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Efficient Computation of Option Price Sensitivities Using Homogeneity and other Tricks

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Abstract

No front-office software can survive without providing derivatives of options prices with respect to underlying market or model parameters, the so called Greeks. We present a list of common Greeks and exploit homogeneity properties of financial markets to derive relationships between Greeks out of which many are model-independent. We apply the results to European style options, rainbow options, as well as options priced in Heston's stochastic volatility model and avoid exorbitant and time-consuming computations of derivatives which even strong symbolic calculators fail to produce.

1 Introduction

The computation of sensitivities of option prices, the so-called "Greeks", is often cumbersome - both for the mathematician and for symbolic calculators. This paper provides methods to avoid differentiation as much as possible. Many Greeks are related among each other. These relations are based on model-independent homogeneity of time and price level of a financial product on the one hand and model dependent relations such as the partial differential equation the value function must satisfy and relations implied by the assumed distribution of the underlying. The basic market model we use is the Black-Scholes model with stocks paying a continuous dividend yield and a riskless cash bond. This model supports the homogeneity properties which are valid in general, but its structure is so simple, that we can concentrate on the essential statements of this paper. We will also discuss how to extend our work to more general market models.

We list the commonly used Greeks and their symbols. We do not claim this list to be complete, because one can always define more derivatives of the option price function.

As special cases we look at the Greeks of European options in the Black-Scholes model in one dimension. It turns out, that one only needs to know two Greeks in order to calculate all the other Greeks without differentiating.

Another interesting example is a European derivative security depending on two assets. For such rainbow options the analysis of the risk due to changing correlation of the two assets is very important. We will show how this risk is related to simultaneous changes of the two underlying securities.

There are several applications of these homogeneity relations.

1. It helps saving time in computing derivatives.
2. It produces a robust implementation compared to Greeks via difference quotients.
3. It allows to check the quality and consistency of Greeks produced by finite-difference-, tree- or Monte Carlo methods.
4. It admits a computation of Greeks for Monte Carlo based values.
5. It shows relationships between Greeks which wouldn't be noticed merely by looking at difference quotients.

1.1 Notation

S	stock price or stock price process
B	cash bond, usually with risk free interest rate r
r	risk free interest rate
q	dividend yield (continuously paid)
σ	volatility of one stock, or volatility matrix of several stocks
ρ	correlation in the two-asset market model
t	date of evaluation ("today")
T	date of maturity
$\tau = T - t$	time to maturity of an option
x	stock price at time t
$f(\cdot)$	payoff function
$v(x, t, \ldots)$	value of an option
k	strike of an option
l	level of an option
v_x	partial derivation of v with respect to x (and analogous)

The standard normal distribution and density functions are defined by

$$n(t) \triangleq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \tag{1}$$

$$\mathcal{N}(x) \triangleq \int_{-\infty}^{x} n(t)\, dt \tag{2}$$

$$n_2(x, y; \rho) \triangleq \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) \tag{3}$$

$$\mathcal{N}_2(x, y; \rho) \triangleq \int_{-\infty}^{x} \int_{-\infty}^{y} n_2(u, v; \rho)\, du\, dv \tag{4}$$

See http://www.MathFinance.de/frontoffice.html for a source code to compute $\mathcal{N}_2$.

1.2 The Greeks

Delta	Δ	v_x	
Gamma	Γ	v_{xx}	
Theta	Θ	v_t	
Rho	ρ	v_r	in the one-stock model
Rhor	ρ_r	v_r	in the two-stock model
Rhoq	ρ_q	v_q	
Vega	Φ	v_σ	
Kappa	κ	v_ρ	correlation sensitivity (two-stock model)

Greeks, not so commonly used:

Leverage	λ	$\frac{x}{v}v_x$	sometimes Ω, sometimes called "gearing"
Vomma	Φ'	$v_{\sigma\sigma}$	
Speed		v_{xxx}	
Charm		v_{xt}	
Color		v_{xxt}	
Cross		$v_{x\sigma}$	
Forward Delta	Δ^F	v_F	
Driftless Delta	Δ^{dl}	$\Delta e^{q\tau}$	
Dual Theta	$\mathrm{Dual}\Theta$	v_T	
Strike Delta	Δ^k	v_k	
Strike Gamma	Γ^k	v_{kk}	
Level Delta	Δ^l	v_l	
Level Gamma	Γ^l	v_{ll}	
Beta	β_{12}	$\frac{\sigma_1}{\sigma_2}\rho$	two-stock model

2 Fundamental Properties

2.1 Homogeneity of Time

In most cases the price of the option is not a function of both the current time t **and** the maturity time T, but rather **only** a function of the time to maturity $\tau = T - t$ implying the relations

$$\Theta = v_t = -v_\tau = -v_T = -\mathrm{Dual}\Theta. \tag{5}$$

This relationship extends naturally to the situation of options depending on several intermediate times such as compound or Bermuda options.

2.2 Scale-Invariance of Time

We present the principle of the scale-invariance of time in this section, because this principle holds in general. In a market model parameters may be quoted on an annual basis. We illustrate this idea in a Black-Scholes framework, in which the volatility is such a model parameter. The same idea can easily be applied to other market models.

We may want to measure time in units other than years in which case interest rates and volatilities, which are normally quoted on an annual basis, must be changed according to the following rules for all $a > 0$.

$$\begin{aligned} \tau &\to \frac{\tau}{a} \\ r &\to ar \\ q &\to aq \\ \sigma &\to \sqrt{a}\sigma \end{aligned} \tag{6}$$

The option's value must be invariant under this rescaling, i.e.,

$$v(x, \tau, r, q, \sigma, \ldots) = v(x, \frac{\tau}{a}, ar, aq, \sqrt{a}\sigma, \ldots) \tag{7}$$

We differentiate this equation with respect to a and obtain for $a = 1$

$$0 = \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi, \tag{8}$$

a general relation between the Greeks *theta, rho, rhoq* and *vega.* Based on the relation

$$\begin{aligned} v(x_1, ..., x_n, \tau, r, q_1, ..., q_n, \sigma_{11}, ..., \sigma_{nn}) = \\ v(x_1, ..., x_n, \frac{\tau}{a}, ar, aq_1, ..., aq_n, \sqrt{a}\sigma_{11}, ..., \sqrt{a}\sigma_{nn}) \end{aligned} \tag{9}$$

we obtain

Theorem 1 *(scale invariance of time)*

$$0 = \tau\Theta + r\rho + \sum_{i=1}^{n} q_i \rho_{q_i} + \frac{1}{2} \sum_{i,j=1}^{n} \Phi_{ij}\sigma_{ij}, \tag{10}$$

where Φ_{ij} denotes the differentiation of v with respect to σ_{ij}.

2.3 Scale Invariance of Prices

The general idea is that value of securities may be measured in a different unit, just like values of European stocks are now measured in Euro instead of in-currencies. Option contracts usually depend on strikes and barrier levels. Rescaling can have different effects on the value of the option. Essentially we may consider the following types of homogeneity classes. Let $v(x,k)$ be the value function of an option, where x is the spot (or a vector of spots) and k the strike or barrier or a vector of strikes or barriers. Let a be a positive real number.

Definition 1 (homogeneity classes) *We call a value function k-homogeneous of degree n if for all $a > 0$*

$$v(ax, ak) \quad = \quad a^n v(x,k). \tag{11}$$

We call an options whose value function is strike-homogeneous of degree 1 *a* strike-defined option *and similarly an option whose value function is level-homogeneous of degree* 0 *a* level-defined option.

The value function of a European call or put option with strike K is then K-homogeneous of degree 1, a digital option which pays a fixed amount if the stock price is higher than a level L is L-homogeneous of degree 0. The path-independent barrier call option paying $(S-k)^+ I_{\{S>K\}}$ is (k,K)-homogeneous of degree 1. A power call with cap paying $\min(C, ((S-K)^+)^2)$ has a homogeneity structure of $v(aS, aK, a^2C) = a^2 v(S,K,C)$.

We show how such a scale invariance can be used to determine some relations among the Greeks. We explain this with two examples. In the first example we analyze a strike-defined option and in the second one we concentrate on a level defined option. The generalization to options with some more parameters like the mentioned path-independent barrier call or power-call can easily be done. For the barrier call one can use the results from the multi-dimensional strike-defined option (26) and (27).

2.3.1 Strike-Delta and Strike-Gamma

For a strike-defined value function we have for all $a, b > 0$

$$abv(x,k) \quad = \quad v(abx, abk). \tag{12}$$

We differentiate with respect to a and get for $a = 1$

$$bv(x,k) \quad = \quad bxv_x(bx, bk) + bkv_k(bx, bk). \tag{13}$$

We now differentiate with respect to b get for $b = 1$

$$\begin{aligned} v(x,k) &= xv_x + xv_{xx}x + xv_{xk}k + kv_k + kv_{kx}x + kv_{kk}k & (14)\\ &= x\Delta + x^2\Gamma + 2xkv_{xk} + k\Delta^k + k^2\Gamma^k. & (15) \end{aligned}$$

If we evaluate equation (13) at $b = 1$ we get

$$v = x\Delta + k\Delta^k. \tag{16}$$

We differentiate this equation with respect to k and obtain

$$\begin{aligned} \Delta^k &= x v_{kx} + \Delta^k + k\Gamma^k, && (17) \\ k x v_{kx} &= -k^2 \Gamma^k. && (18) \end{aligned}$$

Together with equation (15) we conclude

$$x^2 \Gamma = k^2 \Gamma^k. \tag{19}$$

2.3.2 Level-Delta and Level-Gamma

For a level-defined value function we have for all $a, b > 0$

$$v(x, l) = v(abx, abl). \tag{20}$$

We differentiate with respect to a and get at $a = 1$

$$0 = v_x(bx, bl)bx + v_l(bx, bl)bl. \tag{21}$$

If we set $b = 1$ we get the relation

$$\Delta x + \Delta^l l = 0. \tag{22}$$

Now we differentiate equation (21) with respect to b and get at $b = 1$

$$0 = v_{xx} x^2 + 2 v_{xl} x l + v_{ll} l^2. \tag{23}$$

One the other hand we can differentiate the relation between delta and level-delta with respect to l and get

$$v_{xl} x + l\Gamma^l + \Delta^l = 0. \tag{24}$$

Together with equation (23) we conclude

$$x^2 \Gamma + x\Delta = l^2 \Gamma^l + l\Delta^l. \tag{25}$$

In general we obtain

Theorem 2 *(price homogeneity)*

$$v = \sum_{i=1}^n x_i \Delta_i + \sum_{j=1}^m k_j \Delta_j^k \tag{26}$$

$$\sum_{i,j=1}^n x_i x_j \Gamma_{ij} = \sum_{i,j=1}^m k_i k_j \Gamma_{ij}^k \tag{27}$$

for strike-defined options and

$$0 = \sum_{i=1}^n x_i \Delta_i + \sum_{j=1}^m l_j \Delta_j^l \tag{28}$$

$$\sum_{i,j=1}^n x_i x_j \Gamma_{ij} + \sum_{i=1}^n x_i \Delta_i = \sum_{i,j=1}^m l_i l_j \Gamma_{ij}^l + \sum_{i=1}^m l_i \Delta_i^l \tag{29}$$

for level-defined options.

3 European Options in the Black-Scholes Model

We start with relations among Greeks for European claims in the n-dimensional Black-Scholes model

$$\begin{aligned} dS_i(t) &= S_i(t)[(r-q_i)\,dt + \sigma_i\, dW_i(t)], \quad i = 1, \ldots, n & (30) \\ \mathbf{Cov}(W_i(t), W_j(t)) &= \rho_{ij} t, & (31) \end{aligned}$$

where r is the risk-free rate, q_i the dividend rate of asset i or foreign interest rate of exchange rate i, σ_i the volatility of asset i and $(W_1, \ldots, W_n)$ a standard Brownian motion (under the risk-neutral measure) with correlation matrix ρ. Let v denote today's value of the payoff $f(S_1(T), \ldots, S_n(T))$ at maturity T. Then it is known that v satisfies the *Black-Scholes partial differential equation*

$$0 = -v_\tau - rv + \sum_{i=1}^{n} x_i(r-q_i)v_{x_i} + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma \circ \sigma^T)_{ij} x_i x_j v_{x_i x_j}. \tag{32}$$

3.1 Relations among Greeks Based on the Log-Normal Distribution

The value function v has a representation given by the n-fold integral

$$v = e^{-r\tau} \int f\left(\ldots, S_i(0)e^{\sigma_i\sqrt{\tau}x_i + \mu_i\tau}, \ldots\right) g(\vec{x}, \rho)\, d\vec{x}, \tag{33}$$

where $\mu_i = r - q_i - \frac{1}{2}\sigma_i^2$ and $g(\vec{x}, \rho)$ is the n-variate standard normal density with correlation matrix ρ. Since we do not want to assume differentiability of the payoff f, but we know that the transition density g is differentiable, we define a change the variables $y_i \triangleq S_i(0)e^{\sigma_i\sqrt{\tau}x_i + \mu_i\tau}$, which leads to

$$v = e^{-r\tau} \int f(\ldots, y_i, \ldots) g\left(\frac{\ln \frac{y_i}{S_i(0)} - \mu_i\tau}{\sigma_i\sqrt{\tau}}, \rho\right) \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}}. \tag{34}$$

3.1.1 Properties of the Normal Distribution

We collect some properties of the multivariate normal density function g. We suppose that the vector X of n random variables with means zero and unit variances has a nonsingular normal multivariate distribution with probability density function

$$g(x_1, \ldots, x_n; c_{11}, \ldots, c_{nn}) = (2\pi)^{-\frac{1}{2}n} |\mathbf{C}|^{\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{C}\mathbf{x}\right). \tag{35}$$

Here $\mathbf{C}$ is the inverse of the covariance matrix of X, which is denoted by ρ. Then the following identity published in [3] can be proved easily by writing the density in terms of its characteristic function.

Theorem 3 *(Plackett's Identity)*

$$\frac{\partial g}{\partial \rho_{ij}} = \frac{\partial^2 g}{\partial x_i \partial x_j}. \tag{36}$$

In the two-dimensional case this reads as

$$\frac{\partial n_2(x,y;\rho)}{\partial \rho} = \frac{\partial^2 n_2(x,y;\rho)}{\partial x \partial y}, \tag{37}$$

which can be extended readily to the corresponding cumulative distribution function, i.e.,

$$\frac{\partial \mathcal{N}_2(x,y;\rho)}{\partial \rho} = \frac{\partial^2 \mathcal{N}_2(x,y;\rho)}{\partial x \partial y} = n_2(x,y;\rho). \tag{38}$$

3.1.2 Correlation Risk and Cross-Gamma

Using the abbreviation $g_{jk} \triangleq \frac{\partial^2 g}{\partial x_j \partial x_k}$ the cross-gamma and correlation risk are

$$\begin{aligned} \frac{\partial^2 v}{\partial S_j(0) \partial S_k(0)} &= e^{-r\tau} \frac{1}{S_j(0) S_k(0) \sigma_j \sigma_k \tau} \int f(\ldots, y_i, \ldots) g_{jk} \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}}, && (39) \\ \frac{\partial v}{\partial \rho_{jk}} &= e^{-r\tau} \int f(\ldots, y_i, \ldots) g_{\rho_{jk}} \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}}. && (40) \end{aligned}$$

Invoking Plackett's identity (36) saying that $g_{\rho_{jk}} = g_{jk}$ leads to

Theorem 4 *(cross-gamma-correlation-risk relationship)*

$$\frac{\partial v}{\partial \rho_{jk}} = S_j(0) S_k(0) \sigma_j \sigma_k \tau \frac{\partial^2 v}{\partial S_j(0) \partial S_k(0)}. \tag{41}$$

3.1.3 Interest Rate Risk and Delta

A similar computation yields

Theorem 5 *(delta-rho relationship)*

$$\begin{aligned} \frac{\partial v}{\partial q_j} &= -S_j(0) \tau \frac{\partial v}{\partial S_j(0)}, && (42) \\ \frac{\partial v}{\partial r} &= -\tau \left(v - \sum_{j=1}^{n} S_j(0) \frac{\partial v}{\partial S_j(0)} \right). && (43) \end{aligned}$$

3.1.4 Volatility Risk and Gamma

The first and second derivative of the density g satisfy

$$g_j = -g \sum_{i=1}^{n} x_i C_{ij}, \tag{44}$$

$$g_{jk} = g \sum_{i=1}^{n} x_i C_{ij} \sum_{i=1}^{n} x_i C_{ik} - g C_{kj}. \tag{45}$$

For the j-th vega we find thus

$$\sigma_j \frac{\partial v}{\partial \sigma_j} = e^{-r\tau} \int f \cdot g \cdot \left(\sum_{i=1}^{n} x_i C_{ij} x_j^- - 1 \right) \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}}, \tag{46}$$

$$x_j^- \triangleq \frac{\ln \frac{y_i}{S_i(0)} - (r - q_i + \frac{1}{2}\sigma_i^2)\tau}{\sigma_i \sqrt{\tau}} = x_j - \sigma_j \sqrt{\tau}, \tag{47}$$

where we omit the arguments of f and g to simplify the notation. For the cross gammas we derive

$$\sigma_j \sigma_k S_j(0) S_k(0) \tau \frac{\partial^2 v}{\partial S_j(0) \partial S_k(0)} = e^{-r\tau} \int f \cdot g \cdot B_{jk} \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}}, \tag{48}$$

$$B_{jk} \triangleq \sum_{i=1}^{n} x_i C_{ij} \sum_{i=1}^{n} x_i C_{ik} - C_{kj} - \sum_{i=1}^{n} x_i C_{ij} \sigma_k \sqrt{\tau} \delta_{jk}. \tag{49}$$

We now multiply by ρ_{jk}, sum over k, remember that ρ is the inverse matrix of $\mathbf{C}$ and obtain

$$\sum_{k=1}^{n} \rho_{jk} \sigma_j \sigma_k S_j(0) S_k(0) \tau \frac{\partial^2 v}{\partial S_j(0) \partial S_k(0)} = e^{-r\tau} \int f \cdot g \cdot D_j \frac{d\vec{y}}{\prod y_i \sigma_i \sqrt{\tau}}, \tag{50}$$

$$D_j \triangleq \sum_{i=1}^{n} x_i C_{ij} x_j - 1 - \sum_{i=1}^{n} x_i C_{ij} x_j + \sum_{i=1}^{n} x_i C_{ij} x_j^-. \tag{51}$$

In summary we obtain

Theorem 6 *(gamma-vega relationship)*

$$\sigma_j \frac{\partial v}{\partial \sigma_j} = \sum_{k=1}^{n} \rho_{jk} \sigma_j \sigma_k S_j(0) S_k(0) \tau \frac{\partial^2 v}{\partial S_j(0) \partial S_k(0)}. \tag{52}$$

In dimension one the gamma-vega and delta-rho relationships are also mentioned in [4]. Shaw shows there that $v_\sigma - \sigma\tau S^2(t) v_{S(t)S(t)}$ satisfies the Black-Scholes partial differential equation and is hence identically zero for path-independent options. We note that the gamma-vega relationship does not hold for barrier options, simply because gamma and vega are not equal at the barrier.

4 The One-Dimensional Case

4.1 Results for European Claims in the Black-Scholes Model

We list several relations for European options.

$$\begin{aligned}
0 &= \tau\Theta + r\rho + q\rho_q + \frac{1}{2}\sigma\Phi && \text{scale invariance of time} && (53)\\
v &= x\Delta + k\Delta^k && \text{price homogeneity and strikes} && (54)\\
x^2\Gamma &= k^2\Gamma^k && \text{price homogeneity and strikes} && (55)\\
x\Delta &= -l\Delta^l && \text{price homogeneity and levels} && (56)\\
x^2\Gamma + x\Delta &= l^2\Gamma^l + l\Delta^l && \text{price homogeneity and levels} && (57)\\
\rho &= -\tau(v - x\Delta) && \text{delta-rho relationship} && (58)\\
\rho + \rho_q &= -\tau v && \text{rates symmetry} && (59)\\
rv &= \Theta + (r-q)x\Delta + \frac{1}{2}\sigma^2 x^2\Gamma && \text{Black-Scholes PDE} && (60)\\
qv &= \Theta + (q-r)k\Delta^k + \frac{1}{2}\sigma^2 k^2\Gamma^k && \text{dual Black-Scholes (strike)} && (61)\\
rv &= \Theta + (q-r+\sigma^2)l\Delta^l + \frac{1}{2}\sigma^2 l^2\Gamma^l && \text{dual Black-Scholes (level)} && (62)\\
\rho_q &= -\tau x\Delta && \text{delta-rho relationship} && (63)\\
\rho &= -\tau k\Delta^k && \text{combination of (63) and (54)} && (64)\\
\Phi &= \sigma\tau x^2\Gamma && \text{gamma-vega relationship} && (65)
\end{aligned}$$

An interpretation of equation (65) can be found in [6]. We would like to point out that this relationship is based on a fact concerning the normal distribution function, namely defining

$$n(t,\sigma) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}, \tag{66}$$

$$\mathcal{N}(x,\sigma) \triangleq \int_{-\infty}^{x} n(t,\sigma)\,dt, \tag{67}$$

one can verify that

$$\sigma\partial^2_{xx}\mathcal{N}(x,\sigma) = \partial_\sigma\mathcal{N}(x,\sigma). \tag{68}$$

There are surely more relations one can prove, but the next theorem will give a deeper insight into the relations of the Greeks.

Theorem 7 *If the price and two Greeks g_1, g_2 of a European option are given with*

$$g_1 \in G_1 = \{\Delta, \Delta^k, \Delta^l, \rho, \rho_q\}, \tag{69}$$

$$g_2 \in G_2 = \{\Gamma, \Gamma^k, \Gamma^l, \Phi, \Theta\}, \tag{70}$$

then all the other Greeks ($\in G_1 \cup G_2$) can be calculated. Furthermore, if Θ and another Greek from G_2 is given, it is also possible, to determine all other Greeks.

Proof. The relations (53) to (60) are independent of each other. The relations (61) to (63) are conclusions. To get an overview over all these relations, we list the appearance of each Greek in all these relations. With X or O we denote, that the marked Greek appears in the relation. The relations marked with X show, that there is a relation between Greeks of G_1 and G_2 and the O shows, that this relation concerns only the Greeks of one set.

		$Greeks \in G_1$					$Greeks \in G_2$				
equation	v	Δ	Δ^k	Δ^l	ρ	ρ_q	Γ	Γ^k	Γ^l	Φ	Θ
(53)					X	X				X	X
(54)	O	O	O								
(55)							O	O			
(56)		O		O							
(57)		X		X			X		X		
(58)	O	O			O						
(59)	O				O	O					
(60)	X	X					X				X
(61)	X		X					X			X
(62)	X			X					X		X
(64)			O		O						
(65)							O			O	
(63)		O				O					

Let us now assume the option price and one Greek from the set G_1 are given. Then a look at the table shows that all Greeks of the set G_1 can be evaluated. If all Greeks of the set G_1 are known and additionally one Greek of the set G_2 is given, all other Greeks can be determined. One the other hand, only eight equations are independent, so the knowledge of two Greeks is also the minimum knowledge one needs to determine all ten Greeks. This is the proof of the first statement.

If Θ and another Greek from G_2 is given, then it is always possible to determine one Greek of the set G_1 and one applies the part of this theorem already proved. If Γ, Γ^k or Γ^l is given, one can use one of the Black-Scholes equations (60) to (62). If vega Φ is given, one can use (65) to get Γ. □

We conclude this section with an example. In the special case of plain vanilla calls and puts in a foreign exchange market all relations for the Greeks presented above are valid. These formulas are well known and can be found in [7].

4.2 A Path-Independent Barrier Call

4.2.1 Value

The payoff of a path-independent down-and-out barrier call is given by

$$f(S_T, k, K) = (S - k)^+ \cdot I_{\{S_T > K\}} \tag{71}$$

We assume $k < K$ - otherwise it would be a plain vanilla call - and therefore the payoff can be written as $(S_T - k)I_{\{S_T > K\}}$. We claim that k and K are strikes, because this option has the scaling behavior $f(aS_T, ak, aK) = af(S_T, k, K)$. Intuitively one would call K a level; but we defined a level by its scaling behavior in section 2.3.2, which is not valid in this case. Therefore the path-independent barrier call is an example for a strike-defined option.

Using the abbreviation

$$d_\pm \quad \triangleq \quad \frac{\ln(\frac{S_0}{K}) + (r-q)\tau \pm \frac{1}{2}\sigma^2\tau}{\sqrt{\sigma^2\tau}}, \tag{72}$$

the value of a path-independent down-and-out barrier call is given by

$$\begin{aligned} v(S_0, k, K) &= e^{-r\tau}\int_K^\infty \frac{s-k}{s\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(\ln(\frac{s}{S_0}) - (r-q)\tau + \frac{1}{2}\sigma^2\tau)^2}{2\sigma^2\tau}\right) ds \\ &= S_0 e^{-q\tau}\mathcal{N}(d_+) - ke^{-r\tau}\mathcal{N}(d_-). \end{aligned} \tag{73}$$

We now want to calculate all Greeks of this option. We show that Theorem 7 can be used to organize the calculation of the Greeks.

4.2.2 Greeks

Delta. Since differentiation cannot be avoided entirely, we choose the derivative with respect to k, which is obviously

$$v_k \quad = \quad -e^{-r\tau}\mathcal{N}(d_-). \tag{74}$$

Next we differentiate the integral representation of v with respect to K and obtain

$$\begin{aligned} v_K &= e^{-r\tau}\frac{k-K}{K\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(\ln(\frac{K}{S_0}) - (r-q)\tau + \frac{1}{2}\sigma^2\tau)^2}{2\sigma^2\tau}\right) \\ &= \frac{k-K}{K}\frac{1}{\sqrt{\sigma^2\tau}}e^{-r\tau}n(d_-). \end{aligned} \tag{75}$$

In Theorem 7 we had assumed only one strike. In our example we have two strikes, and therefore we need two Greeks from the set G_1 to determine all other Greeks of this set. >From the price homogeneity we know that the relation

$$v \quad = \quad S_0 v_{S_0} + kv_k + Kv_K \tag{76}$$

holds, whence we obtain for the spot delta

$$v_{S_0} \quad = \quad e^{-q\tau}\mathcal{N}(d_+) + \frac{K-k}{S_0}\frac{1}{\sqrt{\sigma^2\tau}}e^{-r\tau}n(d_-). \tag{77}$$

Rho. We use relations (58) and (63) and obtain

$$v_r = \tau k e^{-r\tau}\mathcal{N}(d_-) + \tau\frac{K-k}{\sqrt{\sigma^2\tau}}e^{-r\tau}n(d_-), \tag{78}$$

$$v_q = -\tau S_0 e^{-q\tau}\mathcal{N}(d_+) - \tau\frac{K-k}{\sqrt{\sigma^2\tau}}e^{-r\tau}n(d_-). \tag{79}$$

Gamma. We have calculated all Greeks in G_1. To determine some other Greeks without differentiation we need at least one Greek of the set G_2. In the theorem above we assumed, that the option will be described by one strike, but the option we analyze depends on two strikes. So we have to differentiate trice to get all dual gammas.

$$v_{kk} = 0 \tag{80}$$

$$v_{kK} = \frac{1}{K}\frac{1}{\sqrt{\sigma^2\tau}}e^{-r\tau}n(d_-) \tag{81}$$

$$v_{KK} = -\frac{k}{K^2}\frac{e^{-r\tau}}{\sqrt{\sigma^2\tau}}n(d_-) + \frac{k-K}{K^2}\frac{e^{-r\tau}}{\sigma^2\tau}n(d_-)d_- \tag{82}$$

The extension of (55) to the case of one stock and two strikes is the equation (27) with $n=1$ and $m=2$. In our example this relation is given by

$$S_0^2\Gamma = k^2\Gamma^{kk} + 2kK\Gamma^{kK} + K^2\Gamma^{KK}. \tag{83}$$

>From this relation, which follows from the homogeneity of v, we obtain for the spot gamma without differentiation

$$v_{S_0S_0} = \frac{ke^{-r\tau}}{S_0^2\sqrt{\sigma^2\tau}}n(d_-) + \frac{k-K}{S_0^2}\cdot\frac{e^{-r\tau}}{\sigma^2\tau}n(d_-)d_-. \tag{84}$$

Vega. >From (65) we get

$$v_\sigma = \sqrt{\tau}ke^{-r\tau}n(d_-) - (K-k)e^{-r\tau}\frac{1}{\sigma}n(d_-)d_-. \tag{85}$$

Theta. >From the scale invariance of time (53) we obtain

$$\begin{aligned} v_t = -v_\tau &= -rke^{-r\tau}\mathcal{N}(d_-) + qS_0e^{-q\tau}\mathcal{N}(d_+) \\ &\quad -(r-q)\frac{K-k}{\sqrt{\sigma^2\tau}}e^{-r\tau}n(d_-) - \frac{\sigma}{2\sqrt{\tau}}ke^{-r\tau}n(d_-) \\ &\quad +\frac{1}{2\tau}(K-k)e^{-r\tau}n(d_-)d_- \end{aligned} \tag{86}$$

5 A European Claim in the Two-Dimensional Black-Scholes Model

5.1 Pricing of a European Option

Rainbow options are financial instruments which depend on several risky assets. Many of them are very sensitive to changes of correlation. We call kappa (κ) the derivative of the option value v with respect to the correlation ρ.

The computational effort to compute the kappa is hard, even in a simple framework, but in the Black-Scholes model with two stocks and one cash bond we can use the cross-gamma-correlation-risk relationship which can be used easily to find kappa.

Let the stock price processes S_1 and S_2 be described by

$$\ln \frac{S_1(\tau)}{S_1(0)} = (r - q_1 - \frac{1}{2}\sigma_1^2)\tau + \sigma_1 W_\tau^1, \tag{87}$$

$$\ln \frac{S_2(\tau)}{S_2(0)} = (r - q_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2 \rho W_\tau^1 + \sigma_2 \sqrt{1-\rho^2} W_\tau^2. \tag{88}$$

W^1 and W^2 are two independent Brownian motions under the risk neutral measure. The probability density for the distribution of $S_1(\tau)$ is denoted by $h_1(x)$ and is given by the log-normal density

$$h_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2\tau}} \frac{1}{x} \exp\left(-\frac{A^2}{2\sigma_1^2\tau}\right), \tag{89}$$

$$A \triangleq \ln\left(\frac{x}{S_1(0)}\right) - r\tau + q_1\tau + \frac{1}{2}\sigma_1^2\tau. \tag{90}$$

The equation for the second stock price process can be written as

$$\ln \frac{S_2(\tau)}{S_2(0)} = (r - q_2 - \frac{1}{2}\sigma_2^2)\tau + \frac{\sigma_2\rho}{\sigma_1}\left(\ln\left(\frac{S_1(\tau)}{S_1(0)}\right) - (r - q_1 - \frac{1}{2}\sigma_1^2)\tau\right)$$
$$+\sigma_2\sqrt{1-\rho^2} W_\tau^2. \tag{91}$$

The conditional distribution of $S_2(\tau)$ given $S_1(\tau)$ is thus log-normal with density

$$h_{2|1}(y|x) = \frac{1}{y\sqrt{2\pi\sigma_2^2(1-\rho^2)\tau}} \exp\left(-\frac{B^2}{2\sigma_2^2(1-\rho^2)\tau}\right), \tag{92}$$

$$B \triangleq \left[\ln\left(\frac{y}{S_2(0)}\right) - r\tau + q_2\tau + \frac{1}{2}\sigma_2^2\tau - \frac{\sigma_2\rho}{\sigma_1}A\right]. \tag{93}$$

The joint distribution of $S_1(\tau)$ and $S_2(\tau)$ is given by the product of h_1 and h_2

$$h(x,y) = h_1(x) \cdot h_{2|1}(y|x). \tag{94}$$

A European option with maturity τ and payoff $f(S_1(\tau), S_2(\tau))$ will be priced by

$$v = e^{-r\tau} \int_0^\infty \int_0^\infty h(x,y) \cdot f(x,y) dx dy. \tag{95}$$

This integral has exactly the structure of the integrals studied in section 3.1. Using the results provided above, one can collect several relationships for the Greeks in the two-dimensional case. Additional, the fundamental symmetry "scale invariance of time" is valid too. Because we concentrate on European options, the two dimensional Black-Scholes-PDE also holds.

5.2 Relations among the Greeks

We specialize the relationships among the Greeks found in n dimensions. Some results are

$$0 = \rho_{q_1} + S_1(0)\tau\Delta_1, \tag{96}$$

$$0 = \rho_{q_2} + S_2(0)\tau\Delta_2, \tag{97}$$

$$0 = q_1\rho_{q_1} + q_2\rho_{q_2} + \frac{1}{2}\sigma_1\Phi_1 + \frac{1}{2}\sigma_2\Phi_2 + r\rho_r + \tau\Theta, \tag{98}$$

$$\begin{aligned} 0 = {} & \Theta - rv + (r-q_1)S_1(0)\Delta_1 + (r-q_2)S_2(0)\Delta_2 \\ & + \frac{1}{2}\sigma_1^2 S_1(0)^2\Gamma_{11} + \rho\sigma_1\sigma_2 S_1(0)S_2(0)\Gamma_{12} + \frac{1}{2}\sigma_2^2 S_2(0)^2\Gamma_{22}, \end{aligned} \tag{99}$$

$$\kappa = \sigma_1\sigma_2\tau S_1(0)S_2(0)\Gamma_{12}, \tag{100}$$

$$0 = \rho\kappa - \sigma_1\Phi_1 + \sigma_1^2\tau S_1(0)^2\Gamma_{11}, \tag{101}$$

$$0 = \rho\kappa - \sigma_2\Phi_2 + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, \tag{102}$$

$$0 = \sigma_1\Phi_1 - \sigma_2\Phi_2 - \sigma_1^2\tau S_1(0)^2\Gamma_{11} + \sigma_2^2\tau S_2(0)^2\Gamma_{22}, \tag{103}$$

$$\rho_r = -\tau\left(v - S_1(0)\Delta_1 - S_2(0)\Delta_2\right), \tag{104}$$

$$0 = \tau v + \rho_{q_1} + \rho_{q_2} + \rho_r. \tag{105}$$

Of course one can get more relations by combining some relations above. The relations we have chosen to present are either similar to the one-dimensional case or have another natural interpretation.

- (96) and (97). These relations are a justification for the rough way to deal with dividends. One subtracts the dividends from the actual spot price and prices the option with this price and without dividends. This relation is not effected by the two-dimensionality of the problem.

- (98). This is the two-dimensional version of the general invariance under time scaling.

- (99). This is the Black-Scholes differential equation. This relation must hold, because we concentrated on European claims. It turns out, that the dynamic of an option price is described by the market model and that the price of the option is defined as a boundary problem.

- (100). This is the cross-gamma-correlation-risk relationship; it is remarkable, that this relationship has such a simple structure.
- (101) and (102). These are the gamma-vega relationships. Notice that one can determine κ only by knowledge of some derivatives with respect to parameters which concern only one stock. Of course, there is no difference between the first and the second stock. These relations are valid in the one-dimensional case with $\kappa \equiv 0$.
- (103) follows from (100).
- (104). This is the delta-rho relationship. The interest rate risk is well known to be the negative product of duration and the amount of money invested. The term in the parentheses is exactly the amount of money one would have to invest in the cash bond in order to delta-hedge the option.
- (105). This relation is the two-dimensional rates symmetry, an extension of equation (59). It follows from (104), (96) and (97).

In the following we treat one example in full detail. Further examples such as outside barrier options and spread options are available in [7].

5.3 European Options on the Minimum/Maximum of Two Assets

We consider the payoff

$$[\phi\,(\eta \min(\eta S_1(T), \eta S_2(T)) - K)]^+ . \tag{106}$$

This is a European put or call on the minimum ($\eta = +1$) or maximum ($\eta = -1$) of the two assets $S_1(T)$ and $S_2(T)$ with strike K. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put. Its value function has been published in [5] and can be written as

$$\begin{aligned} & v(t, S_1(t), S_2(t), K, T, q_1, q_2, r, \sigma_1, \sigma_2, \rho, \phi, \eta) && (107)\\ &= \phi \Big[S_1(t)e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1) \\ &\quad + S_2(t)e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2) \\ &\quad - Ke^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho)\right)\Big], \\ \sigma^2 &\triangleq \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2, && (108) \end{aligned}$$

$$\rho_1 \triangleq \frac{\rho\sigma_2 - \sigma_1}{\sigma}, \tag{109}$$

$$\rho_2 \triangleq \frac{\rho\sigma_1 - \sigma_2}{\sigma}, \tag{110}$$

$$d_1 \triangleq \frac{\ln(S_1(t)/K) + (r - q_1 + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \tag{111}$$

$$d_2 \triangleq \frac{\ln(S_2(t)/K) + (r - q_2 + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \tag{112}$$

$$d_3 \triangleq \frac{\ln(S_2(t)/S_1(t)) + (q_1 - q_2 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{113}$$

$$d_4 \triangleq \frac{\ln(S_1(t)/S_2(t)) + (q_2 - q_1 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{114}$$

5.3.1 Greeks

Delta. Space homogeneity implies that

$$v = S_1(t)\frac{\partial v}{\partial S_1(t)} + S_2(t)\frac{\partial v}{\partial S_2(t)} + K\frac{\partial v}{\partial K}. \tag{115}$$

Using this equation one only has to differentiate twice in order to get all deltas. It turns out, that the value function is given in the natural representation, which is presented in the appendix, and one is allowed to read off the deltas:

$$\frac{\partial v}{\partial S_1(t)} = \phi e^{-q_1\tau}\mathcal{N}_2(\phi d_1, \eta d_3; \phi\eta\rho_1), \tag{116}$$

$$\frac{\partial v}{\partial S_2(t)} = \phi e^{-q_2\tau}\mathcal{N}_2(\phi d_2, \eta d_4; \phi\eta\rho_2), \tag{117}$$

$$\frac{\partial v}{\partial K} = -\phi e^{-r\tau}\left(\frac{1-\phi\eta}{2} + \phi\mathcal{N}_2(\eta(d_1 - \sigma_1\sqrt{\tau}), \eta(d_2 - \sigma_2\sqrt{\tau}); \rho)\right). \tag{118}$$

Gamma. Computing the gammas is actually the last situation where differentiation is needed. We use the identities

$$\frac{\partial}{\partial x}\mathcal{N}_2(x, y; \rho) = n(x)\mathcal{N}\left(\frac{y - \rho x}{\sqrt{1-\rho^2}}\right), \tag{119}$$

$$\frac{\partial}{\partial y}\mathcal{N}_2(x, y; \rho) = n(y)\mathcal{N}\left(\frac{x - \rho y}{\sqrt{1-\rho^2}}\right), \tag{120}$$

and obtain

$$
\begin{aligned}
\frac{\partial^2 v}{\partial (S_1(t))^2} = \frac{\phi e^{-q_1\tau}}{S_1(t)\sqrt{\tau}} \Bigg[& \frac{\phi}{\sigma_1} n(d_1) \mathcal{N}\left(\eta\sigma \frac{d_3 - d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right) \\
& - \frac{\eta}{\sigma} n(d_3) \mathcal{N}\left(\phi\sigma \frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\Bigg], \qquad (121)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial (S_2(t))^2} = \frac{\phi e^{-q_2\tau}}{S_2(t)\sqrt{\tau}} \Bigg[& \frac{\phi}{\sigma_2} n(d_2) \mathcal{N}\left(\eta\sigma \frac{d_4 - d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right) \\
& - \frac{\eta}{\sigma} n(d_4) \mathcal{N}\left(\phi\sigma \frac{d_2 - d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\Bigg], \qquad (122)
\end{aligned}
$$

$$
\frac{\partial^2 v}{\partial S_1(t)\partial S_2(t)} = \frac{\phi\eta e^{-q_1\tau}}{S_2(t)\sigma\sqrt{\tau}} n(d_3) \mathcal{N}\left(\phi\sigma \frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right). \qquad (123)
$$

Kappa. The sensitivity with respect to correlation is directly related to the cross-gamma

$$
\frac{\partial v}{\partial \rho} = \sigma_1\sigma_2\tau S_1(t)S_2(t)\frac{\partial^2 v}{\partial S_1(t)\partial S_2(t)}. \qquad (124)
$$

Vega. We refer to (101) and (102) to get the following formulas for the vegas,

$$
\begin{aligned}
\frac{\partial v}{\partial \sigma_1} &= \frac{\rho v_\rho + \sigma_1^2\tau(S_1(t))^2 v_{S_1(t)S_1(t)}}{\sigma_1} \qquad (125) \\
&= S_1(t)e^{-q_1\tau}\sqrt{\tau}\Bigg[\rho_1\phi\eta n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right) \\
&\quad + n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3 - d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\Bigg], \qquad (126) \\
\frac{\partial v}{\partial \sigma_2} &= \frac{\rho v_\rho + \sigma_2^2\tau(S_2(t))^2 v_{S_2(t)S_2(t)}}{\sigma_2} \qquad (127) \\
&= S_2(t)e^{-q_2\tau}\sqrt{\tau}\Bigg[\rho_2\phi\eta n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2 - d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right) \\
&\quad + n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4 - d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\Bigg]. \qquad (128)
\end{aligned}
$$

Rho. Looking at (96), (97) and (104) the rhos are given by

$$
\begin{aligned}
\frac{\partial v}{\partial q_1} &= -S_1(t)\tau\frac{\partial v}{\partial S_1(t)}, \qquad (129) \\
\frac{\partial v}{\partial q_2} &= -S_2(t)\tau\frac{\partial v}{\partial S_2(t)}, \qquad (130) \\
\frac{\partial v}{\partial r} &= -K\tau\frac{\partial v}{\partial K}. \qquad (131)
\end{aligned}
$$

Theta. Among the various ways to compute theta one may use the one based on (98).

$$\frac{\partial v}{\partial t} = -\frac{1}{\tau}\left[q_1 v_{q_1} + q_2 v_{q_2} + r v_r + \frac{\sigma_1}{2} v_{\sigma_1} + \frac{\sigma_2}{2} v_{\sigma_2}\right]. \quad (132)$$

6 Generalization to Higher Dimensions and other Market Models

6.1 Beyond Black-Scholes

Up to now we illustrated our ideas in the Black-Scholes model and in some parts we used specific properties of this model. Nevertheless there are some properties, which are so fundamental, that they should hold in any realistic market model. These fundamental properties are the homogeneity of time, the scale invariance of time and the scale invariance of prices. For every market model one uses, one should check, if the model fulfills these properties.

An example for a market model with non-deterministic volatility is Heston's stochastic volatility model [2].

In this more general framework one needs to clarify the notion of vega. A change of volatility could mean a change of the entire underlying volatility process. If the pricing formula depends on input parameters such as initial volatility, volatility of volatility, mean reversion of volatility, then one can consider derivatives with respect to such parameters. It turns out that our strategy to compute Greeks can still be applied successfully in a stochastic volatility model.

6.2 Heston's Stochastic Volatility Model

$$dS_t = S_t\left[\mu\, dt + \sqrt{v(t)}\, dW_t^{(1)}\right], \quad (133)$$

$$dv_t = \kappa(\theta - v_t)\, dt + \sigma\sqrt{v(t)}\, dW_t^{(2)}, \quad (134)$$

$$\mathbf{Cov}\left[dW_t^{(1)}, dW_t^{(2)}\right] = \rho\, dt, \quad (135)$$

$$\Lambda(S, v, t) = \lambda v. \quad (136)$$

The model for the variance v_t is the same as the one used by *Cox, Ingersoll and Ross* for the short term interest rate, see [1]. We think of $\theta > 0$ as the long term variance, of $\kappa > 0$ as the rate of mean-reversion. The quantity $\Lambda(S, v, t)$ is called the market price of volatility risk.

Heston provides a closed-form solution for European vanilla options paying

$$\left[\phi\,(S_T - K)\right]^+. \quad (137)$$

As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put, K the strike in units of the domestic currency, q the risk free rate of asset S, r the domestic risk free rate and T the expiration time in years.

6.2.1 Abbreviations

$$a \triangleq \kappa\theta \tag{138}$$

$$u_1 \triangleq \frac{1}{2} \tag{139}$$

$$u_2 \triangleq -\frac{1}{2} \tag{140}$$

$$b_1 \triangleq \kappa + \lambda - \sigma\rho \tag{141}$$

$$b_2 \triangleq \kappa + \lambda \tag{142}$$

$$d_j \triangleq \sqrt{(\rho\sigma\varphi i - b_j)^2 - \sigma^2(2u_j\varphi i - \varphi^2)} \tag{143}$$

$$g_j \triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{b_j - \rho\sigma\varphi i - d_j} \tag{144}$$

$$\tau \triangleq T - t \tag{145}$$

$$D_j(\tau, \varphi) \triangleq \frac{b_j - \rho\sigma\varphi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right] \tag{146}$$

$$\begin{aligned} C_j(\tau, \varphi) \triangleq\ & (r - q)\varphi i\tau \\ & + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\varphi i + d)\tau - 2\ln\left[\frac{1 - g_j e^{d_j\tau}}{1 - e^{d_j\tau}}\right]\right\} \end{aligned} \tag{147}$$

$$f_j(x, v, t, \varphi) \triangleq e^{C_j(\tau,\varphi) + D_j(\tau,\varphi)v + i\varphi x} \tag{148}$$

$$P_j(x, v, \tau, y) \triangleq \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-i\varphi y} f_j(x, v, \tau, \varphi)}{i\varphi}\right] d\varphi \tag{149}$$

$$p_j(x, v, \tau, y) \triangleq \frac{1}{\pi}\int_0^\infty \Re\left[e^{-i\varphi y} f_j(x, v, \tau, \varphi)\right] d\varphi \tag{150}$$

$$P_+(\phi) \triangleq \frac{1 - \phi}{2} + \phi P_1(\ln S_t, v_t, \tau, \ln K) \tag{151}$$

$$P_-(\phi) \triangleq \frac{1 - \phi}{2} + \phi P_2(\ln S_t, v_t, \tau, \ln K) \tag{152}$$

This notation is motivated by the fact that the numbers P_j are the cumulative distribution functions (in the variable y) of the log-spot price after time τ starting at x for some drift μ. The numbers p_j are the respective densities.

6.2.2 Value

The value function for European vanilla options is given by

$$V = \phi\left[e^{-q\tau} S_t P_+(\phi) - K e^{-r\tau} P_-(\phi)\right] \tag{153}$$

The value function takes the form of the Black-Scholes formula for vanilla options. The probabilities $P_{\pm}(\phi)$ correspond to $\mathcal{N}(\phi d_{\pm})$ in the constant volatility case.

6.2.3 Greeks

We use the homogeneity of prices, to obtain the deltas. But we must show, that the price is given in its natural representation. So we use the following strategy.

We assume, that equation (153) gives the natural price representation, which is defined in appendix A. Under this assumption we can read off the deltas, and from the deltas we derive the gammas. Using Theorem 8 we show that the assumption of (153) giving the natural price representation was correct.

Spot delta.

$$\Delta \triangleq \frac{\partial V}{\partial S_t} = \phi e^{-q\tau} P_{+}(\phi) \tag{154}$$

Dual delta.

$$\Delta^K \triangleq \frac{\partial V}{\partial K} = -\phi e^{-r\tau} P_{-}(\phi) \tag{155}$$

Gamma. Under the condition, that the deltas are correct, we obtain for the gammas by differentiation:

Spot Gamma.

$$\Gamma \triangleq \frac{\partial \Delta}{\partial S_t} = \frac{\partial \Delta}{\partial x}\frac{\partial x}{\partial S_t} = \frac{e^{-q\tau}}{S_t} p_1(\ln S_t, v_t, \tau, \ln K) \tag{156}$$

Dual Gamma.

$$\Gamma^K \triangleq \frac{\partial \Delta^K}{\partial K} = \frac{\partial \Delta^K}{\partial y}\frac{\partial y}{\partial K} = \frac{e^{-r\tau}}{K} p_1(\ln S_t, v_t, \tau, \ln K) \tag{157}$$

Proof of the natural representation assumption >From Theorem 8 we know, that our initial guess for the deltas is correct, if the relation

$$S_t^2 \Gamma \quad = \quad K^2 \Gamma^K \tag{158}$$

holds. In fact, this equation is given by

$$S_t e^{-q\tau} p_1(\ln S_t, v_t, \tau, \ln K) = K e^{-r\tau} p_2(\ln S_t, v_t, \tau, \ln K), \tag{159}$$

and this statement is true. So our calculation for the deltas and gammas has been finished.

Rho. Rho is connected to delta via equations (64) and (63).

$$\frac{\partial V}{\partial r} = \phi K e^{-r\tau}\tau P_{-}(\phi), \tag{160}$$

$$\frac{\partial V}{\partial q} = -\phi S_t e^{-q\tau}\tau P_{+}(\phi). \tag{161}$$

Theta. Theta can be computed using the partial differential equation for the Heston vanilla option

$$V_t + (r-q)SV_S + \frac{1}{2}\sigma v V_{vv} + \frac{1}{2}vS^2 V_{SS} + \rho\sigma v S V_{vS} - qV \\ +[\kappa(\theta - v) - \lambda]V_v = 0, \tag{162}$$

where the derivatives with respect to initial variance v must be evaluated numerically.

7 Summary

We have learned how to employ homogeneity-based methods to compute analytical formulas of Greeks for analytically known value functions of options in a one-and higher-dimensional market. Restricting the view to the Black-Scholes model there are numerous further relations between various Greeks which are of fundamental interest. The method helps saving computation time for the mathematician who has to differentiate complicated formulas as well as for the computer, because analytical results for Greeks are usually faster to evaluate than finite differences involving at least twice the computation of the option's value. Knowing how the Greeks are related among each other can speed up finite-difference-, tree-, or Monte Carlo-based computation of Greeks or lead at least to a quality check. Many of the results are valid beyond the Black-Scholes model. Most remarkably some relations of the Greeks are based on properties of the normal distribution refreshing the active interplay between mathematics and financial markets.

A The Natural Price Representation for Homogeneous Options

We analyze the following problem. Let $v(x, k)$ be the value of an option and $v(x, k)$ is homogeneous of degree 1. After Evaluating the integral to determine the option-price, one obtains the following formula:

$$v(x, k) = xf(x, k) + kg(x, k) \tag{163}$$

One the other hand, we know from the homogeneity of v:

$$v(x, k) = xv_x(x, k) + kv_k(x, k) \tag{164}$$

So the question is: Can we conclude $f(x,k) = v_x(x,k)$ and $g(x,k) = v_y(x,k)$? The answer is: **No, not in general**.

Because of v being homogeneous of degree 1 we know, that $f(x,k)$,$g(x,k)$,$v_x(x,k)$ and $v_k(x,k)$ are homogeneous of degree 0. Therefore we know that $f(x,k)$ has the representation $f(\frac{x}{k})$ and so on. Introducing the notation $u = \frac{x}{k}$ we find from (163) and (164) that

$$uf(u) + g(u) = uv_x(u) + v_y(u) \tag{165}$$

We define $h(u) = v_x(u) - f(u)$. The answer to the question above would be yes, if and only if $h(u) = 0$ for all u. One can easily show, that the function $h(u)$ has the following properties:

$$\lim_{u \to 0} h(u) = 0 \tag{166}$$
$$\lim_{u \to \infty} h(u) = 0 \tag{167}$$
$$h(u) = uf'(u) + g'(u) \tag{168}$$

So we come to the following definition:

Definition 2 (Natural Representation of Homogeneous Functions)
Let $v(x,k)$ be a homogeneous function of degree 1. Then there is a unique representation

$$v(x,k) = xf\left(\frac{x}{k}\right) + kg\left(\frac{x}{k}\right) \qquad \textit{with} \tag{169}$$
$$0 = uf'(u) + g'(u) \tag{170}$$

We call this the natural representation.

Of course, the definition of the natural representation can be extended to higher dimensions. The question of this section can now be answered more exactly. One can read off the deltas if and only if the price formula is given in its natural representation. This statement also holds in higher dimensions. For the two dimensional case, we summarize:

Theorem 8 *Let $v(x,k)$ be a homogeneous function of degree 1. The representation*

$$v(x,k) = xf(x,k) + kg(x,k) \tag{171}$$

is the natural representation if and only if

$$x^2 \partial_x f(x,k) = k^2 \partial_k g(x,k) \tag{172}$$

Theorem 9 *Let $v(x,k) = xf(x,k) + kg(x,k)$ be the natural representation of a homogeneous function of degree 1. Then the following equations hold:*

$$\partial_x v(x,k) = f(x,k) \tag{173}$$
$$\partial_k v(x,k) = g(x,k) \tag{174}$$

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