely:

$$p^{\pm} := P(X_x^{\pm}(\tau^{\pm}) = x - L), \; q^{\pm} := P(X_x^{\pm}(\tau^{\pm}) = x + L),$$

and $\tau^{\pm}$ are the first exit times of $X_x^{\pm}(t)$ from the interval $(x-L, x+L)$. Because $\rho^{\pm} := 2\int_0^L \frac{f(x) \pm A}{\sigma^2} dx = \pm \frac{2AL}{\sigma^2}$, we get from Theorem 1: $p^+ = q^-$ and $q^+ = p^-$. Let T be so large that $E\tau^{\pm} << T/2$. Evidently, in this case the transport cannot be unidirectional. Indeed, the condition of closeness of q^+ to 1 is necessary for unidirectionality of the transport. So, for the first half period the transport is positively unidirectional and for the second half period is negatively unidirectional and, consequently, it is not unidirectional as a whole. Consider another case. If $v^+ > v^-$, then $E\tau^+ < E\tau^-$ (see (2.14)). Now let T be such that $E\tau^+ << T/2$, $E\tau^- >> T/2$, and as before q^+ is close to 1. Clearly, then one can expect the unidirectional transport in (3.1) in positive direction. Analogously, if T is such that $E\tau^- << T/2$, $E\tau^+ >> T/2$, and p^- is close to 1, one can expect the unidirectional transport in negative direction. These qualitative sufficient conditions of the unidirectional transport are fairly constructive because the magnitudes $E\tau^{\pm}$ can be found by quadratures (see (2.13)).

Example 2. Consider model (3.1) with a simple ratchet potential [3, 8]

$$F(x) = \begin{cases} \dfrac{h}{l}x, \; 0 \le x < l, \\ \dfrac{h}{L-l}(L-x), \; l \le x < L. \end{cases} \tag{3.4}$$

From (2.13) we get

$$E\tau^+ = \frac{\sigma^2}{2} \cdot \frac{L^2h^2}{(Al-h)^2(A(L-l)+h)^2} \tag{3.5}$$

$$\times \frac{\exp(-\frac{2}{\sigma^2}(A(L-l)+h)) + \exp(-\frac{2}{\sigma^2}(Al-h)) - \exp(-\frac{2AL}{\sigma^2}) - 1}{\exp(-\frac{2AL}{\sigma^2}) + 1}$$

$$+ \left(\frac{l^2}{Al-h} + \frac{(L-l)^2}{A(L-l)+h}\right) \cdot \frac{1-\exp(-\frac{2AL}{\sigma^2})}{1+\exp(-\frac{2AL}{\sigma^2})}.$$

The value $E\tau^-$ is obtained by substituting $-A$ in (3.5) instead of A. Note that the indeterminacy in (3.5) if $Al = h$ (or in the corresponding formula for $E\tau^-$ if $A(L-l) = h$) can be evaluated. For instance,

$$E\tau^+ = \frac{l^2}{\sigma^2} + \frac{L^2 - l^2}{AL} \cdot \frac{1-\exp(-\frac{2AL}{\sigma^2})}{1+\exp(-\frac{2AL}{\sigma^2})} \;\; \text{if} \;\; Al = h.$$

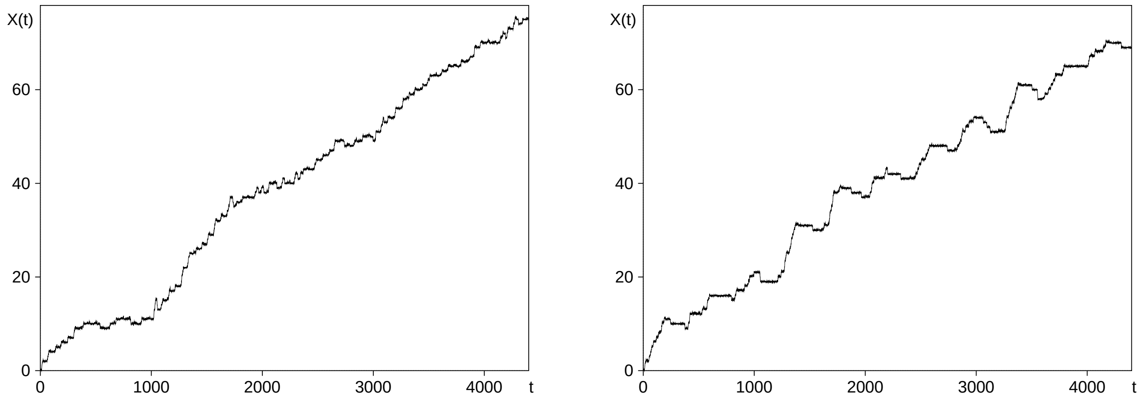

FIGURE 2. Forced thermal ratchets. Sample trajectories of the solution to (3.1) with the potential of (3.4) for $L = 1$, $l = 0.75$, $h = 1/6$, $A = 0.2$, $\sigma = 0.2$, and $T = 60$ (left) and $T = 400$ (right).

Let us take $l = \dfrac{3}{4}L$, $h = \dfrac{10}{9}Al = \dfrac{5}{6}AL$, $\dfrac{2AL}{\sigma^2} >> 1$. Then

$$E\tau^+ \doteq \frac{7200\sigma^2}{169A^2}(\exp(\frac{AL}{6\sigma^2}) - 1) - \frac{87L}{13A},$$

$$E\tau^- \doteq \frac{7200\sigma^2}{17\,689A^2}(\exp(\frac{7AL}{6s^2}) - 1) - \frac{33L}{133A}\ .$$

If, for example, $\dfrac{2AL}{\sigma^2} = 10$, $L = 1$, $A = 0.2$ (and, consequently, $\sigma = 0.2$), then $E\tau^+ \approx 22$ and $E\tau^- \approx 137$. Figure 2 gives sample trajectories of the solution $X(t)$ of (3.1) with the potential $F(x)$ from (3.4) and with the parameters described above. Figure 2 (left) corresponds to $T/2 = 30$ that is less than $E\tau^-$ and greater than $E\tau^+$. In this case the retrograde steps are infrequent and we can consider the transport as unidirectional. When we take T such that $E\tau^- << T/2$ (see figure 2 (right)), the retrograde steps become quite frequent. Let us note that guiding by the qualitative conditions it is quite difficult to indicate parameters corresponding to the unidirectional transport, in particular, due to a large variance of τ^-. Besides, these conditions do not give us a measure of unidirectionality. To get more exhaustive answers, we involve into consideration other characteristics for detailed description of the noise-induced transport.

A general condition of unidirectionality. Our aim is to give an appropriate characteristic of the unidirectional transport in forced thermal ratchets and to propose a universal (numerical) tool for indicating domains of parameters, where the unidirectional transport is realized. Here we use the technique proposed in [21] for finding domains of parameters where noise-induced regular oscillations are observed.

Denote the solution of (3.1) starting at the moment s from the point x as $X_{s,x}(t)$, $t \geq s$. Introduce the probabilities:

the probability $P^{>}_{0,x}$ (the probability $P^{>}_{T/2,x}$) that the trajectory $X_{0,x}(t)$ (the trajectory $X_{T/2,x}(t)$) shifts in negative direction not more than on L during the first half-period (the second half-period) of the periodic forcing

$$P^{>}_{0,x} := P(X_{0,x}(t) > x - L,\ 0 \le t \le T/2),$$
$$P^{>}_{T/2,x} := P(X_{T/2,x}(t) > x - L,\ T/2 \le t \le T);$$

the probability $P^{<}_{0,x}$ (the probability $P^{<}_{T/2,x}$) that the trajectory $X_{0,x}(t)$ (the trajectory $X_{T/2,x}(t)$) shifts in positive direction not more than on L during the first half-period (the second half-period) of the periodic forcing

$$P^{<}_{0,x} := P(X_{0,x}(t) < x + L,\ 0 \le t \le T/2),$$
$$P^{<}_{T/2,x} := P(X_{T/2,x}(t) < x + L,\ T/2 \le t \le T).$$

Denote

$$Q^{>}_{0,x} := 1 - P^{>}_{0,x},\ Q^{>}_{T/2,x} := 1 - P^{>}_{T/2,x},$$
$$Q^{<}_{0,x} := 1 - P^{<}_{0,x},\ Q^{<}_{T/2,x} := 1 - P^{<}_{T/2,x}\ .$$

It is clear that, for example, $Q^{<}_{T/2,x}$ is the probability for $X_{T/2,x}(t)$ to reach the level $x + L$ at least one time during the second half-period.

It is not difficult to see that for $A > 0$

(3.6) $$Q^{>}_{T/2,x} > Q^{>}_{0,x} \text{ and } Q^{<}_{0,x} > Q^{<}_{T/2,x}\ .$$

Therefore, if both $P^{>}_{T/2,x}$ is close to one (i.e., $Q^{>}_{T/2,x}$ is close to zero) and $Q^{<}_{0,x}$ is close to one for all $x \in [0, L]$, then during each period of the periodic forcing the trajectory $X(t)$ moves in positive direction and does not move in negative direction with a probability close to 1. Analogously, if both $P^{<}_{0,x}$ is close to one (i.e., $Q^{<}_{0,x}$ is close to zero) and $Q^{>}_{T/2,x}$ is close to one, then we have the unidirectional transport to the left with a probability close to 1.

So, closeness of one of the following products

$$\Pi^+ = \Pi^+_x(A, T, \sigma) := P^{>}_{T/2,x} \cdot Q^{<}_{0,x},\ \Pi^- = \Pi^-_x(A, T, \sigma) := P^{<}_{0,x} \cdot Q^{>}_{T/2,x}$$

to 1 *for all* x *is a sufficient condition for the unidirectional transport.*

For definiteness, below we are interested in the transport in positive direction, i.e., when Π^+ is close to 1.

The further analysis essentially rests on the possibility to evaluate the probability $P^{>}_{T/2,x}$ (and $Q^{<}_{0,x}$) in a constructive way. To this end, we introduce the function

$$u(s, y) = u_{x-L}(s, y) := P(X^-_{s,y}(t) > x - L,\ s \le t \le T/2),$$
$$0 \le s \le T/2,\ y \ge x - L,$$

where $X^-_{s,x}(t)$ is a solution to (3.3). Since the distribution of $X_{T/2,x}(t)$, $T/2 \le t \le T$, coincides with the distribution of $X^-_{0,x}(t)$, $0 \le t \le T/2$, one can see that

$$P^{>}_{T/2,x} = u_{x-L}(0, x).$$

The function $\tilde{u}(s,y) := P(X^-_{s,y}(t) > x - L)$ obeys the corresponding Cauchy problem for the backward Kolmogorov equation (3.7). It is well known [17, 18, 19] that the function $u(s,y)$ satisfies the boundary value problem in half-band for the same equation:

$$\frac{\partial u}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial y^2} + (f(y) - A)\frac{\partial u}{\partial y} = 0,\ 0 \le s < T/2,\ y > x - L, \tag{3.7}$$

$$u(T/2, y) = 1,\ y > x - L;\ u(s, x - L) = 0,\ 0 \le s \le T/2. \tag{3.8}$$

The solution to the problem (3.7)-(3.8) has the following probabilistic representation:

$$u(s,y) = u_{x-L}(s,y) = E\varphi(\tau_{s,y}(x-L), X^-_{s,y}(\tau_{s,y}(x-L))), \tag{3.9}$$

where $(\tau_{s,y}(x-L), X^-_{s,y}(\tau_{s,y}(x-L)))$ is the first exit point of the space-time diffusion $(t, X^-_{s,y}(t))$, $t > s$, from the domain $[0, T/2) \times (x - L, +\infty)$ and

$$\varphi(s,y) = \begin{cases} 1,\ s = T/2,\ y > x - L, \\ 0,\ 0 \le s \le T/2,\ y = x - L. \end{cases}$$

The probability $Q^<_{0,x}$ can be evaluated analogously. We obtain that $P^<_{0,x} = 1 - Q^<_{0,x}$ is equal to

$$P^<_{0,x} = v_{x+L}(0,x),$$

where $v(s,y) = v_{x+L}(s,y)$ is a solution to the boundary value problem

$$\frac{\partial v}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial y^2} + (f(y) + A)\frac{\partial v}{\partial y} = 0,\ 0 \le s < T/2,\ y < x + L, \tag{3.10}$$

$$v(T/2, y) = 1,\ y < x + L;\ v(s, x + L) = 0,\ 0 \le s \le T/2. \tag{3.11}$$

The solution of this problem has the following probabilistic representation:

$$v(s,y) = v_{x+L}(s,y) = E\psi(\tau_{s,y}(x+L), X^+_{s,y}(\tau_{s,y}(x+L))), \tag{3.12}$$

where $(\tau_{s,y}(x+L), X^+_{s,y}(\tau_{s,y}(x+L)))$ is the first exit point of the space-time diffusion $(t, X^+_{s,y}(t))$, $t > s$, from the domain $[0, T/2) \times (-\infty, x + L)$ and

$$\psi(s,y) = \begin{cases} 1,\ s = T/2,\ y < x + L, \\ 0,\ 0 \le s \le T/2,\ y = x + L. \end{cases}$$

As a result we have got the following theorem.

Theorem 2. *A sufficient condition for the positive unidirectional transport in model* (3.1) *consists in closeness of the product* $\Pi^+ = u(0,x) \cdot (1 - v(0,x))$ *to* 1 *for all* $0 \le x < L$. *Here* $u(0,x)$, $v(0,x)$ *are values of the functions* $u(s,y)$, $v(s,y)$ at $(s,y) = (0,x)$, *where the functions are solutions of the boundary value problems* (3.7)-(3.8) *and* (3.10)-(3.11). *The individual values* $u(0,x)$ *and* $v(0,x)$ *can be found as probabilistic representations* (3.9) *and* (3.12) *for fixed* $(s,y) = (0,x)$. *An analogous assertion is true for the negative unidirectional transport.*

Remark 3. Analogously, we can state boundary value problems for evaluating the probabilities $P(X_{T/2,x}(t) > x - L^-,\ T/2 \le t \le T)$ and $P(X_{0,x}(t) < x + L^+,\ 0 \le t \le T/2)$

with $L^- = mL$, $L^+ = nL$, m, n are positive integers. These probabilities can be used for a detailed description of the transport much as the probabilities $P^>_{T/2,x}$ and $P^<_{0,x}$ are employed above.

It is possible to prove that $u_{x-L}(0, x; A, T, \sigma)$ is decreasing with respect to A, T, and σ, and $v_{x-L}(0, x; A, T, \sigma)$ is increasing with respect to A, T, and σ. Then the product Π^+ has a maximum in T for fixed A and σ. Analogously, the product Π^+ has a maximum in σ for fixed A and T. The remarkable feature of the phenomenon considered here, is that for some ratchet potentials $F(x)$ the product Π^+ is close to 1 for a sufficiently wide range of parameters A, T, σ. In particular, this is confirmed in our tests (see Fig. 3). We take the amplitude A of the periodic forcing less than A^* so that there is no transport in the system (3.1) for $A < A^*$ and $\sigma = 0$.

Remark 4. If the probability $P^>_{T/2,x}$ is close to 1, there is practically no transport in negative direction, and if, in addition, $E\tau^+ << T/2$, then the transport is positively unidirectional. So, closeness of the probability $P^>_{T/2,x}$ to 1 and $E\tau^+ << T/2$ give us the other condition for the unidirectional transport in positive direction. This condition is less general than the one of Theorem 2 but it is essentially easier to evaluate $E\tau^+$ than $Q^<_{0,x}$. Note in passing that in this case the mean shift $\bar{\Delta}$ of $X(t)$ during the single period T of the periodic forcing is approximately estimated as $\bar{\Delta} \doteq \bar{v}^+ T/2$.

Numerical results. In a general case to find domains of parameters corresponding to the unidirectional transport, one should solve the problems (3.7)-(3.8) and (3.10)-(3.11) numerically. We perform some numerical experiments. We take the following ratchet potential $F(x)$

$$F(x) = -\frac{L}{2\pi}(\sin\frac{2\pi x}{L} + \frac{1}{4}\sin\frac{4\pi x}{L}),\ L > 0, \tag{3.13}$$

that is used for some tests, e.g., in [13, 14].

Figure 3 gives level curves of the product Π^+. In accordance with our tests the probabilities $P^>_{T/2,x}$ and $Q^<_{0,x}$ depend only weakly on x, and for definiteness we take x in the presented tests such that the potential $F(x)$ has a local minimum at this point. One can see that the domain of parameters corresponding to the noise-induced unidirectional transport is sufficiently large. Let us mark that there is no unidirectional transport for both sufficiently large and small noise intensities. Figure 4 demonstrates typical trajectories of the solution $X(t)$ to (3.1) with the potential of (3.13). Figure 4 (left) corresponds to the regime of the unidirectional transport. One can see that during the first half-period of the periodic forcing the trajectory moves to the right on a distance of $5 \div 10$ periods of the potential $F(x)$. At the same time, during the second-half period of the periodic forcing the trajectory increment is practically equal to zero. If we increase the noise intensity, the mean shift $\bar{\Delta}$ during the single period of the periodic forcing increases but the transport becomes non-unidirectional (see Fig. 4 (right)).

Let us note that the approach proposed above is universal. For instance, it can easily be carried over to the model with sinusoidal forcing.

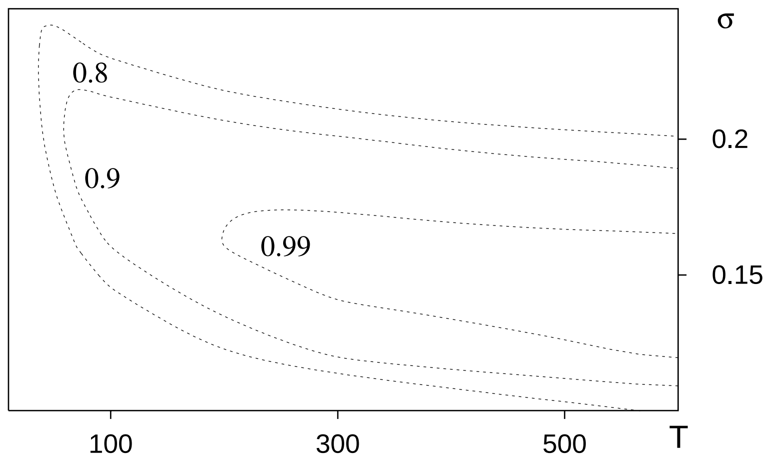

FIGURE 3. Forced thermal ratchets. Level curves of the product Π^+ in the plane (T, σ) for $A = 0.6$ and the potential $F(x)$ of (3.13) with $L = 1$.

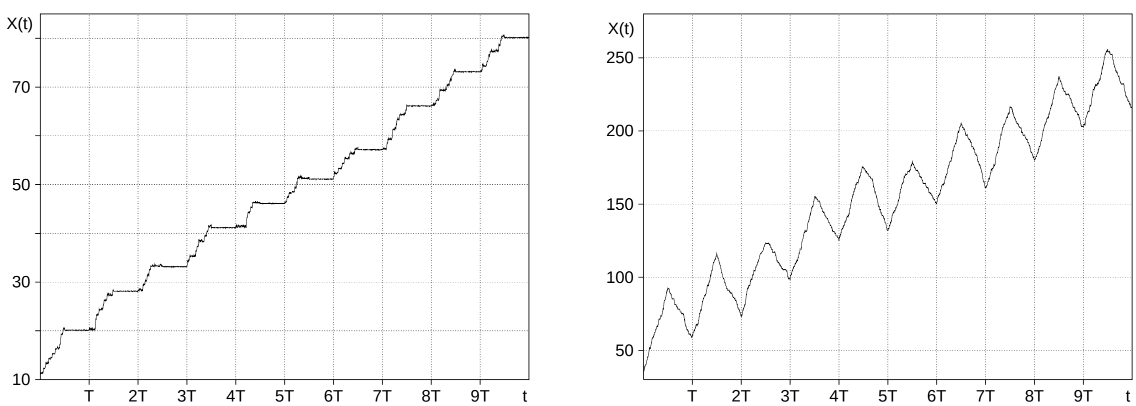

FIGURE 4. Forced thermal ratchets. Sample trajectories of the solution to (3.1) with the potential $F(x)$ of (3.13) for $L = 1$, $A = 0.6$, $T = 400$, and $\sigma = 0.15$ (left) and $\sigma = 0.4$ (right).

In our tests we use both finite-difference schemes and probability methods for solving the problems (3.7)-(3.8) and (3.10)-(3.11). Let us observe that we need in the individual values $u_{x-L}(0, x)$ and $v_{x+L}(0, x)$ only and in such a case the probabilistic approach with the Monte Carlo technique is most relevant. In one-dimensional case probabilistic algorithms require computational effort comparable with finite-difference schemes. But the Monte Carlo approach will be more effective for multi-dimensional models. To simulate sample trajectories, we use mean-square numerical methods for SDEs. Some details on numerical analysis of stochastic models are available, e.g., in [22, 21, 23].

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