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## The Hausdorff dimension of certain attractors


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CONTENTS



ABSTRACT. For the solid torus $V = S^1 \times \mathbb{D}^2$ and a $C^1$ embedding $f : V \to V$ given by

$$f(t, x_1, x_2) = (\varphi(t), \lambda_1(t) \cdot x_1 + z_1(t), \lambda_2(t) \cdot x_2 + z_2(t))$$

with $\frac{d\varphi}{dt} > 1$, $0 < \lambda_i(t) < 1$ the attractor $\Lambda = \bigcap_{i=0}^{\infty} f^i(V)$ is a solenoid, and for each disk $D(t) = \{t\} \times \mathbb{D}^2$ $(t \in S^1)$ the intersection $\Lambda(t) = \Lambda \cap D(t)$ is a Cantor set. It is the aim of the paper to find conditions under which the Hausdorff dimension of $\Lambda(t)$ is independent of $t$ and determined by

$$\dim_H \Lambda(t) = \max(p_1, p_2) \tag{0.1}$$

where the real numbers $p_i$ are characterized by the condition that the pressure of the function $\log \lambda_i^{p_i} : S^1 \to \mathbb{R}$ with respect to the expanding mapping $\varphi : S^1 \to S^1$ becomes zero. (There are two further characterizations of these numbers.)
It is proved that (0.1) holds provided $\lambda_1, \lambda_2$ are sufficiently small and $\Lambda$ satisfies a condition called intrinsic transverseness. Then it is shown that in the space of all embeddings $f$ with $\sup \lambda_i < \Theta^{-2}$ ($\Theta$ the mapping degree of $\varphi$) the subset of those $f$ which have an intrinsically transverse attractor $\Lambda$ is open and dense with respect to the $C^1$ topology.

## 1. Results and some problems

Let $S^1 = \mathbb{R}$ (modulo 1) be the unit circle, and let $\mathbb{D}^2$ be the unit disk in $\mathbb{R}^2$. Then $V = S^1 \times \mathbb{D}^2$ is a solid torus, and $A = S^1 \times \mathbb{I}$ ($\mathbb{I} = [-1,1]$) is an annulus. The natural projections $\pi : V \to S^1$, $\rho_1, \rho_2 : V \to A$ are defined by $\pi(t,x,y) = t$, $\rho_1(t,x,y) = (t,x)$, $\rho_2(t,x,y) = (t,y)$, and the disks $\{t\} \times \mathbb{D}^2 = \pi^{-1}(t)$ ($t \in S^1$) will be denoted by $D(t)$. In this paper we consider $C^1$ embeddings $f : V \to V$ which have the form

$$f(t,x,y) = (\varphi(t), \lambda_1(t)x + z_1(t), \lambda_2(t)y + z_2(t)), \tag{1.1}$$

where

$$\varphi : S^1 \to S^1, \quad \lambda_1, \lambda_2 : S^1 \to (0,1), \quad z_1, z_2 : S^1 \to (-1,1)$$

are $C^1$ mappings, and $\varphi$ is expanding in the sense that

$$\dot{\varphi} = \frac{d\varphi}{dt} > 1.$$

This last condition implies that the mapping degree $\Theta$ of $\varphi$ is at least 2 and that $f$ stretches the torus $V$ in the direction of $S^1$. Since $0 < \lambda_1, \lambda_2 < 1$ the disks $D(t)$ are contracted. So the image $f(V)$ is thinner but longer than $V$, and it is wrapped around in $V$ exactly $\Theta$ times. For each $t \in S^1$ the intersection $f(V) \cap D(t)$ consists of $\Theta$ mutually disjoint ellipses. The set

$$\Lambda = \bigcap_{j=0}^{\infty} f^j(V)$$

is the attractor of $f$. This attractor has a relatively simple structure: it is a solenoid and its local structure can be described as follows. For each $t_0 \in S^1$ the intersection $\Lambda(t_0) = \Lambda \cap D(t_0)$ is a Cantor set, and for any arc $B$ in $S^1$ containing $t_0$ there is a homeomorphismus

$$h : B \times \Lambda(t_0) \to \Lambda \cap \pi^{-1}(B) = \Lambda \cap (B \times \mathbb{D}^2)$$

which can be chosen so that

$$\pi h(t,x) = t, \quad h(t_0, x) = x \qquad (t \in B, \quad x \in \Lambda(t_0)).$$

For each $x \in \Lambda(t_0)$ the embedding $h_x = h(\cdot, x) : B \to V$ is of class $C^1$, and $h_x$ depends, with respect to the $C^1$ topology, continuously on $x$.

In this paper we show how in some cases the Hausdorff dimension $\dim_H \Lambda(t)$ of the sets $\Lambda(t)$ is determined by the mappings $\varphi : S^1 \to S^1$ and $\lambda_1, \lambda_2 : S^1 \to (0,1)$. (In these cases this dimension will not depend on $z_1$ and $z_2$.) Besides $\dim_H \Lambda(t)$ we shall consider the dimensions $\dim_H \rho_1(\Lambda(t))$, $\dim_H \rho_2(\Lambda(t))$. The following proposition defines numbers $p_1, p_2$ which will be related to the dimensions in question.

**Proposition 1.1.** For $i = 1,2$ there is exactly one real number $p_i$ for which the functional equation

$$\sum_{t' \in \varphi^{-1}(t)} \lambda_i(t')^{p_i} \xi(t') = \xi(t) \tag{1.2}$$

has a positive continuous solution $\xi : S^1 \to \mathbb{R}$.

**Remark 1.2** Using elementary properties of the pressure $P(\psi)$ of functions $\psi : S^1 \to \mathbb{R}$ with respect to the mapping $\varphi : S^1 \to S^1$ (see e.g.[1] or [4]) it is not hard to see that $p_i$ is the unique number satisfying $P(p_i \log \lambda_i) = 0$.

**Remark 1.3** Lemma 2.2 in Section 2 shows that the number $p_i$ can be obtained as the limit

$$p_i = \lim_{k \to \infty} p(k),$$

where the numbers $p(k)$ $(k = 1, 2, \dots)$ are defined for an arbitrary point $t \in S^1$ by

$$\sum_{t' \in \varphi^{-k}(t)} [\prod_{j=1}^{k} \lambda_i(\varphi^{j-1}(t'))]^{p(k)} = 1.$$

We note that the product in brackets equals the length of the axis of the ellips $f^k(D(t'))$ which points in the direction of the $i$–th coordinate in $D(t)$.

The space of all $C^1$ embeddings $f : V \to V$ as described above equipped with the $C^1$ topology will be denoted by $\mathcal{F}$. It is easily proved (see Section 2), that for $f \in \mathcal{F}, \quad i = 1, 2$ the inequalities

$$\dim_H \rho_i(\Lambda(t)) \leq p_i \tag{1.3}$$

$$\dim_H \Lambda(t) \leq \max(p_1, p_2) \tag{1.4}$$

hold for all $t \in S^1$. Our aim is to find conditions under which we get equality in (1.3) and in (1.4). The following subsets $\mathcal{F}_i^\times, \quad \mathcal{F}_i', \quad \mathcal{F}_i''$ of $\mathcal{F}$ $i = 1, 2$ will be crucial.

**Definition 1.1.** For $i = 1, 2$ $\mathcal{F}_i^\times$ is the set of all $f \in \mathcal{F}$ which have the following property: For any arc $B$ in $S^1$ and any two components $B_1, B_2$ of $\Lambda \cap \pi^{-1}(B)$ the arcs $\rho_i(B_1), \rho_i(B_2)$ are transverse in $A$ at each point of $\rho_i(B_i) \cap \rho_i(B_2)$. The attractors of the mappings $f \in \mathcal{F}_i^\times$ will be called *intrinsically transverse with respect to* $\rho_i$.

As easily seen, a mapping $f \in \mathcal{F}$ belongs to $\mathcal{F}_i^\times$ provided for any two arcs $B_1, B_2$ as in the definition above which lie in different components of $\pi^{-1}(B) \cap f(V)$ the projection $\rho_i(B_1), \rho_i(B_2)$ are transverse. This implies that $\mathcal{F}_i^\times$ is open in $\mathcal{F}$.

The set $\mathcal{F}_i', \mathcal{F}_i''$ are defined by

$$\mathcal{F}_i' = \{f \in \mathcal{F} | \sup \lambda_i < \Theta^{-2}\}$$

$$\mathcal{F}_i'' = \{f \in \mathcal{F} | \sup \lambda_i < \inf \dot{\varphi} \sup \dot{\varphi}^{-4 \log \inf \lambda_i / \log \sup \lambda_i}\}.$$

Obviously $\mathcal{F}_i', \mathcal{F}_i''$ are open in $\mathcal{F}$, and $\mathcal{F}_i'' \subset \mathcal{F}_i'$.

Now we state the main results.

**Theorem A.** If $i = 1,2$ and $f$ belongs to $\mathcal{F}_i^\times \cap \mathcal{F}_i''$ and $t \in S^1$, then

$$\dim_H \rho_i(\Lambda(t)) = p_i \tag{1.5}$$

**Theorem B.** The set $\mathcal{F}_i^\times \cap \mathcal{F}_i'$ is open and dense in $\mathcal{F}_i'$.

**Corollary.** $\mathcal{F}_i^\times \cap \mathcal{F}_i''$ is open and dense in $\mathcal{F}_i''$, and (1.5) holds generically in $\mathcal{F}_i''$.

Let $\mathcal{H}$ denote the set of all $f \in \mathcal{F}$ for which $\dot{\varphi} \equiv \Theta$ and $\lambda_i$ are constant functions on $S^1$. We define for $i = 1,2$

$$\mathcal{H}_i^\times = \mathcal{F}_i^\times \cap \mathcal{H}, \quad \mathcal{H}_i' = \mathcal{F}_i' \cap \mathcal{H}, \quad \mathcal{H}_i'' = \mathcal{F}_i'' \cap \mathcal{H}.$$

Then

$$\mathcal{H}_i'' = \{f \in \mathcal{H} | \lambda_i < \Theta^{-3}\}$$

**Theorem C.** $\mathcal{H}_i^\times \cap \mathcal{H}_i'$ is open and dense in $\mathcal{H}_i'$.

**Corollary.** $\mathcal{H}_i^\times \cap \mathcal{H}_i''$ is open and dense in $\mathcal{H}_i''$, and (1.5) holds generically in $\mathcal{H}_i''$. In this case $p_i = -\log\Theta/\log\lambda_i$.

**Theorem D.** If $p_{i_0} = \max(p_1, p_2)$, then for any $f \in \mathcal{F}_{i_0}^\times \cap \mathcal{F}_{i_0}''$ and each $t \in S^1$

$$\dim_H \Lambda(t) = p_{i_0}. \tag{1.6}$$

**Corollary:** If $p_{i_0} = \max(p_1, p_2)$ then (1.6) holds generically in $\mathcal{F}_{i_0}''$ and in $\mathcal{H}_{i_0}''$.

The following questions remain open.

**Question A.** Theorem B states that any $C^r$ mapping $f$ in $\mathcal{F}_i'$ $(r \geq 1)$ can be $C^1$ approximated by $C^1$ mappings and even by $C^\infty$ mappings in $\mathcal{F}_i'$ which have an intrinsically transverse attractor. Can $f$ be $C^r$ approximated by such mappings?

**Question B.** Let $f$ be a mapping in $\mathcal{F}_i'$ and let $\mathcal{U}$ be a neighbourhood of $f$ in $\mathcal{F}_i'$ $(i = 1,2)$. Is there always a mapping $g \in \mathcal{U} \cap \mathcal{F}_i^\times$ which is defined by the same mappings $\varphi, \lambda_1, \lambda_2, z_{i'}, (i' \neq i)$ as $f$; i.e. can $\Lambda$ be made intrinsically transverse with respect to $\rho_i$ by a small perturbation of $z_i$ without changing $\varphi, \lambda_1, \lambda_2, z_{i'}$?

The main results are contained in Theorem A and Theorem B. Their proofs are carried out in Section 3 and Section 4, respectively. Since there is no distinction between the cases $i = 1$ and $i = 2$ it is sufficient to consider one fixed index $i$, and we shall write $\lambda_i = \lambda, \quad \rho_i = \rho, \quad p_i = p$ in these sections. The proofs of the following facts are collected in Section 2: 1. Proposition 1.1; 2. two lemmas which will be used later and the second of which implies Remark 1.3; 3. the easy parts $\dim_H \rho_i(\Lambda(t)) \leq p_i \quad (i = 1, 2)$, $\dim_H \Lambda(t) \leq p_{i_0}$ of Theorem A and Theorem D. The remaining part of Theorem D is an immediate consequence of Theorem A, Theorem B and the fact that the projection $\rho_{i_0}$ being Lipschtz continuous, can not raise the Hausdorff dimension of $\Lambda(t)$. The proof of Theorem C follows easily from the proof of Theorem B in Section 4.

Let us mention that the corollary to Theorem D may possibly be helpful to solve a problem concerning the Hausdorff dimension of a class of 1–dimensional hyperbolic attractors and so to generalize a result of McCluskey and Mannings [3] about the Hausdorff dimension of basic sets in surfaces to certain attractors in higher dimensional manifolds. To describe this class and to formulate the problem we need the following definitions.

Let $f : M \to M$ be a $C^r$ diffeomorphism of an m-dimensional compact manifold $M$ without boundary $(r \geq 1, m \geq 2)$. We say that a compact subset $\Lambda$ of $M$ is a *hyperbolic* attractor of $f$ if it has the following properties.
(1) there is a neighborhood $U$ of $\Lambda$ in $M$ such that $f(U) \subset U$ and

$$\Lambda = \bigcap_{i=0}^{\infty} f^i(U).$$

(2) $\Lambda$ is topologically transitive in the sense that there is a dense orbit in $\Lambda$.
(3) For the restriction $T_\Lambda$ of the tangent bundle of $M$ to $\Lambda$ there is a splitting $T_\Lambda = T^s \oplus T^u$ in two continuous df–invariant subbundles $T^s, T^u$ such that with suitably chosen numbers $0 < \lambda < 1, c > 0$ we have

$$|df^k(v)| \leq c\lambda^k |v| \qquad (v \in T^s, \ \ k = 1, 2, \dots),$$

$$|df^k(w)| \geq c^{-1}\lambda^{-1} |w| \qquad (w \in T^u, \ \ k = 1, 2, \dots).$$

(Here $|v|$ denotes the length of $v$ with respect to an arbitrarily chosen Riemannian metric in $M$. Whether $\Lambda$ is a hyperbolic attractor or not does not depend on his metric.)

Under these conditions for each $x \in \Lambda$ the sets

$$W_x^s = \{y \in M | \lim_{k \to \infty} d(f^k(y), f^k(x)) = 0\},$$

$$W_x^u = \{y \in M | \lim_{k \to -\infty} d(f^k(y), f^k(x)) = 0\}$$

are the images of one–to–one immersions $w_x^s : \mathbb{R}^{n'} \to M, w_x^u : \mathbb{R}^n \to M$ of class $C^r$, where $n', n$ are the dimensions of the fibres in $T^s, T^u$, respectively. Therefore the sets $W_x^s, W_x^u$ called the *stable* or *unstable manifold* of $x$. Since $\Lambda$ is an attractor, the unstable manifolds are contained in $\Lambda$, i.e. $\Lambda$ is the union of its unstable manifolds.

We assume that the topological dimension of $\Lambda$ is 1. This implies that the unstable manifolds are 1–dimensional and that the intersection of $\Lambda$ with a stable manifold $W_x^s$ is totally disconnected. Even more: for each $x \in \Lambda$ there is an $(m-1)$–dimensional compact manifold $Q$ (with boundary) in $W_x^s$, a Cantor set $C$ in Int$Q$ and a homeomorphism $h$ of $Q \times \mathbb{I}$ $(\mathbb{I} = [-1, 1])$ onto a neighborhood $V$ of $x$ in $M$ such that

$$V \cap \Lambda = h(C \times \mathbb{I})$$

and for $c \in C, \quad t \in \mathbb{I}, \quad y = h(c, t) \in \Lambda$ the manifolds $h(\{c\} \times \mathbb{I}), \quad h(Q \times \{t\})$ are pieces of the unstable and stable manifolds of $y$, respectively. The set $h(C \times \{t\})$ is a neighbourhood of $y$ in $W_y^s \cap \Lambda$ with respect to the intrinsic topology of $W_y^s$, i.e. the topology in $W_y^s$ which is defined by the topology of $R^{m-1}$ via the mapping $w_y^s : \mathbb{R}^{m-1} \to W_y^s$.

We are interested in the Hausdorff dimensions of the Cantor sets $h(C \times \{t\})$. Since it is not clear that these dimensions do not depend on $t$ and $Q$ we define the local transverse Hausdorff dimension of $\Lambda$ at a point $x \in \Lambda$ to be infimum of the Hausdorff dimensions of all Cantor sets which are neighbourhoods of $x$ in $W_x^s \cap \Lambda$ with respect to the intrinsic topology in $W_x^s$. If these dimensions are independent of $x$ their common value will be called the *transverse Hausdorff dimension of* $\Lambda$.

To determine these dimensions seems to be a hard problem. Therefore we restrict this problem to a class to attractors $\Lambda$ which are related to the attractors considered in the theorems above. This means that we are interested in 1-dimensional hyperbolic attractors $\Lambda$ whose stable bundle has a splitting $T^s = T^{ws} \oplus T^{ss}$ in two continuous $df$–invariant subbundles, where $T^{ws}$ (the bundles of weakest attraction) is 1-dimensional and for $v \in T^{ws}, \quad w \in T_x^{ss}, \quad |v| = |w| = 1$ we have $|d_x f(v)| > |d_x f(w)|$. In this case we say that $\Lambda$ is an attractor with a bundle of weakest attraction. For these attractors we define a function $\lambda : \Lambda \to \mathbb{R}$ by

$$|d_x f(v)| = \lambda(x) \cdot |v| \qquad\qquad (x \in \Lambda, v \in T_x^{ws}).$$

Then there is a unique number $p_\Lambda$ such that the topological pressure $P(p_\Lambda \cdot \log \lambda)$ of the function $p_\Lambda \cdot \log \lambda$ is 0. Now we pose our problem as follows: Find a condition under which 1–dimensional hyperbolic attractors $\Lambda$ with a bundle of weakest attraction generalically have the transverse Hausdorff dimension $p_\Lambda$.

To explain the meaning of the word "generalically" in this context some definitions are necessary. Let $M$ be a compact Riemannian manifold and let $\mathcal{A}$ be the set of all pairs $(f, \Lambda)$, where $f : M \to M$ is a $C^1$ diffeomorphism and $\Lambda$ is a 1-dimensional hyperbolic attractor of $f$ with a bundle of weakest attraction. We define a distance between elements $(f, \Lambda), (f', \Lambda')$ of $\mathcal{A}$ by

$$d((f, \Lambda), (f', \Lambda')) = \max(d(f, f'), d(\Lambda, \Lambda')),$$

where $d(f, f')$ denotes the $C^1$ distance between $f$ and $f'$ (with respect to an embedding of $M$ in a high dimensional space $\mathbb{R}^N$) and $d(\Lambda, \Lambda')$ is the infimum of all numbers $\varepsilon$ for which there is a conjugating homeomorphism $h : \Lambda \to \Lambda'$ satisfying $d(h(x), x) \leq \varepsilon$ for all $x \in \Lambda$. (If no such homeomorphism exists then $d(\Lambda, \Lambda') = \infty$.) By the stability theorem for hyperbolic sets [2] the natural projection $(f, \Lambda) \to f$ is a finite–to–one local homeomorphism of $\mathcal{A}$ into $\mathrm{Diff}^1(M)$.

Now our problem can be formulated as follows: Find a condition for 1-dimensional hyperbolic attractors $\Lambda$ with a bundle of weakest attraction which defines an open set $\mathcal{A}''$ of $\mathcal{A}$ such that the set of all $(f, \Lambda) \in \mathcal{A}$ for which the transverse Hausdorff dimension of $\Lambda$ exists and equals $p_\Lambda$ contains an open and dense subset of $\mathcal{A}''$.

## 2. Preliminaries

Proof of Proposition 1.1. Let $\lambda : S^1 \to \mathbb{R}$ be a positive $C^1$ function, and let $L$ be the Banach space of all continuous functions $\xi : S^1 \to \mathbb{R}$ with the maximum norm. For $\xi \in L$ we define the functions $\xi' : S^1 \to [0, \infty]$, $\bar{\xi} \in L$ by

$$\xi'(t) = \limsup_{\delta \to 0} |\xi(t + \delta) - \xi(t)| \,/\, |\delta|,$$

$$\bar{\xi}(t) = \sum_{t' \in \varphi^{-1}(t)} \xi(t').$$

Moreover, $A_\lambda : L \to L$ will denote the operator which is defined by

$$A_\lambda \xi = \overline{\lambda \xi} = \sum_{t' \in \varphi^{-1}(t)} \lambda(t') \xi(t').$$

The first step in the proof is to find a strictly positive $\xi \in L$ such that $A_\lambda \xi = \mu \xi$ holds for some real $\mu > 0$. To this aim we define a convex cone $K$ in $L$ (i.e. a non–empty convex subset $K$ of $L$ such that $\xi \in K$, $s > 0$ implies $s\xi \in K$) which satisfies $A_\lambda K \subset K$ and has a compact base $B$ (i.e. $B$ has to be a compact intersection of $K$ with a hyperplane in $L$ such that for each $\xi \in K$ there is a positive $s \in \mathbb{R}$ such that $s\xi \in B$). Then by a well known generalization of the Perron–Frobenius theorem for positive matrices (see [5] p. 267) there is an eigenfunction $\xi$ of $A_\lambda$ in $K$ with an eigenvalue $\mu \geq 0$. Since our cone $K$ will only contain functions $\xi \geq 0$, $\xi \not\equiv 0$, this together with $\lambda > 0$ implies $\mu > 0$, and, using the fact that $\varphi$ is expanding, it is easy to see that the eigenfunction $\xi$ is strictly positive.

To define $K$ we choose a positive real $a$ such that

$$\lambda' \leq a(\beta - 1)\lambda,$$

where $\beta = \inf \frac{d\varphi}{dt}$. (Since $\lambda$ is a $C^1$ function, $\lambda'$ is the absolute value of its derivative.) Then

$$K = \{\xi \in L | \ \xi \geq 0, \ \ \xi \not\equiv 0, \ \ \xi' \leq a\xi\}$$

is obviously a convex cone and

$$B = \{\xi \in K | \ \int_{S^1} \xi(t)dt = 1\}$$

is a base of $K$.
To prove $A_\lambda K \subset K$ it is sufficient to show that $(A_\lambda \xi)' \leq a A_\lambda \xi$ holds for all $\xi \in K$.

$$\begin{aligned} (A_\lambda \xi)' = \overline{\lambda\xi}' &\leq \beta^{-1}\overline{(\lambda\xi)'} \leq \beta^{-1}\overline{\lambda'\xi + \lambda\xi'} \\ &\leq \beta^{-1}\overline{a(\beta - 1)\lambda\xi + a\lambda\xi} \ \leq \ a\overline{\lambda\xi} = aA_\lambda\xi. \end{aligned}$$

It remains to prove that $B$ is compact. Let $\{t_1, t_2, \dots\}$ be a countable dense subset of $S^1$. If $\xi_1, \xi_2, \dots$ is any sequence in $B$, then by a diagonal selection process it is not hard to find a subsequence which converges on each $t_i$. Since $\xi' \leq a\xi$ we see that this subsequence convergens with respect to the maximum norm.

Now let $\lambda_1, \lambda_2 : S^1 \to (0,1)$ be positive $C^1$ functions and let $\mu_1, \xi_1, \mu_2, \xi_2$ be positive eigenvalues and positive eigenfunctions of $A_{\lambda_1}, A_{\lambda_2}$, respectively. If $\lambda_1 < \lambda_2$, then $\mu_1 < \mu_2$. (To see this let $\vartheta = \xi_2/\xi_1$. Then

$$\overline{\lambda_1\xi_1\vartheta} < \overline{\lambda_2\xi_1\vartheta} = \mu_2\xi_1\vartheta = \mu_2\mu_1^{-1}\overline{\lambda_1\xi_1}\vartheta$$

and if $t \in S^1$ is chosen so that $\vartheta(t) = \inf\vartheta$ we get

$$\overline{\lambda_1\xi_1\vartheta} \geq \overline{\lambda_1\xi_1}\vartheta(t)$$

and therefore $\mu_2\mu_1^{-1} > 1$.) This monotone dependence of $\mu$ from $\lambda$ easily implies that $\mu = \mu(\lambda)$ is uniquely determined by $\lambda$ and that $\mu(\lambda)$ depends continuously on $\lambda$. For a fixed $C^1$ function $\lambda : S^1 \to (0,1)$ we have

$$\lim_{p\to 0} \mu(\lambda^p) > 1, \ \ \lim_{p\to\infty} \mu(\lambda^p) = 0$$

so that there is a unique $p > 0$ with $\mu(\lambda^p) = 1$, and the proposition is proved. ∎

The following lemma will be applied several times in this paper. Its proof is easy and can be left to the reader.

**Lemma 2.1.** Let $\mu_{m,j} \ \ (m = 1, 2, \dots; \ \ i = 1, \dots, j(m))$ be positive real numbers such that

$$\lim_{m\to\infty} \sup_{1\leq j\leq j(m)} \mu_{m,j} = 0.$$

If the numbers $p^*(m)$ are determined by

$$\sum_{j=1}^{j(m)} \mu_{m,j}^{p^*(m)} = 1$$

and if $p^{**}(m)$ are numbers such that

$$c_1 < \sum_{j=1}^{j(m)} \mu_{m,j}^{p^{**}(m)} < c_1,$$

where $0 < c_1 < c_2$ and $c_1, c_2$ do not depend on $m$, then

$$\lim_{m\to\infty} (p^*(m) - p^{**}(m)) = 0.$$

■

Now we prove a lemma which is a generalization of Remark 1.3.

**Lemma 2.2.** If for $i = 1, 2$ and an arbitrary chosen sequence $s_1, s_2, \ldots$ of points in $S^1$ the numbers $q_i(1),\ q_i(2), \ldots$ are defined by

$$\sum_{t' \in \varphi^{-k}(s_k)} \prod_{j=1}^{k} \lambda_i(\varphi^{j-1}(t'))^{q_i(k)} = 1,$$

then

$$\lim_{k\to\infty} q_i(k) = p_i,$$

where $p_i$ is defined by Proposition 1.1.

PROOF: By repeated application of the defining equation (1.2) for $p_i$ we see that

$$\sum_{t' \in \varphi^{-k}(s_k)} \prod_{j=1}^{k} \lambda_i(\varphi^{j-1}(t'))^{p_i} \xi(t') = \xi(s_k)$$

holds for $k = 1, 2, \ldots$. Since

$$\frac{\inf \xi}{\sup \xi} \le \sum_{t' \in \varphi^{-k}(s_k)} \prod_{j=1}^{k} \lambda_i(\varphi^{j-1}(t'))^{p_i} \le \frac{\sup \xi}{\inf \xi},$$

we merely have to apply Lemma 2.1. ■

The following lemmas prove the easy parts of Theorem A and Theorem D.

**Lemma 2.3.** If $t \in S^1$, then

$$\dim_H \rho_i(\Lambda(t)) \le p_i \quad (i = 1, 2).$$

PROOF: For $k \ge 1$ the sets $\rho_i(\Lambda(t))$ is covered by the intervals $\rho_i f^k(t')\quad (t' \in \varphi^{-k}(t))$ whose lengths are $2\prod_{j=1}^{k} \lambda_i(\varphi^{j-1}(t'))$.

■

**Lemma 2.4.** If $t \in S^1$, $\quad p_{i_0} = \max(p_1, p_2)$, then

$$\dim_H \Lambda(t) \le p_{i_0}.$$

PROOF: We define for $t' \in \varphi^{-k}(t)$

$$\mu_1(t') = \prod_{j=1}^{k} \lambda_1(\varphi^{j-1}(t')), \quad \mu_2(t') = \prod_{j=1}^{k} \lambda_2(\varphi^{j-1}(t'))$$

$$\mu(t') = \tfrac{1}{2}\mathrm{diam} f^k(D(t')) = \max(\mu_1(t'), \mu_2(t')).$$

If the numbers $q_1(k), q_2(k), \bar{p}(k)$ are defined by

$$\sum_{t' \in \varphi^{-k}(t)} \mu_1(t')^{q_1(k)} = \sum_{t' \in \varphi^{-k}(t)} \mu_2(t')^{q_2(k)} = \sum_{t' \in \varphi^{-k}(t)} \mu(t')^{\bar{p}(k)} = 1,$$

then we have

$$\dim_H \Lambda(t) \leq \liminf_{k \to \infty} \bar{p}(k),$$

and Lemma 2.2. implies

$$p_i = \lim_{k \to \infty} q_i(k) \quad (i = 1, 2).$$

If we assume

$$p_{i_0} < \liminf_{k \to \infty} \bar{p}(k),$$

then we get

$$\lim_{k \to \infty} \sum_{t' \in \varphi^{-k}(t)} \mu_i(t')^{\bar{p}(k)} = 0 \qquad (i = 1, 2),$$

$$\lim_{k \to \infty} \sum_{t' \in \varphi^{-k}(t)} \mu(t')^{\bar{p}(k)} \leq \lim_{k \to \infty} \sum_{t' \in \varphi^{-k}(t)} (\mu_1(t')^{\bar{p}(k)} + \mu_2(t')^{\bar{p}(k)}) = 0$$

which contradicts the definition of $\bar{p}(k)$.

■

## 3. PROOF OF THEOREM A

We fix an index $i \in \{0, 1\}$ and consider a mapping $f \in \mathcal{F}_i^\times \cap \mathcal{F}_i''$. As announced in the introduction instead of $\lambda_i$, $\rho_i$, $p_i$ we shall write $\lambda$, $\rho$, $p$, respectively. Moreover we shall use the following notations:

$$\underline{\lambda} = \inf \lambda \quad , \quad \bar{\lambda} = \sup \lambda,$$

$$\underline{\beta} = \inf \dot{\varphi} \quad , \quad \bar{\beta} = \sup \dot{\varphi}.$$

The proof consists of two parts. The first one in 3.A. is devoted to proving the following lemma.

**Lemma 3.A.** For each sufficiently large integer $m$ there is a Cantor set $C' = C'(m)$ in $S^1$ and a subset $\Lambda' = \Lambda'(m)$ of $\pi^{-1}(C'(m)) \cap \Lambda$ such that

$$\lim_{m\to\infty} \inf_{t'\in C'(m)} \dim_H \rho(\Lambda'(m) \cap D(t')) \geq p.$$

Moreover, the restriction of $\rho$ to any set $\Lambda'(t')$ is one–to–one.

In the second part 3.B. for an integer $m$ to which this lemma applies we consider two arbitrary points $t' \in C'(m), t \in S^1$ and an arc $B \subset S^1$ with end points $t', t$. Then for each $x \in \Lambda'(t') = \Lambda'(m) \cap D(t')$ there is a unique arc $B_x \subset \Lambda$ with one end point $x$ and $\pi(B_x) = B$. If the second end point of $B_x$ is denoted by $\tilde{h}(x)$, we get the mapping $\tilde{h} : \Lambda'(t') \to \Lambda(t)$. By Lemma 3.A the projection $\rho$ is one–to–one on $\Lambda'(t')$, and we can define the mapping

$$h = \rho\tilde{h}\rho^{-1} : \rho(\Lambda'(t')) \to \rho(\Lambda(t)).$$

The following lemma will be proved in 3.B.

**Lemma 3.B.** There is a finite partition $\rho(\Lambda'(t')) = E_1 \cup \cdots \cup E_r$ in disjoint compact subsets such that the restrictions $h|_{E_i}$ $(i = 1, \ldots, r)$ are one–to–one and have Lipschitz continuous inverses.

These two lemmas prove Theorem A: Let $\varepsilon > 0$ and $t \in S^1$ be given. By the first lemma we can choose an integer $m$ such that for $t' \in C'(m)$ we have

$$\dim_H \Lambda'(t') \geq p - \varepsilon.$$

If $\rho(\Lambda'(t')) = E_1 \cup \cdots \cup E_r$ is a partion with the properties mentioned in the second lemma we have $\dim_H E_i \geq p - \varepsilon$ for at least one set $E_i$. Since the Lipschitz mapping $(h|_{E_i})^{-1}$ can not raise the Hausdorff dimension we get $\dim_H h(E_i) \geq p - \varepsilon$, and then $h(E_i) \subset \rho(\Lambda(t))$ together with the fact that $\varepsilon > 0$ is arbitrary implies $\dim_H \rho(\Lambda(t)) \geq p$. The opposite inequality has been proved in Lemma 2.3.

## 3.A. Proof of Lemma 3.A.

The proof is divided in four steps 3.A.1. – 3.A.4. (see also Corollary 3.3.). The remaining parts 3.A.5. – 3.A.8. contain proofs of lemmas which are applied in the main proof.
Since $f \in \mathcal{F}_i''$ we have

$$\overline{\lambda} < \underline{\beta}\,\overline{\beta}^{-4\log\lambda/\log\overline{\lambda}}$$

or, equivalently,

$$\frac{2\log\overline{\beta}}{\log\underline{\beta} - \log\underline{\lambda}} < \frac{\log\overline{\lambda}}{2\lg\underline{\lambda}},$$

and we get the subinterval

$$\mathcal{J} = \left(\frac{2\log\overline{\beta}}{\log\underline{\beta} - \log\underline{\lambda}}, \frac{\log\overline{\lambda}}{2\log\underline{\lambda}}\right) \tag{3.0}$$

of $(0, \frac{1}{2})$ which frequently will be used during the proof.

**3.A.1.** SOME COMBINATORIAL CONCEPTS. For $n = 1, 2, \dots$ or $n = \infty$ let $\mathcal{E}_n$ be the set of almost all sequences $\underline{e} = (e_1, e_2, \dots)$ of length $n$, where $e_i \in \{0, \dots, \Theta - 1\}$ ($\Theta$ is degree of the mapping $\varphi : S^1 \to S^1$). Here "almost" means that the infinite sequences $(e_1, e_2, \dots, e_j, \Theta - 1, \Theta - 1, \dots)$ with $e_i = \Theta - 1$ for almost all $i$ are excluded from $\mathcal{E}_\infty$.We say that a sequence $\underline{e} = (e_1, e_2, \dots) \in \mathcal{E}_m$ appears in a sequence $\underline{e}' = (e'_1, e'_2, \dots) \in \mathrm{E}_n$, if $n \geq m$ and if for some $k \in \{0, \dots, n - m\}$ we have $e'_{k+i} = e_i$ $(i = 1, \dots, m)$. The projections

$$\pi_m : \bigcup_{m \leq n \leq \infty} \mathcal{E}_n \to \mathcal{E}_m, \quad \sigma : \mathcal{E}_n \to \mathcal{E}_{n-1}$$

are defined by

$$\pi_m(e_1, e_2, \dots) = (e_1, \dots, e_m), \qquad \sigma(e_1, e_2, \dots) = (e_2, e_3, \dots).$$

Since $\dot{\varphi} > 1$, the mapping $\varphi : S^1 \to S^1$ is expanding, and there is a homeomorphism $h : S^1 \to S^1$ such that $h^{-1}\varphi h(t) = \Theta t$. We define the mapping $\tau : \mathcal{E}_\infty \to S^1$ by

$$\tau(\underline{e}) = h(\sum_{i=1}^{\infty} e_i \Theta^{-i}) \quad (\underline{e} = (e_1, e_2, \dots)).$$

This mapping satisfies $\tau(\mathcal{E}_\infty) = S^1$ and

$$\varphi(\tau(\underline{e})) = \tau(\sigma(\underline{e})).$$

For $\underline{e} \in \mathcal{E}_n$ $(1 \leq n < \infty)$ the set

$$T'_{\underline{e}} = \{\tau(\underline{e}') | \underline{e}' \in \mathcal{E}_\infty, \quad \pi_n(\underline{e}') = \underline{e}\}$$

is an arc in $S^1$ with the upper end point missing. The closure of $T'_{\underline{e}}$ will be denoted by $T_{\underline{e}}$. If $n$ is fixed, then the family $\mathcal{T}_n$ of all arcs $T_{\underline{e}}$ $(\underline{e} \in \mathcal{E}_n)$ is a partition of $S^1$ in arcs which have at most end points in common. The lengths $l$ of these arcs are bounded by $(\sup \dot{\varphi})^{-n} \leq l \leq (\inf \dot{\varphi})^{-n}$. If $\underline{e} \in \mathcal{E}_n$, $1 < n < \infty$, then $\varphi(T_{\underline{e}}) = T_{\sigma(\underline{e})}$, and for $1 \leq m < n < \infty$, $\underline{e} \in \mathcal{E}_m$ the arc $T_{\underline{e}}$ is the union of all arcs $T_{\underline{e}'}$, where $\underline{e}' \in \mathcal{E}_n$, $\pi_m(\underline{e}') = \underline{e}$.

**3.A.2.** THE GEOMETRIC LEMMA The aim of this lemma is, roughly speaking, to find large compact subsets $S$ of $S^1$ and small integers $k > 1$ such that for any $t \in S^1$ and any two different components $D_1, D_2$ of $f(V) \cap D(t)$ the sets $f^k(V_S) \cap D_1$, $f^k(V_S) \cap D_2$ are mapped by $\rho$ to separate subsets of the interval $\rho(D(t)) = t \times [-1, 1]$ in $A = S^1 \times [-1, 1]$, where $V_S$ denotes the parts $\pi^{-1}(S) = S \times \mathbb{D}^2$ of $V$. This property of $S, k$ is equivalent to the fact that for any $t_1, t_2 \in S$ satisfying $\varphi^{k-1}(t_1) \neq \varphi^{k-1}(t_2)$, $\varphi^k(t_1) = \varphi^k(t_2)$ we have $\rho(f^k(D_{t_1})) \cap \rho(f^k(D_{t_2})) = \emptyset$.

**Lemma 3.1.** GEOMETRIC LEMMA Let

$$\gamma \in \left( \frac{2 \log \bar{\beta}}{\log \underline{\beta} - \log \bar{\lambda}}, 1 \right) \tag{3.1}$$

be given. (Obviously the interval in (3.1) contains the interval $\mathcal{J}$ of (3.0).) Then to each sufficiently large integer $n$ we can find an integer $k$ satisfying $1 < k < \frac{n}{2}$ and a proper compact subset $S$ of $S^1$ which is the union of at least $\Theta^n - \Theta^{\gamma n}$ arcs belonging to $\mathcal{T}_n$ such that for any two points $t_1, t_2 \in S$ satisfying $\varphi^{k-1}(t_1) \neq \varphi^{k-1}(t_2)$, $\varphi^k(t_1) = \varphi^k(t_2)$ we have

$$\rho(f^k(D(t_1)) \cap \rho(f^k(D(t_2)) = \emptyset. \tag{3.2}$$

The last condition can be expressed by saying that for each $t \in S^1$ and any two components $D_1, D_2$ of $f(V) \cap D(t)$ we have

$$\rho(D_1 \cap f^k(\pi^{-1}(S))) \cap \rho(D_2 \cap f^k(\pi^{-1}(S))) = \emptyset. \tag{3.3}$$

Since $S$ is compact there is a $\delta > 0$ (depending on $k$ and $S$) such that (3.2) and (3.3) can be replaced by

$$\text{dist } (\rho(f^k(D(t_1)), \rho(f^k(D(t_2))) \geq \delta, \tag{3.4}$$

$$\text{dist } (\rho(D_1 \cap f^k(\pi^{-1}(S))), \rho(D_2 \cap f^{-k}(\pi^{-1}(S)))) \geq \delta, \tag{3.5}$$

respectively.

Proof. Let $B$ be a closed subarc of $S^1$ and let for some $j > 1$ two components of $f^j(V) \cap \pi^{-1}(B)$ be denoted by $Z_1, Z_2$. We say that the pair $Z_1, Z_2$ is an overcrossing of $f^j(V)$ if $Z_1, Z_2$ lie in different components of $f(V) \cap \pi^{-1}(B)$ and the set $\rho Z_1 \cap \rho Z_2$ is a curve–linear quadrangle in $A$ (as shown in Fig. 1), whose projection to $S^1$ is $B$ and whose four edges

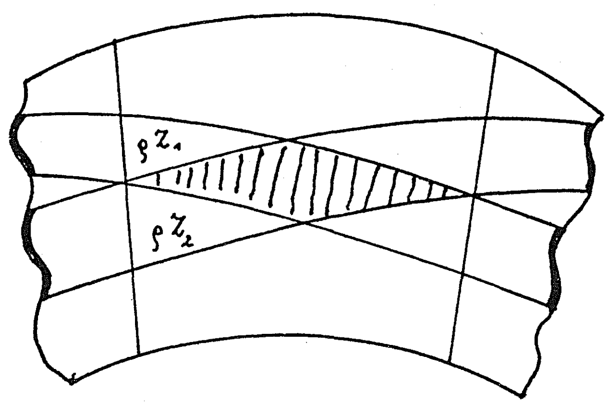


Fig.1

are smooth arcs with transverse intersection at the end points. It is assumed that $\rho(Z_1)$ really crosses $\rho(Z_2)$ in the sense that opposite edges of $\rho(Z_1) \cap \rho(Z_2)$ lie in opposite edges of one of the quadrangles $\rho(Z_1), \rho(Z_2)$ (i.e. that the intersection of $\rho(Z_1)$ and $\rho(Z_2)$ is not as shown in Fig.2.

The arc $B$ will be called the $\pi$–projection of the overcrossing.

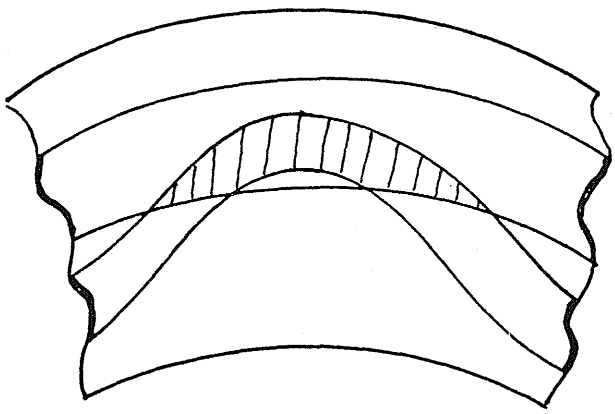

Fig.2

Since $\Lambda$ is intrinsically transverse with respect to $\rho$, for each $\alpha > \overline{\lambda}$ there is a positive integer $k_0 > 1$ with the following three properties:

(1) If $k \geq k_0$ and $D_1, D_2$ are components of $f^k(V) \cap D(t)$ (t any point in $S^1$), which lie in different components of $f(V) \cap D(t)$ and for which $\rho(D_1) \cap \rho(D_2) \neq \emptyset$, then there is an overcrossing $Z_1, Z_2$ in $f^k(V)$ with $D_1 \subset Z_1$, $D_2 \subset Z_2$.

(2) If $k \geq k_0$, then the length of the $\pi$–projection of any overcrossing in $f^k(V)$ is at most $\alpha^{k-k_0}$. (This is implied by the fact that for points $p, q \in \Lambda \cap D(t)$ which lie in different components of $f(V) \cap D(t)$ and have same $\rho$–image $\rho(p) = \rho(q)$, there is a positive lower bound for the angle at $\rho(p)$ between the $\rho$–images of the fibres of $\Lambda$ passing through $p$ and $q$, respectively.)

(3) If $k \geq k_0$, then the number of overcrossings in $f^k(V)$ is exactly $\Theta^{2(k-k_0)}c$ where $c$ denotes the number of overcrossings in $f^{k_0}(V)$. (This holds since each overcrossing in $f^{k_0}(V)$ contains exactly $\Theta^{2(k-k_0)}$ overcrossings of $f^k(V)$.)

Now we choose a real $\alpha > \overline{\lambda}$ which is so close to $\overline{\lambda}$ that $\gamma > 2\log\overline{\beta}/(\log\underline{\beta} - \log\alpha)$ and fix an integer $k_0$ satisfying (1), (2), (3). For an integer $k \geq k_0$ the arcs in $S^1$, which are $\pi$–projections of overcrossings in $f^k(V)$, will be denoted by $B_1, \ldots, B_s$. Then $s = \Theta^{2k'}c$, where $k' = k - k_0$. For each arc $B_i$ $(1 \leq i \leq s)$ we choose a component $B_i^*$ of $\varphi^{-k}(B_i)$ for which $f^k(\pi^{-1}(B_i^*))$ belongs to the overcrossing corresponding to $B_i$. The union $B_1^* \cup \cdots \cup B_s^*$ will be denoted by $B^*$. The construction of $B^*$ implies that for any $t \in S^1$ the parts of $f^k(\pi^{-1}(S^1 \backslash B^*))$ lying in the $\Theta$ components of $f(V) \cap \pi^{-1}(t)$ have disjoint $\rho$–images.

The total length of $B^*$ is at most

$$\underline{\beta}^{-k} s \alpha^{k'} = \underline{\beta}^{-k} c(\Theta^2\alpha)^{k'}$$

and the number of components of $B^*$ does not exceed $s = \Theta^{2k'}c$. If $n > 1$ is an integer, then there are at most $\overline{\beta}^n \underline{\beta}^{-k} c(\Theta^2\alpha)^{k'}$ arcs in $\mathcal{T}_{n'}$ which lie in $B^*$, and at most $2\Theta^{2k'}c$ arcs in $\mathcal{T}_n$, which intersect $B^*$, but are not contained in $B^*$. Since $\mathcal{T}_n$ consists of $\Theta^n$ arcs, the number of arcs in $\mathcal{T}_n$, which do not intersect $B^*$, is at least

$$\Theta^n - \overline{\beta}^n \underline{\beta}^{-k} c(\Theta^2\alpha)^{k'} - 2\Theta^{2k'}c = \Theta^n - \overline{\beta}^n \underline{\beta}^{-k_0} c(\Theta^2 \underline{\beta}^{-1}\alpha)^{k'} - 2\Theta^{2k'}c.$$

If $S$ denotes the union of these arcs, then (3.3) holds for any $t \in S^1$ and any two components $D_1, D_2$ of $f(V) \cap \pi^{-1}(t)$. Therefore, to prove the lemma it is sufficient to show that for each sufficiently large integer $n$ we can find an integer $k = k_0 + k' \geq k_0$ such that $k < \frac{n}{2}$ and

$$\overline{\beta}^n \underline{\beta}^{-k_0} c(\Theta^2 \underline{\beta}^{-1} \alpha)^{k'} + 2\Theta^{2k'} c < \Theta^{n\gamma}. \tag{3.6}$$

This last inequality holds if

$$(\Theta^2 \underline{\beta}^{-1} \alpha)^{k'} < \frac{\Theta^{n\gamma} \underline{\beta}^{k_0}}{2\overline{\beta}^n c}, \quad \Theta^{2k'} < \frac{\Theta^{n\gamma}}{4c}$$

or, since by our choice above $\alpha$ is so small that $\Theta^2 \underline{\beta}^{-1} \alpha < 1$,

$$n \frac{\gamma \log \Theta - \log \overline{\beta}}{\log(\Theta^2 \underline{\beta}^{-1} \alpha)} + \frac{\log(\underline{\beta}^{k_0}/2c)}{\log(\Theta^2 \underline{\beta}^{-1} \alpha)} < k' < n\frac{\gamma}{2} - \frac{\log(4c)}{2 \log \Theta}. \tag{3.7}$$

Once more using our choice of $\alpha$ we get

$$\frac{\gamma \log \Theta - \log \overline{\beta}}{\log(\Theta^2 \underline{\beta}^{-1} \alpha)} < \frac{\gamma}{2}.$$

Therefore, if $n$ is sufficiently large the right–hand side of (3.7) exceeds the left–hand side by more than 1, and we can find an integer $k'$ satisfying (3.7) and therefore (3.6) too. With $k = k_0 + k'$ the second inequality in (3.7) can be written as

$$k < \frac{n}{2} - \frac{n(1-\gamma)}{2} - \frac{\log(4c)}{2 \log \Theta} + k_0$$

and, since $\gamma < 1$, we get $k < \frac{n}{2}$, provided $n$ is sufficiently large. ■

**3.A.3. The sets $\mathcal{E}'_{um}$, $C'(m)$, $\Lambda'(m)$.** let $\gamma \in \mathcal{J}$ be fixed (see (3.0)). In this section we shall define for each sufficiently large integer $m$ a subset $\mathcal{E}'_m$ of $\mathcal{E}_m$ which consists of $r = r(m)$ elements, where

$$\Theta^m - \Theta^{2\gamma m} < r < \Theta^m, \; r \geq 2. \tag{3.8}$$

(Since $\gamma \in \mathcal{J}$ we have $2\gamma < 1$.) For $u = 1, 2, \ldots$ we consider the subsets $\mathcal{E}'_{um}$ of $\mathcal{E}_{um}$ which are given by

$$\mathcal{E}'_{um} = \mathcal{E}'_m \times \cdots \times \mathcal{E}'_m \; (u \text{ factors}),$$

where $\mathcal{E}_{um} = \mathcal{E}_m \times \cdots \times \mathcal{E}_m$ in the obvious sense and their limit

$$\mathcal{E}'_{\infty m} = \{\underline{e} \in \mathcal{E}_\infty | \pi_{um}(\underline{e} \in \mathcal{E}'_{um} \text{ for } u = 1, 2, \ldots\}.$$

Then, as easily seen,

$$C' = C'(m) = \bigcap_{u=1}^{\infty} \bigcup_{\underline{e} \in \mathcal{E}'_{um}} T_{\underline{e}} = \tau(\mathcal{E}_{\infty,m})$$

is a Cantor set, and $\varphi^m(C') = C'$. Morover, $\varphi^m : C' \to C'$ is a $r$–to–1 map. We define

$$V' = V'(m) = C'(m) \times \mathbb{D}^2 = \pi^{-1}(C'(m)).$$

By $\varphi^m(C') = C'$ we have $f^m(V') \subset V'$, and we define

$$\Lambda' = \Lambda'(m) = \bigcap_{u=1}^{\infty} f^{um}(V').$$

This set $\Lambda'$ is a subset of $\Lambda \cap V'$, and for each $t \in C'$

$$\Lambda'(t) = \Lambda'(m,t) = \Lambda'(m) \cap D(t)$$

is a Cantor set.

Later in this section we shall prove the following lemma.

**Lemma 3.2.** There is a positive $\delta' = \delta'(m)$ with the following property. If $t_1, t_2$ are points in $C'(m)$ such that $\varphi^m(t_1) \neq \varphi^m(t_2)$, $\varphi^{2m}(t_1) = \varphi^{2m}(t_2)$, then the subintervals $\rho f^{2m}(D(t_1))$, $\rho f^{2m}(D(t_2))$ of $\rho(D(\varphi^{2m}(t_1)) = \rho(D(\varphi^{2m}(t_2))$ are disjoint, and their distance is at least $\delta'(m)$.
This lemma implies the following Corollaries.

**Corollary 3.3.** If $t_1' \neq t_2'$ belong to $C'$ and $\varphi^m(t_1') = \varphi^m(t_2')$, then the distance between $\rho(f^m(\Lambda'(t_1'))$ and $\rho(h^m(\Lambda'(t_2'))$ is at least $\delta'$ and if $t \in C'$, then $\rho|_{\Lambda'(t')}$ is one–to–one.

Let $t = \tau(\underline{e}') \in C'(m)$ be fixed ($\underline{e}' \in \mathcal{E}_\infty$), and let for any $\underline{e} \in \mathcal{E}'_{um}$ the subinterval $I_{\underline{e}}$ of $\rho(D(t))$ be defined by

$$I_{\underline{e}} = \rho f^{um}(D(\tau(\underline{e}, \underline{e}'))). \tag{3.9}$$

Obviously

$$\rho(\Lambda'(m) \cap D(t)) = \bigcap_{u=1}^{\infty} \bigcup_{\underline{e} \in \mathcal{E}'_{um}} I_{\underline{e}}, \tag{3.10}$$

and Lemma 3.2. states that for $\underline{e}_1, \underline{e}_2 \in \mathcal{E}_{2m}$ satisfying $\sigma^m(\underline{e}_1) \neq \sigma(\underline{e}_2)$ the distance between $I_{\underline{e}_1}$ and $I_{\underline{e}_2}$ is at least $\delta'$. This implies the following corollary.

**Corollary 3.4.** If $\underline{e}_1, \underline{e}_2 \in \mathcal{E}_{um}$ $(u \geq 2)$ and if

$$\sigma^{(j-1)m}(\underline{e}_1) \neq \sigma^{(j-1)m}(\underline{e}_2), \quad \sigma^{jm}(\underline{e}_1) = \sigma^{j(m)}(\underline{e}_2)$$

holds for some $j \in \{2, \ldots, u\}$, then

$$\operatorname{dist}(I_{\underline{e}_1}, I_{\underline{e}_2}) \geq \delta' \cdot \ell(I_{\sigma^{jm}(\underline{e}_1)}),$$

where $\ell(I_{\sigma^{jm}(\underline{e}_1)})$ denotes the length of $I_{\sigma^{jm}(\underline{e}_1)}$.

DEFINITION of $\mathcal{E}'_m$. Let $\gamma' \in \mathcal{J}$ be smaller than $\gamma$, and let $m$ be so large that the Geometric Lemma 3.1. applies to $\gamma'$ and $n = 2m - 1$. Then by this lemma we get $k = k(m)$, $S = S(m)$ and $\delta = \delta(m)$. Since $\gamma' < \gamma$ the conclusion of this lemma holds also for $\gamma, n, k, S, \delta$. We consider the elements $\underline{e}_1, \dots, \underline{e}_s$ of $\mathcal{E}_n$ for which the arcs $T_{\underline{e}_1}, \dots, T_{\underline{e}_s}$ of $\mathcal{I}_n$ do not belong to $S$. The number $s$ of these arcs is bounded by

$$1 \leq s \leq \Theta^n - \Theta^{\gamma' n}.$$

We define

$$\mathcal{E}'_m = \{\underline{e} \in \mathcal{E}_m | \underline{e} \neq (\Theta - 1, \dots, \Theta - 1), \underline{e} \text{ does not appear in any } \underline{e}_i \ (i = 1, \dots, s)\}.$$

The number $r = r(m)$ of elements in $\mathcal{E}'_m$ satisfies

$$\Theta^m - ms - 1 \leq r < \Theta^m.$$

If $m$ is sufficiently large, then $ms + 1 \leq m\Theta^{\gamma'(2m-1)} + 1 \leq \Theta^{2\gamma m}$, and therefore

$$\Theta^m - \Theta^{2\gamma m} < r < \Theta.$$

The following crucial fact is easy to prove: If $\underline{e} \in \mathcal{E}'_{\infty m}$ then none of the sequences $\underline{e}_1, \dots, \underline{e}_s$ can appear in $\underline{e}$ and therefore

$$\varphi^j(C'(m)) \subset S \quad (j \geq 0). \tag{3.11}$$

PROOF OF LEMMA 3.2. Let $j$ be the minimal exponent such that

$$\varphi^j(t_1) = \varphi^j(t_2).$$

Then $m < j \leq 2m$. Since $k < m$ the points

$$t_1^* = \varphi^{j-k}(t_1), \quad t_2^* = \varphi^{j-k}(t_2)$$

are defined, and by (3.11) $t_1^*, t_2^*$ belong to $S$. If we apply the Geometric Lemma to these points we get

$$\begin{aligned}
\delta &< \mathrm{dist}(\rho f^k(D(t_1^*)),\ \rho f^k(D(t_2^*))) \\
&\leq \mathrm{dist}(\rho f^j(D(t_1)),\ \rho f^j(D(t_2))) \\
&\leq \underline{\lambda}^{-(2m-j)} \mathrm{dist}(\rho f^{2m}(D(t_1)),\ \rho f^{2m}(D(t_2))).
\end{aligned}$$

Therefore $\delta' = \delta \underline{\lambda}^m$ has the property required in the lemma.

■

**3.A.4.** PROOF OF THE MAIN PART OF LEMMA 3.A. For each integer $m$ which is sufficiently large the constructions in 3.A.3. yield sets $\mathcal{E}'_{um}$ $(u = 1, 2, \dots, \infty)$, a Cantor set $C' = C'(m)$ in $S^1$ and a set $\Lambda' = \Lambda'(m)$ in $\pi^{-1}(C'(m)) \cap \Lambda$ such that Lemma 3.2. holds for these sets. In each $C'(m)$ we fix an arbitrarily chosen point $t_m$. The corresponding sequence in $\mathcal{E}'_{\infty m}$ will be denoted by $\underline{e}'_m$, i.e. $t_m = \tau(\underline{e}'_m)$.

Now for each integer $\ell \geq 1$ we consider the piecewise constant function $\tilde{\lambda}_\ell : S^1 \to \mathbb{R}$ which on each arc $T'_{\underline{e}}$ ($\underline{e} \in \mathcal{E}_\ell$) has the constant value $\inf_{t \in T_{\underline{e}}} \lambda(t)$. (As defined in 3.A.1. $T'_{\underline{e}}$ is $T_{\underline{e}}$ minus the upper end point.)

If $\ell \geq 1$ and $m > \ell$ is so large that $C'(m)$ is defined and $t_m$, $\underline{e}'_m$ are fixed we consider for each $\underline{e} \in \mathcal{E}_m$ the sequence $(\underline{e}, \underline{e}'_m) \in \mathcal{E}_\infty$ and define

$$\begin{aligned} \mu_m(\underline{e}) = & \quad \textstyle\prod_{i=1}^m \lambda\tau(\sigma^{i-1}(\underline{e}, \underline{e}'_m)) \\ \tilde{\mu}_{m,\ell}(\underline{e}) = & \quad \textstyle\prod_{i=1}^m \tilde{\lambda}_\ell\tau(\sigma^{i-1}(\underline{e}, \underline{e}'_m)) \\ \hat{\mu}_{m,\ell}(\underline{e}) = & \quad [\textstyle\prod_{i=1}^{m-\ell} \tilde{\lambda}_\ell(\sigma^{i-1}(\underline{e}, \underline{e}'_m))] \cdot \underline{\lambda}^\ell. \end{aligned} \tag{3.12}$$

These definitions obviously imply

$$\hat{\mu}_{m,\ell}(\underline{e}) \leq \tilde{\mu}_{m,\ell}(\underline{e}) \leq \mu_m(\underline{e}), \tag{3.13}$$

and the following remark is a simple consequence of $\tilde{\lambda} \leq \lambda$ and that fact that for $\underline{e}_1, \underline{e}_2 \in \mathcal{E}_\infty$ satisfying $\pi_\ell(\underline{e}_1) = \pi_\ell(\underline{e}_2)$ we have $\tilde{\lambda}(\tau(\underline{e}_1)) = \tilde{\lambda}(\tau(\underline{e}_2))$.

**Remark 3.5** If $\underline{e}' \in \mathcal{E}_\infty$, $\quad \underline{e} = \pi_m(\underline{e}')$, $\quad t = \tau(\underline{e}')$, then

$$\hat{\mu}_{m,\ell}(\underline{e}) \leq \lambda(t) \cdot \lambda(\varphi(t)) \ldots \lambda(\varphi^{m-1}(t)).$$

The positive numbers $p(m, \ell)$, $\tilde{p}(m, \ell)$, $\hat{p}(m, \ell)$, $\hat{p}'(m, \ell)$ are defined by

$$\begin{aligned} \sum_{\underline{e} \in \mathcal{E}_m} \mu_m(\underline{e})^{p(m)} &= 1 \\ \sum_{\underline{e} \in \mathcal{E}_m} \tilde{\mu}_{m,\ell}(\underline{e})^{\tilde{p}(m,\ell)} &= 1 \\ \sum_{\underline{e} \in \mathcal{E}_m} \hat{\mu}_{m,\ell}(\underline{e})^{\hat{p}(m,\ell)} &= 1 \\ \sum_{\underline{e} \in \mathcal{E}'_m} \hat{\mu}_{m,\ell}(\underline{e})^{\hat{p}'(m,\ell)} &= 1 \end{aligned} \tag{3.14}$$

Then (3.13) and $\mathcal{E}'_m \subset \mathcal{E}_m$ imply

$$\hat{p}'(m, \ell) \leq \hat{p}(m, \ell) \leq \tilde{p}(m, \ell) \leq p(m), \tag{3.15}$$

and Lemma 2.2 implies

$$\lim_{m \to \infty} p(m) = p. \tag{3.16}$$

Since the points $t_m$ were arbitrarily chosen in the sets $C'(m)$, to verify the main part

$$\lim_{m\to\infty} \inf_{t'\in C'(m)} \dim_H \rho(\Lambda'(m)\cap D(t')) \geq p$$

of Lemma 3.A. it is sufficient to prove

$$\lim_{m\to\infty} \dim_H \rho(\Lambda'(m)\cap D(t_m)) \geq p$$

This inequality is a consequence of (3.16) and the following four lemmas which will be proved in 3.A.5. – 3.A.8.

**Lemma 3.6.** If $\ell > 0$ and if $m \geq \ell$ is so large that $C'(m), t_m$ are defined, then

$$\dim_H \rho(\Lambda'(m)\cap D(t_m)) \geq \hat{p}'(m,\ell).$$

**Lemma 3.7.** If $\ell > 0$ is fixed, then

$$\lim_{m\to\infty} (\hat{p}(m,\ell) - \hat{p}'(m,\ell)) = 0.$$

**Lemma 3.8.** If $\ell > 0$ is fixed, then

$$\lim_{m\to\infty} (\tilde{p}(m,\ell) - \hat{p}(m,\ell)) = 0.$$

**Lemma 3.9.**

$$\lim_{\ell\to\infty} \limsup_{m\to\infty} (p(m) - \tilde{p}(m,\ell)) = 0.$$

**3.A.5.** Proof of Lemma 3.6. The proof will use some elementary facts concerning the Hausdorff dimension of certain Cantor sets which are defined as follows.
Let $I$ be an interval, and let $\mathcal{I}_u$ $(u = 1,2,\dots)$ be the set of all sequences $\underline{i} = (i_1,\dots,i_u)$ of integers $i_j \in \{1,\dots,r\}$ $(r > 1$ fixed). We assume that for each $\underline{i} \in \mathcal{I}_u$ a subinterval $I_{\underline{i}} = I_{i_1,\dots,i_u}$ of $I$ is defined and that these subintervals together with positive numbers $\hat{\mu}_1,\dots,\hat{\mu}_r$ satisfying $\hat{\mu}_1 + \cdots + \hat{\mu}_r < 1$ have the following properties:

(1) $I_{i_1,\dots,i_{u+1}} \subset I_{i_1,\dots,i_u}$.
(2) $\mathrm{diam} I_{i_1,\dots,i_u} \geq \mathrm{diam} I \cdot \prod_{j=1}^u \hat{\mu}_{i_j}$.
(3) $\lim_{u\to\infty}(\max_{\underline{i}\in\mathcal{I}_u} \mathrm{diam} I_u) = 0$.
We do not require that for $\underline{i} \neq \underline{i}'$ in $\mathcal{I}_u$ the intervals $I_{\underline{i}}, I_{\underline{i}'}$, are disjoint. What we do assume is the following weaker condition.
(4) There is a positive real $\delta'$ with the following property: If for two sequences $\underline{i} = (i_1,\dots,i_u)$, $\underline{i}' = (i'_1,\dots,i'_u) \in \mathcal{I}_u$ $(u \geq 2)$ the maximal index $j$ for which $(i_1,\dots,i_j) = (i'_1,\dots,i'_j)$, is at most $u-2$, then $I_{\underline{i}} \cap I_{\underline{i}'} = \emptyset$ and

$$\mathrm{dist}(I_{\underline{i}}, I_{\underline{i}'}) \geq \delta' \cdot \mathrm{diam} I_{i_1,\dots,i_j}.$$

Under these conditions we get the Cantor set

$$C = \bigcap_{u=1}^{\infty} \bigcup_{\underline{i} \in \mathcal{I}_u} I_{\underline{i}}.$$

**Sublemma 3.10.** If the real number $\hat{p}$ is determined by $\hat{\mu}_1^{\hat{p}} + \cdots + \hat{\mu}_r^{\hat{p}} = 1$, then

$$\dim_H C \geq \hat{p}.$$

PROOF OF THE SUBLEMMA Let $J_1, \dots, J_r$ be disjoint subintervals of $I$, where $\mathrm{diam} J_i = \hat{\mu}_i \cdot \mathrm{diam} I$. For each $J_i$ the increasing affine mapping of $I$ to $J_i$ will be denoted by $\alpha_i$, and for $\underline{i} = (i_1, \dots, i_u) \in \mathcal{I}_u$ we define

$$J_{\underline{i}} = \alpha_1 \dots \alpha_u(I).$$

Then

$$C^* = \bigcap_{u=1}^{\infty} \bigcup_{\underline{i} \in \mathcal{I}_u} J_{\underline{i}}$$

is a Cantor set, and methods already used by Hausdorff to determine $\dim_H C_0$ for the classical Cantor discontinuum $C_0$, it can be shown that $\dim_H C^* = \hat{p}$. Since Lipschitz continuous mappings do not raise the Hausdorff dimension to prove the sublemma it is sufficient to find a Lipschitz continuousmapping $h$ of $C$ onto $C^*$.

For each point $c \in C$ there is a unique infinite sequence $i_1, i_2, \dots$ such that $c \in I_{i_1,\dots,i_u}$ $(u = 1, 2, \dots)$. Then we define $h(c)$ to be the unique point in $\bigcap_{u=1}^{\infty} J_{i_1,\dots,i_u}$. If $c \neq c'$ are points in $C$ and if $u$ is the maximal index for which $h(c)$, $h(c')$ lie in the same interval $J_{i_1,\dots,i_u}$, then

$$\mathrm{dist}(h(c), h(c')) \leq \mathrm{diam}\, I \cdot \prod_{j=1}^{u} \hat{\mu}_{i_j}$$

and by (d)

$$\mathrm{dist}(c, c') \geq \delta' \cdot \mathrm{diam}\, I_{i_1,\dots,i_u} \geq \delta' \cdot \mathrm{diam} \cdot \prod_{j=1}^{u} \mu_{i_j}.$$

Therefore $\delta'^{-1}$ is a Lipschitz number for $h$.

∎

PROOF OF LEMMA 3.6. Let $\underline{e}_1, \dots, \underline{e}_r$ be the elements in $\mathcal{E}'_m$, and let for $\underline{i} = (i_1, \dots, i_u) \in \{1, \dots, r\}^u$ and $\underline{e} = (\underline{e}_{i_1}, \dots, \underline{e}_{i_u}) \in \mathcal{E}'_{um}$ the interval $I_{\underline{e}}$ in (3.9) with $t = t_m$, $\underline{e}' = \underline{e}'_m$ be denoted by $I_{\underline{i}}$. Then it is easy to see that these intervals have the properties (1) and (3) mentioned above and that (2) holds with $\hat{\mu}_{\underline{i}} = \hat{\mu}_{m,t}(\underline{e}_{i_1}, \dots, \underline{e}_{i_u})$ as defined in (3.12). Condition (4) is a consequence of Corollary 3.4. Then, together with (3.10) and the fourth equation in (3.14) the sublemma implies Lemma 3.6.

■

**3.A.6.** PROOF OF LEMMA 3.7. Since $\hat{p}(m,\ell) \geq \hat{p}'(m,\ell)$ (see (3.15)), it is sufficient to show that the assumption

$$\limsup_{m\to\infty}(\hat{p}(m,\ell) - \hat{p}'(m,\ell) > 0 \tag{3.17}$$

leads to a contradiction. By Lemma 2.1. this assumption implies

$$\liminf_{m\to\infty} \sum_{\underline{e}\in\mathcal{E}'_m} \hat{\mu}(\underline{e})^{\hat{p}(m,\ell)} = 0$$

and therefore by (3.14)

$$\limsup_{m\to\infty} \sum_{\underline{e}\in\mathcal{E}_m\setminus\mathcal{E}'_m} \hat{\mu}(\underline{e})^{\hat{p}(m,\ell)} = 1.$$

Since by (3.8) the set $\mathcal{E}_m\setminus\mathcal{E}'_m$ contains at most $\Theta^{2m\gamma}$ elements and $\hat{\mu}(\underline{e}) \leq \overline{\lambda}^m$, we get

$$\limsup_{m\to\infty} \Theta^{2m\gamma}\overline{\lambda}^{m\hat{p}(m,\ell)} \geq 1$$

$$\liminf_{m\to\infty} \hat{p}(m,\ell) \leq 2\gamma\frac{\log\Theta}{-\log\overline{\lambda}}.$$

This together with the fact that $\gamma$ was chosen in the interval $\mathcal{J}$ of (3.0) and therefore less than $\log\overline{\lambda}/(2\log\underline{\lambda})$ implies

$$\liminf_{m\to\infty} \hat{p}(m,\ell) < \frac{\log\Theta}{-\log\underline{\lambda}}. \tag{3.18}$$

On the other hand, since $\mathcal{E}'_m$ contains at least $\Theta^m - \Theta^{2m\gamma}$ elements (see (3.8)) and since $\hat{\mu}(\underline{e}) \geq \underline{\lambda}^m$ we get by the definition (3.14) of $\hat{p}'(m,\ell)$

$$(\Theta^m - \Theta^{2m\gamma})\underline{\lambda}^{m\hat{p}'(m,\ell)} \leq 1,$$

$$\begin{aligned}\hat{p}'(m,\ell) &\geq \frac{\log(\Theta^m - \theta^{2m\gamma})}{-m\log\underline{\lambda}} \\ &= \frac{\log\Theta}{-\log\underline{\lambda}} + \frac{\log(1-\Theta^{m(2\gamma-1)})}{-m\log\underline{\lambda}},\end{aligned}$$

$$\liminf_{m\to\infty} \hat{p}'(m,\ell) \geq \frac{\log\Theta}{-\log\underline{\lambda}}. \tag{3.19}$$

Looking at (3.19) and $\hat{p}(m,\ell) \geq \hat{p}'(m,\ell)$ we see the contradiction to (3.18).

■

**3.A.7.** PROOF OF LEMMA 3.8. Since $\tilde{\mu} \leq \bar{\lambda}^m$ and since $\mathcal{E}_m$ consists of $\Theta^m$ elements,

$$\Theta^m \overline{\lambda}^{m\tilde{p}(m,\ell)} \geq 1$$

$$\tilde{p}(m,\ell) \leq \frac{\log\Theta}{-\log\overline{\lambda}}. \tag{3.20}$$

Moreover (3.12) implies fo each $\underline{e} \in \mathcal{E}_m$

$$\tilde{\mu}(\underline{e}) \leq (\overline{\lambda}/\underline{\lambda})^\ell \hat{\mu}(\underline{e})$$

and by (3.14) we get

$$1 = \sum_{\underline{e}\in\mathcal{E}_m} \tilde{\mu}(\underline{e})^{\tilde{p}(m,\ell)} \leq (\overline{\lambda}/\underline{\lambda})^{\tilde{p}(m,\ell)\cdot\ell} \sum_{\underline{e}\in\mathcal{E}_m} \hat{\mu}(\underline{e})^{\tilde{p}(m,\ell)}.$$

By (3.20) the factor $(\frac{\overline{\lambda}}{\underline{\lambda}})^{\tilde{p}(m,\ell)\cdot\ell}$ is bounded with respect to $m$ ($\ell$ fixed), and therefore

$$\liminf_{m\to\infty} \sum_{\underline{e}\in\mathcal{E}_m} \hat{\mu}(\underline{e})^{\tilde{p}(m,\ell)} > 0.$$

Now a straight forward application of Lemma 2.1. proves Lemma 3.8.

■

**3.A.8.**Proof of Lemma 3.9. If the lemma would be false we could find a positive $\varepsilon$ and arbitrarily large integers $\ell$ for each of which there are integers $m \geq \ell$ such that

$$p(m) - \tilde{p}(m,\ell) \geq \varepsilon. \tag{3.21}$$

To lead this assumption to a contradiction we define for $\ell \geq 1$

$$\eta_\ell = \sup_{t\in S^1} \frac{\lambda(t)}{\tilde{\lambda}_\ell(t)}.$$

Then for $m \geq \ell$ satisfying (3.21) we have

$$\begin{aligned} 1 = \sum_{\underline{e}\in\mathcal{E}_m} \mu(\underline{e})^{p(m)} &\leq \eta_\ell^{mp(m)} \sum_{\underline{e}\in\mathcal{E}_m} \tilde{\mu}(\underline{e})^{p(m)} \\ &\leq \eta_\ell^{mp(m)} \sum_{\underline{e}\in\mathcal{E}_m} \tilde{\mu}(\underline{e})^{\tilde{p}(m,\ell)+\varepsilon} \\ &\leq \eta_\ell^{mp(m)} \overline{\lambda}^{m\varepsilon} \\ &= (\eta_\ell^{p(m)} \overline{\lambda}^{\varepsilon})^m. \end{aligned}$$

Since $\lambda$ is continuous we have $\lim_{\ell\to\infty} \eta_\ell = 1$. This together with $\overline{\lambda}^\varepsilon < 1$ and $p(m) \leq -\log\Theta/\log\overline{\lambda}$ shows that the last inequality is impossible for large $\ell$.

■

**3.B. Proof of Lemma 3.B.**

Let $m$, $t' \in C'(m)$, $t \in S^1$ and $B$ be as in this lemma. By the proof of Lemma 3.A. we know that $\varphi^m(C'(m)) = C'(m)$, and by Corollary 3.3. there is a positive $\delta_1$ such that for any $t^* \in C'(m)$ and any two points $x, y \in \Lambda'(t^*)$ satisfying $\pi(f^{-m}(x)) \neq \pi(f^{-m}(y))$ we have

$$d(\rho(W^u_{\delta_1}(x)), \quad \rho(W^u_{\delta_1}(y))) \geq \delta_1, \tag{3.22}$$

where $W^u_{\delta_1}(x)$ is the arc in $\Lambda$ containing $x$ whose projection $\pi(W^u_{\delta_1}(x))$ is the closed $\delta_1$–neighbourhood of $\pi(x)$ in $S^1$, and $W^u_{\delta_1}(y)$ is defined in the same way. (We assume $\delta_1 < \frac{1}{2}$.)

Let $k_1$ be a positive integer which is so large that each of the $\Theta^{k_1 m}$ components of $\varphi^{-k_1 m}(B)$ has length at most $\delta_1$. Then, if $t''_1, \ldots, t''_r$ are the points in $\varphi^{-k_1 m}(t') \cap C'(m)$, we define

$$E_i = \rho(f^{k_1 m}(\Lambda'(t''_i))) \quad (i = 1, \ldots, r).$$

Since by Lemma 3.A. the restriction of $\rho$ to $\Lambda'(t')$ is one–to–one, $E_1, \ldots, E_r$ is a partition of $\rho(\Lambda'(t'))$ in disjoint subsets.

Now we choose two different points $x', y' \in \rho(\Lambda'(t'))$ which belong to the same set $E_i$. There are unique points $\tilde{x}', \tilde{y}' \in \Lambda'(t')$ such that $\rho(\tilde{x}') = x'$, $\rho(\tilde{y}') = y'$. Besides these four points we consider

$$\begin{aligned}
&\tilde{x} = \tilde{h}(\tilde{x}'), && \tilde{y} = \tilde{h}(\tilde{y}'),\\
&x = \rho(\tilde{x}) = h(x'), && y = \rho(\tilde{y}) = h(y'),\\
&\tilde{x}'_j = f^{-jm}(\tilde{x}'), && \tilde{y}'_j = f^{-jm}(\tilde{y}'),\\
&\tilde{x}_j = f^{-jm}(\tilde{x}), && \tilde{y}_j = f^{-jm}(\tilde{y}), \quad (j = 0, 1, 2, \ldots)\\
&t'_j = \pi(\tilde{x}'_j) = \pi(\tilde{y}'_j), && t_j = \pi(\tilde{x}_j) = \pi(\tilde{y}_j).
\end{aligned}$$

We shall show that

$$d(x', y') \leq \frac{2}{\delta_1} \prod_{j=1}^{\infty} (1 + \underline{\lambda}^{-1} l \underline{\beta}^{-mj}) d(x, y)$$

where $l$ is a Lipschitz constant of $\lambda$. (Since $\underline{\beta} > 1$ the product is convergent.) This will prove that $h|_{E_i}$ is one–to–one with Lipschitz continuous inverse.

Let $k$ be the maximal integer for which $\pi f^{-km}(\tilde{x}') = \pi f^{-km}(\tilde{y}')$. Since $x', y'$ belong to the same set $E_i$ we have $k \geq k_1$, and this implies $d(t'_k, t_k) \leq \delta_1$, $\tilde{x}_k \in W^u_{\delta_1}(\tilde{x}'_k)$, $\tilde{y}_k \in W^u_{\delta_1}(\tilde{y}'_k)$. Then by (3.22)

$$d(x_k, y_k) = d(\rho(\tilde{x}_k), \rho(\tilde{y}_k)) \geq \delta_1,$$

and

$$d(x,y) \geq \delta_1 \prod_{j=1}^{k} \lambda(t_j).$$

Similarly $d(x_k', y_k') \leq 2$ implies

$$d(x', y') \leq 2 \prod_{j=1}^{k} \lambda(t_j'),$$

$$d(x', y') \leq \frac{2}{\delta_1} \prod_{j=1}^{k} \frac{\lambda(t_j')}{\lambda(t_j)} d(x,y).$$

Obviously $d(t_j', t_j) \leq \underline{\beta}^{-mj} d(t', t) \leq \underline{\beta}^{-mj}$, and we get

$$\begin{aligned} \lambda(t_j') &\leq \lambda(t_j) + l\underline{\beta}^{-mj} \\ d(x', y') &\leq \tfrac{2}{\delta_1} \prod_{j=1}^{k} (1 + \lambda(t_j)^{-1} l\underline{\beta}^{-mj}) d(x,y) \\ &\leq \tfrac{2}{\delta_1} \prod_{j=1}^{\infty} (1 + \underline{\lambda}^{-1} l\underline{\beta}^{-mj}) d(x,y). \end{aligned}$$

■

## 4. Proof of Theorem B and Theorem C

Here we prove Theorem B. How this proof covers Theorem C as well will be indicated at the end of the section. As in the preceding section we assume that an index $i \in \{1, 2\}$ is fixed and write $\rho_i = \rho$, $\mathcal{F}_i' = \mathcal{F}'$, $\mathcal{F}_i^\times = \mathcal{F}^\times$. The natural projection of the annulus $A = S^1 \times \mathbb{I}$ to $S^1$ will be denoted by $\sigma : A \to S^1$.
Let $\mathcal{G}$ be the space of all $C^0$ mappings $g : A \to A$ which can be written as

$$g(t, x) = (\psi(t), \kappa(t)x + v(t)),$$

where $\psi = \psi_g : S^1 \to S^1$, $\kappa = \kappa_g : S^1 \to (0,1)$, $v = v_g : S^1 \to (-1,1)$ for some finite decomposition of $S^1$ in arcs $B_1, \ldots, B_m$ are $C^1$ on each $B_i$, and $\dot\psi = \frac{d\psi}{dt} > 1$ holds on each $B_i$. We assume that $B_i \cap B_{i+1}$ consists of a common end point of these arcs which will be denoted by $t_i$ while $B_i \cap B_j = \emptyset$ for $|i - j| > 1$. (Indices are counted modulo $m$.) If $\Theta = \Theta_g$ is the mapping degree of $\psi$, then the decomposition can, and will, be chosen so that $m > \Theta$, $\Theta$ is a divisor of $m$ and $\psi(B_i) = B_{(i-1)\Theta+1} \cup \cdots \cup B_{i\Theta}$. These partitions will be called Markov partitions of $S^1$. Since the mappings in $\mathcal{G}$ are piecewise $C^1$ we have the $C^1$ topology in $\mathcal{G}$, and the natural coordinates in $A = S^1 \times \mathbb{I}$ ($\mathbb{I} = [-1,1]$) define a $C^1$ distance in $\mathcal{G}$. For each point $t_i$ the functions $\psi, \kappa, v$ have two derivatives (a left one and a right one) which will be denoted by $\dot\psi^-, \dot\psi^+, \dot\kappa^-, \dot\kappa^+, \dot v^-, \dot v^+$, respectively. The value

$$\delta = \delta_g = \max_{1\le i\le m} \quad \max(|\dot\psi^+(t_i) - \dot\psi^-(t_i)|, \ |\dot\kappa^+(t_i) - \dot\kappa^-(t_i)|, \ |\dot v^+(t_i) - \dot v^-(t_i)|) \tag{4.1}$$

measures to what extent $g$ differs from a $C^1$ mapping. In particular, $g$ is of class $C^1$ if and only if $\delta = 0$. The space of all $C^1$ mappings in $\mathcal{G}$ will be denoted by $\mathcal{G}^1$.

The projection $\rho : V \to A$ defines a continuous projection $\mathcal{F} \to \mathcal{G}^1$. The image of $\mathcal{F}'$ is

$$\mathcal{G}' = \{g \in \mathcal{G}^1 | \sup \kappa_g < \Theta_g^{-2}\}. \tag{4.2}$$

The set

$$\Xi = \Xi_g = \bigcap_{k=0}^{\infty} g^k(A)$$

is the attractor of $g$. If $g \in \mathcal{G}^1$ is the projection of $f \in \mathcal{F}$, then $\Xi_g = \rho(\Lambda_f)$, where $\Lambda_f$ is the attractor of $f$ in $V$.
To characterize those arcs in $\Xi$ which are related to the dynamics in $A$ we consider sequences $B_0^*, B_1^*, \ldots$ of arcs in $S^1$ satisfying $\psi(B_i^*) = B_{i-1}^* \quad (i = 1, 2, \ldots)$. Then for each of these sequences the set

$$B = \bigcap_{k=0}^{\infty} g^k(B_k^* \times \mathbb{I})$$

is an arc in $\Xi$ which together with the defining sequence $B_i^*$ will be called an *admissible arc* in $\Xi$. (It may happen that two admissible arcs differ only in their defining sequences!)
If $g \in \mathcal{G}$ is the projection of $f \in \mathcal{F}$, then each arc $B$ in the attractor $\Lambda_f$ of $f$ which is short in the sense that $\pi(B) \neq S^1$ by $B' = \rho(B), \quad B_i^* = \pi(f^{-i}(B))$ defines an admissible arc in $\Xi$, and all admissible arcs in $\Xi$ can be obtained in this way. If $g \in \mathcal{G}^1$, then all admissible arcs in $\Xi$ are of class $C^1$, and $\Xi_g$ will be called intrinsically transverse (abbreviated i. tr.) if any two different admissible arcs $B, B'$ with defining sequences $B_i^*, B_i'^*$ satisfying $B_0'^* = B_0^*$ are transverse at any point $p \in B \cap B'$. The set of all $g \in \mathcal{G}^1$ with i.tr. $\Xi_g$ will be denoted by $\mathcal{G}^\times$.

**Remark 4.3.** As easily seen, $\Xi_g$ is i.tr. provided any two different admissible arcs $B, B'$ with $B_0^* = B_0'^*, \quad B_1^* \neq B_1'^*$ are transverse.

If $g \in \mathcal{G}^1$ is the projection of $f \in \mathcal{F}$, then $\Xi$ is i. tr. if and only if $\Lambda_f$ has this property; i.e. the projection $\mathcal{F} \to \mathcal{G}^1$ maps $\mathcal{F}^\times$ to $\mathcal{G}^\times$. Therefore to prove Theorem

B it is sufficient to prove the following proposition.

**Proposition 4.1.** The set $\mathcal{G}^{\times}$ is dense in $\mathcal{G}'$.

In the proof piecewise projective mappings in $\mathcal{G}$ will play a crucial role.

**Definition 4.2.** A mapping $g \in \mathcal{G}$ is called piecewise projective if there is a Markov partition $B_1, \ldots, B_m$ of $S^1$ such that the restriction of $g$ to any rectangle $Q_i = B_i \times \mathbb{I}$ is a projective mapping to a quadrangle in the rectangle $Q_i' = \psi_g(B_i) \times \mathbb{I}$. (Here the sets $Q_i, Q_i'$ are regarded, in the obvious way, as subsets of the plane $\mathbb{R}^2$.)
If $g \in \mathcal{G}$ is piecewise projective, then each admissible arc $B$ in $\Xi_g$ with $\sigma(B) = \psi_g(B_i)$ $(1 \le i \le m)$ is a straight segment in the rectangle $Q_i'$.
For $g \in \mathcal{G}$ and a fixed corresponding Markov partition $B_1, \ldots, B_m$ with partitioning points $t_1, \ldots, t_r$ such that $B_i = [t_{i-1}, t_i]$ we denote $\kappa_g(t_i)$ by $\kappa_i$ and consider the set

$$\begin{aligned} Z = Z_g &= \{(u_1, \ldots, u_m) \in \mathbb{R}^m | [u_i - \kappa_i, \ \ u_i + \kappa_i] \in \mathbb{I} \ \ (i = 1, \ldots, m)\} \\ &= \times_{i=1}^{m} [\kappa_i - 1, 1 - \kappa_i]. \end{aligned} \tag{4.3}$$

For each $u = (u_1, \ldots, u_m) \in Z$ there is a unique mapping $g_u \in \mathcal{G}$ which is piecewise projective with respect to our Markov partition and which satisfies

$$g_u(t_i, x) = (\psi(t_i), \kappa_i x + u_i) \quad (i = 1, \ldots, m). \tag{4.4}$$

We shall use homogeneous coordinates in $Q_i = B_i \times \mathbb{I}$ and in $Q_i' = \psi(B_i) \times \mathbb{I} = [t_{\Theta(i-1)}, t_{\Theta i}] \times \mathbb{I}$ which are determined by

$$(t_{i-1}, 0) \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, (t_i, 0) \sim \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, (t_{i-1}, 1) \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, (t_i, 1) \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

in $Q_i$ and by

$$(t_{\Theta(i-1)}, 0) \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, (t_{\Theta i}, 0) \sim \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$(t_{\Theta(i-1)}, 1) \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, (t_{\Theta i}, 1) \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

in $Q_i'$. With respect to these coordinates the projective mapping $g_u|_{Q_i}$ correspondents to the matrix

$$\begin{pmatrix} \kappa_i & 0 & 0 \\ \kappa_i u_{i+1} - \kappa_{i+1} u_i & \kappa_i \kappa_{i+1} & \kappa_{i+1} u_i \\ \kappa_i - \kappa_{i+1} & 0 & \kappa_{i+1} \end{pmatrix}.$$

Using some elementary facts from projective geometry we find that $g_u$ can be written as

$$g_u(t,x) = (\psi_u(t), \kappa_u(t)x + v_u(t)),$$

where for $t = t_i + s(t_{i+1} - t_i) \in B_i$ and $t'_i = \psi_u(t_i) = \psi_g(t_i) = t_{\Theta i}, \quad t'_{i+1} = \psi_u(t_{i+1}) = \psi_g(t_{i+1}) = t_{\Theta(i+1)}$ the mappings $\psi_u, \kappa_u, v_u$ are given by

$$\begin{aligned}
\psi_u(t) = &\ t'_i + s(t'_{i+1} - t'_i)\tfrac{\kappa_i}{\kappa_{i+1}+s(\kappa_i-\kappa_{i+1})} \\
\kappa_u(t) = &\ \tfrac{\kappa_i\kappa_{i+1}}{\kappa_{i+1}+s(\kappa_i-\kappa_{i+1})} \\
v_u(t) = &\ \tfrac{\kappa_{i+1}u_i+s(\kappa_i u_{i+1}-\kappa_{i+1}u_i)}{\kappa_{i+1}+s(\kappa_i-\kappa_{i+1})} \\
\kappa_u(t) = &\ \kappa_i + \tfrac{\psi_u(t)-t'_i}{t'_{i+1}-t'_i}(\kappa_{i+1} - \kappa_i) \\
v_u(t) = &\ u_i + \tfrac{\psi_u(t)-t'_i}{t'_{i+1}-t'_i}(u_{i+1} - u_i).
\end{aligned}$$

**Remark 4.4.** The first equation together with $\dot{\psi}_g > 1$ and $\psi_u(B_i) = \psi_g(B_i)$ implies $\dot{\psi}_u > 1$ and hence $g_u \in \mathcal{G}$.

**Remark 4.5.** The mappings $\psi_u$ and $\kappa_u$ do not depend on $u$, and if $g \in \mathcal{G}'$, then

$$\sup \kappa_u \leq \sup \kappa_g < \Theta^{-2}.$$

**Remark 4.6.** If $t \in S^1$ is fixed, then $v_u(t)$ is a linear function of $u$ and

$$\left|\frac{\partial v_u(t)}{\partial u_j}\right| \leq 1. \quad (1 \leq j \leq m).$$

In $Z_g$ we consider open subsets

$$Z^*_{g,1} \subset Z^*_{g,2} \subset \dots, \quad Z^*_g = \bigcup_{k=1}^{\infty} Z^*_{g,k}, \tag{4.5}$$

where $Z^*_{g,k}$ consists of all $u \in Z$ satisfying

$$g_u(g^k_u(A) \cap \sigma^{-1}(t_i)) \cap g_u(g^k_u(A) \cap \sigma^{-1}(t_{i'})) = \emptyset,$$

where $1 \leq i < i' \leq m, \psi_u(t_i) = \psi_u(t_{i'})$. This definition implies that for any $u \in Z^*_g$ the attractor $\Xi_u$ of $g_u$ is, in a certain sense, intrinsically transverse. The following remark states this fact precisely.

**Remark 4.7.** For each $u \in Z^*_{g,k}$ there is a positive $\alpha = \alpha_u$ with the following two properties:

(1) If $1 \le i < i' \le m$ and $\psi_u(t_i) = \psi_u(t_{i'})$ then the distance between the two sets

$$g_u(g_u^k(A) \cap \sigma^{-1}([t_i - \alpha, t_i + \alpha])), \ g_u(g_u^k(A) \cap \sigma^{-1}([t_{i'} - \alpha, t_{i'} + \alpha]))$$

is at least $\alpha$.

(2) If $B, B'$ are admissible arcs of $\Xi_u$ such that $\sigma(B) = B_i,$; $\sigma(B') = B_{i'}$ $(1 \le i < i' \le m)$, $\sigma(g_u(B)) = \sigma(g_u(B')) = \psi_u(B_i) = \psi_u(B_{i'})$, then either

$$d(g_u(B), g_u(B')) \ge \alpha$$

or $g_u(B) \cap g_u(B')$ consists of a point and the angle between the straight segments $g_u(B), g_u(B')$ is a least $\alpha$.

We shall prove the following three lemmas:

**Lemma 4.8.** If $g \in \mathcal{G}'$ and $\varepsilon > 0$ are given, then there is a corresponding Markov partition of $S^1$ and an element $u^*$ of $Z_g$ such that the $C^1$ distance between $g$ and $g_{u^*}$ is at most $\varepsilon$.

**Lemma 4.9.** For any $g \in \mathcal{G}'$ and any corresponding Markov partition of $S^1$ the set $Z_g^*$ is dense in $Z_g$ (see (4.3), (4.5))

**Lemma 4.10.** For any $u \in Z_g^*$ there is a mapping $g^* \in \mathcal{G}^\times$ for which the $C^1$ distance between $g_u$ and $g^*$ is at most $\delta_{g_u}$ (see (4.1)).

Using these lemmas Proposition 4.1. is easily proved: Starting with a mapping $g \in \mathcal{G}'$ and $\varepsilon > 0$ we find (by Lemma 4.8.) a Markov partition of $S^1$ and an element $u^* \in Z_g$ such that $g_{u^*}$ is $\varepsilon$–close to $g$. Then applying Lemma 4.9. we get an element $u \in Z_g^*$ such that $g_u$ is $\varepsilon$–close to $g_{u^*}$ and hence $2\varepsilon$–close to $g$. Since $g_u$ is $2\varepsilon$–close to the $C^1$ mapping $g$ we have $\delta_{g_u} \le 4\varepsilon$, and by Lemma 4.10. there is a mapping $g^* \in \mathcal{G}^\times$ which is $4\varepsilon$–close to $g_u$ and, therefore, $6\varepsilon$–close to $g$.

The proof of Lemma 4.8. is so easy that it can be omitted.

Proof of Lemma 4.9. Let $g \in \mathcal{G}'$ and a corresponding Markov partition $B_1, \dots, B_m$ with partitioning points $t_1, \dots, t_m$ be fixed. Since $\psi_u$ $(u \in Z_g)$ is independent of $u$ (see Remark 4.5.) for each $t \in S^1$ the point set $\psi_u^{-1}(t)$ also does not depend on $u$, and we denote its points by $0 \le t(1) < t(2) < \cdots < t(\Theta) < 1$. Moreover for $\underline{j} = (j_1, \dots, j_k) \in \{1, \dots, \Theta\}^k$ and $0 \le \ell < k$ we shall write $\underline{j}^\ell = (j_1, \dots, j_{k-\ell})$ and

$$t(\underline{j}) = t(\dots((t(j_1))(j_2))\dots)(j_k).$$

Then $\psi_u^\ell(t(\underline{j})) = t(\underline{j}^\ell)$ and

$$\psi_u^{-k}(t) = \{t(\underline{j}) | \ \underline{j} \in \{1, \dots, \Theta\}^k\}.$$

For each point $t_i$ $(1 \le i \le m)$ and $k > 0$ the set $f^k(S^1 \times \{0\}) \cap \sigma^{-1}(t_i)$ consists of the $\Theta^k$ points $(t_i, x_i(\underline{j}))$ where $\underline{j} \in \{1, \ldots, m\}^k$ and $x_i(\underline{j})$ is given by

$$x_i(\underline{j}) = \kappa_u(t_i(\underline{j}^{k-1}))[\kappa_u(t_i(\underline{j}^{k-2}))[\ldots[\kappa_u(t_i(\underline{j}^1))[\kappa_u(t_i(\underline{j}))v_u(t_i(\underline{j}))]+$$
$$+v_u(t_i(\underline{j}^1))]\ldots] + v_u(t_i(\underline{j}^{k-2}))] + v_u(t_i(\underline{j}^{k-1})). \tag{4.6}$$

By Remark 4.6. each value $v_u(t_i(\underline{j}^\ell))$ is a linear function of $u$, and so for $i, \underline{j}$ fixed $u \mapsto x_i(\underline{j})$ defines a linear functional

$$x_i(\underline{j}) : \mathbb{R}^m \to \mathbb{R}.$$

We are interested in the functionals

$$L(i, \underline{j}, \underline{j}') = x_i(\underline{j}) - x_i(\underline{j}') : \mathbb{R}^m \to \mathbb{R} \quad (i = 1, \ldots, m; \quad \underline{j}, \underline{j}' \in \{1, \ldots, \Theta\}^k).$$

Let $\mathcal{L}_k$ be the set of all $L(i\Theta, \underline{j}, \underline{j}')$ $(i = 1, \ldots, m/\Theta; \underline{j}, \underline{j}' \in \{1, \ldots, \Theta\}^k)$ for which $j_1 = \underline{j}^{k-1} \neq j_1' = \underline{j}'^{k-1}$. If $u \in Z_g$ satisfies for each $L \in \mathcal{L}_k$ the inequality

$$|L(u)| > 2(\sup \kappa_u)^k,$$

then $u \in Z^*_{g,k}$.
Each of the opposite inequalities

$$|L(u)| \le 2(\sup \kappa_u)^k$$

defines in $\mathbb{R}^m$ the complement of the $\delta_L$–neighbourhood $S_L$ of a hyperplane, where

$$\delta_L = 2(\sup \kappa_u)^k / |\nabla L|$$

($\nabla L$ the gradient of $L$), and

$$Z^*_{g,k} \supset Z_g \setminus \bigcup_{L \in \mathcal{L}_k} S_L. \tag{4.7}$$

Let $L = L(i\Theta, \underline{j}, \underline{j}') \in \mathcal{L}_k$. Since $j_1 = \underline{j}^{k-1} \neq j_1' = \underline{j}'^{k-1}$ we have for $t_{i\Theta}(j_1) = t_\ell, t_{i\Theta}(j_1') = t_{\ell'}$

$$v_u(t_{i\Theta}(j_1)) = u_\ell, \;\; v_u(t_{i\Theta}(j_1')) = u_{\ell'}, \;\; \ell \neq \ell',$$

and therefore

$$\frac{\partial v_u(t_{i\Theta}(\underline{j}^{k-1}))}{\partial u_\ell} = 1, \;\; \frac{\partial v_u(t_{i\Theta}(\underline{j}'^{k-1}))}{\partial u_\ell} = 0.$$

By (4.6), Remark 4.5. and Remark 4.6.

$$\tfrac{\partial L}{\partial u_\ell} \ge 1 - 2(\sup \kappa_u + (\sup \kappa_u)^2 + \ldots)$$

$$\ge 1 - \tfrac{2 \sup \kappa_u}{1 - \sup \kappa_u} = \tfrac{1 - 3 \sup \kappa_u}{1 - \sup \kappa_u}.$$

Then our assumptions $\sup \kappa_u \le \sup \kappa_g < \Theta^{-2}$ and $\Theta \ge 2$ show that the right hand side is positive, and we get

$$|\triangle L| \geq \frac{1-3\sup\kappa_u}{1-\sup\kappa_u}, \quad \delta_L \leq 2(\sup\kappa_u)^k \frac{1-\sup\kappa_u}{1-3\sup\kappa_u}.$$

Since $Z_g$ is bounded there is a real $K$ such that for each $L \in \mathcal{L}_k$ the volume of $S_L \cap Z_g$ is at most $K \cdot (\sup\kappa_u)^k$. The set $\mathcal{L}_k$ contains less than $m\Theta^{2k}$ elements, and if we use once more $\sup\kappa_u < \Theta^{-2}$, we see that for $k \to \infty$ the volume of the set

$$Z_g \cap \bigcup_{L\in\mathcal{L}_k} S_L$$

tends to 0, and the volume of (4.7) converges to the volume of $Z_g$. This implies that $Z_g^*$ is dense in $Z_g$.

■

PROOF OF LEMMA 4.9. We assume $u \in Z_{g,k}^*$. Let $B_i = [t_{i-1}, t_i]$ be the arcs of the corresponding Markov partition. Then for each positive $\eta$ which is smaller than $\delta_{g_u}$ and smaller than half the minimal length of the arcs $B_i$ we define $C^1$ functions $\psi_\eta : S^1 \to S^1, \quad \kappa_\eta : S^1 \to (0,1), \quad v_\eta : S^1 \to \mathbb{I}$ which have the following properties:

(1) $\psi_\eta(t_i) = \psi_{g_u}(t_i) = t_{i\Theta} \quad (i = 0, \ldots, m-1)$, and $\dot{\psi}_\eta > 1$.
(2) $|v_\eta + \kappa_\eta| < 1$
(3) $\psi_\eta = \psi_{g_u}, \quad \kappa_\eta = \kappa_{g_u}, \quad v_\eta = v_{g_u}$ on each arc $[t_i+\eta, t_{i+1}-\eta] \quad (i = 0, \ldots, m-1)$.
(4) $|\psi_\eta - \psi_{g_u}| < \eta, \quad |\kappa_\eta - \kappa_{g_u}| < \eta, \quad |v_\eta - v_{g_u}| < \eta$
$|\dot{\psi}_\eta - \dot{\psi}_{g_u}| < \delta_{g_u}, \quad |\dot{\kappa}_\eta - \dot{\kappa}_{g_u}| < \delta_{g_u}, \quad |\dot{v}_\eta - \dot{v}_{g_u}| < \delta_{g_u}$
on each arc $[t_i - \eta, \ t_i + \eta] \quad (i = 1, \ldots, m)$.

To these functions there corresponds a mapping $g_\eta \in \mathcal{G}^1$ such that $\psi_\eta = \psi_{g_\eta}, \quad \kappa_\eta = \kappa_{g_\eta}, \quad v_\eta = v_{g_\eta}$. By $\eta < \delta_{g_u}$ the $C^1$ distance between $g-u$ and $g_\eta$ is less than $\delta_{g_u}$, and to prove the lemma it is sufficient to show that for $\eta$ sufficiently small $g_\eta$ belongs to $\mathcal{G}^\times$. using Remark 4.7. this can be done by a straightforward argument.

■

It remains to show how this proof can be modified to a proof of Theorem C. To this aim we consider the subset $\mathcal{K}$ of $\mathcal{G}$ which consists of all those $g \in \mathcal{G}$ for which $\dot{\psi}_g$ and $\kappa_g$ are constant and define $\mathcal{K}' = \mathcal{K} \cap \mathcal{G}', \quad \mathcal{K}^\times = \mathcal{K} \cap \mathcal{G}^\times$. Then it is sufficient to prove Proposition 4.1. with $\mathcal{G}^1, \mathcal{G}^\times$ replaced by $\mathcal{K}^1, \mathcal{K}^\times$, respectively. Since for $g \in \mathcal{K}^1$ the mapping $g_u$ belongs to $\mathcal{K}$ the lemmas 4.8., 4.9., 4.10. hold with $\mathcal{K}', \mathcal{K}^\times$ instead of $\mathcal{G}', \mathcal{G}^\times$, and the proof of Theorem C is complete.

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