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# Interpolation for function spaces related to mixed boundary value problems

***Dedicated to Helga Rothkirch***

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## Abstract

Interpolation theorems are proved for Sobolev spaces of functions on nonsmooth domains with vanishing trace on a part of the boundary.


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## Contents

# 1 Introduction

During the last years the negatively indexed Sobolev spaces $W^{-1,p}$ proved to be an adequate class for the study of reaction–diffusion equations in nonsmooth situations, see e.g. [14], [7] or [6], as occuring in the mathematical modeling of semiconductor devices and their manufacturing processes. This is due to the fact that in the $W^{-1,p}$ spaces one can cope with jumping coefficients of the differential operators and mixed boundary conditions under weak assumptions on the spatial domain $\Omega$ and the Neumann part $\Gamma$ of its boundary. The concept of regular sets $\Omega \cup \Gamma$, see Gröger [11], turned out to be a powerful tool in the definition of function spaces which are appropriate for the treatment of mixed boundary value problems related to reaction–diffusion equations. This is due to regularity results for the corresponding second order elliptic operators, see [11], [10], [13], [9]. In dealing with evolution equations (but not only in this context) it is desirable to have interpolation results between the function space serving as the domain of the corresponding elliptic operator and the range space, see e.g. [1], [2] and [16] and the references cited there. In the case of smooth boundary operators these things are well elaborated, see [21], [18], [15], but for mixed boundary conditions nothing seems to be established. It is the aim of this paper to investigate the interpolation between spaces of Bessel potentials $\{H^{s,p}\}_s$ including boundary behaviour. The sets $\Omega \subset \mathbb{R}^d$ under consideration are locally defined via Lipschitz diffeomorphisms to so called standard sets, which allow to transform only spaces with derivative order from the interval $s \in [-1, 1]$ and integrability exponent $1 < p < \infty$. Thus, one can only expect results for this subclass.

# 2 Notations, Definitions, Preliminaries

## 2.a Spatial domain

In the sequel $\Omega$ will always be a bounded domain in $\mathbb{R}^d$ and $\Gamma$ a part of its boundary $\partial\Omega$. The open unit ball in $\mathbb{R}^d$ we denote by $B$ and the halfball $\{y \in B : y_1 < 0\}$ by $B^-$. For the equatorial plate $\{y \in B : y_1 = 0\}$ we use the symbol $\Gamma_0$ and for its half $\{y \in \Gamma_0 : y_2 > 0\}$ the symbol $\Gamma_0^+$. Throughout this paper we make the following assumption:

**2.1 Assumption.** For every point $x \in \partial\Omega$ there exist two open sets $U, V \subset \mathbb{R}^d$ and a bi–Lipschitz transformation $\Phi$ from $U$ onto $V$ such that $x \in U$, and $\Phi(U \cap (\Omega \cup \Gamma))$ coincides with one of the two model sets $B$ and $B^- \cup \Gamma_0$.

For some of our results we shall replace Assumption 2.1 by the following slightly more restrictive

**2.2 Assumption.** For every point $x \in \partial\Omega$ there exist two open sets $U, V \subset \mathbb{R}^d$ and a bi–Lipschitz transformation $\Phi$ from $U$ onto $V$ with a. e. constant absolute value of the functional determinant such that $x \in U$, and $\Phi(U \cap (\Omega \cup \Gamma))$ coincides with one of the two model sets $B$ and $B^- \cup \Gamma_0$.

**2.3 Remark.** The class of sets mentioned in Assumption 2.1 is in fact the class of regular sets in the sense of Gröger, see [11], [12], [8]. The subclass of regular sets characterized by Assumption 2.2 still contains all Lipschitz domains, see [22, Ch. 1.2, Thm. 2.5], and it is broad enough to cover the domains arising from the applications we have in mind, see §1. For certain technical reasons we do not employ the model sets from Gröger's concept in this work, namely

$$E_1 = B^-, \qquad E_2 = B^- \cup \Gamma_0, \qquad E_3 = B^- \cup \Gamma_0^+.$$

In fact, the two concepts of model sets are equivalent, as follows from our first theorem which will be proved in the appendix.

**2.4 Theorem.** *There are bi–Lipschitzian transformations $\Psi_d$ and $\Psi_d^+$ from $\mathbb{R}^d$ onto itself, such that $\Psi_d$ maps the open unit ball $B$ onto the open unit halfball $B^-$ and $\Psi_d^+$ maps the set $B^- \cup \Gamma_0^+$ onto $B^- \cup \Gamma_0$. Additionally, the mappings $\Psi_d$ and $\Psi_d^+$ may be taken such that their functional determinant is constant.*

## 2.b Function spaces

Throughout the remaining part of the paper $1 < p < \infty$ denotes any number and $p'$ its conjugate defined by $1/p + 1/p' = 1$.

**2.5 Definition.** (See e.g. [21, Ch. 2.3.3, Ch. 4.2.1] or [20].) For $s \geq 0$ we denote by $H^{s,p}(\Omega)$ the space of complex valued functions which are restrictions to $\Omega$ of functions from the space

$$H^{s,p}(\mathbb{R}^d) \stackrel{\text{def}}{=} \bigl\{\psi \in L^p(\mathbb{R}^d) : \bigl\|F^{-1}\bigl((1+|\cdot|_d^2)^{s/2}\, F\psi\bigr)\bigr\|_{L^p(\mathbb{R}^d)} < \infty\bigr\},$$

where $F$ denotes the Fourier transform and $|\cdot|_d$ the Euclidean norm in $\mathbb{R}^d$. The norm in $H^{s,p}(\Omega)$ is given by the expression

$$\|\psi\|_{H^{s,p}(\Omega)} \stackrel{\text{def}}{=} \inf\bigl\{\|\varphi\|_{H^{s,p}(\mathbb{R}^d)} : \varphi \in H^{s,p}(\mathbb{R}^d),\ \varphi|_\Omega = \psi\bigr\}.$$

**2.6 Remark.** The space $H^{1,p}(\Omega)$ coincides with the usual Sobolev space

$$W^{1,p}(\Omega) \stackrel{\text{def}}{=} \left\{u \in L^p(\Omega) : \frac{\partial u}{\partial x_l} \in L^p(\Omega),\ l \in \{1,\dots,d\}\right\},$$

see [21, Ch. 4.2.1]. Any domain satisfying Assumption 2.1 is an extension domain for $L^p(\Omega)$ as well as for $W^{1,p}(\Omega)$, see Remark 2.3 and [5, Ch. 3.4, Thm. 3.10]. If $\Omega$ is an extension domain simultaneously for both, $L^p(\Omega)$ and $W^{1,p}(\Omega)$, then for $s \in [0,1]$ we could have defined $H^{s,p}(\Omega)$ also as the complex interpolation space $[L^p(\Omega), W^{1,p}(\Omega)]_s$. This follows from the Retraction–Coretraction Theorem, see [21, Ch. 1.2.4], where the restriction operator to $\Omega$ acts as retraction and the extension operator to $\mathbb{R}^d$ acts as the coretraction operator.

First we notice that the spaces $H^{s,p}(\Omega)$ allow bi–Lipschitz transformations from one domain to another one.

**2.7 Theorem.** *Let $\Omega_1$ and $\Omega_2$ be two open subsets of $\mathbb{R}^d$. If $\Phi$ is a bi–Lipschitz transformation from $\Omega_1$ onto $\Omega_2$, then, for every $s \in [0,1]$, the mapping*

$$u \longmapsto u \circ \Phi^{-1}, \quad u \in H^{s,p}(\Omega_1),$$

*is a topological isomorphism from $H^{s,p}(\Omega_1)$ onto $H^{s,p}(\Omega_2)$.*

*Proof.* For $s = 0$ the claim directly follows from the change of variables formula under Lipschitz transformations, see [3, Ch. 3.3.3, Thm.2]. The case $s = 1$ is proved in [17, Ch. 2.3.3.1, Lem. 3.2]. The remaining cases are obtained via interpolation, see Remark 2.6. $\square$

**2.8 Definition.** For $s \in [0,1]$ we define $H^{s,p}_\Gamma(\Omega)$ as the closure in $H^{s,p}(\Omega)$ of the set

$$C_c^\infty(\Omega \cup \Gamma) \stackrel{\text{def}}{=} \{u|_\Omega \,:\, u \in C_c^\infty(\mathbb{R}^d),\, \operatorname{supp}(u) \cap (\partial\Omega \setminus \Gamma) = \varnothing\}$$

and $H^{-s,p}_\Gamma(\Omega)$ as the dual of $H^{s,p'}_\Gamma(\Omega)$. If $\Gamma$ is empty and $s \in [-1,1]$, then we write $H^{s,p}_0(\Omega)$ instead of $H^{s,p}_\Gamma(\Omega)$. If $\Omega$ is the unit ball $B$, then we abbreviate $H^{s,p}(B)$ by $H^{s,p}$, $H^{s,p}_0(B)$ by $H^{s,p}_0$, and $L^p(B)$ by $L^p$.

**2.9 Remark.** From Assumption 2.1 follows that $\Gamma$ is relatively open in $\partial\Omega$. Referring to mixed boundary value problems we can identify $\Gamma$ with the Neumann and $\partial\Omega \setminus \Gamma$ with the Dirichlet part of the boundary $\partial\Omega$.

**2.10 Theorem.** *For $i \in \{1,2\}$ let $\Omega_i \subset \mathbb{R}^d$ be a bounded domain and $\Gamma_i$ a part of its boundary such that Assumption 2.1 is satisfied. Moreover, let $\Phi : \Omega_1 \cup \Gamma_1 \longrightarrow \Omega_2 \cup \Gamma_2$ be a bi–Lipschitz transformation. Then, for every $s \in [0,1]$, the mapping*

$$u \longmapsto u \circ \Phi^{-1}, \quad u \in H^{s,p}_{\Gamma_1}(\Omega_1),$$

*is a topological isomorphism from $H^{s,p}_{\Gamma_1}(\Omega_1)$ onto $H^{s,p}_{\Gamma_2}(\Omega_2)$.*

*Proof.* If $v \in H^{s,p}_{\Gamma_2}(\Omega_2)$, then there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ with

$$\operatorname{supp}(v_j) \cap (\partial\Omega_2 \setminus \Gamma_2) = \varnothing \quad \text{and} \quad \lim_{j\to\infty} \|v - v_j|_{\Omega_2}\|_{H^{s,p}(\Omega_2)} = 0.$$

Theorem 2.7 implies

$$v \circ \Phi,\, v_j \circ \Phi \in H^{s,p}(\Omega_1) \quad \text{and} \quad \lim_{j\to\infty} \|v \circ \Phi - v_j \circ \Phi\|_{H^{s,p}(\Omega_1)} = 0. \tag{2.1}$$

Moreover,

$$\operatorname{supp}(v_j \circ \Phi) \cap (\partial\Omega_1 \setminus \Gamma_1) = \Phi^{-1}(\operatorname{supp}(v_j) \cap (\partial\Omega_2 \setminus \Gamma_2)) = \varnothing.$$

Because $\partial\Omega_1 \setminus \Gamma_1$ and $\operatorname{supp}(v_j \circ \Phi)$ are compact sets, there must be a positive distance between them, and we denote

$$\delta_j := \frac{1}{4}\, \operatorname{dist}(\partial\Omega_1 \setminus \Gamma_1, \operatorname{supp}(v_j \circ \Phi)).$$

Now, $v_j \circ \Phi$ can be extended to a function $u_j \in W^{1,p}(\mathbb{R}^d)$. Applying the convolution with suitable mollifiers to $u_j$ we construct functions $w_{ij} \in C_c^\infty(\mathbb{R}^d)$ fulfilling

$$\operatorname{supp}(w_{ij}) \cap (\partial\Omega_1 \setminus \Gamma_1) = \varnothing \quad \text{for all } i,j \in \mathbb{N},\ i > 1/\delta_j$$

and

$$\lim_{i\to\infty} \|v_j \circ \Phi - w_{ij}|_{\Omega_1}\|_{W^{1,p}(\Omega_1)} = 0, \tag{2.2}$$

see [17, Ch. 2.2.2.1, Thm. 2.1]. Relation (2.2) implies that

$$\lim_{i\to\infty} \|v_j \circ \Phi - w_{ij}|_{\Omega_1}\|_{H^{s,p}(\Omega_1)} = 0.$$

Therefore, we have $v_j \circ \Phi \in H^{s,p}_{\Gamma_1}(\Omega_1)$ for all $j \in \mathbb{N}$. Because $H^{s,p}_{\Gamma_1}(\Omega_1)$ is a closed subspace of $H^{s,p}(\Omega_1)$, (2.1) yields $v \circ \Phi \in H^{s,p}_{\Gamma_1}(\Omega_1)$.

Analogously, it follows that $u \circ \Phi^{-1} \in H^{s,p}_{\Gamma_2}(\Omega_2)$, if $u \in H^{s,p}_{\Gamma_1}(\Omega_1)$. □

As the reader will see below, the interpolation between the spaces $H^{s_1,p}_\Gamma(\Omega)$ and $H^{s_2,p}_\Gamma(\Omega)$ can be obtained from the special case $(\Omega,\Gamma) = (B,\varnothing)$ (unit ball with pure Dirichlet boundary). The idea, which enables such a reduction is to define an adequate retraction–coretraction to an $n$-fold Cartesian product of spaces on the unit ball with homogeneous Dirichlet boundary conditions.

## 2.c Retractions and coretractions

**2.11 Definition.** We define the continuous operators

$$P : L^{p'} \longrightarrow L^{p'}(B^-) \quad \text{and} \quad Q : L^{p'}(B^-) \longrightarrow L^{p'}$$

setting

$$(Pv)(y) \overset{\text{def}}{=} \frac{1}{2}(v(y) + v(\sigma y)), \qquad y \in B^-,$$

and

$$(Qv)(y) \overset{\text{def}}{=} v(y_-), \qquad y \in B;$$

here and in the sequel $\sigma y \overset{\text{def}}{=} (-y_1, y_2, \dots, y_d)$ and $y_- \overset{\text{def}}{=} (-|y_1|, y_2, \dots, y_d)$ for $y = (y_1, \dots, y_d)$.

Note that $P$ maps $H^{1,p'}$ (resp. $H^{1,p'}_0$) continuously into $H^{1,p'}(B^-)$ (resp. $H^{1,p'}_{\Gamma_0}(B^-)$) and that $Q$ maps $H^{1,p'}(B^-)$ (resp. $H^{1,p'}_{\Gamma_0}(B^-)$) continuously into $H^{1,p'}$ (resp. $H^{1,p'}_0$). As a consequence, for every $s \in [0,1]$, $P$ maps $H^{s,p'}$ continuously into $H^{s,p'}(B^-)$ and $Q$ maps $H^{s,p'}(B^-)$ continuously into $H^{s,p'}$.

**2.12 Definition.** For $\Omega$ and $\Gamma$ we fix an open covering $U_1, \dots, U_m$ of $\overline{\Omega}$ and bi–Lipschitz transformations

$$\begin{aligned} \Phi_k :\ & U_k \cap (\Omega \cup \Gamma) && \longrightarrow && B^- \cup \Gamma_0 && \text{if } k \in \{1, \dots, j\}, \\ \Phi_k :\ & U_k \cap (\Omega \cup \Gamma) && \longrightarrow && B && \text{if } k \in \{j+1, \dots, n\}. \end{aligned}$$

This is possible due to Assumption 2.1. We define linear continuous mappings

$$\begin{array}{llcll} T_k: & L^{p'}(B^-) & \longrightarrow & L^{p'}(U_k \cap \Omega) & \text{if } k \in \{1, \dots, j\}, \\ T_k: & L^{p'} & \longrightarrow & L^{p'}(U_k \cap \Omega) & \text{if } k \in \{j+1, \dots, n\}, \end{array}$$

by

$$(T_k v)(x) \overset{\text{def}}{=} v(\Phi_k(x)), \qquad x \in \Omega \cap U_k.$$

By Theorem 2.7 the operator $T_k$ maps $H^{1,p'}(B^-)$ (resp. $H^{1,p'}$) continuously and isomorphically onto $H^{1,p'}(U_k \cap \Omega)$. As a consequence, for every $s \in [0,1]$, the operator $T_k$ maps $H^{s,p'}(B^-)$ (resp. $H^{s,p'}$) continuously and isomorphically onto $H^{s,p'}(U_k \cap \Omega)$. By Theorem 2.10 the image of $H^{s,p'}_{\Gamma_0}(B^-)$ (resp. $H^{s,p'}_0$) under $T_k$ is $H^{s,p'}_{U_k \cap \Gamma}(U_k \cap \Omega)$.

**2.13 Definition.** We fix a $C^\infty$-partition of unity $\eta_1, \dots, \eta_n$ subordinate to the open covering $U_1, \dots, U_n$ of $\overline{\Omega}$ and define the mappings

$$R : \left[H^{-1,p}_0\right] \longrightarrow H^{-1,p}_\Gamma(\Omega) \quad \text{and} \quad S : H^{-1,p}_\Gamma(\Omega) \longrightarrow \left[H^{-1,p}_0\right]^n \tag{2.3a}$$

by

$$\langle Rg, u \rangle \overset{\text{def}}{=} \sum_{k=1}^{j} \left\langle g_k, QT_k^{-1}(\eta_k u)\right\rangle + \sum_{k=j+1}^{n} \left\langle g_k, T_k^{-1}(\eta_k u)\right\rangle,$$
$$g = (g_1, \dots, g_n) \in \left[H^{-1,p}_0\right]^n, \quad u \in H^{1,p'}_\Gamma(\Omega), \tag{2.3b}$$

and

$$\langle Sf, v \rangle \overset{\text{def}}{=} \left\langle f, \sum_{k=1}^{j} T_k P v_k + \sum_{k=j+1}^{n} T_k v_k \right\rangle,$$
$$f \in H^{-1,p}_\Gamma(\Omega), \quad v = (v_1, \dots, v_n) \in \left[H^{1,p'}_0\right]^n. \tag{2.3c}$$

In (2.3b) $\eta_k u$ is to be regarded as an element of $H^{1,p'}_{U_k \cap \Gamma}(U_k \cap \Omega)$. The functions $T_k P v_k$ and $T_k v_k$ in (2.3c) are to be interpreted in a natural way as elements of $H^{1,p'}_\Gamma(\Omega)$ (extension by zero).

**2.14 Lemma.** *The mappings $R$ and $S$ defined above have the following properties:*

(1) $RS$ is the identity mapping of $H^{-1,p}_\Gamma(\Omega)$.

(2) For each $s \in [0,1]$ the operator $R$ maps $\left[H^{-s,p}_0\right]^n$ onto $H^{-s,p}_\Gamma(\Omega)$.

(3) For each $s \in [0,1]$ the operator $S$ maps $H^{-s,p}_\Gamma(\Omega)$ into $\left[H^{-s,p}_0\right]^n$.

*Proof.* The assertions follow immediately from the definitions of the operators $R$ and $S$ and the properties of the operators $P$, $Q$ and $T_k$ mentioned above. $\square$

**2.15 Lemma.** *Let Assumption 2.2 be satisfied and let $R$ and $S$ be defined as before but this time by means of bi–Lipschitz transformations $\Phi_k$ with a.e. constant functional determinant. Then the operator $R$ maps $\left[H_0^{1,p}\right]^n$ onto $H_\Gamma^{1,p}(\Omega)$, and the operator $S$ maps $H_\Gamma^{1,p}(\Omega)$ into $\left[H_0^{1,p}\right]^n$.*

*Proof.* 1. If $g \in \left[H_0^{1,p}\right]^n$, then

$$\begin{aligned}
\langle Rg, u\rangle &= \int_B \left\{ \sum_{k=1}^{j} g_k(y)(\eta_k u)(\Phi_k^{-1}(y_-)) + \sum_{k=j+1}^{n} g_k(y)(\eta_k u)(\Phi_k^{-1}(y)) \right\} \mathrm{d}y \\
&= \sum_{j=1}^{k} \int_{B^-} \big(g_k(y) + g_k(\sigma y)\big)(\eta_k u)(\Phi_k^{-1}(y))\,\mathrm{d}y + \sum_{k=j+1}^{n} \int_B g_k(y)(\eta_k u)(\Phi_k^{-1}(y))\,\mathrm{d}y \\
&= \sum_{j=1}^{k} \int_{U_k\cap\Omega} 2(Pg_k)(\Phi_k(x))(\eta_k u)(x) d_k\,\mathrm{d}x + \sum_{k=j+1}^{n} \int_{U_k\cap\Omega} g_k(\Phi_k(x))(\eta_k u)(x) d_k\,\mathrm{d}x \\
&= \int_\Omega \left\{ \sum_{j=1}^{k} 2 d_k \eta_k(x)(T_k P g_k)(x) + \sum_{k=j+1}^{n} d_k \eta_k(x)(T_k g_k)(x) \right\} u(x)\,\mathrm{d}x.
\end{aligned}$$

Here $d_k$ denotes the (constant) functional determinant of the transformation $\Phi_k$ and $\eta_k T_k P g_k$ (resp. $\eta_k T_k g_k$) is to be regarded as a function on $\Omega$ vanishing outside $U_k \cap \Omega$. Thus, the functional $Rg$ is represented by the function

$$\sum_{k=1}^{j} 2 d_k \eta_k T_k P g_k + \sum_{k=j+1}^{n} d_k \eta_k T_k g_k \in H_\Gamma^{1,p}(\Omega).$$

2. If $f \in H_\Gamma^{1,p}(\Omega)$, then

$$\begin{aligned}
\langle Sf, v\rangle &= \frac{1}{2}\sum_{k=1}^{j} \int_{U_k\cap\Omega} f(x)\big(v_k(\Phi_k(x)) + v_k(\sigma\Phi_k(x))\big)\,\mathrm{d}x + \sum_{k=j+1}^{n} \int_{U_k\cap\Omega} f(x) v_k(\Phi_k(x))\,\mathrm{d}x \\
&= \frac{1}{2}\sum_{k=1}^{j} \int_{B^-} f(\Phi_k^{-1}(y))\big(v_k(y) + v_k(\sigma y)\big) d_k^{-1}\,\mathrm{d}y + \sum_{k=j+1}^{n} \int_B f(\Phi_k^{-1}(y)) v_k(y) d_k^{-1}\,\mathrm{d}y \\
&= \int_B \left\{ \frac{1}{2}\sum_{k=1}^{j} f(\Phi_k^{-1}(y_-)) v_k(y) d_k^{-1} + \sum_{k=j+1}^{n} f(\Phi_k^{-1}(y)) v_k(y) d_k^{-1} \right\} \mathrm{d}y \\
&= \int_B \left\{ \frac{1}{2}\sum_{k=1}^{j} d_k^{-1}(QT_k^{-1} f)(y) v_k(y) + \sum_{k=j+1}^{n} d_k^{-1}(T_k^{-1} f)(y) v_k(y) \right\} \mathrm{d}y.
\end{aligned}$$

Thus, the functional $Sf$ is represented by

$$\left(\frac{1}{2} d_1^{-1} Q T_1^{-1} f, \ldots, \frac{1}{2} d_j^{-1} Q T_j^{-1} f, d_{j+1}^{-1} T_{j+1}^{-1} f, \ldots, d_n^{-1} T_n^{-1} f\right) \in \left[H_0^{1,p}\right]^n.$$

3. The fact that $R$ maps $\left[H_0^{1,p}\right]^n$ *onto* $H_\Gamma^{1,p}(\Omega)$ follows from the preceding steps of the proof and the relation $RSf = f$ for $f \in H_\Gamma^{1,p}(\Omega)$ (see Lemma 2.14). ☐

# 3 Interpolation spaces

We are now going to formulate our interpolation results.

**3.1 Theorem.** *Let $0 < s < 1$ and $1 < p_0, p_1 < \infty$. Suppose that $1/p = (1-s)/p_0 + s/p_1$ and $s \neq 1/p'$. Then*

$$H_\Gamma^{-s,p}(\Omega) = \big[L^{p_0}(\Omega), H_\Gamma^{-1,p_1}(\Omega)\big]_s \quad \textit{and} \quad H_\Gamma^{s,p'}(\Omega) = \big[L^{p_0'}(\Omega), H_\Gamma^{1,p_1'}(\Omega)\big]_s.$$

*Proof.* The second result is well known if $(\Omega, \Gamma) = (B, \varnothing)$, see [21, Ch. 4.3.2, Thm. 1 and 2]. The first result for the same case $(\Omega, \Gamma) = (B, \varnothing)$ is a consequence of the corresponding second result and the Duality Theorem for complex interpolation, see [21, Ch. 1.11.3]. The proof of the first result for the general case follows from the special case $(\Omega, \Gamma) = (B, \varnothing)$ by the Retraction–Coretraction Theorem, see [21, Ch. 1.2.4]. It is just the result of Lemma 2.14 which shows that the Retraction–Coretraction Theorem is applicable and leads to the desired result. The general case of the second assertion then follows from the first assertion by another application of the Duality Theorem for complex interpolation. □

**3.2 Theorem.** *Let $s_0, s_1 \in [0,1]$, $1 < p_0, p_1 < \infty$ and $s_i \neq 1/p_i'$, $i \in \{0,1\}$. Furthermore, suppose that $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $s = (1-\theta)s_0 + \theta s_1 \neq 1/p'$. Then*

$$\big[H_\Gamma^{-s_0,p_0}(\Omega), H_\Gamma^{-s_1,p_1}(\Omega)\big]_\theta = H_\Gamma^{-s,p}(\Omega) \quad \textit{and} \quad \big[H_\Gamma^{s_0,p_0'}(\Omega), H_\Gamma^{s_1,p_1'}(\Omega)\big]_\theta = H_\Gamma^{s,p'}(\Omega).$$

*Proof.* The theorem is an immediate consequence of Theorem 3.1 and the Reiteration Theorem for complex interpolation, see [21, Ch. 1.9.3, Rem. 1]. □

In the following part of this section we shall deal with the interpolation of spaces $H_\Gamma^{s_0,p}(\Omega)$ and $H_\Gamma^{s_1,p}(\Omega)$ for $s_0, s_1 \in [-1,1]$. In the sequel, we suppose that Assumption 2.2 is always satisfied.

**3.3 Theorem.** *Let $1 < p_0, p_1 < \infty$, $1/p_i - 1 < s_i \leq 1$ and $s_i \neq 1/p_i$, $i \in \{0,1\}$. Moreover, suppose that $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $s = (1-\theta)s_0 + \theta s_1 \neq 1/p$. Then*

$$\big[H_\Gamma^{s_0,p_0}(\Omega), H_\Gamma^{s_1,p_1}(\Omega)\big]_\theta = H_\Gamma^{s,p}(\Omega). \tag{3.1}$$

*Proof.* 1. In case that $(\Omega, \Gamma) = (B, \varnothing)$ the assertion (3.1) follows immediately from [21, Ch. 4.3.2, Thm. 1 and 2].

2. Applying the Retraction-Coretraction Theorem (see [21, Ch. 1.2.4]), Lemma 2.14, and Lemma 2.15 to the special case $(\Omega, \Gamma) = (B, \varnothing)$ we arrive at the desired result for the general case. □

**3.4 Lemma.** *There holds*

$$\big[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\big]_{1/2} = L^p(\Omega). \tag{3.2}$$

*Proof.* Again, we first treat the special case $(\Omega, \Gamma) = (B, \varnothing)$, i.e. we prove

$$\big[H_0^{-1,p}, H_0^{1,p}\big]_{1/2} = L^p. \tag{3.3}$$

In the following we abbreviate the space $H^{2,p}(B)$ (see Definition 2.5) by $H^{2,p}$. Let further $-\Delta$ simultaneously denote the negative Laplace operator

$$-\Delta : H_0^{1,p} \longrightarrow H_0^{-1,p}$$

and its restriction to $H^{2,p} \cap H_0^{1,p}$

$$-\Delta : H^{2,p} \cap H_0^{1,p} \longrightarrow L^p.$$

Both mappings are linear, topological isomorphisms between the corresponding spaces, see Simader [19, Thm. 4.6] and [21, Ch. 5.5.1], respectively. According to [21, Ch. 4.9.2, Equ. 3]

$$(-\Delta)^{1/2} : H_0^{1,p} \longrightarrow L^p \tag{3.4}$$

also is a linear, topological isomorphism, as well as

$$(-\Delta)^{1/2} : L^p \longrightarrow H_0^{-1,p}$$

and

$$(-\Delta)^{1/2} : H^{2,p} \cap H_0^{1,p} \longrightarrow H_0^{1,p},$$

and by complex interpolation, the operator

$$(-\Delta)^{1/2} : \big[L^p, H^{2,p} \cap H_0^{1,p}\big]_{1/2} \longrightarrow \big[H_0^{-1,p}, H_0^{1,p}\big]_{1/2}$$

also must be a linear, topological isomorphism. As $\big[L^p, H^{2,p} \cap H_0^{1,p}\big]_{1/2}$ is the space $H_0^{1,p}$, see [21, Ch. 4.3.3, Equ. 7], $(-\Delta)^{1/2}$ provides a linear, topological isomorphism between $H_0^{1,p}$ and $\big[H_0^{-1,p}, H_0^{1,p}\big]_{1/2}$, which, thus, must coincide with $L^p$ by (3.4).

The general case follows from (3.3) by means of the Retraction–Coretraction Theorem, see [21, Ch. 1.2.4], Lemma 2.14, and Lemma 2.15. □

**3.5 Theorem.** *Let $s_0, s_1 \in [-1, 1]$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. Suppose that*

$$s_0, s_1, s \notin \{1/p, -1/p'\}. \tag{3.5}$$

*Then $\big[H_\Gamma^{s_0,p}(\Omega), H_\Gamma^{s_1,p}(\Omega)\big]_\theta = H_\Gamma^{s,p}(\Omega)$.*

*Proof.* Because of Theorem 3.2 we may assume that $s_0 > 0$ and $s_1 < 0$. From Theorem 3.1 and Lemma 3.4 one obtains by means of the Reiteration Theorem, see e.g. [21, Ch. 1.9.3 Rem. 1],

$$\begin{aligned} H_\Gamma^{s_0,p}(\Omega) &= \big[L^p(\Omega), H_\Gamma^{1,p}(\Omega)\big]_{s_0} = \Big[\big[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\big]_{1/2}, H_\Gamma^{1,p}(\Omega)\Big]_{s_0} \\ &= \big[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\big]_{(1-s_0)/2}. \end{aligned} \tag{3.6}$$

Analogously one finds

$$H_\Gamma^{s_1,p}(\Omega) = \left[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\right]_{(1-s_1)/2}. \tag{3.7}$$

Hence, using once more the Reiteration Theorem, we get

$$\begin{aligned}
\left[H_\Gamma^{s_0,p}(\Omega), H_\Gamma^{s_1,p}(\Omega)\right]_\theta &= \Big[\left[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\right]_{(1-s_0)/2}, \left[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\right]_{(1-s_1)/2}\Big]_\theta \\
&= \left[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\right]_{(1-\theta)(1-s_0)/2+\theta(1-s_1)/2} \\
&= \left[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\right]_{(1-s)/2} = H_\Gamma^{s,p}(\Omega).
\end{aligned}$$

In the last stage we made use of (3.6) or (3.7), depending on the sign of $s$. The condition (3.5) has been imposed to avoid forbidden indices in the above calculations. □

**3.6 Remark.** Real interpolation between the $\{H^{s,p}\}_s$ spaces leads to the usual Besov spaces including trace conditions: one uses Lemma 3.4, an iteration formula between complex and real interpolation, see [21, Ch. 1.10.3, Thm. 2], and quite similar considerations as carried out in Theorem 3.1.

## A Appendix: A technical lemma

The proof of Theorem 2.4 rests upon the following lemma.

**A.1 Lemma.** *For any integer $d \geq 1$ there is a bi–Lipschitzian mapping $\Lambda_{d+1}$ from $\mathbb{R}^{d+1}$ onto itself with constant functional determinant, such that $\Lambda_{d+1}$ maps the unit ball $B^{d+1}$ of $\mathbb{R}^{d+1}$ onto the cylinder with height 2 and radius 1:*

$$\Lambda_{d+1} : B^{d+1} \xrightarrow{onto} \big\{(x,y) : x \in B^d,\ y \in \mathbb{R},\ -1 \leq y \leq 1\big\}.$$

**A.2 Remark.** We write the coordinates in $\mathbb{R}^{d+1}$ as $(x_1, ..., x_d, y)$, for short $(x,y)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, and abbreviate as before $\|x\|_{\mathbb{R}^d}$ by $|x|_d$.

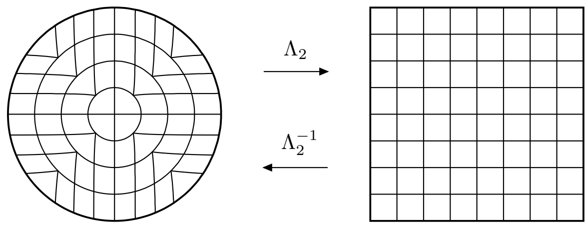
*Proof.* In the two dimensional case the mapping $\Lambda_2$ defined by

$$\Lambda_2(x,y) \stackrel{\text{def}}{=} \begin{cases} (0\,,\,0) & \text{if } x = y = 0, \\ \left(\sqrt{x^2+y^2}\,,\ \frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{y}{x}\right) & \text{if } |y| \leq x,\ x > 0, \\ \left(-\sqrt{x^2+y^2}\,,\ -\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{y}{x}\right) & \text{if } |y| \leq -x,\ x < 0, \\ \left(\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{x}{y}\,,\ \sqrt{x^2+y^2}\right) & \text{if } |x| \leq y,\ y > 0, \\ \left(-\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{x}{y}\,,\ -\sqrt{x^2+y^2}\right) & \text{if } |x| \leq -y,\ y < 0, \end{cases}$$

with the inverse

$$\Lambda_2^{-1}(x,y) = \begin{cases} (0\,,\,0) & \text{if } x=y=0, \\ \left(x\,\cos\frac{\pi}{4}\frac{y}{x}\,,\,x\,\sin\frac{\pi}{4}\frac{y}{x}\right) & \text{if } |y|\le|x|,\ x\neq 0, \\ \left(y\,\sin\frac{\pi}{4}\frac{x}{y}\,,\,y\,\cos\frac{\pi}{4}\frac{x}{y}\right) & \text{if } |x|\le|y|,\ y\neq 0, \end{cases}$$

meets the requirements, see also the following illustration:




Now, we will regard the problem in $\mathbb{R}^{d+1}$ with $d \geq 2$. We demand that the halfspace $\mathbb{R}^{d+1}_+ \stackrel{\text{def}}{=} \{(x,y)\,:\,x\in\mathbb{R}^d,\ y\geq 0\}$ is mapped into itself and the hyperplane defined by $y=0$ is mapped onto itself. Additionally, the mappings we are looking for shall map hyperplanes which contain zero and whose normal vectors are orthogonal to the vector $(0,\dots,0,1)$ into themselves and, therefore, are of the form

$$(x,y) \longmapsto (x_1 g(x,y),\dots,x_d\, g(x,y), h(x,y)), \tag{A.1a}$$

for all $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ and $y\in\mathbb{R}$. Moreover, we demand

$$(x_1 g(x,y),\dots,x_d\, g(x,y),h(x,y)) = (x_1 g(x,-y),\dots,x_d\, g(x,-y),-h(x,-y)). \tag{A.1b}$$

Thus, it suffices to define $\Lambda_{d+1}$ on the upper halfspace $\mathbb{R}^{d+1}_+$. In order to do so, we divide $\mathbb{R}^{d+1}_+$ into two (overlapping) subsets, namely

$$\begin{aligned} C_\gamma^{\triangledown} &\stackrel{\text{def}}{=} \big\{(x,y) : x\in\mathbb{R}^d,\ y\geq\gamma|x|_d\big\}, \\ C_\gamma^{\bowtie} &\stackrel{\text{def}}{=} \big\{(x,y) : x\in\mathbb{R}^d,\ 0\leq y\leq\gamma|x|_d\big\}, \end{aligned}$$

and define $\Lambda_{d+1}$, thereby observing (A.1), by the bi–Lipschitzian mappings

$$\Lambda_\gamma^{\triangledown} : C_\gamma^{\triangledown} \longrightarrow C_1^{\triangledown}, \qquad \Lambda_\gamma^{\bowtie} : C_\gamma^{\bowtie} \longrightarrow C_1^{\bowtie}, \tag{A.2}$$

such that the restrictions of these mappings coincide on the common boundary

$$\partial C_\gamma^{\triangledown} = \big\{(x,y) : x\in\mathbb{R}^d,\ y=\gamma|x|_d\big\},$$

of $C_\gamma^{\triangledown}$ and $C_\gamma^{\bowtie}$. Here, $\gamma>0$ is a constant, which we will specify later.

First, we construct a mapping on $C_\gamma^\nabla$. Defining $h^\nabla(x,y) \stackrel{\text{def}}{=} \sqrt{|x|_d^2 + y^2}$ we are looking for a function $g^\nabla(x,y) = v\big(|x|_d/y\big)$ such that the functional determinant

$$\left|\frac{\partial(g^\nabla(x,y)\,x, h^\nabla(x,y))}{\partial(x,y)}\right| = \begin{vmatrix} v + \frac{x_1^2}{y\,|x|_d}v' & \frac{x_1x_2}{y|x|_d}v' & \dots & \frac{x_1x_d}{y\,|x|_d}v' & -\frac{x_1|x|_d}{y^2}v' \\ \frac{x_1x_2}{y\,|x|_d}v' & v + \frac{x_2^2}{y\,|x|_d}v' & \dots & \frac{x_2x_d}{y\,|x|_d}v' & -\frac{x_2\,|x|_d}{y^2}v' \\ \dots & \dots & \dots & \dots & \dots \\ \frac{x_1x_d}{y\,|x|_d}v' & \frac{x_2x_d}{y\,|x|_d}v' & \dots & v + \frac{x_d^2}{y\,|x|_d}v' & -\frac{x_d\,|x|_d}{y^2}v' \\ \frac{x_1}{\sqrt{|x|_d^2+y^2}} & \frac{x_2}{\sqrt{|x|_d^2+y^2}} & \dots & \frac{x_d}{\sqrt{|x|_d^2+y^2}} & \frac{y}{\sqrt{|x|_d^2+y^2}} \end{vmatrix}$$

is equal to 1. Applying suitable algebraic calculations we get

$$1 = \frac{1}{\sqrt{|x|_d^2+y^2}} \begin{vmatrix} v + \frac{x_1^2}{y\,|x|_d}v' & \frac{x_1x_2}{y|x|_d}v' & \dots & \frac{x_1x_d}{y\,|x|_d}v' & -\frac{x_1|x|_d}{y^2}v' \\ -\frac{x_2}{x_1}v & v & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{x_d}{x_1}v & 0 & \dots & v & 0 \\ x_1 & x_2 & \dots & x_d & y \end{vmatrix}$$

$$= \frac{1}{\sqrt{|x|_d^2+y^2}} \begin{vmatrix} v & 0 & \dots & 0 & -\frac{x_1|x|_d}{y^2}v' \\ -\frac{x_2}{x_1}v & v & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{x_d}{x_1}v & 0 & \dots & v & 0 \\ x_1\Big(1+\frac{y^2}{|x|_d^2}\Big) & x_2\Big(1+\frac{y^2}{|x|_d^2}\Big) & \dots & x_d\Big(1+\frac{y^2}{|x|_d^2}\Big) & y \end{vmatrix}$$

$$= \frac{1}{\sqrt{|x|_d^2+y^2}} \begin{vmatrix} v & 0 & \dots & 0 & -\frac{x_1|x|_d}{y^2}v' \\ 0 & v & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v & 0 \\ \frac{|x|_d^2}{x_1}\Big(1+\frac{y^2}{|x|_d^2}\Big) & x_2\Big(1+\frac{y^2}{|x|_d^2}\Big) & \dots & x_d\Big(1+\frac{y^2}{|x|_d^2}\Big) & y \end{vmatrix}.$$

Developing the last determinant, one ends up with the following ordinary differential equation for $v$:

$$\frac{1}{\sqrt{\frac{|x|_d^2}{y^2}+1}}\, v^d\Big(\frac{|x|_d}{y}\Big) + \Big(\frac{|x|_d^2}{y^2}+1\Big)\frac{1}{\sqrt{\frac{y^2}{|x|_d^2}+1}}\, v^{d-1}\Big(\frac{|x|_d}{y}\Big)\, v'\Big(\frac{|x|_d}{y}\Big) = 1,$$

which transforms under the substitution $v^d = w$, $\frac{|x|_d}{y} = \xi$ equivalently into

$$w' + \frac{d}{\xi(\xi^2+1)} w = \frac{d}{\xi\sqrt{\xi^2+1}}.$$

The general solution of this equation is

$$\xi \mapsto d\,\frac{(\xi^2+1)^{d/2}}{\xi^d}\left(\int_0^\xi \frac{\alpha^{d-1}}{(\alpha^2+1)^{(d+1)/2}}\,\mathrm{d}\alpha + c\right) = d\,\frac{(\xi^2+1)^{d/2}}{\xi^d}\left(\int_0^{\arctan\xi} \sin^{d-1}\alpha\,\mathrm{d}\alpha + c\right),$$

where $c$ is an arbitrary real constant. As one has to avoid a singularity in $\xi = 0$, one chooses $c = 0$. Thus, one obtains for $g^\triangledown$:

$$g^\triangledown(x,y) = \begin{cases} \sqrt{\frac{y^2}{|x|_d^2}+1}\left(d\int_0^{\arctan(|x|_d/y)} \sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{1/d} & \text{if } x \in \mathbb{R}^d,\ x \neq 0, \\ 1 = \lim\limits_{\xi\mapsto 0} \sqrt{\frac{y^2}{\xi^2}+1}\left(d\int_0^{\arctan(\xi/y)} \sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{1/d} & \text{if } x \in \mathbb{R}^d,\ x = 0. \end{cases}$$

We will now define the corresponding functions on $C_\gamma^{\bowtie}$. Because spheres have to pass into cylinder surfaces we define $g^{\bowtie}(x,y) = \sqrt{1+\frac{y^2}{|x|_d^2}}$. Now we are looking for a function $h^{\bowtie}(x,y) = u\big(|x|_d, y\big)$ such that the functional determinant

$$\left|\frac{\partial(g^{\bowtie}(x,y)\,x, h^{\bowtie}(x,y))}{\partial(x,y)}\right| = \begin{vmatrix} g^{\bowtie} - \frac{x_1^2y^2}{g^{\bowtie}\,|x|_d^2} & -\frac{x_1x_2y^2}{g^{\bowtie}\,|x|_d^2} & \dots & -\frac{x_1x_dy^2}{g^{\bowtie}\,|x|_d^2} & \frac{x_1y}{g^{\bowtie}\,|x|_d^2} \\ -\frac{x_1x_2y^2}{g^{\bowtie}\,|x|_d^2} & g^{\bowtie} - \frac{x_2^2y^2}{g^{\bowtie}\,|x|_d^2} & \dots & -\frac{x_2x_dy^2}{g^{\bowtie}\,|x|_d^2} & \frac{x_2y}{g^{\bowtie}\,|x|_d^2} \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{x_1x_dy^2}{g^{\bowtie}\,|x|_d^2} & -\frac{x_2x_dy^2}{g^{\bowtie}\,|x|_d^2} & \dots & g^{\bowtie} - \frac{x_d^2y^2}{g^{\bowtie}\,|x|_d^2} & \frac{x_dy}{g^{\bowtie}\,|x|_d^2} \\ \frac{x_1}{|x|_d}\partial_1 u & \frac{x_2}{|x|_d}\partial_1 u & \dots & \frac{x_d}{|x|_d}\partial_1 u & \partial_2 u \end{vmatrix}$$

becomes 1. It will turn out that this condition on the functional determinant together with the requirement that $u$ should vanish on the set $\big\{(x,y) : x \in \mathbb{R}^d,\ y = 0\big\}$ in fact determines $u$

uniquely. Now, the condition on the functional determinant can be evaluated as follows

$$1 = \begin{vmatrix} g^{\bowtie} - \dfrac{x_1^2 y^2}{g^{\bowtie}\,|x|_d^2} & -\dfrac{x_1 x_2 y^2}{g^{\bowtie}\,|x|_d^2} & \dots & -\dfrac{x_1 x_d y^2}{g^{\bowtie}\,|x|_d^2} & \dfrac{x_1 y}{g^{\bowtie}\,|x|_d^2} \\ -\dfrac{x_2}{x_1} g^{\bowtie} & g^{\bowtie} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\dfrac{x_d}{x_1} g^{\bowtie} & 0 & \dots & g^{\bowtie} & 0 \\ \dfrac{x_1}{|x|_d}\partial_1 u & \dfrac{x_2}{|x|_d}\partial_1 u & \dots & \dfrac{x_d}{|x|_d}\partial_1 u & \partial_2 u \end{vmatrix}$$

$$= \begin{vmatrix} g^{\bowtie} & 0 & \dots & 0 & \dfrac{x_1 y}{g^{\bowtie}\,|x|_d^2} \\ -\dfrac{x_2}{x_1} g^{\bowtie} & g^{\bowtie} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\dfrac{x_d}{x_1} g^{\bowtie} & 0 & \dots & g^{\bowtie} & 0 \\ \dfrac{x_1}{|x|_d}\left(\partial_1 u + \dfrac{y}{|x|_d}\partial_2 u\right) & \dfrac{x_2}{|x|_d}\left(\partial_1 u + \dfrac{y}{|x|_d}\partial_2 u\right) & \dots & \dfrac{x_d}{|x|_d}\left(\partial_1 u + \dfrac{y}{|x|_d}\partial_2 u\right) & \partial_2 u \end{vmatrix}$$

$$= \begin{vmatrix} g^{\bowtie} & 0 & \dots & 0 & \dfrac{x_1 y}{g^{\bowtie}\,|x|_d^2} \\ 0 & g^{\bowtie} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g^{\bowtie} & 0 \\ \dfrac{|x|_d}{x_1}\left(\partial_1 u + \dfrac{y}{|x|_d}\partial_2 u\right) & \dfrac{x_2}{|x|_d}\left(\partial_1 u + \dfrac{y}{|x|_d}\partial_2 u\right) & \dots & \dfrac{x_d}{|x|_d}\left(\partial_1 u + \dfrac{y}{|x|_d}\partial_2 u\right) & \partial_2 u \end{vmatrix}.$$

Developing the last determinant, one easily obtains the condition

$$\left(1 + \frac{y^2}{|x|_d^2}\right)^{(d-2)/2} \left(\partial_2 u - \frac{y}{|x|_d}\partial_1 u\right) = 1.$$

The substitution $\dfrac{|x|_d}{y} = \xi$ yields the partial differential equation

$$-y\,\frac{\partial u(\xi, y)}{\partial \xi} + \xi\,\frac{\partial u(\xi, y)}{\partial y} = \frac{\xi^{d-1}}{\left(\xi^2 + y^2\right)^{(d-2)/2}}$$

with the boundary condition

$$u(\xi, 0) = 0 \quad \text{for } 0 \le \xi < \infty.$$

By the method of characteristics, see [4], one finds the solution

$$(\xi, y) \longmapsto \sqrt{\xi^2 + y^2} \int_0^{\arctan(y/\xi)} \cos^{d-1} \alpha \, \mathrm{d}\alpha,$$

and ends up with

$$h^{\bowtie}(x, y) = \sqrt{|x|_d^2 + y^2} \int_0^{\arctan(y/|x|_d)} \cos^{d-1} \alpha \, \mathrm{d}\alpha.$$

Up to now we have constructed two mappings

$$(x_1 g^{\triangledown}, \ldots, x_d \, g^{\triangledown}, h^{\triangledown}) : \mathbb{R}_+^{d+1} \xrightarrow{\text{onto}} \mathbb{R}_+^{d+1},$$
$$(x_1 g^{\bowtie}, \ldots, x_d \, g^{\bowtie}, h^{\bowtie}) : \mathbb{R}_+^{d+1} \xrightarrow{\text{onto}} \mathbb{R}_+^{d+1},$$

with functional determinant 1. N.B. these mappings do not depend on $\gamma$. We are now going to modify both mappings such that they coincide on the set

$$\partial C_\gamma^{\triangledown} = \{(x, y) \, : \, x \in \mathbb{R}^d, \ y = \gamma |x|_d\},$$

for some $\gamma > 0$. To that end we introduce the functions

$$\tau : \lambda \longmapsto \left( d \int_0^{\arctan(1/\lambda)} \sin^{d-1} \alpha \, \mathrm{d}\alpha \right)^{1/d},$$

$$\rho : \lambda \longmapsto \int_0^{\arctan \lambda} \cos^{d-1} \alpha \, \mathrm{d}\alpha,$$

and define mappings (A.2) by

$$\begin{aligned}
\Lambda_\gamma^{\triangledown}(x, y) &\stackrel{\text{def}}{=} \left( x_1 \frac{g^{\triangledown}(x, y)}{\tau(\gamma)}, \ldots, x_d \frac{g^{\triangledown}(x, y)}{\tau(\gamma)}, h^{\triangledown}(x, y) \right), \\
\Lambda_\gamma^{\bowtie}(x, y) &\stackrel{\text{def}}{=} \left( x_1 \, g^{\bowtie}(x, y), \ldots, x_d \, g^{\bowtie}(x, y), \frac{h^{\bowtie}(x, y)}{\rho(\gamma)} \right),
\end{aligned}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}_+$, with functional determinants

$$\left| \frac{\partial \Lambda^{\triangledown}}{\partial(x, y)} \right| = \left( \frac{1}{\tau(\gamma)} \right)^d \quad \text{and} \quad \left| \frac{\partial \Lambda^{\bowtie}}{\partial(x, y)} \right| = \frac{1}{\rho(\gamma)}, \tag{A.3}$$

respectively. If

$$d \int_0^{\arctan(1/\gamma)} \sin^{d-1} \alpha \, \mathrm{d}\alpha \, = \int_0^{\arctan \gamma} \cos^{d-1} \alpha \, \mathrm{d}\alpha, \tag{A.4}$$

then the values of the functional determinants (A.3) are equal. There is exactly one $\gamma > 0$ which satisfies (A.4), and in the sequel $\gamma$ shall be this value. From the monotonicity properties of $\tau$ and $\rho$ one deduces that, in accordance with (A.2), $\Lambda_\gamma^{\triangledown}$ maps the set $C_\gamma^{\triangledown}$ onto $C_1^{\triangledown}$ and that $\Lambda_\gamma^{\bowtie}$ maps $C_\gamma^{\bowtie}$ onto $C_1^{\bowtie}$.

The inverse mappings to $\Lambda_\gamma^\triangledown$ and $\Lambda_\gamma^\bowtie$ are given by

$$\frac{1}{y}\sqrt{1+\Big(\tau^{-1}\Big(\frac{\tau(\gamma)|x|_d}{y}\Big)\Big)^2}\,(\Lambda_\gamma^\triangledown)^{-1}(x,y)=\Big(\frac{x_1}{|x|_d},\ldots,\frac{x_d}{|x|_d},\tau^{-1}\Big(\frac{\tau(\gamma)|x|_d}{y}\Big)\Big),$$

$$\sqrt{1+\Big(\rho^{-1}\Big(\frac{\rho(\gamma)y}{|x|_d}\Big)\Big)^2}\,(\Lambda_\gamma^\bowtie)^{-1}(x,y)=\Big(x_1,\ldots,x_d,|x|_d\,\rho^{-1}\Big(\frac{\rho(\gamma)y}{|x|_d}\Big)\Big),$$

for all $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ and $y\in\mathbb{R}_+$. From the monotonicity properties of $\tau$ and $\rho$ follows that $(\Lambda_\gamma^\triangledown)^{-1}$ maps $C_1^\triangledown$ onto $C_\gamma^\triangledown$ and $(\Lambda_\gamma^\bowtie)^{-1}$ maps $C_1^\bowtie$ onto $C_\gamma^\bowtie$. With respect to the solution $\gamma$ of (A.4) we now define

$$\Lambda_{d+1}(x,y)\overset{\text{def}}{=}\begin{cases}\Lambda_\gamma^\triangledown(x,y) & \text{if } (x,y)\in C_\gamma^\triangledown,\\ \Lambda_\gamma^\bowtie(x,y) & \text{if } (x,y)\in C_\gamma^\bowtie.\end{cases}$$

The Lipschitz properties of $\Lambda_{d+1}$ and its inverse, easily follow from the fact that a function is Lipschitzian, if and only if it belongs to $W^{1,\infty}$, see [3, Ch. 4.2, Thm. 5]. □

**A.3 Remark.** In the three dimensional case the constant $\gamma$ equals to $2/\sqrt{5}$ and the mapping $g^\triangledown$ is given by

$$g^\triangledown(x,y,z)=\sqrt{\frac{3}{1+\Big(1+\frac{x^2+y^2}{z^2}\Big)^{-1/2}}}$$

and $h^\bowtie$ is simply the mapping $(x,y,z)\mapsto 3z/2$. Thus,

$$\Lambda_3(x,y,z)=\begin{cases}(0,0,0) & \text{if } x=y=z=0,\\ \Big(x\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}},y\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}},\frac{3}{2}z\Big) & \text{if } \frac{\sqrt5}{2}|z|\le\sqrt{x^2+y^2},\\ \Big(xg^\triangledown(x,y,z),yg^\triangledown(x,y,z),\sqrt{x^2+y^2+z^2}\Big) & \text{if } \frac{\sqrt5}{2}z>\sqrt{x^2+y^2},\\ \Big(xg^\triangledown(x,y,z),yg^\triangledown(x,y,z),-\sqrt{x^2+y^2+z^2}\Big) & \text{if } -\frac{\sqrt5}{2}z>\sqrt{x^2+y^2}.\end{cases}$$

The inverse of $\Lambda_3$ is given by

$$\Lambda_3^{-1}(x,y,z)=\begin{cases}(0,0,0) & \text{if } x=y=z=0,\\ \Big(x\sqrt{1-\frac{4}{9}\frac{z^2}{x^2+y^2}},y\sqrt{1-\frac{4}{9}\frac{z^2}{x^2+y^2}},\frac{2}{3}z\Big) & \text{if } \frac{\sqrt5}{2}|z|\le\sqrt{x^2+y^2},\\ \Big(\frac{x}{\sqrt3}\sqrt{2-\frac{x^2+y^2}{3z^2}},\frac{y}{\sqrt3}\sqrt{2-\frac{x^2+y^2}{3z^2}},z-\frac{x^2+y^2}{3z}\Big) & \text{if } \frac{\sqrt5}{2}z>\sqrt{x^2+y^2},\\ \Big(\frac{x}{\sqrt3}\sqrt{2-\frac{x^2+y^2}{3z^2}},\frac{y}{\sqrt3}\sqrt{2-\frac{x^2+y^2}{3z^2}},\frac{x^2+y^2}{3z}-z\Big) & \text{if } -\frac{\sqrt5}{2}z>\sqrt{x^2+y^2}.\end{cases}$$

# B Appendix: Proof of Theorem 2.4

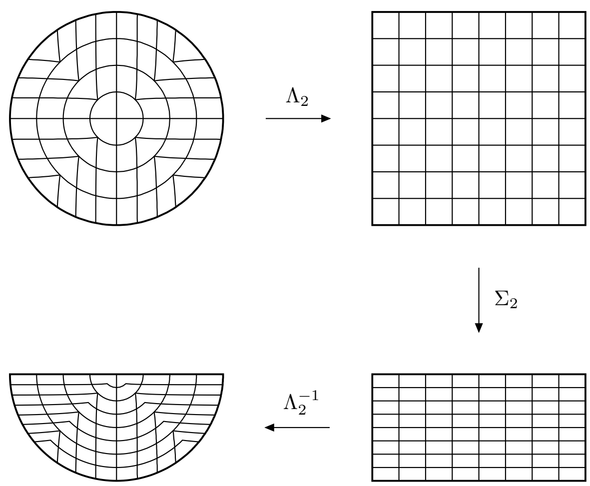
By means of Lemma A.1 we are now able to prove Theorem 2.4. For integers $d \geq 1$ we define the affine compression

$$\Sigma_{d+1} : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d+1}, \quad \Sigma_{d+1}(x,y) \stackrel{\text{def}}{=} \left(x, \frac{y-1}{2}\right) \quad \text{for } x \in \mathbb{R}^d,\ y \in \mathbb{R}.$$

The mapping

$$\Psi_d \stackrel{\text{def}}{=} \Lambda_d^{-1} \Sigma_d \Lambda_d,$$

$d \geq 2$, maps the open unit ball $B$ onto the open unit halfball $B^-$, is bi–Lipschitzian, and the absolute value of its functional determinant is constant. The following figure illustrates the action of the mapping $\Psi_2$ in the two dimensional case.



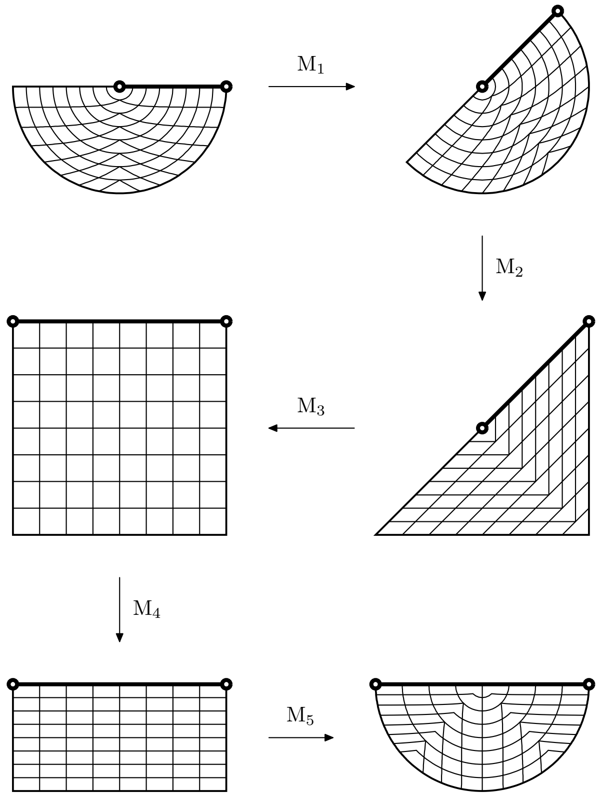

We prove the second assertion of Theorem 2.4 by induction on the space dimension. Let us first regard the mapping in $\mathbb{R}^2$. A mapping $\Psi_2^+$ with the stated properties can be defined as the composition

$$\Psi_2^+ \stackrel{\text{def}}{=} \mathrm{M}_5 \mathrm{M}_4 \mathrm{M}_3 \mathrm{M}_2 \mathrm{M}_1$$

of five mappings which are defined as follows:

$$\begin{aligned}
&\mathrm{M}_1(x,y) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(x-y, x+y) \quad \text{for } (x,y) \in \mathbb{R}^2, && \text{(rotation)},\\
&\mathrm{M}_2(x,y) \stackrel{\text{def}}{=} \Lambda_2(x,y) && \text{(halfball to triangle)},\\
&\mathrm{M}_3(x,y) \stackrel{\text{def}}{=} \begin{cases} (2x-1, y-x+1) & \text{if } y \geq -x, \\ (x-y-1, 2y+1) & \text{if } y < -x, \end{cases} && \text{(triangle to square)},\\
&\mathrm{M}_4(x,y) \stackrel{\text{def}}{=} \Sigma_2(x,y) && \text{(compression)},\\
&\mathrm{M}_5(x,y) \stackrel{\text{def}}{=} \Lambda_2^{-1}(x,y) && \text{(halfsquare to halfball)},
\end{aligned}$$

see also the following figure:




If for some integer $d$, $\Psi_d^+$ is a mapping as stated in Theorem 2.4, then the mapping

$$\Psi_{d+1}^+ \stackrel{\text{def}}{=} \Lambda_{d+1}^{-1}\Theta_{d+1}\Lambda_{d+1},$$

with

$$\Theta_{d+1} : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d+1}, \quad \Theta_{d+1}(x, y) \stackrel{\text{def}}{=} (\Psi_d^+ x, y) \quad \text{for } x \in \mathbb{R}^d,\ y \in \mathbb{R},$$

meets the requirements of Theorem 2.4 for $d + 1$.

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