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Numerical solution of the Neumann problem for nonlinear parabolic equations by probability approach

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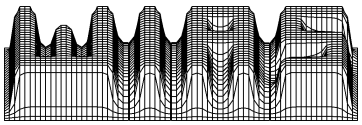

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ABSTRACT. A number of new layer methods solving the Neumann problem for semilinear parabolic equations is constructed by using probabilistic representations of their solutions. The methods exploit the ideas of weak sense numerical integration of stochastic differential equations in bounded domain. In spite of the probabilistic nature these methods are nevertheless deterministic. Some convergence theorems are proved. Numerical tests are presented.

1. INTRODUCTION

A probability approach to constructing layer methods for solving nonlinear partial differential equations (nonlinear PDE) of parabolic type is proposed in [13, 14, 15]. The papers [13, 14] are devoted to layer approximation methods for the Cauchy problem for semilinear parabolic equations and the paper [15] deals with the nonlinear Dirichlet problem. The aim of this paper is to develop such methods for nonlinear problems with Neumann boundary conditions.

Let G be a bounded domain in $\mathbf{R}^d$, $Q = [t_0, T) \times G$ be a cylinder in $\mathbf{R}^{d+1}$, $\Gamma = \overline{Q} \setminus Q$. The set Γ is a part of the boundary of the cylinder Q consisting of the upper base and the lateral surface. Consider the Neumann problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x,u)\frac{\partial u}{\partial x^i} + g(t,x,u) = 0,\ (t,x) \in Q, \tag{1.1}$$

with the initial condition

$$u(T,x) = \varphi(x) \tag{1.2}$$

and the boundary condition

$$\frac{\partial u}{\partial \nu} = \psi(t,x,u),\ t \in [t_0, T],\ x \in \partial G, \tag{1.3}$$

where ν is the direction of the internal normal to the boundary ∂G at the point $x \in \partial G$.

The form of equation (1.1) is relevant to a probabilistic approach, i.e., the equation is considered under $t < T$, and the "initial" conditions are prescribed at $t = T$. Using the well known probabilistic representation of the solution to (1.1)-(1.3) (see [5, 4]), we get

$$u(t,x) = \mathbf{E}(\varphi(X_{t,x}(T)) + Z_{t,x,0}(T)). \tag{1.4}$$

In (1.4) $X_{t,x}(s),\ Z_{t,x,z}(s),\ t_0 \le t < T,\ s \ge t,\ x \in \overline{G}$, is a solution of the Cauchy problem to the Ito system of stochastic differential equations (SDE)

$$dX = b(s,X,u(s,X))I_G(X)ds + \sigma(s,X,u(s,X))I_G(X)dw(s) + \nu(X)I_{\partial G}(X)d\mu(s), \tag{1.5}$$

$$X(t) = x,$$

$$dZ = g(s,X,u(s,X))I_G(X)ds + \psi(s,X,u(s,X))I_{\partial G}(X)d\mu(s),\ Z(t) = z,$$

where $w(s) = (w^1(s), ..., w^d(s))^\top$ is a standard Wiener process, $b(s,x,u) = (b^1(s,x,u), ..., b^d(s,x,u))^\top$ is a column vector, the matrix $\sigma = \sigma(s,x,u)$ is obtained from the equation

$$\sigma\sigma^\top = a,\ \sigma = \{\sigma^{ij}(s,x,u)\},\ a = \{a^{ij}(s,x,u)\},\ i,j = 1, ..., d,$$

$\mu(s)$ is a local time of the process X on ∂G, and $I_A(x)$ is the indicator of a set A.

Introduce a time discretization, for definiteness the equidistant one:

$$T = t_N > t_{N-1} > \cdots > t_0,\ h := \frac{T - t_0}{N}\ .$$

The proposed here methods give an approximation $\bar{u}(t_k, x)$ of the solution $u(t_k, x)$, $k = N, ..., 0$, $x \in \overline{G}$, i.e., step by step everywhere in the domain $\overline{G}$. They exploit the ideas of weak sense numerical integration of SDE in bounded domain [11, 12] (see also [9, 6, 16]). As a result, we express $\bar{u}(t_k, x)$ recurrently in terms of $\bar{u}(t_{k+1}, x)$, $k = N-1, ..., 0$, i.e., we construct some layer methods which are discrete in the variable t only. In spite of their probabilistic nature these methods are nevertheless deterministic.

In Section 2, a few layer methods for the nonlinear Neumann problem are constructed. Using probabilistic type arguments, a convergence theorem is proved in Section 3. To realize a layer method in practice, a discretization in the variable x with interpolation at every step is needed to turn the method into an algorithm. Such numerical algorithms are given in Section 4. A majority of ideas can be demonstrated at $d = 1$, and we restrict ourselves to this case in Sections 2-4. The case $d \geq 2$ is discussed in Section 5. Numerical tests are presented in the last section.

Traditional numerical analysis of nonlinear PDE is available, e.g., in [17, 18, 19, 22]. Other probability approaches are considered in [7, 20]. The probability approach to boundary value problems for linear parabolic equations is treated in [10, 11, 12, 2].

2. Construction of layer methods

The Neumann boundary value problem in the one-dimensional case has the form

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 u}{\partial x^2} + b(t,x,u)\frac{\partial u}{\partial x} + g(t,x,u) = 0,\ \ t_0 \leq t < T,\ \alpha < x < \beta\ , \tag{2.1}$$

$$u(T,x) = \varphi(x),\ \alpha \leq x \leq \beta; \tag{2.2}$$

$$\frac{\partial u}{\partial x}(t,\alpha) = \psi_1(t, u(t,\alpha)),\ \frac{\partial u}{\partial x}(t,\beta) = \psi_2(t, u(t,\beta)),\ t_0 \leq t \leq T. \tag{2.3}$$

In this case Q is a partly open rectangle: $Q = [t_0, T) \times (\alpha, \beta)$, and Γ consists of the upper base $\{T\} \times [\alpha, \beta]$ and two vertical intervals: $[t_0, T) \times \{\alpha\}$ and $[t_0, T) \times \{\beta\}$. We assume that $\sigma(t,x,u) \geq \sigma_* > 0$ for $(t,x) \in \overline{Q}$, $-\infty < u < \infty$.

Let $u = u(t,x)$ be a solution to the problem (2.1)-(2.3) which is supposed to exist, to be unique, and to be sufficiently smooth. Theoretical results on this topic are available in [8, 21] (see also references therein).

1. Let us suppose for a while that it is possible to extend the coefficients of equation (2.1) so that the new equation has a solution $u(t,x)$ on $[t_0, T) \times \mathbf{R}$ which is an extension of the solution to the boundary value problem (2.1)-(2.3). The function $u(t,x)$ is nothing

but a solution of a Cauchy problem for equation (2.1). To construct methods, we use the representation

$$u(t_k,x)=\mathbf{E}(u(t_{k+1},X_{t_k,x}(t_{k+1}))+Z_{t_k,x,0}(t_{k+1})), \tag{2.4}$$

where $X_{t,x}(s),\ Z_{t,x,z}(s),\ t_0\le t<T,\ s\ge t,\ x\in\overline{G}$, is a solution of the Cauchy problem to the Ito system of SDE

$$dX=b(s,X,u(s,X))ds+\sigma(s,X,u(s,X))dw(s),\ X(t)=x,$$

$$dZ=g(s,X,u(s,X))ds,\ Z(t)=z. \tag{2.5}$$

Applying the explicit weak Euler scheme with the simplest simulation of noise to system (2.5), we get

$$\bar{X}_{t_k,x}(t_{k+1})=x+b(t_k,x,u(t_k,x))h+\sigma(t_k,x,u(t_k,x))\sqrt{h}\xi\ , \tag{2.6}$$

$$\bar{Z}_{t_k,x,0}(t_{k+1})=g(t_k,x,u(t_k,x))h\ , \tag{2.7}$$

where ξ is distributed by the law: $P(\xi=\pm1)=\dfrac{1}{2}$.

Using (2.4) (we suppose the layer $u(t_{k+1},x)$ to be known) , we get to within $O(h^2)$:

$$\begin{aligned}u(t_k,x)&\simeq\mathbf{E}(u(t_{k+1},\bar{X}_{t_k,x}(t_{k+1}))+\bar{Z}_{t_k,x,0}(t_{k+1}))\\&=\frac{1}{2}u(t_{k+1},x+b(t_k,x,u(t_k,x))h-\sigma(t_k,x,u(t_k,x))\sqrt{h})\\&+\frac{1}{2}u(t_{k+1},x+b(t_k,x,u(t_k,x))h+\sigma(t_k,x,u(t_k,x))\sqrt{h})+g(t_k,x,u(t_k,x))h.\end{aligned} \tag{2.8}$$

Now we can obtain an implicit relation for an approximation of $u(t_k,x)$. Applying the method of simple iteration to the implicit relation and taking $u(t_{k+1},x)$ as a null iteration, we get the following explicit one-step approximation $v(t_k,x)$ of $u(t_k,x)$:

$$v(t_k,x)=\frac{1}{2}u(t_{k+1},x+hb_k-h^{1/2}\sigma_k)+\frac{1}{2}u(t_{k+1},x+hb_k+h^{1/2}\sigma_k)+hg_k\ , \tag{2.9}$$

where $b_k,\ \sigma_k,\ g_k$ are the coefficients $b,\ \sigma,\ g$ calculated at the point $(t_k,x,u(t_{k+1},x))$.

But in reality we know the layer $u(t_{k+1},x)$ for $\alpha\le x\le\beta$ only. At the same time the argument $x+hb_k-h^{1/2}\sigma_k$ for x close to α is less than α and the argument $x+hb_k+h^{1/2}\sigma_k$ for x close to β is more than β. Thus we need to extend the layer $u(t_{k+1},x)$ in a constructive manner.

To this end let us use the formula

$$\begin{aligned}u(t,\alpha-\Delta x)&=u(t,\alpha+\Delta x)-2\frac{\partial u}{\partial x}(t,\alpha)\cdot\Delta x+O(\Delta x^3)\\&=u(t,\alpha+\Delta x)-2\psi_1(t,u(t,\alpha))\cdot\Delta x+O(\Delta x^3).\end{aligned} \tag{2.10}$$

If $\Delta x=O(\sqrt{h})$, the accuracy of formula (2.10) is $O(h^{3/2})$. Therefore

$$\begin{aligned}u(t_{k+1},x+hb_k-h^{1/2}\sigma_k)&=u(t_{k+1},2\alpha-x-hb_k+h^{1/2}\sigma_k)\\&-2\psi_1(t_{k+1},u(t_{k+1},\alpha))\cdot(\alpha-x-hb_k+h^{1/2}\sigma_k)+O(h^{3/2}),\end{aligned}$$

i.e., in the case $x + hb_k - h^{1/2}\sigma_k < \alpha$ we get the explicit one-step approximation $v(t_k, x)$ for $u(t_k, x)$ of accuracy $O(h^{3/2})$:

$$v(t_k, x) = \frac{1}{2}u(t_{k+1}, 2\alpha - x - hb_k + h^{1/2}\sigma_k)$$
$$-\psi_1(t_{k+1}, u(t_{k+1}, \alpha)) \cdot (\alpha - x - hb_k + h^{1/2}\sigma_k)$$
$$+\frac{1}{2}u(t_{k+1}, x + hb_k + h^{1/2}\sigma_k) + hg_k.$$

The analogous formula can be written for the right end β. As a result, we obtain the following method

(2.11)
$$\bar{u}(t_N, x) = \varphi(x),\ x \in [\alpha, \beta],$$
$$\bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k) + \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k) + h\bar{g}_k,$$
$$x + h\bar{b}_k \pm h^{1/2}\bar{\sigma}_k \in [\alpha, \beta],$$
$$\bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, 2\alpha - x - h\bar{b}_k + h^{1/2}\bar{\sigma}_k) - \psi_1(t_{k+1}, \bar{u}(t_{k+1}, \alpha)) \cdot (\alpha - x - h\bar{b}_k + h^{1/2}\bar{\sigma}_k)$$
$$+\frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k) + h\bar{g}_k,\ x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k < \alpha,$$
$$\bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k) + \frac{1}{2}\bar{u}(t_{k+1}, 2\beta - x - h\bar{b}_k - h^{1/2}\bar{\sigma}_k)$$
$$+\psi_2(t_{k+1}, \bar{u}(t_{k+1}, \beta)) \cdot (x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k - \beta) + h\bar{g}_k,$$
$$x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k > \beta,\ k = N-1, ..., 1, 0,$$

where $\bar{b}_k,\ \bar{\sigma}_k,\ \bar{g}_k$ are the coefficients $b,\ \sigma,\ g$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$.

The method (2.11) is an explicit layer method for solving the Neumann problem (2.1)-(2.3). Its one-step error near the boundary is $O(h^{3/2})$ and for internal points is $O(h^2)$ (see Lemma 3.1). Apparently, this method has order of convergence $O(h)$ (see Remark 3.2).

2. Applying a slightly modified weak scheme with one-step boundary order $O(h^{3/2})$ from [11, 12] to system (1.5), it is not difficult to obtain

(2.12)
$$X_{t_k,x}(t_{k+1}) \simeq \bar{X}_{t_k,x}(t_{k+1}) = x + h\tilde{b}_k + h^{1/2}\tilde{\sigma}_k\xi_k$$
$$Z_{t_k,x,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,z}(t_{k+1}) = z + h\tilde{g}_k,\ x + h\tilde{b}_k \pm h^{1/2}\tilde{\sigma}_k \in [\alpha, \beta];$$
$$\bar{X}_{t_k,x}(t_{k+1}) = x + (\alpha - x) + \sqrt{h\tilde{\sigma}_k^2 + (\alpha - x)^2}$$
$$\bar{Z}_{t_k,x,z}(t_{k+1}) = z + h\tilde{g}_k - \psi_1(t_k, u(t_k, \alpha)) \cdot (\alpha - x - h\tilde{b}_k + \sqrt{h\tilde{\sigma}_k^2 + (\alpha - x)^2}),$$
$$x + h\tilde{b}_k - h^{1/2}\tilde{\sigma}_k < \alpha;$$
$$\bar{X}_{t_k,x}(t_{k+1}) = x + (\beta - x) - \sqrt{h\tilde{\sigma}_k^2 + (\beta - x)^2}$$
$$\bar{Z}_{t_k,x,z}(t_{k+1}) = z + h\tilde{g}_k - \psi_2(t_k, u(t_k, \beta)) \cdot (\beta - x - h\tilde{b}_k - \sqrt{h\tilde{\sigma}_k^2 + (\beta - x)^2}),$$
$$x + h\tilde{b}_k + h^{1/2}\tilde{\sigma}_k > \beta.$$

Here $\tilde{b}_k$, $\tilde{\sigma}_k$, $\tilde{g}_k$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ calculated at the point $(t_k, x, u(t_k,x))$ and $\xi_{N-1}, \xi_{N-2}, \ldots, \xi_0$ are i.i.d. random variables with the law $P(\xi = \pm 1) = 1/2$.

One can see that using approximation (2.12) and representation (2.4), we get an implicit one-step approximation for $u(t_k,x)$. Applying the method of simple iteration to this implicit approximation with $u(t_{k+1},x)$ as a null iteration, we come to the explicit one-step approximation $v(t_k,x)$ of $u(t_k,x)$:

$$
\begin{aligned}
(2.13)\qquad v(t_k,x) &= \frac{1}{2}u(t_{k+1}, x + hb_k - h^{1/2}\sigma_k) + \frac{1}{2}u(t_{k+1}, x + hb_k + h^{1/2}\sigma_k) + hg_k,\\
&x + hb_k \pm h^{1/2}\sigma_k \in [\alpha,\beta];\\
&v(t_k,x) = u(t_{k+1}, \alpha + \sqrt{h\sigma_k^2 + (\alpha - x)^2})\\
&-\psi_1(t_{k+1}, u(t_{k+1},\alpha)) \cdot (\alpha - x - hb_k + \sqrt{h\sigma_k^2 + (\alpha - x)^2}) + hg_k,\\
&x + hb_k - h^{1/2}\sigma_k < \alpha;\\
&v(t_k,x) = u(t_{k+1}, \beta - \sqrt{h\sigma_k^2 + (\beta - x)^2})\\
&-\psi_2(t_{k+1}, u(t_{k+1},\beta)) \cdot (\beta - x - hb_k - \sqrt{h\sigma_k^2 + (\beta - x)^2}) + hg_k,\\
&x + hb_k + h^{1/2}\sigma_k > \beta,\ k = N-1, \ldots, 1, 0,
\end{aligned}
$$

where b_k, σ_k, g_k are the coefficients b, σ, g calculated at the point $(t_k, x, u(t_{k+1},x))$. Let us observe that within the limits of considered accuracy it is very often possible to take t_{k+1} instead of t_k. That is why, one can take, for instance, $\psi_1(t_{k+1}, u(t_{k+1},\alpha))$ instead of $\psi_1(t_k, u(t_{k+1},\alpha))$ in (2.13).

The corresponding explicit layer method for solving the Neumann problem (2.1)-(2.3) has the form

$$
\begin{aligned}
(2.14)\qquad &\bar{u}(t_N,x) = \varphi(t_N,x),\ x \in [\alpha,\beta],\\
\bar{u}(t_k,x) &= \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k) + \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k) + h\bar{g}_k,\\
&x + h\bar{b}_k \pm h^{1/2}\bar{\sigma}_k \in [\alpha,\beta];\\
&\bar{u}(t_k,x) = \bar{u}(t_{k+1}, \alpha + \sqrt{h\bar{\sigma}_k^2 + (\alpha - x)^2})\\
&-\psi_1(t_{k+1}, \bar{u}(t_{k+1},\alpha)) \cdot (\alpha - x - h\bar{b}_k + \sqrt{h\bar{\sigma}_k^2 + (\alpha - x)^2}) + h\bar{g}_k,\\
&x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k < \alpha;\\
&\bar{u}(t_k,x) = \bar{u}(t_{k+1}, \beta - \sqrt{h\bar{\sigma}_k^2 + (\beta - x)^2})\\
&-\psi_2(t_{k+1}, \bar{u}(t_{k+1},\beta)) \cdot (\beta - x - h\bar{b}_k - \sqrt{h\bar{\sigma}_k^2 + (\beta - x)^2}) + h\bar{g}_k,\\
&x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k > \beta;\\
&k = N-1, ..., 1, 0,
\end{aligned}
$$

where $\bar{b}_k = \bar{b}_k(x) = b(t_k, x, \bar{u}(t_{k+1}, x))$, $\bar{\sigma}_k = \bar{\sigma}_k(x) = \sigma(t_k, x, \bar{u}(t_{k+1}, x))$, $\bar{g}_k = \bar{g}_k(x) = g(t_k, x, \bar{u}(t_{k+1}, x))$.

This layer method has the one-step error near the boundary estimated by $O(h^{3/2})$ and for internal points estimated by $O(h^2)$ (see Lemma 3.1). We prove that its order of convergence is $O(h)$ when boundary condition does not depend on the solution (see Theorem 3.1). Apparently, this is so in the general case as well (see Remark 3.1).

3. Applying a weak scheme with one-step boundary order $O(h)$ from [11, 12] to system (1.5), it is not difficult to obtain

$$X_{t_k,x}(t_{k+1}) \simeq \bar{X}_{t_k,x}(t_{k+1}) = x + h\tilde{b}_k + h^{1/2}\tilde{\sigma}_k\xi_k \tag{2.15}$$

$$Z_{t_k,x,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,z}(t_{k+1}) = z + h\tilde{g}_k,\ x + h\tilde{b}_k \pm h^{1/2}\tilde{\sigma}_k \in [\alpha, \beta];$$

$$\bar{X}_{t_k,x}(t_{k+1}) = x + qh^{1/2},\ \bar{Z}_{t_k,x,z}(t_{k+1}) = z - \psi_1(t_k, u(t_k, \alpha))qh^{1/2},\ x + h\tilde{b}_k - h^{1/2}\tilde{\sigma}_k < \alpha;$$

$$\bar{X}_{t_k,x}(t_{k+1}) = x - qh^{1/2},\ \bar{Z}_{t_k,x,z}(t_{k+1}) = z + \psi_2(t_k, u(t_k, \beta))qh^{1/2},\ x + h\tilde{b}_k + h^{1/2}\tilde{\sigma}_k > \beta.$$

Here $\tilde{b}_k$, $\tilde{\sigma}_k$, $\tilde{g}_k$ are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ calculated at the point $(t_k, x, u(t_k, x))$, $\xi_{N-1}, \xi_{N-2}, \ldots, \xi_0$ are i.i.d. random variables with the law $P(\xi = \pm 1) = 1/2$, and q is a positive number (see Remark 3.3, where a discussion on choosing q is given). As before, we obtain the following explicit one-step approximation $v(t_k, x)$ of $u(t_k, x)$:

$$v(t_k, x) = \frac{1}{2}u(t_{k+1}, x + hb_k - h^{1/2}\sigma_k) + \frac{1}{2}u(t_{k+1}, x + hb_k + h^{1/2}\sigma_k) + hg_k, \tag{2.16}$$

$$x + hb_k \pm h^{1/2}\sigma_k \in [\alpha, \beta];$$

$$v(t_k, x) = u(t_{k+1}, x + qh^{1/2}) - \psi_1(t_{k+1}, u(t_{k+1}, \alpha))qh^{1/2},\ x + hb_k - h^{1/2}\sigma_k < \alpha;$$

$$v(t_k, x) = u(t_{k+1}, x - qh^{1/2}) + \psi_2(t_{k+1}, u(t_{k+1}, \beta))qh^{1/2},\ x + hb_k + h^{1/2}\sigma_k > \beta;$$

$$k = N - 1, ..., 1, 0.$$

The corresponding explicit layer method for solving the Neumann problem (2.1)-(2.3) has the form

$$\bar{u}(t_N, x) = \varphi(t_N, x),\ x \in [\alpha, \beta], \tag{2.17}$$

$$\bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k) + \frac{1}{2}\bar{u}(t_{k+1}, x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k) + h\bar{g}_k,$$

$$x + h\bar{b}_k \pm h^{1/2}\bar{\sigma}_k \in [\alpha, \beta];$$

$$\bar{u}(t_k, x) = \bar{u}(t_{k+1}, x + qh^{1/2}) - \psi_1(t_{k+1}, \bar{u}(t_{k+1}, \alpha))qh^{1/2},\ x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k < \alpha;$$

$$\bar{u}(t_k, x) = \bar{u}(t_{k+1}, x - qh^{1/2}) + \psi_2(t_{k+1}, \bar{u}(t_{k+1}, \beta))qh^{1/2},\ x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k > \beta;$$

$$k = N - 1, ..., 1, 0,$$

where $\bar{b}_k = \bar{b}_k(x) = b(t_k, x, \bar{u}(t_{k+1}, x))$, $\bar{\sigma}_k = \bar{\sigma}_k(x) = \sigma(t_k, x, \bar{u}(t_{k+1}, x))$, $\bar{g}_k = \bar{g}_k(x) = g(t_k, x, \bar{u}(t_{k+1}, x))$.

This layer method is simpler but less accurate than (2.11) and (2.14). Its one-step error near the boundary is $O(h)$ and for internal points is $O(h^2)$ (see Lemma 3.3). We prove

that its order of convergence is $O(h^{1/2})$ when boundary condition does not depend on the solution (see Theorem 3.2). Apparently, this is so in the general case as well.

4. In [2] another weak scheme for SDE with reflection is proposed and applied to solving the linear Neumann problem. The authors state that the scheme has the weak order of convergence $O(h^{1/2})$. On the base of this scheme, the layer method for the nonlinear problem can be constructed:

$$\bar{u}(t_N,x)=\varphi(t_N,x),\ x\in[\alpha,\beta], \tag{2.18}$$

$$\bar{u}(t_k,x)=\frac{1}{2}\bar{u}(t_{k+1},x+h\bar{b}_k-h^{1/2}\bar{\sigma}_k)+\frac{1}{2}\bar{u}(t_{k+1},x+h\bar{b}_k+h^{1/2}\bar{\sigma}_k)+h\bar{g}_k,$$

$$x+h\bar{b}_k\pm h^{1/2}\bar{\sigma}_k\in[\alpha,\beta];$$

$$\bar{u}(t_k,x)=\frac{1}{2}\bar{u}(t_{k+1},\alpha)+\frac{1}{2}\bar{u}(t_{k+1},x+h\bar{b}_k+h^{1/2}\bar{\sigma}_k)$$

$$-\frac{1}{2}\psi_1(t_{k+1},\bar{u}(t_{k+1},\alpha))(\alpha-x-h\bar{b}_k+h^{1/2}\bar{\sigma}_k)+h\bar{g}_k,\ x+h\bar{b}_k-h^{1/2}\bar{\sigma}_k<\alpha;$$

$$\bar{u}(t_k,x)=\frac{1}{2}\bar{u}(t_{k+1},x+h\bar{b}_k-h^{1/2}\bar{\sigma}_k)+\frac{1}{2}\bar{u}(t_{k+1},\beta)$$

$$-\frac{1}{2}\psi_2(t_{k+1},\bar{u}(t_{k+1},\beta))(\beta-x-h\bar{b}_k-h^{1/2}\bar{\sigma}_k)+h\bar{g}_k,\ x+h\bar{b}_k+h^{1/2}\bar{\sigma}_k>\beta;$$

$$k=N-1,...,1,0,$$

where $\bar{b}_k=\bar{b}_k(x)=b(t_k,x,\bar{u}(t_{k+1},x))$, $\bar{\sigma}_k=\bar{\sigma}_k(x)=\sigma(t_k,x,\bar{u}(t_{k+1},x))$, $\bar{g}_k=\bar{g}_k(x)=g(t_k,x,\bar{u}(t_{k+1},x))$.

Apparently, this layer method has order of convergence $O(h^{1/2})$.

Remark 2.1. Combining methods from [15] and from this section, we can solve mixed boundary value problems, i.e., when we have the Dirichlet condition on a part of the boundary ∂G and the Neumann condition on the rest of ∂G.

The methods (2.11), (2.14), (2.17), and (2.18) are deterministic though the probabilistic approach is used for their construction.

3. Convergence theorems

We shall keep the following assumptions.

(i) There exists a unique solution $u(t,x)$ of problem (2.1)-(2.3) such that

$$-\infty\le u_\circ<u_*\le u(t,x)\le u^*<u^\circ\le\infty,\ t_0\le t\le T,\ x\in[\alpha,\beta], \tag{3.1}$$

where $u_\circ$, u_*, u^*, u° are some constants, and there exist the uniformly bounded derivatives:

$$\left|\frac{\partial^{i+j}u}{\partial t^i\partial x^j}\right|\le K,\ i=0,\ j=1,2,3,4;\ i=1,\ j=0,1,2;\ i=2,\ j=0;\ t_0\le t\le T,\ x\in[\alpha,\beta]. \tag{3.2}$$

(ii) The coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ are uniformly bounded and uniformly satisfy the Lipschitz condition with respect to x and u:

$$|b| \le K,\ |\sigma| \le K,\ |g| \le K, \tag{3.3}$$

$$|b(t,x_2,u_2) - b(t,x_1,u_1)| + |\sigma(t,x_2,u_2) - \sigma(t,x_1,u_1)| + |g(t,x_2,u_2) - g(t,x_1,u_1)| \le K\left(|x_2 - x_1| + |u_2 - u_1|\right),$$

$$t_0 \le t \le T,\ x \in [\alpha,\beta],\ u_\circ < u < u^\circ.$$

Below we use the letters K and C without any index for various constants which do not depend on h, k, x.

Let us evaluate the one-step error $\rho(t_k,x)$ of methods (2.11) and (2.14).

Lemma 3.1. *Under assumptions (i) and (ii), the one-step error $\rho(t_k,x)$ of methods (2.11) and (2.14) is estimated as*

$$|\rho(t_k,x)| = |v(t_k,x) - u(t_k,x)| \le Ch^2,\ x + hb_k \pm h^{1/2}\sigma_k \in [\alpha,\beta]; \tag{3.4}$$

$$|\rho(t_k,x)| = |v(t_k,x) - u(t_k,x)| \le Ch^{3/2},\ x + hb_k - h^{1/2}\sigma_k < \alpha \text{ or } x + hb_k + h^{1/2}\sigma_k > \beta, \tag{3.5}$$

where $v(t_k,x)$ is the corresponding one-step approximation, C does not depend on h, k, x.

Proof. If both the points $x + hb_k \pm h^{1/2}\sigma_k$ belong to $[\alpha,\beta]$, we have

$$v(t_k,x) = \frac{1}{2}u(t_{k+1}, x + hb_k - h^{1/2}\sigma_k) + \frac{1}{2}u(t_{k+1}, x + hb_k + h^{1/2}\sigma_k) + hg_k. \tag{3.6}$$

Expanding the terms of (3.6) at the point (t_k,x) and taking into account that $u(t,x)$ is the solution of problem (2.1)-(2.3), we get (3.4) (see also [13, 14], where similar assertions are proved in detail).

Let us consider the case when the point $x + hb_k - h^{1/2}\sigma_k < \alpha$. The relation (3.5) for method (2.11) follows from Section 2.1. Let us prove this relation for method (2.14). Due to (2.13),

$$v(t_k,x) = u(t_{k+1}, x + \Delta X^\alpha) - \psi_1(t_{k+1}, u(t_{k+1},\alpha)) \cdot (\Delta X^\alpha - hb_k) + hg_k, \tag{3.7}$$

where

$$\Delta X^\alpha := \alpha - x + \sqrt{h\sigma_k^2 + (\alpha - x)^2}\,.$$

It is clear that

$$|\alpha - x| \le Ch^{1/2},\ \ |\Delta X^\alpha| \le Ch^{1/2}. \tag{3.8}$$

Taking into account that $\psi_1(t_{k+1}, u(t_{k+1},\alpha)) = u'_x(t_{k+1},\alpha)$ (see (2.3)), then expanding the functions $u(t_{k+1}, x + \Delta X^\alpha)$ and $u'_x(t_{k+1}, x + (\alpha - x))$ at the point (t_k,x), and using assumptions (i), (ii) and inequalities (3.8), we get

$$v(t_k,x) = u + \frac{\partial u}{\partial t}h + \frac{\partial u}{\partial x}\Delta X^\alpha + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\Delta X^\alpha)^2 - \frac{\partial u}{\partial x}(\Delta X_\alpha - hb_k) - \frac{\partial^2 u}{\partial x^2}(\alpha - x)\Delta X^\alpha + g_k h + O(h^{3/2}) \tag{3.9}$$

$$= u + h(\frac{\partial u}{\partial t} + b_k \frac{\partial u}{\partial x} + g_k) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta X^\alpha(\Delta X^\alpha - 2(\alpha - x)) + O(h^{3/2}),$$

where the function u and its derivatives are calculated at the point (t_k, x).

The expression $\Delta X^\alpha(\Delta X^\alpha - 2(\alpha - x))$ is equal to $h\sigma_k^2$.

Due to assumptions (i) and (ii), we obtain

$$b_k = b(t_k, x, u(t_{k+1}, x)) = \tilde{b}_k + O(h),\ \sigma_k^2 = \tilde{\sigma}_k^2 + O(h),\ g_k = \tilde{g}_k + O(h),$$

where $\tilde{b}_k$, $\tilde{\sigma}_k$, $\tilde{g}_k$ are calculated at the point $(t_k, x, u(t_k, x))$.

Then we get from (3.9):

$$v(t_k, x) = u + h(\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + g) + O(h^{3/2}). \tag{3.10}$$

Since $u(t, x)$ is the solution of problem (2.1)-(2.3), the relation (3.10) implies

$$v(t_k, x) = u(t_k, x) + O(h^{3/2}).$$

The case $x + hb_k + h^{1/2}\sigma_k > \beta$ can be considered analogously. Lemma 3.1 is proved.

A discussion concerning convergence of method (2.11) see in Remark 3.2. The theorem on global convergence for method (2.14) is given in the specific case of the Neumann problem (2.1)-(2.3), when the functions $\psi_1(t, u)$ and $\psi_2(t, u)$ do not depend on u (see a discussion concerning the general case in Remark 3.1). To prove the theorem, we need some auxiliary constructions.

In connection with the layer method (2.14), we introduce the random sequence X_i, Z_i :

$$X_k = x,\ Z_k = 0, \tag{3.11}$$

$$X_{i+1} = X_i + h\breve{b}_i + h^{1/2}\breve{\sigma}_i\xi_i,\quad Z_{i+1} = Z_i + h\breve{g}_i,$$

$$X_i + h\breve{b}_i \pm h^{1/2}\breve{\sigma}_i \in [\alpha, \beta];$$

$$X_{i+1} = X_i + \Delta X_i^\alpha,\ Z_{i+1} = Z_i + h\breve{g}_i - \psi_1(t_{i+1})\cdot(\Delta X_i^\alpha - h\breve{b}_i),$$

$$\Delta X_i^\alpha := (\alpha - X_i) + \sqrt{h\breve{\sigma}_i^2 + (\alpha - X_i)^2},$$

$$X_i + h\breve{b}_i - h^{1/2}\breve{\sigma}_i < \alpha;$$

$$X_{i+1} = X_i + \Delta X_i^\beta,\ Z_{i+1} = Z_i + h\breve{g}_i - \psi_2(t_{i+1})\cdot(\Delta X_i^\beta - h\breve{b}_i),$$

$$\Delta X_i^\beta := (\beta - X_i) - \sqrt{h\breve{\sigma}_i^2 + (\beta - X_i)^2},$$

$$X_i + h\breve{b}_i + h^{1/2}\breve{\sigma}_i > \beta;$$

$$i = k, \ldots, N-1,\ k \geq 0.$$

Here ξ_i are i.i.d. random variables with the law $P(\xi = \pm 1) = \frac{1}{2}$ and $\breve{b}_i = \bar{b}_i(X_i) = b(t_i, X_i, \bar{u}(t_{i+1}, X_i))$, $\breve{\sigma}_i = \bar{\sigma}_i(X_i) = \sigma(t_i, X_i, \bar{u}(t_{i+1}, X_i))$, $\breve{g}_i = \bar{g}_i(X_i) = g(t_i, X_i, \bar{u}(t_{i+1}, X_i))$. Let us note that the function $\bar{u}(t_i, x)$, $i = 0, \ldots, N$, $x \in [\alpha, \beta]$, is uniquely defined by (2.14).

Evidently, the sequence (t_i, X_i) is a Markov chain.

Introduce the boundary layer $\partial\Gamma \in \overline{Q}$: for all the points $(t_k, x) \in \overline{Q} \setminus \partial\Gamma$ both the points $x + h\bar{b}_k(x) \pm h^{1/2}\bar{\sigma}_k(x)$ belong to $[\alpha, \beta]$ and for the points $(t_k, x) \in \partial\Gamma$ either $x + h\bar{b}_k(x) - h^{1/2}\bar{\sigma}_k(x) \notin [\alpha, \beta]$ or $x + h\bar{b}_k(x) + h^{1/2}\bar{\sigma}_k(x) \notin [\alpha, \beta]$.

Lemma 3.2. *Under assumptions* (i) *and* (ii), *the mean of the number of steps* $\varkappa(t_k, x)$, *which the Markov chain* (t_i, X_i), $i = k, \ldots, N$, $k \geq 0$, $X_k = x$, *spends in the layer* $\partial\Gamma$, *is estimated as*

$$E\varkappa(t_k, x) \leq \frac{C}{\sqrt{h}},$$

where C *does not depend on* h, k, x.

The proof of Lemma 3.2 differs only little from the proof of the corresponding lemma on the mean number of steps in the case of the linear Neumann problem given in [12] and is therefore omitted.

Theorem 3.1. *Let the Neumann problem for equation* (2.1) *with condition* (2.2) *have the following boundary conditions*

$$\frac{\partial u}{\partial x}(t, \alpha) = \psi_1(t), \ \frac{\partial u}{\partial x}(t, \beta) = \psi_2(t), \ t_0 \leq t \leq T. \tag{3.12}$$

Under assumptions (i) *and* (ii), *the method* (2.14) *has the first order of convergence with respect to* h, *i.e.,*

$$|\bar{u}(t_k, x) - u(t_k, x)| \leq Kh,$$

where K *does not depend on* h, k, x.

Proof. Here we exploit ideas of proving convergence theorems for probabilistic methods solving linear boundary value problems [10, 11, 12].

Let X_i, Z_i, $i = k, \ldots, N$, $X_k = x$, $Z_k = 0$, be the sequence defined by (3.11). It is clear that

$$\bar{u}(t_k, x) = E\left[\bar{u}(t_N, X_N) + Z_N\right] = E\left[\varphi(t_N, X_N) + Z_N\right] = E\left[u(t_N, X_N) + Z_N\right].$$

Introduce the notation $R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x)$. Then we have

$$\begin{aligned} R(t_k, x) &= E\left[u(t_N, X_N) + Z_N\right] - u(t_k, x) \\ &= \sum_{i=k}^{N-1} E\left[u(t_{i+1}, X_{i+1}) - u(t_i, X_i) + Z_{i+1} - Z_i\right] \\ &= \sum_{i=k}^{N-1} EI_{\overline{Q}\setminus\partial\Gamma}(t_i, X_i)\left[u(t_{i+1}, X_{i+1}) - u(t_i, X_i) + Z_{i+1} - Z_i\right] \\ &+ \sum_{i=k}^{N-1} EI_{\partial\Gamma}(t_i, X_i)\left[u(t_{i+1}, X_{i+1}) - u(t_i, X_i) + Z_{i+1} - Z_i\right]. \end{aligned} \tag{3.13}$$

Denote the first sum in the right-hand side of (3.13) by $R^{(1)}(t_k,x)$ and the second one by $R^{(2)}(t_k,x)$. We have

$$R^{(1)}(t_k,x)=\sum_{i=k}^{N-1} EI_{\overline{Q}\setminus\partial\Gamma}(t_i,X_i)\left[u(t_{i+1},X_{i+1})-u(t_i,X_i)+Z_{i+1}-Z_i\right] \tag{3.14}$$

$$=\sum_{i=k}^{N-1} E\left(I_{\overline{Q}\setminus\partial\Gamma}(t_i,X_i)E\left[u(t_{i+1},X_{i+1})-u(t_i,X_i)+Z_{i+1}-Z_i\diagup X_i,Z_i\right]\right).$$

According to (3.11), we obtain for $(t_i,X_i)\in\overline{Q}\setminus\partial\Gamma$:

$$A_i:=E\left[u(t_{i+1},X_{i+1})-u(t_i,X_i)+Z_{i+1}-Z_i\diagup X_i,Z_i\right] \tag{3.15}$$

$$=\frac{1}{2}u(t_{i+1},X_i+h\breve{b}_i-h^{1/2}\breve{\sigma}_i)+\frac{1}{2}u(t_{i+1},X_i+h\breve{b}_i+h^{1/2}\breve{\sigma}_i)+h\breve{g}_i.$$

Expanding the functions $u(t_{i+1},X_i+h\breve{b}_i\pm h^{1/2}\breve{\sigma}_i)$ at the point (t_i,X_i), we get

$$u(t_{i+1},X_i+h\breve{b}_i\pm h^{1/2}\breve{\sigma}_i)=u(t_i,X_i)+\frac{\partial u}{\partial t}h+(h\breve{b}_i\pm h^{1/2}\breve{\sigma}_i)\frac{\partial u}{\partial x} \tag{3.16}$$

$$+\frac{\breve{\sigma}_i^2}{2}\frac{\partial^2 u}{\partial x^2}h\pm\breve{b}_i\breve{\sigma}_i\frac{\partial^2 u}{\partial x^2}h^{3/2}\pm\breve{\sigma}_i\frac{\partial^2 u}{\partial t\partial x}h^{3/2}\pm\frac{\breve{\sigma}_i^3}{6}\frac{\partial^3 u}{\partial x^3}h^{3/2}+O(h^2),$$

where the derivatives are calculated at the point (t_i,X_i).

Here we have to suggest for a while that the value $u(t_{i+1},x)+R(t_{i+1},x)$ for $x\in[\alpha,\beta]$ remains in the interval $(u_\circ,u^\circ)$ for a sufficiently small h (see conditions (ii)). Clearly, $R(t_N,x)=0$, and below we prove recurrently that $R(t_i,x)$ is sufficiently small under a sufficiently small h. Thereupon, thanks to (3.1), this suggestion will be justified for such h.

Then due to assumptions (i) and (ii), we obtain

$$\bar{b}_i(x)=b(t_i,x,\bar{u}(t_{i+1},x))=b(t_i,x,u(t_i,x))+\Delta b(t_{i+1},x)+O(h) \tag{3.17}$$

$$:=b_i(x)+\Delta b_i(x)+O(h),$$

where

$$|\Delta b_i(x)|\le K|R(t_{i+1},x)|,\ |O(h)|\le Kh;$$

and analogously

$$\bar{\sigma}_i^2(x)=\sigma_i^2(x)+\Delta\sigma_i^2(x)+O(h),\ \bar{g}_i(x)=g_i(x)+\Delta g_i(x)+O(h), \tag{3.18}$$

$$|\Delta\sigma_i^2(x)|,\ |\Delta g_i(x)|\le K|R(t_{i+1},x)|.$$

Substituting (3.16) in (3.15) and taking into account (3.17)-(3.18), we come to the relation

$$A_i=h\left(\frac{\partial u}{\partial t}+b_i\frac{\partial u}{\partial x}+\frac{\sigma_i^2}{2}\frac{\partial^2 u}{\partial x^2}+g_i\right)+r_i+O(h^2),$$

where

$$|r_i|\le Kh|R(t_{i+1},X_i)|,\ |O(h^2)|\le Ch^2,$$

the derivatives are calculated at the point (t_i, X_i), and b_i, σ_i, g_i are calculated at $(t_i, X_i, u(t_i, X_i))$. Taking into account that $u(t,x)$ is the solution of problem (2.1)-(2.3), this relation implies

$$A_i = r_i + O(h^2).$$

Therefore

$$R^{(1)}(t_k, x) = \sum_{i=k}^{N-1} EI_{\overline{Q}\setminus\partial\Gamma}(t_i, X_i)\left[r_i + O(h^2)\right]. \tag{3.19}$$

Now consider $R^{(2)}(t_k, x)$. Let $(t_i, X_i) \in \partial\Gamma$ be such that X_i is close to α. Then according to (3.11), we obtain

$$\begin{aligned} B_i &:= E\left[u(t_{i+1}, X_{i+1}) - u(t_i, X_i) + Z_{i+1} - Z_i \diagup X_i, Z_i\right] \\ &= u(t_{i+1}, X_i + \Delta X_i^\alpha) - u(t_i, X_i) - \psi_1(t_{i+1})(\Delta X_i^\alpha - h\breve{b}_i) + h\breve{g}_i \\ &= u(t_{i+1}, X_i + \Delta X_i^\alpha) - u(t_i, X_i) - \frac{\partial u}{\partial x}(t_{i+1}, \alpha)\cdot(\Delta X_i^\alpha - h\breve{b}_i) + h\breve{g}_i. \end{aligned} \tag{3.20}$$

Clearly

$$|X_i - \alpha| \le C\sqrt{h},\ |\Delta X_i^\alpha| \le C\sqrt{h}. \tag{3.21}$$

Expanding the terms of (3.20) at the point (t_i, X_i) and taking into account assumptions (i)-(ii), relations (3.17)-(3.18), and (3.21), we obtain (cf. (3.9)-(3.10)):

$$B_i = h(\frac{\partial u}{\partial t} + b_i\frac{\partial u}{\partial x} + \frac{\sigma_i^2}{2}\frac{\partial^2 u}{\partial x^2} + g_i) + \bar{r}_i + O(h^{3/2}),$$

where

$$|\bar{r}_i| \le Kh|R(t_{i+1}, X_i)|,$$

the derivatives are calculated at the point (t_i, X_i), and b_i, σ_i, g_i are calculated at $(t_i, X_i, u(t_i, X_i))$. Since $u(t,x)$ is the solution of problem (2.1)-(2.3), this relation implies

$$B_i = \bar{r}_i + O(h^{3/2}).$$

An analogous relation can be obtained for $(t_i, X_i) \in \partial\Gamma$ with X_i being close to β. Therefore

$$R^{(2)}(t_k, x) = \sum_{i=k}^{N-1} EI_{\partial\Gamma}(t_i, X_i)\left[\bar{r}_i + O(h^{3/2})\right]. \tag{3.22}$$

Substituting (3.19) and (3.22) in (3.13), we get

$$R(t_k, x) = \sum_{i=k}^{N-1} EI_{\overline{Q}\setminus\partial\Gamma}(t_i, X_i)\left[r_i + O(h^2)\right] + \sum_{i=k}^{N-1} EI_{\partial\Gamma}(t_i, X_i)\left[\bar{r}_i + O(h^{3/2})\right]. \tag{3.23}$$

Let $R_k := \max_{x\in[\alpha,\beta]}|R(t_k, x)|$. Due to Lemma 3.2, we obtain from (3.23)

$$R_k \le Kh\sum_{i=k}^{N-1} R_{i+1} + Ch. \tag{3.24}$$

Introduce $\varepsilon_k := Kh\sum_{i=k}^{N-1} R_{i+1} + Ch,\ k = N-1, ..., 0$. From (3.24) $R_k \le \varepsilon_k$ and consequently $\varepsilon_k = KhR_{k+1} + \varepsilon_{k+1} \le (1+Kh)\varepsilon_{k+1},\ k = N-2, ..., 0$. Then (since $\varepsilon_{N-1} = Ch$)

$$R_k \le \varepsilon_k \le Ce^{K(T-t_0)} \cdot h,\ k = N, ..., 0.$$

Theorem 3.1 is proved.

Remark 3.1. Apparently, the conclusion of Theorem 3.1 is true under the boundary conditions (2.3). We do not succeed in proving such a general theorem but we can prove it in the case of the linear boundary conditions

$$\frac{\partial u}{\partial x}(t,\alpha) = \varphi_1(t)u(t,\alpha) + \psi_1(t),\ \frac{\partial u}{\partial x}(t,\beta) = \varphi_2(t)u(t,\beta) + \psi_2(t),\ t_0 \le t \le T, \tag{3.25}$$

(the corresponding proof is more complicated in comparison with case (3.12) and is not given here). Besides numerical experiments confirm just mentioned conjecture (see Section 6).

Remark 3.2. As for convergence of method (2.11) in the case of boundary condition (3.12), we note first that it is not difficult to estimate its global error as $O(\sqrt{h})$ following deterministic type arguments similar to ones used in [13] (see [14, 15] as well). At the same time, it is natural to expect that probabilistic type arguments used in Theorem 3.1 will lead to the first order of convergence in h. However, we do not succeed in getting such a proof because the assertion of Lemma 3.2 is, most probably, not true in this case, and we need some additional auxiliary constructions. Let us also indicate that numerical experiments of Section 6 demonstrate $O(h)$ convergence of (2.11).

It turns out that method (2.17) in the case (3.12) (and in the case (3.25) as well) is convergent with order $O(h^{1/2})$. As above, this fact is apparently true for the general case of boundary conditions. Let us formulate the corresponding results (we note that in connection with method (2.17) one can introduce a Markov chain (t_i, X_i) for which Lemma 3.2 takes place).

Lemma 3.3. *Under assumptions* (i) *and* (ii), *the one-step error* $\rho(t_k, x)$ *of method* (2.17) *is estimated as*

$$|\rho(t_k,x)| = |v(t_k,x) - u(t_k,x)| \le Ch^2,\ x + hb_k \pm h^{1/2}\sigma_k \in [\alpha,\beta];$$

$$|\rho(t_k,x)| = |v(t_k,x) - u(t_k,x)| \le Ch,\ x + hb_k - h^{1/2}\sigma_k < \alpha \text{ or } x + hb_k + h^{1/2}\sigma_k > \beta,$$

where $v(t_k, x)$ *is defined by* (2.16), C *does not depend on* h, k, x.

Theorem 3.2. *Under assumptions* (i) *and* (ii), *the method* (2.17) *for the Neumann problem* (2.1)-(2.3), (3.12) *is of order* $O(h^{1/2})$, *i.e.,*

$$|\bar{u}(t_k,x) - u(t_k,x)| \le Kh^{1/2}, \tag{3.26}$$

where K *does not depend on* h, k, x.

Proofs of Lemma 3.3 and Theorem 3.2 are very similar to that of Lemma 3.1 and Theorem 3.1, and we do not give them here.

Remark 3.3. The layer method (2.17) has the parameter q, which, in principle, may be any positive number. Naturally, the value of q affects the method accuracy: K of (3.26) depends on q. By an extended analysis of the one-step boundary error and of the mean number of steps of the corresponding Markov chain in the boundary layer $\partial\Gamma$, we get

$$K \le C_1 \cdot \left(\frac{1}{q} \max_{(t,x)\in\overline{Q}} |\frac{\partial u}{\partial t}| + \frac{q}{2} \max_{(t,x)\in\overline{Q}} |\frac{\partial^2 u}{\partial x^2}| \right) + C_2,$$

where C_i, $i = 1, 2$, do not depend on h, k, x, and q.

Evidently, both large and small values of q are not appropriate. If we know estimates of derivatives of the solution for a considered problem, it is not difficult to indicate an appropriate q. But generally the choice of q requires special consideration.

Let $b(t,x,u) \equiv 0$ and $g(t,x,u) \equiv 0$. In this case the one-step boundary error $\rho(t_k, x)$ of method (2.17) near α is evaluated as

$$\rho(t_k, x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t_k, x) \cdot \left(q^2 h + 2(x-\alpha)qh^{1/2} - h\sigma_k^2\right) + O(h^{3/2}),\ x - h^{1/2}\sigma_k < \alpha,$$

and analogously near β. Taking $qh^{1/2} = \alpha - x + \sqrt{h\sigma_k^2 + (\alpha - x)^2}$, we obtain $\rho(t_k, x) = O(h^{3/2})$. Substitution of such q (depending on k and x) in (2.17) gives us a method with convergence order $O(h)$, which coincides with the method (2.14). Such an analysis also suggests that it is preferable to take $q \approx \sigma$.

4. Numerical algorithms

To have become numerical algorithms, the layer methods of Section 2 need a discretization in the variable x. Consider the equidistant space discretization with space step h_x (recall that the notation for time step is h): $x_j = \alpha + jh_x,\ j = 0, 1, 2, ..., M,\ h_x = (\beta - \alpha)/M$.

Using, for example, linear interpolation, we construct the following algorithm on the base of method (2.14) (we denote it as $\bar{u}(t_k, x)$ again, since this does not cause any confusion):

$$\bar{u}(t_N, x) = \varphi(t_N, x),\ x \in [\alpha, \beta], \tag{4.1}$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + h\bar{b}_{k,j} - h^{1/2}\bar{\sigma}_{k,j}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j + h\bar{b}_{k,j} + h^{1/2}\bar{\sigma}_{k,j}) + h\bar{g}_{k,j},$$

$$x_j + h\bar{b}_{k,j} \pm h^{1/2}\bar{\sigma}_{k,j} \in [\alpha, \beta];$$

$$\bar{u}(t_k, x_j) = \bar{u}(t_{k+1}, \alpha + \sqrt{h\bar{\sigma}_{k,j}^2 + (\alpha - x_j)^2})$$

$$-\psi_1(t_{k+1}, \bar{u}(t_{k+1}, \alpha)) \cdot (\alpha - x_j - h\bar{b}_{k,j} + \sqrt{h\bar{\sigma}_{k,j}^2 + (\alpha - x_j)^2}) + h\bar{g}_{k,j},$$

$$x_j + h\bar{b}_{k,j} - h^{1/2}\bar{\sigma}_{k,j} < \alpha;$$

$$\bar{u}(t_k, x_j) = \bar{u}(t_{k+1}, \beta - \sqrt{h\bar{\sigma}_{k,j}^2 + (\beta - x_j)^2})$$

$$-\psi_2(t_{k+1},\bar u(t_{k+1},\beta))\cdot(\beta-x_j-h\bar b_{k,j}-\sqrt{h\bar\sigma^2_{k,j}+(\beta-x_j)^2})+h\bar g_{k,j},$$
$$x_j+h\bar b_{k,j}+h^{1/2}\bar\sigma_{k,j}>\beta;\ j=1,2,...,M-1\ ,$$

$$\bar u(t_k,x)=\frac{x_{j+1}-x}{h_x}\,\bar u(t_k,x_j)+\frac{x-x_j}{h_x}\,\bar u(t_k,x_{j+1}),\ x_j<x<x_{j+1}, \tag{4.2}$$
$$j=0,1,2,...,M-1\ ,\ k=N-1,...,1,0,$$

where $\bar b_{k,j}$, $\bar\sigma_{k,j}$, $\bar g_{k,j}$ are the coefficients b, σ, g calculated at the point $(t_k,x_j,\bar u(t_{k+1},x_j))$.

Theorem 4.1. *Consider problem* (2.1)-(2.3), (3.12). *If the value of* h_x *is taken equal to* $\varkappa h$, $\varkappa$ *is a positive constant, then under assumptions* (*i*) *and* (*ii*) *the algorithm* (4.1)-(4.2) *has the first order of convergence, i.e., the approximation* $\bar u(t_k,x)$ *from formulas* (4.1)-(4.2) *satisfies the relation*

$$|\bar u(t_k,x)-u(t_k,x)|\le Kh,$$

where K *does not depend on* x, h, k.

Proof. In connection with the algorithm (4.1)-(4.2), we introduce the random sequence X_i, Z_i, $i=k,...,N$. We put $X_k=x_j$, $Z_k=0$ and then

$$X^{\pm}_{i+1}:=X_i+h\breve b_i\pm h^{1/2}\breve\sigma_i,\ i=k,...,N-1; \tag{4.3}$$

for $i=k,\dots,N-2$:

$$\text{if } X^{\pm}_{i+1}\in[\alpha,\beta],\ \text{then } P(X_{i+1}=x_l)=\frac12\frac{x_{l+1}-X^-_{i+1}}{h_x},\ P(X_{i+1}=x_{l+1})=\frac12\frac{X^-_{i+1}-x_l}{h_x},$$
$$P(X_{i+1}=x_m)=\frac12\frac{x_{m+1}-X^+_{i+1}}{h_x},\ P(X_{i+1}=x_{m+1})=\frac12\frac{X^+_{i+1}-x_m}{h_x},\ Z_{i+1}=Z_i+h\breve g_i,$$
$$\text{where } x_l,\ x_{l+1},\ x_m,\ x_{m+1} \text{ are such that } x_l\le X^-_{i+1}<x_{l+1},\ x_m<X^+_{i+1}\le x_{m+1};$$
$$\text{if } X^-_{i+1}<\alpha,\ \text{then}$$
$$P(X_{i+1}=x_m)=\frac{x_{m+1}-(X_i+\Delta X^\alpha_i)}{h_x},\ P(X_{i+1}=x_{m+1})=\frac{(X_i+\Delta X^\alpha_i)-x_m}{h_x},$$
$$Z_{i+1}=Z_i+h\breve g_i-\psi_1(t_{i+1})\cdot(\Delta X^\alpha_i-h\breve b_i),$$
$$\text{where } \Delta X^\alpha_i=(\alpha-X_i)+\sqrt{h\breve\sigma^2_i+(\alpha-X_i)^2} \text{ and } x_m,\ x_{m+1} \text{ are such that}$$
$$x_m<X_i+\Delta X^\alpha_i\le x_{m+1};$$
$$\text{if } X^+_{i+1}>\beta,\ \text{then}$$
$$P(X_{i+1}=x_l)=\frac{x_{l+1}-(X_i+\Delta X^\beta_i)}{h_x},\ P(X_{i+1}=x_{l+1})=\frac{(X_i+\Delta X^\beta_i)-x_l}{h_x},$$
$$Z_{i+1}=Z_i+h\breve g_i-\psi_2(t_{i+1})\cdot(\Delta X^\beta_i-h\breve b_i),$$
$$\text{where } \Delta X^\beta_i=(\beta-X_i)-\sqrt{h\breve\sigma^2_i+(\beta-X_i)^2} \text{ and } x_l,\ x_{l+1} \text{ are such that}$$
$$x_l\le X_i+\Delta X^\beta_i<x_{l+1};$$

for $i = N-1$:

$$\text{if } X_N^{\pm} \in [\alpha,\beta], \text{ then } P(X_N = X_N^-) = P(X_N = X_N^+) = \frac{1}{2},\ Z_N = Z_{N-1} + h\breve{g}_{N-1};$$

$$\text{if } X_N^- < \alpha, \text{ then}$$

$$X_N = X_{N-1} + \Delta X_{N-1}^{\alpha},\ Z_N = Z_{N-1} + h\breve{g}_{N-1} - \psi_1(t_N)\cdot(\Delta X_{N-1}^{\alpha} - h\breve{b}_{N-1}),$$

$$\text{where } \Delta X_{N-1}^{\alpha} = (\alpha - X_{N-1}) + \sqrt{h\breve{\sigma}_{N-1}^2 + (\alpha - X_{N-1})^2};$$

$$\text{if } X_N^+ > \beta, \text{ then}$$

$$X_N = X_{N-1} + \Delta X_{N-1}^{\beta},\ Z_N = Z_{N-1} + h\breve{g}_{N-1} - \psi_2(t_N)\cdot(\Delta X_{N-1}^{\beta} - h\breve{b}_{N-1}),$$

$$\text{where } \Delta X_{N-1}^{\beta} = (\beta - X_{N-1}) - \sqrt{h\breve{\sigma}_{N-1}^2 + (\beta - X_{N-1})^2}.$$

Here $\breve{b}_i = \bar{b}_i(X_i) = b(t_i, X_i, \bar{u}(t_{i+1}, X_i))$, $\breve{\sigma}_i = \bar{\sigma}_i(X_i) = \sigma(t_i, X_i, \bar{u}(t_{i+1}, X_i))$, and $\breve{g}_i = \bar{g}_i(X_i) = g(t_i, X_i, \bar{u}(t_{i+1}, X_i))$.

It is clear that

$$\bar{u}(t_k, x_j) = E\left[\bar{u}(t_N, X_N) + Z_N\right] = E\left[\varphi(t_N, X_N) + Z_N\right] = E\left[u(t_N, X_N) + Z_N\right].$$

Introduce the notation

$$R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x),\ R_k := \max_{x\in[\alpha,\beta]} |R(t_k, x)|.$$

Using arguments similar to those which lead us to (3.23) and taking into account that the error of linear interpolation is $O(h_x^2)$, we get

$$R(t_k, x_j) = \sum_{i=k}^{N-1} EI_{\overline{Q}\setminus\partial\Gamma}(t_i, X_i)\left[r_i + O(h^2) + O(h_x^2)\right] \tag{4.4}$$

$$+ \sum_{i=k}^{N-1} EI_{\partial\Gamma}(t_i, X_i)\left[\bar{r}_i + O(h^{3/2}) + O(h_x^2)\right],$$

where

$$|r_i|,\ |\bar{r}_i| \le Kh|R(t_{i+1}, X_i)|.$$

A lemma similar to Lemma 3.2 can be proved for the Markov chain (t_i, X_i) defined by (4.3). Then, we obtain from (4.4) for $h_x = \varkappa h$:

$$|R(t_k, x_j)| \le Kh\sum_{i=k}^{N-1} R_{i+1} + Ch. \tag{4.5}$$

We have

$$u(t_k, x) = \frac{x_{j+1} - x}{h_x}\, u(t_k, x_j) + \frac{x - x_j}{h_x}\, u(t_k, x_{j+1}) + O(h_x^2),\ x_j \le x \le x_{j+1}. \tag{4.6}$$

From (4.6) and (4.2):

$$R(t_k, x) = \frac{x_{j+1} - x}{\varkappa h}\, R(t_k, x_j) + \frac{x - x_j}{\varkappa h}\, R(t_k, x_{j+1}) + O(h^2),$$

whence due to (4.5)

$$|R(t_k,x)| \le Kh\sum_{i=k}^{N-1} R_{i+1} + Ch.$$

Consequently we get (3.24). Theorem 4.1 is proved.

Remark 4.1. It is natural to attract cubic interpolation instead of linear one for constructing numerical algorithms. Exploitation of cubic interpolation allows us to take the space step $h_x = \varkappa\sqrt{h}$ (in contrast to $h_x = \varkappa h$ for the linear interpolation) and, thus, to reduce the volume of computations. Unfortunately, we do not succeed in proving a convergence theorem in the case of cubic interpolation. The way of proving Theorem 4.1 gives us some restriction on the type of interpolation procedure which we can use for constructing numerical algorithms. The restriction is such that the sum of the absolute values of the coefficients staying at $\bar u(t_k,\cdot)$ in the interpolation procedure must be not greater than 1. Linear interpolation and B-splines of order $O(h_x^2)$ satisfy this restriction. But cubic interpolation of order $O(h_x^4)$ does not satisfy it. In Section 6 we test algorithms based on cubic interpolation. The tests give fairly good results. See also some theoretical explanations and numerical tests in [13, 14, 15].

On the base of linear interpolation and layer method (2.17), we get the following algorithm:

(4.7)
$$\bar u(t_N,x) = \varphi(t_N,x),\ x\in[\alpha,\beta],$$
$$\bar u(t_k,x_j) = \frac{1}{2}\bar u(t_{k+1},x_j + h\bar b_{k,j} - h^{1/2}\bar\sigma_{k,j}) + \frac{1}{2}\bar u(t_{k+1},x_j + h\bar b_{k,j} + h^{1/2}\bar\sigma_{k,j}) + h\bar g_{k,j},$$
$$x_j + h\bar b_{k,j} \pm h^{1/2}\bar\sigma_{k,j} \in [\alpha,\beta];$$
$$\bar u(t_k,x_j) = \bar u(t_{k+1},x_j + q\sqrt{h}) - \psi_1(t_{k+1},\bar u(t_{k+1},\alpha))\cdot qh^{1/2},$$
$$x_j + h\bar b_{k,j} - h^{1/2}\bar\sigma_{k,j} < \alpha;$$
$$\bar u(t_k,x_j) = \bar u(t_{k+1},x_j - q\sqrt{h}) + \psi_2(t_{k+1},\bar u(t_{k+1},\beta))\cdot qh^{1/2},$$
$$x_j + h\bar b_{k,j} + h^{1/2}\bar\sigma_{k,j} > \beta;\ j = 1,2,...,M-1\ ,$$

(4.8)
$$\bar u(t_k,x) = \frac{x_{j+1}-x}{h_x}\,\bar u(t_k,x_j) + \frac{x-x_j}{h_x}\,\bar u(t_k,x_{j+1}),\ x_j < x < x_{j+1},$$
$$j = 0,1,2,...,M-1\ ,\ k = N-1,...,1,0,$$

where $\bar b_{k,j}$, $\bar\sigma_{k,j}$, $\bar g_{k,j}$ are the coefficients b, σ, g calculated at the point $(t_k,x_j,\bar u(t_{k+1},x_j))$.

Theorem 4.2. *Consider problem* (2.1)-(2.3), (3.12). *If the value of* h_x *is taken equal to* $\varkappa h^{3/4}$, $\varkappa$ *is a positive constant, then under assumptions* (i) *and* (ii) *the algorithm* (4.7)-(4.8) *has order of convergence* $O(\sqrt{h})$, *i.e., the approximation* $\bar u(t_k,x)$ *from formulas* (4.7)-(4.8) *satisfies the relation*

$$|\bar u(t_k,x) - u(t_k,x)| \le K\sqrt{h},$$

where K *does not depend on* $x,\ h,\ k$.

This theorem is proved by the same arguments as Theorem 4.1.

Analogously, one can write down algorithms based on linear interpolation and on layer methods (2.11) and (2.18).

5. Extension to the multi-dimensional Neumann problem

Here we restrict ourselves to extension of the layer method (2.17). But let us note that a generalization of the other layer methods of Section 2 to the multi-dimensional case can be done as well. Though it is not difficult to generalize the layer methods given above for an arbitrary d, we should mention that layer methods are feasible if the dimension d of the domain G is not more than 3. Here we consider the problem (1.1)-(1.3) with $d = 2$. Remind that σ is a 2×2-matrix satisfying the relation $\sigma\sigma^\top = a$.

Introduce the notation ${}_iX_{k+1} := ({}_iX^1_{k+1},\ {}_iX^2_{k+1})$,

$$
\begin{aligned}
{}_iX^1_{k+1} &= x^1 + \bar b^1_k h + \bar\sigma^{11}_k \sqrt{h}\cdot\ {}_i\xi^1 + \bar\sigma^{12}_k\sqrt{h}\cdot\ {}_i\xi^2,\\
{}_iX^2_{k+1} &= x^2 + \bar b^2_k h + \bar\sigma^{21}_k \sqrt{h}\cdot\ {}_i\xi^1 + \bar\sigma^{22}_k\sqrt{h}\cdot\ {}_i\xi^2,\\
&i = 1,2,3,4,\ x = (x^1,x^2) \in \overline{G} \subset R^2,
\end{aligned}
$$

where ${}_1\xi = (-1,-1)$, ${}_2\xi = (-1,1)$, ${}_3\xi = -\,{}_1\xi$, ${}_4\xi = -\,{}_2\xi$ and $\bar b_k = (\bar b^1_k, \bar b^2_k)$, $\bar\sigma_k = \{\bar\sigma^{jl}_k\}$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$ calculated at the point $(t_k, x, \bar u(t_{k+1}, x))$.

If the point $x = (x^1, x^2) \in G$ is sufficiently far from the boundary ∂G (more precisely, if the points ${}_iX_{k+1}$, $i = 1,2,3,4$, belong to $\overline{G}$), the layer method has the form (cf. [13, 15]):

$$
\bar u(t_k, x^1, x^2) = \sum_{i=1}^{4} \frac{1}{4}\bar u(t_{k+1},\ {}_iX^1_{k+1},\ {}_iX^2_{k+1}) + \bar g_k \cdot h, \tag{5.1}
$$

where $\bar g_k$ is the coefficient $g(t,x,u)$ calculated at the point $(t_k, x, \bar u(t_{k+1}, x))$.

If the point $x = (x^1, x^2) \in \overline{G}$ is close or belongs to the boundary ∂G, then some of the points ${}_iX_{k+1} = ({}_iX^1_{k+1},\ {}_iX^2_{k+1})$, $i = 1,2,3,4$, may be outside of the domain $\overline{G}$. In this case let us consider the projection $\bar x$ of the point x on ∂G. Let $\nu = (\nu^1, \nu^2)$ be the unit vector of the internal normal at the point $\bar x$. Clearly, if $x \ne \bar x$, $\nu = (x - \bar x)/|x - \bar x|$. Then we put

$$
\bar u(t_k, x^1, x^2) = \bar u(t_{k+1}, x + qh^{1/2}\nu) - \psi(t_{k+1}, \bar x, \bar u(t_{k+1}, \bar x))\cdot qh^{1/2}. \tag{5.2}
$$

Thus, we obtain the method (5.1)-(5.2): the rule (5.1) is to be for points $x = (x^1, x^2) \in G$ such that all the corresponding points ${}_iX = ({}_iX^1,\ {}_iX^2)$, $i = 1,2,3,4$, belong to $\overline{G}$, and the rule (5.2) is to be otherwise. The error of the one-step approximation corresponding to (5.1) is of order $O(h^2)$ and that corresponding to (5.2) is of order $O(h)$. If the function ψ does not depend on u, we can prove that the layer method (5.1)-(5.2) has the global error estimated by $O(\sqrt{h})$. These assertions can be checked directly without attracting some new ideas in comparison with that from Section 3. In spite of the probabilistic nature the method (5.1)-(5.2) is deterministic.

To construct the corresponding numerical algorithms, we attract linear interpolation as in the previous section. If the domain G is a rectangle Π with corners (x_0^1, x_0^2), $(x_0^1, x_{M_2}^2)$, $(x_{M_1}^1, x_0^2)$, $(x_{M_1}^1, x_{M_2}^2)$, we introduce the equidistant space discretization:

$$\Delta_{M_1,M_2} := \{(x_j^1, x_l^2) :\ x_j^1 = x_0^1 + jh_{x^1},\ x_l^2 = x_0^2 + lh_{x^2},\ j = 0, \dots, M_1,\ l = 0, \dots, M_2\},$$

$$h_{x^1} = \frac{x_{M_1}^1 - x_0^1}{M_1},\ h_{x^2} = \frac{x_{M_2}^2 - x_0^2}{M_2}.$$

The values of $\bar{u}(t_k, x_j^1, x_l^2)$ at the nodes of Δ_{M_1,M_2} are found in accordance with (5.1)-(5.2). Let $x_j^1 < x^1 < x_{j+1}^1,\ x_l^2 < x^2 < x_{l+1}^2$. Then the value of $\bar{u}(t_k, x^1, x^2)$ is evaluated as

$$(5.3)\quad \bar{u}(t_k, x^1, x^2) = \frac{x_{j+1}^1 - x^1}{h_{x^1}} \cdot \frac{x_{l+1}^2 - x^2}{h_{x^2}} \bar{u}(t_k, x_j^1, x_l^2) + \frac{x_{j+1}^1 - x^1}{h_{x^1}} \cdot \frac{x^2 - x_l^2}{h_{x^2}} \bar{u}(t_k, x_j^1, x_{l+1}^2)$$

$$+ \frac{x^1 - x_j^1}{h_{x^1}} \cdot \frac{x_{l+1}^2 - x^2}{h_{x^2}} \bar{u}(t_k, x_{j+1}^1, x_l^2) + \frac{x^1 - x_j^1}{h_{x^1}} \cdot \frac{x^2 - x_l^2}{h_{x^2}} \bar{u}(t_k, x_{j+1}^1, x_{l+1}^2).$$

If the function ψ does not depend on u, we can prove that taking $h_{x^i} = \varkappa^i h^{3/4}$, $i = 1, 2$, $\varkappa^1, \varkappa^2 > 0$, the error of this algorithm is estimated as $O(h^{1/2})$.

The case of an arbitrary domain G requires special consideration. For instance, for a sufficiently wide class of domains G, it is possible to find one-to-one mapping of G onto a domain G' with a rectangular grid (see, e.g., [3] and references therein). Then we can use the above given algorithm in G' and map the results onto G.

6. Numerical tests

In the previous sections, we deal with semilinear parabolic equations with negative direction of time t : the equations are considered under $t < T$ and the "initial" conditions are given at $t = T$. Of course, the proposed methods are adaptable to semilinear parabolic equations with positive direction of time, and this adaptation is particularly easy in the autonomous case. In our numerical tests we use algorithms with positive direction of time for such equations.

Consider the Neumann problem for the one-dimensional Burgers equation:

$$(6.1)\qquad \frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x},\ \ t > 0,\ x \in (-4, 4),$$

$$(6.2)\qquad u(0, x) = -\frac{\sigma^2 \sinh x}{\cosh x + A},\ x \in [-4, 4],$$

$$(6.3)\qquad \frac{\partial u}{\partial x}(t, -4) = u(t, -4)\left(\frac{1}{\sigma^2} u(t, -4) - 1\right) - \frac{\sigma^2 \exp(-4)}{\cosh 4 + A \exp(-\sigma^2 t/2)},$$

$$\frac{\partial u}{\partial x}(t, 4) = u(t, 4)\left(\frac{1}{\sigma^2} u(t, 4) - 1\right) - \frac{\sigma^2 \exp 4}{\cosh 4 + A \exp(-\sigma^2 t/2)},\ t > 0.$$

Here A is a positive constant.

TABLE 1. Dependence of the errors $err^c(t)$ (top position) and $err^l(t)$ (bottom position) in h under $t = 2$, $\sigma = 1.5$, and $A = 2$.

h	algorithm (4.1)-(4.2)	algorithm (4.1), (6.4)	algorithm (2.11), (6.4)	algorithm (4.7), (6.4)	algorithm (2.18), (6.4)
0.16	$5.216 \cdot 10^{-1}$	$7.434 \cdot 10^{-1}$	$5.967 \cdot 10^{-2}$	> 1	$7.333 \cdot 10^{-1}$
	$8.509 \cdot 10^{-2}$	$1.177 \cdot 10^{-1}$	$1.380 \cdot 10^{-2}$	$3.328 \cdot 10^{-1}$	$1.098 \cdot 10^{-1}$
0.01	$3.170 \cdot 10^{-2}$	$1.888 \cdot 10^{-2}$	$3.867 \cdot 10^{-3}$	$3.722 \cdot 10^{-1}$	$1.346 \cdot 10^{-1}$
	$5.748 \cdot 10^{-3}$	$3.737 \cdot 10^{-3}$	$1.224 \cdot 10^{-3}$	$6.161 \cdot 10^{-2}$	$2.192 \cdot 10^{-2}$
0.0016	$4.479 \cdot 10^{-3}$	$3.835 \cdot 10^{-3}$	$7.124 \cdot 10^{-4}$	$1.653 \cdot 10^{-1}$	$4.909 \cdot 10^{-2}$
	$8.149 \cdot 10^{-4}$	$7.444 \cdot 10^{-4}$	$2.127 \cdot 10^{-4}$	$2.750 \cdot 10^{-2}$	$8.172 \cdot 10^{-3}$
0.0001	$2.387 \cdot 10^{-4}$	$2.711 \cdot 10^{-4}$	$4.639 \cdot 10^{-5}$	$4.378 \cdot 10^{-2}$	$1.168 \cdot 10^{-2}$
	$4.479 \cdot 10^{-5}$	$5.213 \cdot 10^{-5}$	$1.357 \cdot 10^{-5}$	$7.307 \cdot 10^{-3}$	$1.968 \cdot 10^{-3}$

The exact solution to this problem has the form [1]

$$u(t,x) = -\frac{\sigma^2 \sinh x}{\cosh x + A \exp(-\sigma^2 t/2)}\,.$$

In the tests we use cubic interpolation (see Remark 4.1)

$$\bar{u}(t_k, x) = \sum_{i=0}^{3} \Phi_{j,i}(x) \bar{u}(t_k, x_{j+i}), \quad x_j < x < x_{j+3},$$

$$\Phi_{j,i}(x) = \prod_{m=0, m\neq i}^{3} \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}}. \tag{6.4}$$

Here we test the following five algorithms: (i) the algorithm (4.1)-(4.2), (ii) the algorithm based on layer method (2.14) and cubic interpolation (6.4), (iii) the algorithm based on layer method (2.11) and cubic interpolation (6.4), (iv) the algorithm based on layer method (2.17) and cubic interpolation (6.4), and (v) the algorithm based on layer method (2.18) and cubic interpolation (6.4). We take the space step $h_x = h$ for linear interpolation and $h_x = \sqrt{h}$ for cubic interpolation. The parameter q of algorithm (4.7), (6.4) is taken being equal to 1.

Table 1 gives numerical results obtained by these algorithms. The errors of the approximate solutions $\bar{u}$ are presented in the discrete Chebyshov norm (top position in the table) and in l^1-norm (bottom position):

$$err^c(t) = \max_{x_i} |\bar{u}(t, x_i) - u(t, x_i)|,$$

$$err^l(t) = \sum_i |\bar{u}(t, x_i) - u(t, x_i)| \cdot h_x\,.$$

In the experiments, the algorithm (4.7), (6.4) and the algorithm (2.18), (6.4) converge as $O(h^{1/2})$, the other algorithms converge as $O(h)$.

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