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On the smoothness of the solution to a boundary value problem for a differential–difference equation

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0 Introduction

This paper deals with the first boundary value problem (BVP) for equations which are a differential with respect to one variable (t) and difference with respect to the other variable (s) in a bounded domain.

The initial value problem for differential–difference equations of this type was studied in [1], [2]. The theory of the BVP under investigation is connected with the theory of the BVP for strongly elliptic differential–difference equations which are difference and differential with respect to the same variable (see [3]).

Some questions of this work were considered earlier in the papers [4], [5], [8].

Section 1 considers the solvability of the BVP for differential–difference equations. In contrast to differential equations the smoothness of the generalized solutions can be broken in the domain Q and is preserved only in some subdomains $Q_r \subset Q$ where $(\cup_r \bar{Q}_r = \bar{Q})$. In section 2 we construct such a set of Q_r. Section 3 deals with the smoothness of generalized solutions in the subdomains Q_r. Section 4 considers the conditions under which the smoothness is preserved when passing the boundaries between neighboring subdomains Q_r.

1 Solvability of a Boundary Value Problem

Consider the equation

$$-(R_1 x_t(t,s))_t + R_2 x(t,s) = f(t,s), \quad (t,s) \in Q, \tag{1.1}$$

with boundary conditions xyz

$$x(t,s) = 0, \quad (t,s) \in \mathbb{R}^2 \backslash Q. \tag{1.2}$$

Here $(t,s) \in \mathbb{R}^2$ and $f(t,s) \in L_2(Q)$ is a real valued function, $Q \subset \mathbb{R}^2$ is a bounded domain with the boundary $\partial Q \in C^\infty$ or a rectangle; $R_k : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ are difference operators

$$(R_k x)(t,s) = \sum_{i \in M} a_k^i(t,s)[x(t,s+i) + x(t,s-i)], \tag{1.3}$$

$k = 1, 2$, $a_1^i(t,s) \in C^{2,0}(\bar{Q})$, $a_2^i(t,s) \in C(\bar{Q})$, and, if $a_k^i(t,s) \neq 0$, then $|a_k^i(t,s)| > \varepsilon > 0$ for $(t,s) \in \bar{Q}$, $M \subset \mathbb{Z}$ ($\mathbb{Z}$ - the set of integers) ($0 \in M$).

Denote by $H^{p,0}(Q)$ the Sobolev space $H^{p,0}(Q) = (x \in L_2(Q) | \frac{\partial^k x}{\partial t^k} \in L_2(Q),\ k = 1, ..., p)$ (see [7], III, [6]), with the inner product

$$(x,y)_p = \sum_{i=0}^{p} \int_Q x_t^{(i)} y_t^{(i)} dt ds.$$

Let H be the closure of the set $C_0^\infty(Q)$ in $H^{1,0}(Q)$. The inner product in H is given by:

$$(x, y)_1 = \int_Q (xy + x_t y_t) dtds.$$

Consider the operators I_Q, P_Q, R_Q^i, $i = 1, 2$,

$$\begin{aligned}
I_Q &: L_2(Q) \to L_2(\mathbb{R}^2), (I_Q x)(t, s) = x(t, s), (t, s) \in Q, (I_Q x)(t, s) = 0, \\
&\quad (t, s) \in \mathbb{R}^2 \backslash Q; \\
P_Q &: L_2(\mathbb{R}^2) \to L_2(Q), \quad (P_Q x)(t, s) = x(t, s), \quad (t, s) \in Q; \\
R_Q^i &: L_2(Q) \to L_2(Q), R_Q^i = P_Q R_i I_Q, i = 1, 2.
\end{aligned}$$

Definition 1.1. We say that the function $x \in H$ is a *solution* of the BVP (1.1), (1.2), if for all $v \in H$

$$(R_Q^1 x_t, v_t) + (R_Q^2 x, v) = (f, v), \tag{1.4}$$

where $(.,.)$ denotes the inner product in $L_2(Q)$.

A bounded self–adjoint operator A in the Hilbert space H is said to be positive definite (non–negative) if for all $y \in H$, $(Ay, y) \geq c(y, y)$, $c = const > 0$ $((Ay, y) \geq 0)$.

Definition 1.2. Let R_Q^1 be positive definite. Then the boundary value problem (1.1), (1.2) is said to be *the first boundary value problem.*

The operator R_Q^1 is positive definite if and only if the matrices $R_{Q_r}^1$ are positive definite (see [3]).

Lemma 1.1. The operators $R_Q^i : L_2(Q) \to L_2(Q)$, $i = 1, 2$, are bounded and self–adjoint.

Lemma 1.2. The operators R_Q^i, $i = 1, 2$, map H (continuously) into $H^{1,0}(Q)$ and $(R_Q^i x)_t = (R_Q^i)_t x + R_Q^i x_t$.

The proofs are evident.

From lemma 1.10, S1.8, 1, [7] it follows

Lemma 1.3. For all $f \in H$

$$||f||_{L_2(Q)} < k_Q ||f_t||_{L_2(Q)}, \tag{1.5}$$

where the constant k_Q depends only on Q.

Let R_Q^2 be a non–negative operator, i.e.,

$$(R_Q^2 x, x) \geq 0. \tag{1.6}$$

Lemma 1.4. Let R_Q^1 be positive definite, R_Q^2 non–negative operators.

Then in H we can introduce an equivalent inner product by the formula

$$(x, y)_H = (R_Q^1 x_t, y_t) + (R_Q^2 x, y). \tag{1.7}$$

Proof. By lemma 1.1 $(x,y)_H = (y,x)_H$. It is sufficient to check that there exist $c_1, c_2 > 0$, such that for all $x \in H$, $c_1(x,x)_1 \leq (x,x)_H \leq c_2(x,x)_1$. The first inequality follows from lemma 1.1, the second from definition 1.1, formula (1.6) and lemma 1.3. ■

Theorem 1.1. Let R_Q^1 be a positive definite operator, R_Q^2 a non–negative operator. Then there exists a unique solution $x \in H$ of BVP (1.1), (1.2),

$$||x||_H \leq c||f||_{L_2(Q)}, \ c > 0. \tag{1.8}$$

Proof. By lemma 1.4 eq. (1.4) is equivalent to

$$(x,v)_H = (f,v). \tag{1.9}$$

For any fixed $f \in L_2(Q)$ the linear functional $\varphi_f(v) = (v,f)$ is bounded in H. According to the Riesz representation theorem, there exists a unique function $F_f \in H$ such that for all $v \in H$, $(f,v) = (F_f,v)_H$, and $||F_f||_H \leq c||f||_{L_2(Q)}$. Hence, there exists a unique solution of (1.9) $x = F_f \in H$. ■

Remark 1.2. Generally speaking the BVP (1.1)–(1.2), does not have classical solutions for smooth f. It is natural to introduce generalized solutions for BVP for differential–difference equations, and in fact, takes place in the theory of the BVP for strongly elliptic differential–difference equations (see [3]).

Example 1.1. Consider the BVP

$$-(R_1 x_t(t,s))_t = 1, \qquad (t,s) \in Q, \tag{1.10}$$
$$x(t,s) = 0, \quad (t,s) \in \mathbb{R}^2 \backslash Q, \tag{1.11}$$

where $(R_1 x)(t,s) = 3x(t,s) + x(t,s+1) + x(t,s-1)$, $Q = ((t,s)|t \in (0,1), s \in (t,t+2))$. It is easy to check that $x \in H$ is a solution of BVP (1.10)–(1.11), where

$$\begin{array}{ll} x(t,s) = -t^2 + st, & s \in (0,1), t \in (0,s), \\ -t^2 + \frac{4-s}{3}t - \frac{s-1}{3}, & s \in (1,2),\ t \in (s-1,1), \\ -t^2 + \frac{5-s}{3}t, & s \in (1,2),\ t \in (0,s-1), \\ -t^2 + (s-1)t - (s-2), & s \in (2,3),\ t \in (s-2,1). \end{array}$$

By theorem 1.1 x is a unique solution of BVP (1.10)–(1.11). For any $s_0 \in (1,2)$, $\lim_{t \to (s_0-1)+0} x_t(t,s_0) = -\frac{7}{3}s_0 + \frac{10}{3} \neq \lim_{t \to (s_0-1)-0} x_t(t,s_0) = -\frac{7}{3}s_0 + \frac{13}{3}$, hence, $x(t,s) \notin H^{2,0}(Q)$, and $x(t,s) \notin C_t^2(Q)$, i.e. BVP (1.10)–(1.11) has no classical solutions for $f = 1 \in C^\infty(Q)$.

2 The Geometrical Constructions

As was shown in section 1, the smoothness of the generalized solutions with respect to t can be broken in domain Q, $x \notin H^{2,0}(Q)$. In this section we construct the partition of the domain Q into disjoint subdomains Q_r, $r = 1,2,...$, $(\bar{Q} = \cup_r \bar{Q}_r)$, such that the solution of BVP (1.1), (1.2) $x \in H^{2,0}(Q_r)$, $r = 1,2,...$.

Consider the ordered set $\beta = (i_k)_{k=0}^{l}$, $i_k \in \hat{M} = M \cup (-M)$, $i_0 = 0$. For any set β we construct the sets $A_\beta^0 = \partial Q$, $A_\beta^1 = A_\beta^0 + h_{i_1}, \ldots$, $A_\beta^k = (A_\beta^{k-1} \cap \bar{Q}) + h_i, \ldots$, $A_\beta \triangleq A_\beta^l = (A_\beta^{l-1} \cap \bar{Q}) + h_{i_l}$, where $h_{i_k} = (0, i_k)$.

Denote $\tau = \cup_{\beta \in B}(A_\beta \cap \bar{Q}), \mathcal{K} = \cup_{\beta_1, \beta_2 \in B} \bar{Q} \cap A_{\beta_1} \cap \overline{(A_{\beta_2} \backslash A_{\beta_1})}$, where B is the set of all β such that $A_\beta \notin \emptyset$.

The sets τ, $\mathcal{K}$ are closed. Suppose that ∂Q satisfies the conditions: $\mu\mathcal{K} = 0$ where $\mu(.)$ is the Lebesgue measure.

Consider the open set $Q \backslash \tau$. Let Q_r, $r = 1, 2, \ldots$, be the open connected disjoint components of the set $\bar{Q} \backslash \tau : \bar{Q} \backslash \tau = \cup_r Q_r$. Let $\mathcal{R}_0$ be the set of all Q_r.

Let also τ_k, $k = 1, 2, \ldots$, be the open connected disjoint components of the set $\tau \backslash \mathcal{K}$, $\tau \backslash \mathcal{K} = \cup \tau_k$. Denote $\mathcal{B}_0$ the set of all τ_k.

It is easy to check that $\tau = \cup_r \partial Q_r$, $\mathcal{K} = \cup_k \partial \tau_k$.

Definition 2.1. Let $v \subset \mathbb{R}^2$ be a bounded connected set. We say that the collection of subsets $\mathcal{R} = (V_k)_k$ of the set V is a partition of V if $V = \cup_k V_k$ (or $\bar{V} = \cup_k \bar{V}_k$) and $V_k \cap V_j = \emptyset$, $k \neq j$.

Definition 2.2. We say that the partition $\mathcal{R} = (V_k)_k$ is regular (with respect to set M) if all V_k are connected and open and for all V_k and $i \in \hat{M}$ there exists $V_j \in \mathcal{R}$ such that $V_k + h_i = V_j$, or $(V_k + h_i) \cap \bar{V} = \emptyset$, where $h_i = (0, i)$.

Lemma 2.1. The set $\mathcal{R}_0$ is a regular partition of the domain Q.

Proof. By the conditions of lemma the sets Q_r are connected and disjoint and $\cup_r \bar{Q}_r = (\bar{Q} \backslash \tau) \cup (\cup_r \partial Q_r) = \bar{Q}$. It is sufficient to prove that for all $Q_r \in \mathcal{R}_0$ and $i \in \hat{M}$ there exists $Q_l \in \mathcal{R}_0$ such that $Q_r + h_i = Q_l$, or $(Q_r + h_i) = \cap \bar{Q} = \emptyset$.

Suppose the opposite: there then exist Q_r, Q_l and $i \in \hat{M}$ such that $(Q_r + h_i) \cap Q_l \neq \emptyset$ and $(Q_r + h_i) \neq Q_l$. Without loss of generality we can assume that $Q_l \backslash (Q_r + h_i) \neq \emptyset$ (if $(Q_r + h_i) \backslash Q_l \neq \emptyset$ the proof is analogous). Since the set Q_l is connected there exists a point $z \in \partial(Q_r + h_i) \cap Q_l$. It is evident that $y = z - h_i \in \partial Q_r$. Since $\partial Q_r \in \tau$ there exists an ordered set $\beta_1 = (i_0, \ldots, i_m) \in B$, such that $y \in A_{\beta_1} \bar{Q}$. Therefore, $z \in A_{\beta_2} \cap \bar{Q}$ where $\beta_2 = (i_0, \ldots, i_m, i) \in B$. Hence $z \in \tau$, and since $\tau \cap Q_l = \emptyset$, we receive a contradiction. ∎

Lemma 2.2. The set $\mathcal{B}_0$ is a regular partition of the set τ. And if $\tau_k \in \mathcal{B}_0$ and $h_i = (0, i)$, $i \in \hat{M}$, $\tau_k + h_i \cap \tau = \emptyset$, then $\tau_k + h_i \cap \bar{Q} = \emptyset$.

Proof. We prove that for all $\tau_k \in \mathcal{B}_0$ and $h_i = (0, i)$, $i \in \hat{M}$, or a $\tau_j \in \mathcal{B}_0$ exists such that $\tau_k + h_i = \tau_j$, or $\tau_k + h_i \cap \bar{Q} = \emptyset$.

Suppose the opposite: let there exist τ_k and h_i such that $\tau_k + h_i \cap \bar{Q} \neq \emptyset$ and $\tau_k + h_i \neq \tau_l$ for any $\tau_l \in \mathcal{B}_0$. By construction of τ, $(\tau_k + h_i) \cap \bar{Q} \subset \tau$ and since $\mu(\mathcal{K}) = \emptyset$ and τ_k is open, there exists $\tau_j \in \mathcal{B}_0$ such that $(\tau_k + h_j) \cap \tau_j \neq \emptyset$. By assumption, $\tau_k + h_j \Delta \tau_j \neq \emptyset$. Without loss of generality $\tau_j \backslash (\tau_k + h_j) \neq \emptyset$. Since τ_j is connected and open there exists $y_0 \in \tau_j$ such that for all $S_L(y_0)$, where $S_L(y_0)$ is the open balls of radius L with center y_0, $(S_L(y_0) \cap (\tau_j \backslash (\tau_k + h_i)) \neq \emptyset$ and $S_L(y_0) \cap \tau_j \cap (\tau_k + h_i) \neq \emptyset$. If $y_0 \in (\tau_k + h_i) \cap \tau_j$ then $y_0 \in \overline{\tau_j \backslash (\tau_k + h_i)}$ and by construction of $\mathcal{K}, y_0 \in \mathcal{K}$. If $y_0 \in \tau_j \backslash (\tau_k + h_i)$, then

$y_0 \in \partial(\tau_k + h_i)$; for $y_0 - h_i \in \partial\tau_k$ then $y_0 - h_i \in \mathcal{K}$, hence $y_0 \in \mathcal{K}$. In both cases we obtain a contradiction, since $\tau_j \cap \mathcal{K} = \emptyset$, hence $y_0 + h_i = \tau_j$. ■

The partition $\mathcal{R}_0(\mathcal{B}_0)$ consists of the classes: subdomains Q_m, where $Q_j\,(\tau_m, \tau_j)$ belongs to the same class if there exists a sequence of shifts $(h_{i_k})_{k=1}^{p}$, $i_k \in \hat{M}$, carrying Q_m into Q_j (τ_m in τ_j) within the boundaries of $Q(\bar{Q})$.

We denote index $s = (r,k)$ (Q_{rk}, τ_{rk}), where $r = 1, 2, ...$ is the number of classes and $k = 1, ..., N_r$ is the number of elements in the r–th class.

Without loss of generality we may suppose that $\tau_{rk} \in Q$, $k = 1, ..., N_0$; $\tau_{rk} \in \partial Q$, $k = N_0 + 1, ..., N_r$.

Lemma 2.3.–2.8 are analogous to lemmas 4.2–4.4 [3].

Using the definition of the set $\mathcal{K}$ one can obtain the following statements.

Lemma 2.3. Let $y \in \partial Q \cap_{i=1}^{k} \partial Q_{r_i l_i}$, $\quad k \geq 2$, $(r_i, l_i) \neq (r_j, l_j)$, $\quad i \neq j$.

Then $y \in \mathcal{K}$.

Lemma 2.4. Let $y \in Q \cap_{i=1}^{k} \partial Q_{r_i l_i}$, $\quad k \geq 3$, $(r_i, l_i) \neq (r_j, l_j)$, $\quad i \neq j$.

Then $y \in \mathcal{K}$.

From lemma 2.3 follows lemma 2.5.

Lemma 2.5. For any $\tau_{vj} \subset \vartheta Q$, $\tau_{vj} \subset \mathcal{B}_0$ there exists a subdomain $Q_{rl} \in \mathcal{R}_0$, such that $\tau_{vj} \subset \partial Q_{rl}$ and $\tau_{vj} \cap \partial Q_{sk} = \emptyset$; $(r,l) \neq (s,k)$.

From lemma 2.4 follows lemma 2.6.

Lemma 2.6. For any $\tau_{vj} \subset Q(\tau_{vj} \subset \mathcal{B}_0)$, there exist subdomains $Q_{r_1 l_1}, Q_{r_2 l_2} \in \mathcal{R}_0$ $(Q_{r_1 l_1} \neq Q_{r_2 l_2})$, such that $\tau_{vj} \subset \partial Q_{r_1 l_1} \cap \partial Q_{r_2 l_2}$ and $\tau_{vj} \cap \partial Q_{r_g l_g} = \emptyset$, $(r_g, l_g) \neq (r_i, l_i)$ $i = 1, 2$.

From lemmas 2.1, 2.2, 2.5, 2.6 follows lemma 2.7.

Lemma 2.7. Any class v of the partition $\mathcal{B}_0$ corresponds to only 2 classes of subdomains of the partition $\mathcal{B}_0$: p and q, such that (after renumbering):

$$\begin{aligned}
&\tau_{vj} \subset \partial Q_{pj} \cap \partial Q_{qj} \cap Q, && j = 1, ..., N_0,\\
&\tau_{vj} \subset \partial Q_{pj} \cap \vartheta Q, && j = N_0 + 1, ..., N_p,\\
&\tau_{vj} \subset \partial Q_{qk} \cap \vartheta Q, && j = N_p + 1, ..., N_v, k = j - (N_p - N_0).
\end{aligned}$$

$\tau_{vj} \cap \partial Q_{rk} = \emptyset$, $r \neq p, q$.

Remark 2.1. In lemma 2.7 we can have the cases:

a) when for $\forall Q_{pj}, \exists k, k \neq j, \exists Q_{pk}, Q_{pj} = Q_{pk}$;
b) $N_0 = 0$ and class v corresponds to only one class p.

Definition 2.3. Let $(\tau_{vi})_{i=1}^{N_v}$ - v–th class of the partition $\mathcal{B}_0$. We name the set of all points $S = (y_{vi})_{i=1}^{N_v}$ $(y_{vi} \in \bar{Q})$, such that $y_{vi} \in \tau_{vi}$, $y_{vi} = y_{v1} + h_i$, where $\tau_{vi} = \tau_{v1} + h_i$, the *complete set of points from classes* v. Points $y_{vi} \in Q$ we name *inner points,* points

$y_{vi} \in \partial Q$ - *boundary points.*

Without loss of generality we suppose that $y_{vi} \in Q$, $i = 1, ..., N_0$, $y_{vi} \in Q$, $i = N_0 + 1, ..., N_v$.

Remark 2.2. By construction of a set τ the complete set S always has boundary points, though may not have inner products, i.e. $0 \leq N_0 < N_v$.

Example 2.1. Let the operator R have the form:
$(Rx)(t,s) = 5x(t,s) + x(t,s+5) + x(t,s-5) + x(t,s+8) + x(t,s-8)$.
$Q = ((t,s)|t \in (0,1),\ s \in (0,1];\ t \in (0,2),\ s \in (1,9))$. Then $M = (0,5,8)$, the partition $\mathcal{R}_0$ consist of 3 classes:

$$
\begin{aligned}
Q_{11} &= (0,1) \times (0,1), \\
Q_{12} &= (0,1) \times (3,4), \\
Q_{13} &= (0,1) \times (5,6), \\
Q_{14} &= (0,1) \times (8,9), \\
Q_{21} &= ((t,s)|t \in (0,2),\ s \in (1,3);\ t \in (1,2),\ s \in [3,4)), \\
Q_{22} &= ((t,s)|t \in (0,2),\ s \in (6,8);\ t \in (1,2),\ s \in [8,9)), \\
Q_{31} &= ((t,s)|t \in (0,2),\ s \in (4,5);\ t \in (1,2),\ s \in [5,6)).
\end{aligned}
$$

The set $\mathcal{K}$ consists of 16 points: $(0,i)$, $i = 0,1,3,4,5,6,8,9)$ (i,j), $j = 1,2$, $i = 1,4,6,9$. In the set $\mathcal{B}_0$ are 9 classes.

Example 2.2. Im example 1.1 $\mathcal{R}_0 = (Q_{11}, Q_{12})$, $Q_{11} = ((t,s)|t \in (0,1),\ s \in (t,t+1))$, $Q_{12} = ((t,s)|t \in (0,1),\ s \in (t+1,t+2))$, $\mathcal{K}$ consists of 6 points: $(0,i)$, $i = 0,1,2$; $(1,i)$, $i = 1,2,3$. In a partition $\mathcal{B}_0$ are 3 classes: $\tau_{1i} = (0) \times i-1, i)$ $i = 1,2$; $\tau_{2i} = (1) \times (i, i+1)$ $i = 1,2$; $\tau_{31} = ((t,s)|t = s-1,\ s \in (1,2))$, $\tau_{32} = \tau_{31} - (0,1)$, $\tau_{33} = \tau_{31} + (0,1)$.

3 The Smoothness of Solutions in Subdomains Q_{rk}

Let $P_r : L_2(Q) \to L_2(\cup_l Q_{rl})$ - be the orthogonal projection operator of $L_2(Q)$ onto $L_2(\cup_l Q_{rl})$, $(l = 1, ..., N_r)$, where $L_2(\cup_l Q_{rl}) = (x(y) \in L_2(Q)|\ x(y) = 0$ for $y \in Q \setminus \cup_l Q_{rl})$.

We introduce the isomorphism of Hilbert spaces U_r:
$L_2(\cup_l Q_{rl}) \to L_2^N(Q_{r1})$ by the formula $(U_r x)_l(y) = x(y + h_{rl})$, $(y \in Q_{r1})$, where $l = 1, ..., N = N_r$; h_{rl} such that

$$
Q_{r1} + h_{rl} = Q_{rl}\ (h_{r1} = 0),\ \ L_2^N(Q_{r1}) = \prod_{l=1}^{N} L_2(Q_{r1}).
$$

Operator $R_{Q_r} : L_2^N(Q_{r1}) \to L_2^N(Q_{r1})$, $R_{Q_r} = U_r R_Q U_r^{-1}$, is the operator of multiplication by an $N \times N$ dimensional matrix R_{Q_r} with the elements

$$
b_{kj}^r(t,s) = \begin{matrix} a_i(t, s+i_k), & \text{if } i = i_j - i_k \in \hat{M}\,, \\ 0 & \text{if } i \notin \hat{M}\,. \end{matrix} \tag{3.1}
$$

Lemma 3.1. Let R_Q - positive defined operator, and $R_Q x \in H^{k,0}(Q_{rj})$, $j = 1, ..., N = N_r$, where Q_{rj} - components of a regular partition $\mathcal{R}_0$.

Then $x \in H^{k,0}(Q_{rj})$, $j = 1, ..., N$.

Proof. Denote $z = U_r R_Q P_r x \in L_2^N(Q_{r1})$. Since $R_Q x \in H^{k,0}(Q_{rj})$, $j = 1, ..., N$, so $z \in \prod_{l=1}^N H^{k,0}(Q_{r1})$, hence

$$z = U_r R_Q P_r x = U_r R_Q U_r^{-1} P_r x = R_{Q_r} U_r P_r x \in \prod_{l=1}^{N} H^{k,0}(Q_{r,1}).$$

Since R_{Q_r} is a positive definite matrix, so $(R_{Q_r})^{-1}$ is bounded, hence $U_r P_r x = (R_{Q_r})^{-1} z \in \sum_{l=1}^N H^{k,0}(Q_{r1})$ and $x \in H^{k,0}(Q_{rj})$, $j = 1, ..., N$. ∎

Theorem 3.1. Let R_Q^1 be a positive definite operator, x - a solution of BVP (1.1), (1.2), $\mathcal{R}_0 = (Q_{rj})$ - a regular partition.

Then $R_Q^1 x \in H^{2,0}(Q)$, $x \in H^{2,0}(Q_{rj})$, $j = 1, ..., N_r$ and x satisfies BVP (1.1), (1.2) almost everywhere.

Proof. Using lemma 1.1, and integrating by part, we obtain for

$$x, v \in H,\ (R_Q^1 x_t, v_t) = ((R_Q^1 x)_t, v_t) - ((R_Q^1)_t x, v_t) = ((R_Q^1 x)_t, v_t) + (((R_Q^1)_t x)_t, v).$$

Denote $y = R_Q^1 x \in H^{1,0}(Q)$, $g = -f + ((R_Q^1)_t x)_t + R_Q^2 x$, $g \in L_2(Q)$. Eq. (1.4) takes form: $(y_t, v_t) = -(g, v)$ for any $v \in H$, i.e. g is a generalized derivative with respect to t of the function y_t in $Q' \subset Q$ (see [6], S3, III) and $y_{tt} = g$ almost everywhere in Q', hence $y_{tt} = g$ almost everywhere in Q and $y \in H^{2,0}(Q)$. Hence $R_Q^1 x \in H^{2,0}(Q)$, and by lemma 3.1 $x \in H^{2,0}(Q_{rj})$, $j = 1, ..., N_r$. ∎

Example 3.1. Consider the BVP: $(R_Q x_t)_t = 24$,

$$\begin{aligned} (Rx)(t,s) &= 8x(t,s) + 4[x(t,s+1] + x(t,s-1)] + 2[x(t,s+2) + x(t,s-2)] + \\ &\quad +[x(t,s+3) + x(t,s-3)], \\ Q &= ((t,s) | t \in (0,2), s \in (0,3); t \in (0,1), s \in [3,4)). \end{aligned}$$

The partition $\mathcal{R}_0$ consists of 7 subdomains: $Q_{1i} = (0,1) \times (i-1, i)$, $i = 1,2,3,4$, $Q_{2,i} = (1,2) \times (i-1, i)$, $i = 1,2,3$.

By theorem 3.1 we have that $x \in H^{2,0}(Q_{ri})$.

The partition $\mathcal{B}_0$ consist of 5 classes. The smoothness of solutions can be broken only for one class: $\tau_{1i} = (1) \times (i-1, i)$, $i = 1,2,3,4$ (another classes of the partition or have not inner points or their components τ_{ri} are parallel to the axis $0t$).

It is easy to check that x is a solution of BVP:

$$x(t,s) = \begin{array}{ll} t^2 - 2t, & s \in (0,1),\ t \in (0,2), \\ t^2/2 - t, & s \in (1,2),\ t \in (0,2), \\ t^2 - 2t, & s \in (2,3),\ t \in (0,1), \\ t^2/2 - 3/2t, & s \in (2,3),\ t \in (1,2), \\ t^2 - t, & s \in (3,4),\ t \in (0,1). \end{array}$$

The function $x \in H^{2,0}(Q')$, where $Q' = (0,2) \times (0,2)$, i.e. on τ_{11}, τ_{12} the smoothness of solution is preserved. However for $s \in (2,3)$ $\lim_{t\to 1-0} x_t(t,s) = 0 \neq \lim_{t\to 1+0} x_t(t,s) = -1/2$, i.e. the smoothness of BVP on the boundary τ_{12} is broken, $x \neq H^{2,0}(Q)$.

4 The Smoothness of the Solutions on the Boundary of the Neighboring Subdomains

In example 3.1 we have shown that the generalized solution of BVP (1.1), (1.2), may not have corresponding smoothness on the boundary of the neighboring subdomains, and at the same time may be smooth at other points of the boundary. This is connected with the fact that the differential–difference operators are non–local. In this section we consider the necessary and sufficient conditions of the smoothness of the solutions on the boundary of the neighboring subdomains. This conditions are similar to the conditions of preserving the smoothness of solutions of the boundary value problem for the strongly elliptic differential–difference equations (see [3, theorem 6.1]). Consider the point $y^* = (t^*, s^*) \in \mathcal{T}$. We obtain conditions of existence $\alpha > 0$ such that the solution of BVP (1.1), (1.2) $x \in H^{2,0}(K_\alpha(y^*) \cap Q)$ for any $f \in L_2(Q)$ i.e. x has a corresponding smoothness in the neighborhood of the point y^* (here $K_\alpha(y^*) = ((t,s) \mid \mid t - t^* \mid < a, \mid s - s^* \mid < a))$.

At first we investigate the case when $y \notin \mathcal{K}$, i.e. $y^* \in \mathcal{T}_{vi} \cap Q$ for some v, i (if $y^* \in \mathcal{T}_{vi} \cap \partial Q$, then from lemma 2.5 $K_\alpha(y*) \cap Q = K_\alpha(y^*) \cap Q_{pi}$ for some a, p, i and by theorem 3.1 $x \in H^{2,0}(K_\alpha(y^*) \cap Q)$.

Consider the complete set $S = (y_l)_{l=1}^{N_v}$, corresponding to the point $y^*(y_l = y^*$ for some $l), y_l \in Q, l = 1, ..., N_0, y_l \in \partial Q, l = N_0 + 1, ..., N_v$ (see definition 2.4). By lemma 2.7 to the class v of the partition $\mathcal{B}_0$ correspond 2 classes: p and q of the partition $\mathcal{R}$, and after renumbering we obtain:

$$
\begin{aligned}
y_l &= y_{pl} \in \partial Q_{pl} \cap \partial Q_{ql} \cap Q, && l = 1, ..., N_0, \\
y_l &= y_{pl} \in \partial Q_{pl} \cap \partial Q, && l = N_0 + 1, ..., N_p, \\
y_l &= y_{qk} \in \partial Q_{qk} \cap \partial Q, && l = N_p + 1, ..., N_v, \ k = l - (N_p - N_0).
\end{aligned}
$$

We can choose $\alpha > 0$ so small that the sets $\partial Q_{rl} \cap K_\alpha(y_{rl})$ $(r = p, q, \ l = 1, ..., N_r)$ are connected and belong to the class C^1,

$$
\begin{aligned}
&K_\alpha(y_{pl}) \subset Q_{pl} \cup Q_{ql} \cup \partial Q_{pl}, && l = 1, ..., N_0, \\
&K_\alpha(y_{rl}) \cap Q = K_\alpha(y_{rl}) \cup Q_{rl}, && l = N_0 + 1, ..., N_r, r = p, q.
\end{aligned}
$$

Denote $\Gamma_l = \partial Q_{pl} \cap K_\alpha(y_{pl})$, $l = 1, ..., N_0$, $\Gamma \triangleq \Gamma_1$, $\tau^t = ((t,s) \in \tau \mid \exists a > 0 : K_\alpha(t,s) \cap \tau = (t - a, t + a) \times (s))$.

Remark 4.1. By theorem 3.1 $x \in H^{2,0}(K_\alpha(y_{rl} \cap Q_{rl})$, besides $x \in H^{1,0}(K_\alpha(y_{rl}) \cap Q)$, $r = p, q$, $l = 1, ..., N_0$. Hence if $y_{pl} \in \tau^t$, then $x \in H^{2,0}(K_\alpha(y_{pl}) \cap Q)$. If $y_{pl} \in \tau \backslash \tau^t$ (an axis $0t$ cuts Γ_l) then $x \in H^{2,0}(K_\alpha(y_{pl} \cap Q)$ if and only if $(x_p(t,s))_{t|\Gamma_l} = (x_q(t,s))_{t|\Gamma_l}$, where $x_r(t,s) = x(t,s)$, $(t,s) \in Q_{rl}$, $r = p, q$.

Below we suppose that $y_{pl} \in \tau \backslash \tau^t$.

Denote

$$\begin{aligned} z^r(t,s) &= ((R_{Q_r}U_rP_rx)_1(t,s),...,(R_{Q_r}U_rP_rx)_{N_0}(t,s)), \\ V^r(t,s) &= ((U_rP_rx)_1(t,s),...,(U_rP_rx)_{N_0}(t,s)), \\ W^r(t,s) &= ((U_rP_rx)_{N_0+1}(t,s),...,(U_rP_rx)_{N_r}(t,s)), \quad r=p,q. \end{aligned}$$

By theorem 3.1 $(R_Qx) \in H^{2,0}(K_\alpha(y_{pl}))$, $l=1,...,N_0$. Hence

$$(Z_t^p - Z_l^q)|_\Gamma = 0. \tag{4.1}$$

Since $x \in H$.

$$V^q|_\Gamma = V^p|_\Gamma, \quad W^q|_\Gamma = 0, \quad W^q|_\Gamma = 0. \tag{4.2}$$

Denote R^0 - $N_0 \times N_0$ - dimensional matrix, obtained from the matrix R_{Q_p} by eliminating the last $N_r - N_0$ columns and lines (since $h_{pl} = h_{ql}$, $l=1,...,N_0$, the matrix, obtained from R_{Q_q}, is equal to R^0). Denote R^r $(r=p,q)$ $(N_0 \times N_r - N_0)$ - dimensional matrix, obtained from the matrix R_{Q_r} by eliminating the first N_0 columns and the lst $N_r - N_0$ lines. By (4.2) eq. (4.1) takes form

$$\begin{aligned} (R^0V_t^p &+ (R^0)_tV^p + R^pW_t^p + (R^p)_tW^p)|_\Gamma - (R^0V_t^q + (R^0)_tV^q + R^qW_t^q \\ &+ (R^q)_tW^q)|_\Gamma = (R^0(V_t^p - V_t^q) + R^pW_t^p - R^qW_t^q)|_\Gamma = 0. \end{aligned} \tag{4.3}$$

Denote $Y = (U_t^p - V_t^q)|_\Gamma$, $W = (W_t^p | W_t^q)|_\Gamma$, $R^{p,q} = (-R^p | R^q)$.

From (4.3) we obtain

$$R^0|_\Gamma Y = R^{p,q}|_\Gamma W, \tag{4.4}$$
$$Y = \Lambda W, \tag{4.5}$$

where $\Lambda = ((R^0)^{-1}R^{pq})|_\Gamma$, $\Lambda = ||\lambda_{lk}(t,s)||$, $k = 1,...,N_p+N_q-2N_0$, $l=1,...,N_0$.

Theorem 4.1. The solution of BVP (1.1)-(1.2) x for the point $y_{pl} \in (Q \cap \partial Q_{pl}) \backslash \mathcal{K}$ for every $f \in L_2(Q)$ belongs to $H^{2,0}(K_\alpha(y_{pl}))$ for some $\alpha > 0$ if and only if

$$\lambda_{lk}(t,s) = 0, \tag{4.6}$$

$k = 1,...,N_p+N_q-2N_0$, $(t,s) \in (K_\alpha(y_{p1} \cap \partial Q_{p1}) \backslash \tau^t$.

Proof. Sufficiency. Let (4.6) be true. Then by (4.5) $y_l = 0$ $(Y = (Y_1,....,Y_l,...,Y_{N_0}))$, i.e.

$$(U_pP_px_t)_{l|_\Gamma} = (U_qP_qx_t)_{l_\Gamma}. \tag{4.7}$$

From theorem 3.1 and (4.7) follows that $x \in H^{2,0}(K_\alpha(y_{pl}))$.

Necessity. Let for every $\alpha > 0$ there exist k $(1 \le k \le N_p+N_q-2N_0)$, and $y^* = (t^*,s^*) \in \Gamma$ such that $\lambda_{lk}(t^*,s^*) \neq 0$.

Then we prove that there exists $x \in H$ such that $-(R_Q^1 x_t)_t + R_Q^2 x = f^* \in L_2(Q)$, while $x \notin H^{2,0}(K_\alpha(v_{pl}))$ for any $\alpha > 0$. We choose α so small that the point y_{p1} divides the curve Γ into two parts $\Gamma = \Gamma^1 \cup \Gamma^2 \cup y_{p1}$. The curve Γ^1 and Γ^2 can ly on one side from the axis $0t$, or on different sides, or one curve coincides with an axis $0t$ (by remark 1.4 even if one curve does not coincide with the axis $0t$, let it be Γ^2). Without loss of generality we assume that Γ^2 lies on the right side from $0t$. By choosing the parameter α sufficiently small, we obtain: $\Gamma^2 : t = \gamma_2(s)$, $\gamma_2 \in C^1(0, \delta_2)$, $0 \le s \le \delta_2 \le \alpha$.

For the curve Γ^1 we can have the cases:

$$
\begin{array}{lll}
\text{a) } t = \gamma_1(s), & \gamma_1 \in C^1(-\delta_1, 0), & -\alpha \le -\delta_1 \le s \le 0,\ \delta_1 > 0; \\
\text{b) } t = \gamma_1(s), & \gamma_1 \in C^1(0, \delta_1), & 0 \le s \le \delta_1 \le \alpha,\ \delta_1 > 0; \\
\text{c) } t \in (0, \alpha_1) & (\text{or } t \in (-\alpha, 0)) & s = 0.
\end{array}
$$

Suppose without loss of generality that $(t, s) \in Q_{p1}$, if $\gamma_2(s) < t < \gamma_2(s) + \varepsilon$, and $(t, s) \in Q_{q1}$, if $\gamma_2(s) - \varepsilon < t < \gamma_2(s)$, for some $\varepsilon > 0$, $s \in (0, \delta_2)$, and in case b) $\gamma_2(s) < \gamma_1(s)$, $0 < s \le \min(\delta_1, \delta_2)$.

By assumption $\lambda_{lk}(y^*) \ne 0$, where $y^* = (t^*, s^*)$, $t^* = \gamma_2(s^*)$, $s^* \in (0, \delta_2)$. Since $\lambda_{lk}(t, s)$, $\gamma_2(s)$ are continuous functions, $\lambda_{lk}(\gamma_2(s), s) \ne 0$ in the some neighborhood of $s^* : s \in (s_1, s_2)$, $0 \le s_1 \le s^* \le s_2 < \delta_2$, $s_1 < s_2$.

Denote $b_1 = \gamma_2(s_1)$, $b_2 = \gamma_2(s_2)$, $b = \max(b_1, b_2)$ $(b < \alpha)$, $\Delta = a - b > 0$, $P = ((t, s) | (t, s) \in Q_{p1},\ s_1 \le s \le s_2,\ \gamma_2(s) \le t \le \alpha)$.

Consider the function

$$
x(t, s) = \begin{cases} U_p^{-1} u^p(t, s), & (t, s) \in \cup_{i=1}^{Np} Q_{pi} \cap K_\alpha(y_{pi}), \\ U_p^{-1} u^q(t, s), & (t, s) \in \cup_{i=1}^{Nq} Q_{qi} \cap K_\alpha(y_{qi}), \\ 0, & (t, s) \in Q \backslash (\cup_{i=1}^{Np} K_\alpha(y_{pi}) \cup_{i=1}^{Nq} K_\alpha(y_{qi})), \end{cases}
$$

where $(t, s) \in Q$, $u^r(t, s) = (u_1^r, \dots, u_i^r, \dots, u_{N_r}^r)$, $r = p, q$,

$$
u_i^p(t, s) = \begin{cases} \lambda_{ik}(t, s)(t - \gamma_2(s))\zeta(t - \gamma_2(s)), & (t, s) \in P,\ i = 1, \dots, N_0, \\ \delta_{ik}(t - \gamma_2(s))\zeta(t - \gamma_2(s)), & (t, s) \in P, i = N_0 + 1, \dots, N_p, \\ 0, & (t, s) \in Q_{p1} \backslash P; \end{cases} \tag{4.8}
$$

$u_i^q(t, s) = 0, \quad (t, s) \in Q_{q1}, \quad i = 1, \dots, N_q.$

Here $\delta_{ik} = 0$, $i \ne k, \delta_{ii} = 1$, $\zeta(t) = 1$, $0 \le t < \Delta/3$, $\zeta(t) = 0$, $(2/3)\Delta < t < \Delta$, $\zeta \in C^\infty(R^1)$.

Obviously $u^r \in \prod_{i=1}^{N_r} H^{2,0}(Q_{r1})$, $r = p, q$, function $x \in H$, $x \in H^{2,0}(Q_{ri})$, $r = p, q$, $i = 1, \dots, N_r$ and satisfy (4.5), hence $(R_Q^1 x)(t, s) \in H^{2,0}(Q)$. Therefore exists a $f^* \in L_2(Q)$ such that $-(R_Q^1 x_t)_t + R_Q^2 x = f^*$. By (4.8)

$$
(u_l^p(t, s))'_t \Big|_{\substack{t = \gamma_2(\Omega) \\ \Omega_1 < \Omega < \Omega_2}} = \lambda_{lk}(t, s) \Big|_{\substack{t = \gamma_2(\Omega) \\ \Omega_1 < \Omega < \Omega_2}} \ne (u_l^q(t, s))'_t \Big|_{\substack{t = \gamma_2(\Omega) \\ \Omega_1 < \Omega < \Omega_2}} = 0,
$$

hence $(U_pP_px_t)(t,s))_{|\Gamma} \neq (U_pP_qx_t)(t,s))_{|\Gamma}$ and $x \notin H^{2,0}(K_\alpha(y_{pl}))$. ■

Example 4.1. Let us apply theorem 4.1 for an investigation of the smoothness of the solutions of the BVP from example 3.1 at the points $y_i \in \tau_{1i}$, $i = 1,2,3$. Here $p = 1, q = 2, N_0 = 3, N_1 = 4, N_2 = 3$. Taking into account the form of the operator R, we construct the matrices

$$R_{Q1} = \begin{bmatrix} 8\,4\,2\,1 \\ 4\,8\,4\,2 \\ 2\,4\,8\,4 \\ 1\,2\,4\,8 \end{bmatrix}, \quad R_{Q2} = \begin{bmatrix} 8\,4\,2 \\ 4\,8\,4 \\ 2\,4\,8 \end{bmatrix},$$

$$R^0 = \begin{bmatrix} 8\,4\,2 \\ 4\,8\,4 \\ 2\,4\,8 \end{bmatrix}, \quad R^{1,2} = -\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Therefore,

$$\Lambda = (R^0)^{-1}R^{1,2} = -\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

Hence, by theorem 4.1 there exist $a_1, a_2 > 0$ such that, the solution $x \in H^{2,0}(K_{\alpha_1}(y_1))$, $x \in H^{2,0}$ $(K_{\alpha_2}(y_2))$, but $x \notin H^{2,0}(K_\alpha(y_3))$ for any $a > 0$. It corresponds to the result, obtained in example 3.1.

Now we investigate the question of preservation of the smoothness of solutions of BVP (1.1)–(1.2) for the points $y^* \in \mathcal{K}$. Since

$$\mathcal{K} = \cup_k \partial\tau_k \quad \text{so } y^* \in \cap_i \bar{\tau}_i, \quad i > 1.$$

Let y^* be an isolated singular point: there exist a neighborhood of $y^* : K_\alpha(y^*)$, containing only one point from $\mathcal{K} - y^*$.

From theorem 4.1 follows corollary 4.1.

Corollary 4.1. Let $y^* \in \mathcal{K}$.

The solution of BVP (1.1)–(1.2) $x \in H^{2,0}(K_\alpha(y^*))$ for some $a > 0$ if for every $\tau_i \in \mathcal{B}_0$ such that $(\tau_i \backslash \tau^t) \cap K_\alpha(y^*) \neq \emptyset$ for points $y \in (\tau_i \backslash \tau^t) \cap K_\alpha(y^*)$ the conditions of theorem 4.1 are fulfilled.

Example 4.2. Let us investigate the smoothness of solutions of BVP from example 3.1 at the points $y_4(1,1)$ and $y_5 = (1,2)$. Obviously, we have that $y_4, y_5 \in \mathcal{K}$, and $y_4 = \bar{\tau}_{11} \cap \bar{\tau}_{12}$, $y_5 = \bar{\tau}_{12} \cap \bar{\tau}_{13}$. As follows from example 4.1 the conditions of corollary 4.1 are fulfilled for τ_{11} and τ_{12}, hence $x \in H^{2,0}(K_\alpha(y_4))$, $a < 1$. For the point y_5 the conditions of corollary 4.1 are fulfilled only for τ_{12} and disturbed for $\bar{\tau}_{13}$, i.e. $x \notin H^{2,0}(K_\alpha(y_5))$ for any $a < 0$.

Corollary 4.2. Let $S = (y_k)_{k=1}^{N_v}$ be a complete set from a class v of the partition $\mathcal{B}_0$, containing inner points. Then there exists $f_s \in L_2(Q)$ such that the smoothness of solutions of BVP (1.1), (1.2) fails at least in one of the points of a set S: for some l $(1 \leq l \leq N_0)$ $x \notin H^{2,0}(K_\alpha(y_l))$ for any $a > 0$.

Proof. By lemma 2.7 class v from $\mathcal{B}_0$ corresponds to some classes p, q from $\mathcal{R}_0$. From definition of the set S and remark 2.2 it follows that there exists an inner point $y_k \in S$; $k \leq N_0$, a boundary point $y_j \in S$, $N_0 < j \leq N_v$ and $i \in \hat{M}$, such that $y_k + (0,i) = y_j$ (y_j can belong or to ∂Q_{pj}, or to ∂Q_{qj}; let $y \in \partial Q_{pj}$). Therefore in (1.3) for the difference operator R_1 element $a_i(y_k) \neq 0$ and in the matrix R_{Q_p} by the formula (3.1) $bkj(y_1) = a_i(y_k) \neq 0$. Then $r_{m|y_1}$ - m–th column ($m = j - N_0$) of the matrix $R^p_{|y_1}$ - is not equal to zero. Since R^0 is a positive definite matrix, so equation $R^0_{|y_1}\lambda_m = r_{m|y_1} \neq 0$ has a solution: $\lambda_m = (R^0_{|y_1})^{-1} r_{m|y_1} \neq 0$, where $\lambda_m = (\lambda_{1m}, ..., \lambda_{N_0 m})$, i.e. for some $l \, (1 \leq l \leq N_0)$, $\lambda_{lm} \neq 0$.

From (4.5) it follows that λ is m–th column of the matrix Λ and by theorem 4.1 the necessary conditions of preservation the smoothness of solutions in points y_l are not fulfilled. Hence $x \notin H^{2,0}(K_\alpha(y_l))$ for any $a > 0$. ∎

Corollary 4.3. For the BVP (1.1), (1.2) always exists $f \in L_2(Q)$ such that the smoothness of solutions is broken in some point $y^* \in \tau : x \notin H^{2,0}(K_\alpha(y^*))$ for any $a > 0$.

Corollary 4.4. For nonrectangular domains the set τ plays a main role in the investigation of solutions smoothness of the BVP (1.1), (1.2). From theorem 4.1 and corollary 4.3 follows that on τ always exists a non–smooth solution; for some $f \in L_2(Q)$, outside of the τ - the solution always preserves smoothness.

For rectangular domains investigation of smoothness is simplified. From theorem 3.1 and remark 4.1 follows assertion 4.1.

Assertion 4.1. Let $Q = (a_1, a_2) \times (b_1, b_2)$. Then the solution of BVP (1.1), (1.2) $x \in H^{2,0}(Q)$.

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41. E. Platen: An approach to bond pricing.

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