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# Two notes on continuous modelling of porous media


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**A note on two-component model for Terzaghi Gedankenexperiment**


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# A note on objectivity of momentum sources in porous materials

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## Summary

The note is devoted to the analysis of contributions of relative accelerations to partial momentum balance equations of multicomponent models of porous materials. We show that such contributions violate the principle of material objectivity. Even if we ignore this principle contributions of relative accelerations are either undistinguishable from other contributions or yield unacceptable modes of propagation of sound waves or both. Consequently we conclude that such contributions should be ignored completely in the construction of macroscopical models of porous materials.


# 1 Introduction

In his classical work on twocomponent macroscopical models of porous materials M.A. Biot [1] considered interactions of components described by a relative acceleration. This type of contributions are still being used in various applications of Biot's model.

We argue in this note that such models violate the material objectivity under rather natural assumptions concerning transformation properties of a continuous model. In addition they do not seem to reflect any essential microscopical properties which should be transferred on the macroscopical level of description. In contrast to theories of rarified gases (e.g. [2], p. 147) nonobjective macroscopical contributions in theories of porous materials seem to be some orders of magnitude smaller than many other effects not appearing at all on the macroscopical level of description or appearing solely in a very crude manner. A typical example is an influence of microscopical vortices in flows of a fluid component which yields an essential contribution to a form of macroscopical boundary conditions on permeable boundaries, and should be incorporated by an additional macroscopical field of - say - a tortuosity.

It is known that interactions between a flow of fluid, and an obstacle are influenced much stronger by vortices than by time dependent relative velocities (compare the classical problem of Euler - d'Alembert paradox). The latter lead in the classical fluid dynamics to an explicit solution and to the notion of added mass, and consequently, to speculations on a macroscopical dependence on relative accelerations in theories of porous materials. This sort of argument does not hold water not only due to the fact that vortices and similar effects are ignored. In addition in the process of averaging microscopical properties to obtain a macroscopical model of porous materials microscopical interactions through contact surfaces are replaced by an extension of the set of constitutive variables and this does not require explicit solutions of any microscopical boundary value problems which are never available anyway.

In the second Section of this note we present some classical transformation properties of continuum mechanics related to the form of momentum balance equations in non-inertial frames of reference. In the third Section we discuss the form of momentum sources under the assumption of material objectivity. We show that in such a model a contribution of relative accelerations cannot appear. The fourth Section contains an example of a model in which the dependence on relative accelerations is incorporated in a way similar to this of M. A. Biot. By means of a solution for a simple steady state flow we show that the influence of this contribution is so small that it cannot be observed in any experiment on a porous material which can be described by a classical continuum model. Finally in the fifth Section we present propagation conditions for sound waves within such a model. It is shown that corrections due to relative accelerations cannot be distinguished from those of constitutive relations for partial stresses, and, in addition, the transversal wave contains a contribution of a fluid component which is assumed to be ideal. This does not seem to be plausible, and it is, certainly, not observable.

The general conclusion of this note is that contributions of relative accelerations in continuum models of porous materials should not appear at all.

## 2 Transformation properties of momentum balance

We check invariance properties of partial momentum balance equations under Euclidean transformation. This type of invariance follows in the classical mechanics from the assumption that the space of configuration is isometric, and the time space is homogeneous (see: e.g. [3]). The latter is of no interest for our purposes. This transformation has the form

$$\mathbf{x}^* = \mathbf{O}(t)\,\mathbf{x} + \mathbf{d}(t), \quad \mathbf{O}^T = \mathbf{O}^{-1}, \quad \mathbf{x}^*, \mathbf{x} \in \Re^3, \quad \mathbf{O} \in Orth, \quad \mathbf{d} \in V^3, \tag{2.1}$$

where $\mathbf{x}^*$, $\mathbf{x}$ denote points of the space of configuration, $t$ is time, $Orth$ is the group of orthogonal transformations, and $V^3$ denotes the space of 3D vectors.

A scalar $\varphi$, a vector $\varpi$, a second order tensor $\boldsymbol{\Upsilon}$ are called *objective* if they satisfy the following transformation rules

$$\forall \mathbf{O} \in Orth: \quad \varphi^* = \varphi, \quad \varpi^* = \mathbf{O}\varpi, \quad \boldsymbol{\Upsilon}^* = \mathbf{O}\boldsymbol{\Upsilon}\mathbf{O}^T. \tag{2.2}$$

It is assumed that mass densities of continuum mechanics are objective scalars, and contact forces are objective vectors. The latter together with the transformation rule for unit vectors perpendicular to material surfaces yields the objectivity of Cauchy stress tensors.

On the other hand kinematic quantities of continuum mechanics are not objective. In the case of a single component body $B$ we have in the so-called Lagrangian description

$$\mathbf{x} = \mathbf{f}(\mathbf{X}, t) \Longrightarrow \mathbf{f}^*(\mathbf{X}, t) = \mathbf{O}\mathbf{f}(\mathbf{X}, t) + \mathbf{d},$$

$$\frac{\partial \mathbf{f}^*}{\partial t}(\mathbf{X}, t) = \mathbf{O}\frac{\partial \mathbf{f}}{\partial t}(\mathbf{X}, t) + \dot{\mathbf{O}}\mathbf{f}(\mathbf{X}, t) + \dot{\mathbf{d}}, \quad \dot{\mathbf{O}} := \frac{d\mathbf{O}}{dt}, \quad \dot{\mathbf{d}} := \frac{d\mathbf{d}}{dt},$$

$$\begin{aligned} \frac{\partial^2 \mathbf{f}^*}{\partial t^2}(\mathbf{X}, t) &= \mathbf{O}\frac{\partial^2 \mathbf{f}}{\partial t^2}(\mathbf{X}, t) + 2\dot{\mathbf{O}}\frac{\partial \mathbf{f}}{\partial t}(\mathbf{X}, \mathbf{t}) + \ddot{\mathbf{O}}\mathbf{f}(\mathbf{X}, t) + \ddot{\mathbf{d}}, \\ \ddot{\mathbf{O}} &:= \frac{d^2\mathbf{O}}{dt^2}, \quad \ddot{\mathbf{d}} := \frac{d^2\mathbf{d}}{dt^2}, \end{aligned}$$

$$\mathrm{Grad}\mathbf{f}^*(\mathbf{X}, t) = \mathbf{O}\mathrm{Grad}\mathbf{f}(\mathbf{X}, t),$$

where $\mathbf{X}$ denotes a material point of the continuous body $B$, and $\mathbf{f}$ is a function of motion in the Lagrangian description of motion of the body $B$. The latter satisfies usual smoothness assumptions. After transformation to the Eulerian description we obtain from the above relations the following transformation rules for the velocity field $\mathbf{v}$, the acceleration field $\mathbf{a}$, and for the right Cauchy-Green deformation tensor $\mathbf{C} := (\mathrm{Grad}\mathbf{f})^T (\mathrm{Grad}\mathbf{f})$

$$\begin{aligned} &\mathbf{v}^*(\mathbf{x}^*, t) = \mathbf{O}\mathbf{v}(\mathbf{x}, t) + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{d}}, \quad \mathbf{x}^* = \mathbf{O}\mathbf{x} + \mathbf{d}, \\ &\mathbf{a}^*(\mathbf{x}^*, t) = \mathbf{O}\mathbf{a}(\mathbf{x}, t) + 2\dot{\mathbf{O}}\mathbf{v}(\mathbf{x}, t) + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{d}}, \quad \mathbf{C}^* = \mathbf{C}. \end{aligned} \tag{2.3}$$

Hence the deformation tensor $\mathbf{C}$ is not objective, but its components behave as they were objective scalars in relation to transformations in the space of configurations. The velocity $\mathbf{v}$, and the acceleration $\mathbf{a}$ are not objective as well but their transformation rules are more complicated because their components are directly related to reference systems in the space of configuration.

We proceed to investigate partial balance equations of a two-component porous body. We rely on Eulerian description, i.e. fields are functions of the space point $\mathbf{x}$ and time $t$.

We assume that there is no mass exchange between components

$$\frac{\partial \rho^F}{\partial t} + \mathrm{div}\left(\rho^F \mathbf{v}^F\right) = 0, \quad \frac{\partial \rho^S}{\partial t} + \mathrm{div}\left(\rho^S \mathbf{v}^S\right) = 0. \tag{2.4}$$

where $\rho^F, \rho^S$ denote the partial mass densities, $\mathbf{v}^F, \mathbf{v}^S$ are velocities of components. This assumption solely simplifies arguments, and can be ignored, if needed, without much additional effort.

For the fluid component, and for the skeleton, respectively, momentum balance equations have the following form in an inertial reference system

$$\begin{aligned} \rho^F \mathbf{a}^F &= \operatorname{div}\mathbf{T}^F + \hat{\mathbf{p}}^F + \rho^F \mathbf{b}^F, \\ \rho^S \mathbf{a}^S &= \operatorname{div}\mathbf{T}^S + \hat{\mathbf{p}}^S + \rho^S \mathbf{b}^S, \end{aligned} \tag{2.5}$$

$$\hat{\mathbf{p}}^F + \hat{\mathbf{p}}^S = 0, \tag{2.6}$$

where $\mathbf{T}^F, \mathbf{T}^S$ are the partial Cauchy stress tensors, $\hat{\mathbf{p}}^F, \hat{\mathbf{p}}^S$ are the momentum sources, and $\mathbf{b}^F, \mathbf{b}^S$ denote partial body forces. After the transformation to a non-inertial *-system the above balance equations have the form

$$\begin{aligned} \rho^F \left(\mathbf{O}\mathbf{a}^F + 2\dot{\mathbf{O}}\mathbf{v}^F + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{d}}\right) &= \mathbf{O}\left(\operatorname{div}\mathbf{T}^F\right) + \hat{\mathbf{p}}^{F*} + \rho^F \mathbf{b}^{F*}, \\ \rho^S \left(\mathbf{O}\mathbf{a}^S + 2\dot{\mathbf{O}}\mathbf{v}^S + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{d}}\right) &= \mathbf{O}\left(\operatorname{div}\mathbf{T}^S\right) + \hat{\mathbf{p}}^{S*} + \rho^S \mathbf{b}^{S*}, \\ \hat{\mathbf{p}}^{F*} + \hat{\mathbf{p}}^{S*} &= 0, \end{aligned} \tag{2.7}$$

where the following relations have been used

$$\operatorname{div}{}^*\mathbf{T}^{F*} = \mathbf{O}\left(\operatorname{div}\mathbf{T}^F\right), \quad \operatorname{grad}{}^*(\cdots) = \mathbf{O}^T \operatorname{grad}(\cdots), \tag{2.8}$$

and similarily for the skeleton.

Bearing these relations in mind we obtain the following identities

$$\begin{aligned} \rho^F \left[\mathbf{b}^{F*} - \mathbf{i}^{F*} - \mathbf{O}\mathbf{b}^F\right] &= -\left(\hat{\mathbf{p}}^{F*} - \mathbf{O}\hat{\mathbf{p}}^F\right), \\ \rho^S \left[\mathbf{b}^{S*} - \mathbf{i}^{S*} - \mathbf{O}\mathbf{b}^S\right] &= \hat{\mathbf{p}}^{F*} - \mathbf{O}\hat{\mathbf{p}}^F, \end{aligned}$$

$$\begin{aligned} \mathbf{i}^{F*} &:= 2\mathbf{W}\left(\mathbf{v}^{F*} - \dot{\mathbf{d}}\right) - \mathbf{W}^2\left(\mathbf{x}^* - \mathbf{d}\right) + \dot{\mathbf{W}}\left(\mathbf{x}^* - \mathbf{d}\right) + \ddot{\mathbf{d}}, \\ \mathbf{i}^{S*} &:= 2\mathbf{W}\left(\mathbf{v}^{S*} - \dot{\mathbf{d}}\right) - \mathbf{W}^2\left(\mathbf{x}^* - \mathbf{d}\right) + \dot{\mathbf{W}}\left(\mathbf{x}^* - \mathbf{d}\right) + \ddot{\mathbf{d}}, \quad \mathbf{W} := \dot{\mathbf{O}}\mathbf{O}^T, \end{aligned} \tag{2.9}$$

where the contributions on the right hand side of $(2.9)_{3,4}$ correspond to the Coriolis acceleration, centrifugal acceleration, Euler acceleration and the relative translational acceleration in both components, respectively. $\mathbf{W}$ denotes the skew-symmetric tensor of angular velocities of two reference systems.

In mechanics of single component systems the right hand side of relations $(2.9)_{1,2}$ is identically zero (conservation of momentum!). If we make a similar assumption for the system of two components we obtain the rules of transformation for body forces

$$\mathbf{b}^{F*} - \mathbf{i}^{F*} = \mathbf{O}\mathbf{b}^F, \quad \mathbf{b}^{S*} - \mathbf{i}^{S*} = \mathbf{O}\mathbf{b}^S. \tag{2.10}$$

i.e. the combinations on the left hand sides (the so-called apparent body forces) are objective. However in mechanics of multicomponent systems there is no argument based on a conservation law which would eliminate contributions from sources. The main aim of this work is to show that such contributions are not plausible. It means we claim that the momentum sources are objective vectors, i.e.

$$\hat{\mathbf{p}}^{F*} = \mathbf{O}\hat{\mathbf{p}}^{F}. \tag{2.11}$$

In the next Section we show simple consequences of this assumption. In Section 4 we present an example of a model in which the relation (2.11) is not assumed to hold. Such is the case in the Biot's model [1] in which momentum sources depend on the relative acceleration of components.

# 3 Objective sources of momentum

We consider a simple case of a poroelastic material undergoing isothermal processes. A larger class of materials can be considered in a similar manner but calculations are more tedious. In a chosen inertial frame constitutive laws for momentum sources are assumed to have the following form

$$\hat{\mathbf{p}}^{F} \equiv -\hat{\mathbf{p}}^{S} = p\left(\rho^{F}, \mathbf{C}^{S}, \mathbf{w}, \mathbf{a}\right), \tag{3.1}$$

where

$$\mathbf{w} := \mathbf{v}^{F} - \mathbf{v}^{S}, \quad \mathbf{a} := \mathbf{a}^{F} - \mathbf{a}^{S}, \tag{3.2}$$

and $\mathbf{C}^{S}$ is the right Cauchy-Green deformation tensor of the skeleton.

We assume the material objectivity, i.e. in any other frame obtained by an orthogonal transformation the constitutive relation must have the form

$$\hat{\mathbf{p}}^{F*} = p\left(\rho^{F*}, \mathbf{C}^{S*}, \mathbf{w}^{*}, \mathbf{a}^{*}\right), \tag{3.3}$$

with the transformation rules

$$\mathbf{C}^{S*} = \mathbf{C}^{S}, \quad \mathbf{w}^{*} = \mathbf{O}\mathbf{w}, \quad \mathbf{a}^{*} = \mathbf{O}\mathbf{a} + 2\dot{\mathbf{O}}\mathbf{w}. \tag{3.4}$$

The latter two follow easily from (2.3) and definitions (3.2). Now the substitution in (2.11) yields

$$\forall \mathbf{O} \in Orth: \quad \mathbf{O}p\left(\rho^{F}, \mathbf{C}^{S}, \mathbf{w}, \mathbf{a}\right) = p\left(\rho^{F}, \mathbf{C}^{S}, \mathbf{O}\mathbf{w}, \mathbf{O}\mathbf{a} + 2\dot{\mathbf{O}}\mathbf{w}\right). \tag{3.5}$$

Let us choose a particular instant of time in which $\mathbf{O} = \mathbf{1}$, and $\dot{\mathbf{O}}$ is arbitrary. Then

$$p\left(\rho^{F}, \mathbf{C}^{S}, \mathbf{w}, \mathbf{a}\right) = p\left(\rho^{F}, \mathbf{C}^{S}, \mathbf{w}, \mathbf{a} + 2\dot{\mathbf{O}}\mathbf{w}\right). \tag{3.6}$$

Certainly, this relation can hold solely if the source is independent of **a.** Then the relation (3.5) reduces to the following condition

$$\forall \mathbf{O} \in Orth: \quad \mathbf{O}p\left(\rho^F, \mathbf{C}^S, \mathbf{w}\right) = p\left(\rho^F, \mathbf{C}^S, \mathbf{O}\mathbf{w}\right), \tag{3.7}$$

which means that the function $p$ should be isotropic with respect to the relative velocity $\mathbf{w}$. Consequently

$$\hat{\mathbf{p}}^F = -\pi\left(\rho^F, \mathbf{C}^S, |\mathbf{w}|\right)\mathbf{w}, \tag{3.8}$$

where the minus sign has been introduced for historical reasons, and $\pi$ is a scalar function not limited any further by objectivity arguments.

# 4 Linear dependence on the relative acceleration

We ignore now the assumption (2.11) and consider a constitutive law for the momentum source which in the inertial frame of reference is linear and isotropic with respect to the relative velocity $\mathbf{w}$ and the relative acceleration $\mathbf{a}$, i.e.

$$\hat{\mathbf{p}}^F = -\pi\mathbf{w} + b\mathbf{a}, \tag{4.1}$$

where the coefficients $\pi$, and $b$ for poroelastic materials may be functions of $\rho^F$, and $\mathbf{C}^S$. In an arbitrary *-frame we have

$$\hat{\mathbf{p}}^{F*} = -\pi\mathbf{w}^* + b\mathbf{a}^* = \mathbf{O}\hat{\mathbf{p}}^F + 2b\dot{\mathbf{O}}\mathbf{w}, \tag{4.2}$$

because the mass density $\rho^F$, and $\mathbf{C}^S$ do not change under this transformation, and, consequently the coefficients $\pi$, and $b$ remain unchanged as well. According to the relations (2.9) the transformation of body forces yields in this case the following relations

$$\rho^F\left[\mathbf{b}^{F*} - \mathbf{i}^{F*} - \mathbf{O}\mathbf{b}^F\right] = -2b\dot{\mathbf{O}}\mathbf{w},$$
$$\rho^S\left[\mathbf{b}^{S*} - \mathbf{i}^{S*} - \mathbf{O}\mathbf{b}^S\right] = 2b\dot{\mathbf{O}}\mathbf{w}.$$

In the explicit form

$$\mathbf{b}^{F*} - 2\mathbf{W}\left(\mathbf{v}^{F*} - \dot{\mathbf{d}} - \frac{b}{\rho^F}\mathbf{w}^*\right) - \mathbf{W}^2\left(\mathbf{x}^* - \mathbf{d}\right) + \dot{\mathbf{W}}\left(\mathbf{x}^* - \mathbf{d}\right) + \ddot{\mathbf{d}} = \mathbf{O}\mathbf{b}^F, \tag{4.3}$$

$$\mathbf{b}^{S*} - 2\mathbf{W}\left(\mathbf{v}^{S*} - \dot{\mathbf{d}} + \frac{b}{\rho^S}\mathbf{w}^*\right) - \mathbf{W}^2\left(\mathbf{x}^* - \mathbf{d}\right) + \dot{\mathbf{W}}\left(\mathbf{x}^* - \mathbf{d}\right) + \ddot{\mathbf{d}} = \mathbf{O}\mathbf{b}^S,$$

where the second relation follows easily from condition (2.6).

Hence, in contrast to the classical nonobjective contributions to body forces the constitutive relation (4.1) leads to body forces in noninertial systems which depend on *material properties* of the system. This does not seem to be very plausible. However such a contribution can be relatively easy verified experimentally if it is essential at all.

In order to discuss this point in some details we consider a simple example of stationary flow through a cylinder. The geometry of the system is shown in Figure 1. It is assumed that a fluid flows into the porous material at the surface $r = r_i$, and it flows out at the surface $r = r_e$. Deformations of the skeleton are assumed to be small which means that the radii of the cylinder approximately do not change. The cylinder rotates with a constant angular velocity $\omega$ as shown in Fig.1.

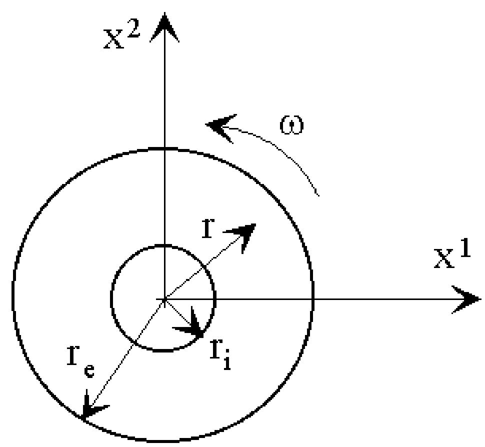


**Figure 1**: *Geometry of the flow through the cylinder*

Under the assumption of a constant porosity, and a constant mass density of the fluid $\rho^F$ (it corresponds to the incompressibility of a real fluid) the problem is described by the following fields

$$\left\{p^F, v_r^F, v_\phi^F, u_r^S, u_\phi^S\right\} \tag{4.4}$$

where $p^F$ denotes the partial pressure of the fluid, $v_r^F, v_\phi^F$ are the radial and circumferential physical components of the velocity of fluid, respectively, and $u_r^S, u_\phi^S$ describe the physical components of displacement of the skeleton. We use the noninertial cylindrical frame of reference rotating together with the cylinder with the angular velocity $\omega$. To simplify the notation we leave out the asterics indicating the noninertial character of the reference system. All above listed fields are functions of the radius $r$ alone. Consequently the velocity of skeleton $\mathbf{v}^S$ and the acceleration $\mathbf{a}^S$ are identically zero, and solely convective contributions to the acceleration $\mathbf{a}^F$ remain.

The field equations in the noninertial frame have the form

- mass conservation of the fluid component

$$\frac{d\left(rv_r^F\right)}{dr} = 0, \tag{4.5}$$

- momentum balance for the fluid component

$$\begin{aligned}
\left(\rho^F - b\right)\left(\frac{1}{2}\frac{dv_r^{F2}}{dr} - \frac{v_\phi^{F2}}{r}\right) &= -\frac{dp^F}{dr} - \pi v_r^F + 2\left(\rho^F - b\right)\omega v_\phi^F - \omega^2 r, \\
\left(\rho^F - b\right) v_r^F \frac{1}{r}\frac{d}{dr}\left(rv_\phi^F\right) + \pi v_\phi^F &= -2\omega v_r^F,
\end{aligned} \tag{4.6}$$

and the momentum balance equation for the skeleton which is immaterial for our present argument.

We can easily construct solutions for components of the partial velocity of the fluid. From equation (4.5) we obtain

$$v_r^F = \frac{C}{r}, \quad C = \text{const.} \tag{4.7}$$

Equation $(4.6)_2$ yields then

$$\frac{C}{r^2}\frac{d}{dr}\left(rv_\phi^F\right) + \frac{\pi}{\rho^F - b}v_\phi^F = -2\omega\frac{C}{r}. \tag{4.8}$$

Hence we obtain

$$v_\phi^F = \frac{1}{r}\left(-\frac{2\omega}{\pi}C\left(\rho^F - b\right) + A\exp\left(-\frac{\pi}{2C\left(\rho^F - b\right)}r^2\right)\right), \quad A = \text{const.} \tag{4.9}$$

We need boundary conditions in order to find constants $A$ and $C$. One of them is obvious - the circumferential velocity $v_\phi^F$ should be zero for $r = r_i$ because the fluid is assumed to enter the cylinder in the normal direction. A condition for the velocity $v_r^F$ at this surface should follow from a condition of the third kind (e.g. see: [4]) which we shall not quote here. It describes an amount of fluid mass which flows into the cylinder per unit time and unit surface. We need solely an order of magnitude of this quantity. The value of the partial velocity corresponding to this condition is denoted by $v_{r_i}^F$. After easy manipulations we obtain

$$\frac{v_\phi^F}{v_r^F} = -\frac{\omega\left(\rho^F - b\right)}{\pi}\left[1 - \exp\left(-\frac{\pi r_i}{2v_{r_i}^F\left(\rho^F - b\right)}\left(\frac{r^2}{r_i^{F2}} - 1\right)\right)\right]. \tag{4.10}$$

For typical values of the mass density $\rho^F 10^3 \frac{kg}{m^3}$, and permeability coefficient $\pi 10^8 \frac{kg}{m^3 s}$ we have to rotate the system with the angular speed $\omega 10^5 \frac{1}{s}$ in order to be able to observe an influence of the relative acceleration on the flow in the cylinder. Certainly this is not reasonable in the case of classical porous and granular materials. Consequently even if we accepted the lack of material objectivity its influence could not be observed in steady state experiments. In the next Section we check dynamical effects of such contributions.

# 5 Influence of nonobjectivity on the propagation of sound waves

Additional contributions of accelerations to momentum balance equations influence not only steady-state flows but primarily dynamical processes such as the propagation of sound waves. We check the propagation condition for such waves in the case of poroelastic materials. Propagation conditions determine speeds of propagation of wave fronts and amplitudes of waves on these fronts.

Sound waves are also called weak discontinuity waves because their fronts are characterized by discontinuity of derivatives of fields but not of fields themselves. It means in our case

$$\begin{aligned} \left[\left[\rho^F\right]\right] &= 0, \quad \left[\left[\rho^S\right]\right] = 0, \quad \left[\left[\mathbf{v}^F\right]\right] = 0, \quad \left[\left[\mathbf{v}^S\right]\right] = 0, \quad \left[\left[\mathbf{e}^S\right]\right] = 0, \\ [[\cdots]] &:= (\cdots)^+ - (\cdots)^-, \end{aligned} \tag{5.1}$$

where

$$\mathbf{e}^S := \operatorname{symgrad} \mathbf{u}^S, \tag{5.2}$$

$\mathbf{u}^S$ denotes the displacement of the skeleton, and the values $(\cdots)^+$, $(\cdots)^-$ are estimated on the positive and negative side of the front, respectively. Time derivatives of the fields do not have to be continuous, and we introduce the following notation

$$\begin{aligned} R^F &:= \left[\left[\frac{\partial \rho^F}{\partial t}\right]\right], \quad R^S := \left[\left[\frac{\partial \rho^S}{\partial t}\right]\right], \\ \mathbf{A}^F &:= \left[\left[\frac{\partial \mathbf{v}^F}{\partial t}\right]\right], \quad \mathbf{A}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t}\right]\right]. \end{aligned} \tag{5.3}$$

These quantities are called the amplitudes of sound wave.

If we denote by $c$ the speed of propagation of the front then we have the following Hadamard kinematical compatibility conditions (e.g. [5])

$$\begin{gathered} c\left[\left[\operatorname{grad} \rho^F\right]\right] = -R^F \mathbf{n}, \quad c\left[\left[\operatorname{grad} \rho^S\right]\right] = -R^S \mathbf{n}, \\ c\left[\left[\operatorname{grad} \mathbf{v}^F\right]\right] = -\mathbf{A}^F \otimes \mathbf{n}, \quad c\left[\left[\operatorname{grad} \mathbf{v}^S\right]\right] = -\mathbf{A}^S \otimes \mathbf{n}, \\ c\left[\left[\operatorname{grad} \mathbf{e}^S\right]\right] = -\left[\left[\frac{\partial \mathbf{e}^S}{\partial t}\right]\right] \otimes \mathbf{n}. \end{gathered} \tag{5.4}$$

In these relations $\mathbf{n}$ denotes the unit vector normal to the surface of the front, and its orientation defines the positive side of the surface.

We proceed to investigate conditions following from field equations. In order to simplify arguments we consider the linear model in which stress tensors appearing in balance equations (2.5) are given by the following constitutive relations

$$\mathbf{T}^F = -\left[p_0^F + \kappa\left(\rho^F - \rho_0^F\right)\right] \mathbf{1}, \quad \mathbf{T}^S = \mathbf{T}_0^S + \lambda^S \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu^S \mathbf{e}^S, \tag{5.5}$$

where the material parameters $,\lambda^S,\mu^S$ are assumed to be constant, and $\mathbf{T}_0^S,p_0^F,\rho_0^F$ denote constant reference values of the partial stress tensor in the skeleton, the partial pressure, and the partial mass density, respectively. Hence the fluid component is ideal (no contributions of the velocity gradient), and the skeleton is elastic. Such porous materials are called linear poroelastic.

After substitution of the above constitutive relations as well as of the relation (4.1) in momentum balance equations we obtain field equations provided we account for the following integrability condition

$$\frac{\partial \mathbf{e}^S}{\partial t}=\mathrm{symgrad}\mathbf{v}^S. \tag{5.6}$$

This relation leads to the following condition on the wave front

$$c^2\left[\left[\mathrm{grad}\mathbf{e}^S\right]\right]=\frac{1}{2}\left(\mathbf{A}^S\otimes\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{A}^S\otimes\mathbf{n}\right). \tag{5.7}$$

We construct the limits of mass, and momentum balance equations on the wave front. This yields immediately the following set of algebraic relations for amplitudes

$$\begin{aligned} R^F\left(c-\mathbf{v}^F\cdot\mathbf{n}\right) &= \rho^F\mathbf{A}^F\cdot\mathbf{n}, \\ R^S\left(c-\mathbf{v}^S\cdot\mathbf{n}\right) &= \rho^S\mathbf{A}^S\cdot\mathbf{n}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} \left(\rho^F-b\right)\left(c-\mathbf{v}^F\cdot\mathbf{n}\right)\mathbf{A}^F+b\left(c-\mathbf{v}^S\cdot\mathbf{n}\right)\mathbf{A}^S &= R^F\mathbf{n}, \\ bc\left(c-\mathbf{v}^F\cdot\mathbf{n}\right)\mathbf{A}^F+\left(\rho^S-b\right)c\left(c-\mathbf{v}^S\cdot\mathbf{n}\right)\mathbf{A}^S &= \left(\lambda^S+\mu^S\right)\mathbf{A}^S\cdot\mathbf{nn}+\mu^S\mathbf{A}^S. \end{aligned} \tag{5.9}$$

It is a homogeneous set of equations for amplitudes $R^F,R^S,\mathbf{A}^F,\mathbf{A}^S$. Consequently its determinant must vanish, and this yields the so called propagation condition determining speeds of propagation, and some relations between components of amplitudes. We consider solutions of this condition under the assumption that the speed of propagation $c$ is much larger than normal velocities $\mathbf{v}^F\cdot\mathbf{n}$, and $\mathbf{v}^S\cdot\mathbf{n}$. Then we obtain easily

$$\begin{aligned} \left[\left(\rho^F-b\right)c^2\mathbf{1}-\rho^F\kappa\mathbf{n}\otimes\mathbf{n}\right]\mathbf{A}^F+bc^2\mathbf{A}^S &= 0, \\ bc^2\mathbf{A}^F+\left[\left(\rho^S-b\right)c^2\mathbf{1}-\left(\lambda^S+\mu^S\right)\mathbf{n}\otimes\mathbf{n}-\mu^S\mathbf{1}\right]\mathbf{A}^S &= 0. \end{aligned} \tag{5.10}$$

To see the structure of solutions it is convenient to split the amplitudes $\mathbf{A}^F,\mathbf{A}^S$ into normal and tangential components

$$\mathbf{A}^F=\mathbf{A}^F\cdot\mathbf{nn}+\mathbf{A}_\perp^F,\quad \mathbf{A}^S=\mathbf{A}^S\cdot\mathbf{nn}+\mathbf{A}_\perp^S,\quad \mathbf{A}_\perp^F\cdot\mathbf{n}\equiv 0,\quad \mathbf{A}_\perp^S\cdot\mathbf{n}\equiv 0. \tag{5.11}$$

The set of two scalar equations for normal components $\mathbf{A}^F\cdot\mathbf{n},\mathbf{A}^S\cdot\mathbf{n}$ has nontrivial solutions if the following condition is satisfied

$$\left[\left(\rho^F-b\right)c^2-\rho^F\kappa\right]\left[\left(\rho^S-b\right)c^2-\left(\lambda^S+2\mu^S\right)\right]-b^2c^4=0. \tag{5.12}$$

Consequently we obtain two modes of propagation - the so-called P1-, and P2-wave which propagate with speeds $\pm c_{P1}$ and $\pm c_{P2}$ being solutions of the above biquadratic equation for $c$. For these speeds to be real one has to fulfil a condition on material coefficients which we do not need to present here.

This is the usual result for twocomponent porous materials [5]. However in the present case the speeds of propagation depend on the value of coefficient $b$ and they differ from the following classical results

$$b \equiv 0 \quad \Longrightarrow \quad c_{P1}^2 = \frac{\lambda^S + 2\mu^S}{\rho^S}, \quad c_{P2}^2 = \kappa. \tag{5.13}$$

Corrections predicted by the relation (5.12) would be still acceptable due to the fact that speeds depend on multiplicative combinations of elastic properties of components with mass densities and the coupling coefficient $b$. This means that measurements of speeds do not yield any information on the coupling coefficient alone, and we can obtain good fitting by correcting material coefficients in (5.13) particularily by a low accuracy of measurements of these speeds for porous materials. This fitting by means of the coupling coefficient is extensively discussed in the works [6,7]. We return to these papers in the last Section.

In contrast to the above modes of propagation the result for the transversal mode seems to contradict all available experimental observations. Namely we obtain the following solution of the propagation condition

$$\mathbf{A}_\perp^F = -\frac{b}{\rho^F - b}\mathbf{A}_\perp^S, \qquad c^2\left[\left(\rho^S - b\right) - \frac{b^2}{\rho^F - b}\right] = \mu^S. \tag{5.14}$$

Consequently for $b \neq 0$ the speed of propagation of transversal waves would be dependent on the mass density of the fluid component in contrast to the classical result $c^2 = \frac{\mu^S}{\rho^S}$, and, in addition, there would exist a transversal component of the amplitude $\mathbf{A}_\perp^F$ of the wave carried by an ideal fluid. Such effects are very unlikely and they have never been reported by experimentalists.

# 6 Some heuristic remarks

The Biot's correction of interactions in momentum balance equations is recently motivated by results for flows through obstacles and dynamical effects in suspensions. In these fields of research the notion of an *added mass* which should support the Biot's corrections is well established and justified by reasonable physical arguments (see: [8], p. 134ff; in this book an extensive literature on the subject is quoted).

However in contrast to multicomponent theories of porous materials solutions of those flow problems are constructed, from the viewpoint of multicomponent theories, on a microscopical level of observation. Interactions through the pressure on surfaces of obstacles are eliminated by means of explicit solutions of mass and momentum conservation equations for the fluid. In this way one can construct an explicit momentum balance equation for obstacles where, as result of elimination of surface interactions, corrections to the mass appear. Certainly, this is not the case in macroscopical theories of porous materials where microscopical interactions are smeared out by averaging. They contribute on the macroscopical level through the correction of the set of constitutive variables. Such corrections are reflected both by a simultaneous appearance of constitutive variables of all components in all constitutive relations, and by additional microstructural variables such as porosity or tortuosity.

It seems to be justified to assume that the tortuosity should be this microstructural variable which replaces an added mass in macroscopical models. However the way in which it is introduced by Gajo and others [6,7] contradicts the classical definition of this notion (see: [9], Sec. 4.8.). According to such a definition the tortuosity describes a geometrical property *additional* to the volume fraction of voids described by the porosity. Consequently it cannot be related to the porosity by any algebraic relation as it is claimed in these and some other papers.

The tortuosity as an additional field seems to be a natural candidate on the macroscopical level to describe effects of dynamical interactions of flows of the fluid component through complex channels of the skeleton without contradicting the principle of objectivity.

Bearing in mind the above analysis we have to accept the consequence that the relative accelerations cannot contribute to momentum balance equations.

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# A note on two-component model for Terzaghi Gedankenexperiment

K.Wilmański


## Summary

K. von Terzaghi and O. K. Froehlich [1] have proposed a simple Gedankenexperiment in which the question of flow of the fluid through a permeable boundary of a porous medium during a consolidation process was addressed. It was shown that an amount of mass transported per unit surface and in unit time depends on a jump of pore pressure. In this note we find an analytic solution of a 1D problem with such a boundary condition. The model under investigations stems from the work [4]. The main purpose of this solution is the construction of a bench-mark problem for both parameter analysis within this model as well as a numerical investigation of various 2D problems.


## 1 Introduction

In the classical book on consolidation problems [1] K. von Terzaghi and O. K. Fröhlich presented a Gedankenexperiment which indicates a natural form of boundary condition for permeable boundaries of porous media. By means of this example one can show that an amount of fluid which flows through an interface should be proportional to the jump of real pore pressure. Such a condition was incorporated for the first time in a continuous multicomponent model of porous materials by H. Deresiewicz [2], and its slightly modified form has been used in the dissertation of K. Runesson [3]. The model applied in the present work with this type of boundary condition has been presented in [4].

This boundary condition, typical for porous and granular materials, has a particular bearing in the theory of surface waves [5], and it is responsible for the existence of additional modes of propagation observed experimentally.

In this note we present an analytical solution of a one-dimensional quasistatic problem corresponding to the Gedankenexperiment of von Terzaghi and Fröhlich. The main purpose of this solution is to construct a bench-mark problem for various two- and threedimensional numerical problems of the multicomponent theory of porous materials. A simple form of this solution enables an arbitrary variation of material parameters, and, consequently, checking of the sensitivity of solution on changes of material properties such as bulk and surface permeabilies.

# 2 The model

We consider a semiinfinite porous medium loaded in the plane $x = 0$ by a constant loading $p_a + q_0$ in the direction perpendicular to the boundary. The first contribution - atmospheric pressure $p_a$ - determines a condition for the flow of fluid through the boundary. The second contribution is an external loading applied at the time $t = 0$, and acting simultaneously on both components of the porous medium.

The model is based on assumptions of small deformations of the skeleton and small changes of mass density of the fluid component. In the one-dimensional case under considerations it means

$$\left|e^S\right| \ll 1, \qquad \left|\frac{\rho^F - \rho_0^F}{\rho_0^F}\right| \ll 1, \tag{2.1}$$

where $e^S$ denotes the extension in the $x$-direction (direction of loading), $\rho^F$ is the current partial mass density of the fluid, and $\rho_0^F$ - its reference value. In such a case the mass density of the skeleton is constant $\rho^S = \text{const}$.

We seek the following fields of the model

$$(x,t) \longmapsto \left\{\rho^F, v^F, u^S\right\} \in \Re^3, \quad 0 \le x < \infty, \quad 0 \le t < \infty, \tag{2.2}$$

where $v^F, u^S$ denote the velocity of the fluid and the displacement of the skeleton, respectively, both in the $x$-direction. The porosity appearing in the general model is assumed to be constant. This seems to be well justified in the case of the linear model of quasistatic processes. We have the following field equations at disposal

- partial mass conservation of the fluid component

$$\frac{\partial \rho^F}{\partial t} + \rho_{in}^F \frac{\partial v^F}{\partial x} = 0, \tag{2.3}$$

- partial momentum balance equations supplemented with linear constitutive laws

$$\frac{\partial p^F}{\partial x} + \pi\left(v^F - v^S\right) = 0, \quad p^F = p_{in}^F + \kappa\left(\rho^F - \rho_{in}^F\right), \quad v^S := \frac{\partial u^S}{\partial t},$$

$$\frac{\partial \sigma^S}{\partial x} + \pi\left(v^F - v^S\right) = 0, \quad \sigma^S = \sigma_{in}^S + \left(\lambda^S + 2\mu^S\right)e^S, \quad e^S := \frac{\partial u^S}{\partial x}, \tag{2.4}$$

where $\kappa$ denotes the effective coefficient of compressibility, $\lambda^S, \mu^S$ are effective Lamé constants and $\rho_{in}^F, p_{in}^F, \sigma_{in}^S$ are initial mass density, initial pressure and initial normal stress in the skeleton, respectively. Let us mention that effective material parameters depend on the porosity. These relations do not have to be specified for the present example.

For the quasistatic Terzaghi problem the initial values of partial stresses are determined by assuming that the consolidation begins after an instantaneous elastic deformation[1].

We proceed in the following way. At the beginning of the process the boundary of the skeleton is open and the system is in a state of static equilibrium ($v^S = v^F \equiv 0, \quad \rho^F = \rho_0^F =$ const.). Then $p^F = np_a, \sigma^S = -(1-n)\, p_a$. Here $p_a$ denotes the external (atmospheric) reference pressure and $n$ is a constant porosity. Now we apply the external load $q_0$ acting downwards. Instantaneously after the application of the load the system deforms elastically. This deformation is described by the compression (negative longitudinal deformation) of the skeleton $-e_{in}^S \left(e_{in}^S > 0\right)$. The corresponding change of the mass density of the fluid component for small deformations is given by the relation $\rho_{in}^F = \rho_0^F \left(1 + e_{in}^S\right)$. Now $e_{in}^S$ can be easily found from the equilibrium condition:

$$\sigma_{in}^S - p_{in}^F = -p_a - q_0. \tag{2.5}$$

We have to use the constitutive relations (2.4) in which $\sigma_{in}^S, p_{in}^F$ are replaced by $\sigma_0^S, p_0^F$ and $e^S = -e_{in}^S, \rho^F = \rho_{in}^F$. We obtain

$$e_{in}^S = \frac{q_0}{(\lambda^S + 2\mu^S) + \kappa\rho_0^F}, \quad \rho_{in}^F = \rho_0^F \left(1 + \frac{q_0}{(\lambda^S + 2\mu^S) + \kappa\rho_0^F}\right). \tag{2.6}$$

Consequently

$$\begin{aligned} p_0^F &= np_a + \frac{\kappa\rho_0^F}{(\lambda^S + 2\mu^S) + \kappa\rho_0^F} q_0, \\ \sigma_0^S &= -(1-n)\, p_a - \frac{\left(\lambda^S + 2\mu^S\right)}{(\lambda^S + 2\mu^S) + \kappa\rho_0^F} q_0. \end{aligned} \tag{2.7}$$

Now we can return to the consolidation problem. We choose the above described state as the initial state of the problem. For the field equations (2.3-4) we have the boundary conditions

$$\begin{aligned} \sigma^S - p^F &= -p_a - q_0, \quad for \quad x = 0, \\ -\rho_{in}^F \left(v^F - v^S\right) &= \alpha \left(p^F - np_a\right), \quad for \quad x = 0, \end{aligned} \tag{2.8}$$

and the Sommerfeld conditions for $x \longrightarrow \infty$.[2] The material parameter $\alpha$ is the so-called surface permeability. The sign on the left-hand side of $(2.8)_2$ follows from

[1] in the case of full dynamic description the external load is taken over from the boundary by the acceleration which means that we do not have to introduce any instantaneous deformations (see: W. Kempa [*Arch.Mech.*, 1997]) and need not be included in initial stresses. In the classical soil mechanics, however, it is done in the way presented above (e.g. see: H.-J. Lang, J. Huder, P. Amann [*Bodenmechanik und Grundbau, Springer,* 1996]).

[2] The boundary conditions in infinity are introduced to simplify the derivation of the solution of the problem. However the solution can be also constructed for the finite domain $0 \leq x \leq l$. In this case $v^F = 0$ and $u^S = 0$ for $x = l$. The solution by means of the Laplace transform (see: further in this note) has then the form of an infinite series rather than a closed form. This is due to the infinite number of poles.

the negative orientation of the normal vector $\mathbf{n}$ to the boundary $\left(\mathbf{n}\cdot\left(\mathbf{v}^F-\mathbf{v}^S\right)=-\left(v^F-v^S\right)\right)$.

Initial conditions are assumed to have the form

$$\rho^F=\rho^F_{in},\quad e^S=0,\quad v^F=0,\quad v^S=0\quad for\quad t=0. \tag{2.9}$$

We solve the above problem reducing the system to a single equation for the mass density $\rho^F$. As we see this yields a simplest linear parabolic equation of the second order with boundary conditions of the third kind. The latter is a typical feature of almost all problems of porous materials with diffusion.

# 3 Governing equation for $\rho^F$

Let us notice that the combination of equations $(2.4)_1$ and $(2.4)_2$ and the boundary condition $(2.8)_1$ yield

$$\sigma^S-p^F=-p_a-q_0=\text{const}. \tag{3.1}$$

Simultaneously it follows from $(2.4)_2$

$$\begin{aligned}\frac{\partial^2 p^F}{\partial x^2}+\pi\left(\frac{\partial v^F}{\partial x}-\frac{\partial e^S}{\partial t}\right)&=\frac{\partial^2 p^F}{\partial x^2}+\pi\left(\frac{\partial v^F}{\partial x}-\frac{1}{\lambda^S+2\mu^S}\frac{\partial\sigma^S}{\partial t}\right)=\\ &=\frac{\partial^2 p^F}{\partial x^2}+\pi\left(\frac{\partial v^F}{\partial x}-\frac{1}{\lambda^S+2\mu^S}\frac{\partial p^F}{\partial t}\right)=\\ &=\kappa\frac{\partial^2\rho^F}{\partial x^2}+\pi\left(-\frac{1}{\rho^F_{in}}\frac{\partial\rho^F}{\partial t}-\frac{\kappa}{\lambda^S+2\mu^S}\frac{\partial\rho^F}{\partial t}\right)=0.\end{aligned}$$

Hence we obtain the equation

$$\frac{\partial\rho^F}{\partial t}-D\frac{\partial^2\rho^F}{\partial x^2}=0,\quad D:=\frac{\kappa\rho^F_{in}}{\pi}\left(1+\frac{\kappa\rho^F_{in}}{\lambda^S+2\mu^S}\right)^{-1}>0. \tag{3.2}$$

The boundary condition for $x=0$ follows from the substitution of the equation $(2.4)_1$ into the condition $(2.8)_2$

$$v^F-v^S=-\frac{\kappa}{\pi}\frac{\partial\rho^F}{\partial x}\quad\Longrightarrow \tag{3.3}$$

$$\begin{aligned}\frac{\kappa\rho^F_{in}}{\pi}\frac{\partial\rho^F}{\partial x}&=\alpha\left[\frac{\kappa\rho^F_0}{(\lambda^S+2\mu^S)+\kappa\rho^F_0}q_0+\kappa\left(\rho^F-\rho^F_{in}\right)\right]\equiv\alpha\kappa\left(\rho^F-\rho^F_0\right)\\ for\quad x&=0\end{aligned} \tag{3.4}$$

and

$$\rho^F=\rho^F_{in}\quad for\quad x\longrightarrow\infty,\qquad \rho^F=\rho^F_{in}\quad for\quad t=0. \tag{3.5}$$

Consequently we have to solve the linear parabolic equation with the boundary condition of the third kind.

# 4 Dimensionless form and values of material parameters

To solve the above problem we use the following dimensionless variables

$$w := \frac{\rho^F - \rho^F_{in}}{\rho^F_{in}}, \quad t \longrightarrow \frac{t}{\tau}, \quad x \longrightarrow \frac{x}{\sqrt{D\tau}}, \tag{4.1}$$

where $\tau$ is a positive constant remaining arbitrary in a semiinfinite problem. In the case of the finite length $l$ of the domain we can identify this constant with, for instance, $\frac{l^2}{D}$. The problem has now the form

$$\begin{aligned} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} &= 0, \quad x > 0, \quad t > 0, \\ \gamma \frac{\partial w}{\partial x} - w &= w_\infty, \quad x = 0, \quad t > 0, \\ w &= 0, \quad x \to \infty, \quad t > 0, \\ w &= 0, \quad x > 0, \quad t = 0, \end{aligned} \tag{4.2}$$

where

$$\gamma := \frac{\rho^F_0}{\alpha\pi\sqrt{D\tau}}, \quad w_\infty := \frac{q_0}{\kappa\rho^F_0 + (\lambda^S + 2\mu^S)}. \tag{4.3}$$

We construct further a numerical example in which we use data for the material parameters typical for soil mechanics (e.g. [6]):

**Table**

| | | | |
|---|---|---|---|
| $\rho^F_0 = 2.5 \times 10^2 \frac{kg}{m^3}$ | $\rho^S = 3 \times 10^3 \frac{kg}{m^3}$ | $n = 0.25$ | $\tau = 10^{-3} s$ |
| $\frac{\lambda^S + 2\mu^S}{\rho^S} = (3 \times 10^3)^2 \frac{m^2}{s^2}$ | $\kappa = 10^6 \frac{m^2}{s^2}$ | $\pi = 10^4 \frac{kg}{m^3 s}$ | $\alpha = 10^{-4} \frac{s}{m}$ |

The characteristic time $\tau$ has been chosen in such a way that the value of the diffusion coefficient $D = 25 \times 10^3 \frac{m^2}{s}$, which follows from the above data would lead the length of the finite domain $l = 5m$. It follows that the coefficient $\gamma$ appearing in the boundary condition $(4.2)_2$ has the value 50.

# 5 Solution

We construct the solution of the problem (4.2) by means of the Laplace transform. We have

$$\frac{d^2\bar{w}}{dx^2} - s\bar{w} = 0, \quad \bar{w} := \int_0^\infty we^{-st}dt, \quad x > 0,$$

$$\begin{aligned} \gamma\frac{d\bar{w}}{dx} - \bar{w} &= \frac{w_\infty}{s}, \quad x = 0, \\ \bar{w} &= 0, \quad x \longrightarrow \infty. \end{aligned} \tag{5.1}$$

It follows

$$\bar{w} = -w_\infty \frac{\frac{1}{\gamma}e^{-\sqrt{s}x}}{s\left(\sqrt{s} + \frac{1}{\gamma}\right)}. \tag{5.2}$$

Inverse transform has the following closed form (e.g. [7], Tabl. 8.4-1)

$$\frac{w}{w_\infty} = \exp(\frac{t + \gamma x}{\gamma^2})\text{erfc}\left(\frac{x}{2\sqrt{t}} + \frac{\sqrt{t}}{\gamma}\right) - \text{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{5.3}$$

Below we present graphs following from this solution.

In Figure 1 we show the function (5.3). It represents a long time behavior of this function. In Figure 2 we show the behaviour of the same function for short times. An artefact following from the incompatibility of the initial condition and the boundary condition at $t = 0$, $x = 0$ can be seen in this Figure.

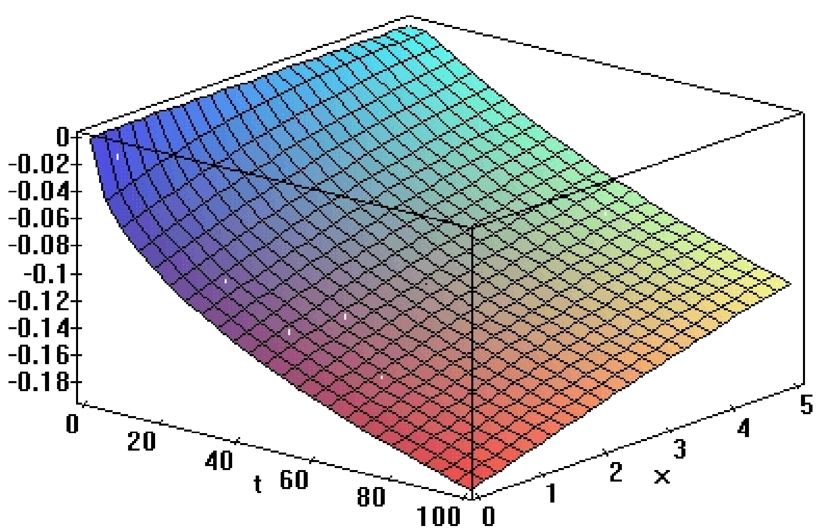



**Figure 1:** *Normalized changes of the partial mass density* $\frac{w}{w_\infty}$ *as a function of time* $t \equiv \frac{t}{\tau}$ *and depth* $x \equiv \frac{x}{\sqrt{D\tau}}$.

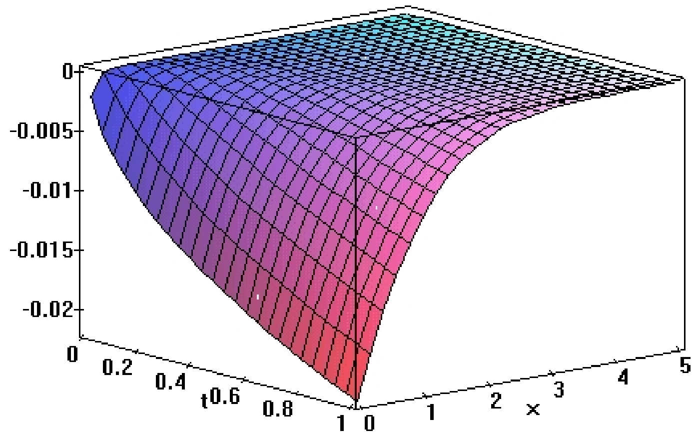



**Figure 2:** *The behaviour of* $\frac{w}{w_\infty}$ *for short times.*

The function $w$ possesses the following asymptotic properties:

1.

$$\lim_{t\to\infty}\frac{w}{w_\infty}=\lim_{t\to\infty}\frac{\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}+\frac{\sqrt{t}}{\gamma}\right)}{\exp(-\frac{t+\gamma x}{\gamma^2})}+\lim_{t\to\infty}\left(-\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)\right)=$$

$$=\lim_{t\to\infty}\left(\frac{2\gamma^2}{\sqrt{\pi}}\right)\frac{\left(\frac{1}{2\gamma\sqrt{t}}-\frac{x}{4t\sqrt{t}}\right)\exp\left[-\left(\frac{\sqrt{t}}{\gamma}+\frac{x}{2\sqrt{t}}\right)^2\right]}{\exp\left(-\frac{t+\gamma x}{\gamma^2}\right)}-1=-1. \tag{5.4}$$

2.

$$\lim_{\gamma\to\infty}\frac{w}{w_\infty}\equiv\lim_{\alpha\to 0}\frac{w}{w_\infty}=0, \tag{5.5}$$

which means that the impermeable boundary ($\alpha=0$) admits solely the solution identical with the (constant) initial state. This is natural because of the choice of the reference configuration.

Substitution of the result (5.4) in the constitutive relation for the partial pressure yields

$$\lim_{t\to\infty}p^F=np_a. \tag{5.6}$$

This is the result which one expects in von Terzaghi Gedankenexperiment.

For the construction of the most representative figure for the partial pressure $p^F$ we have chosen $p_a=0.1MPa$ and $q_0=50MPa$. This corresponds to app. 0.2% deformation of the skeleton.

In the Figure 3 we see the behavior of the partial pressure $p^F$ in $MPa$. The length and time are dimensionless. The shape of this curve is qualitatively identical with those following from the original model of Terzaghi (e.g.: [8]).

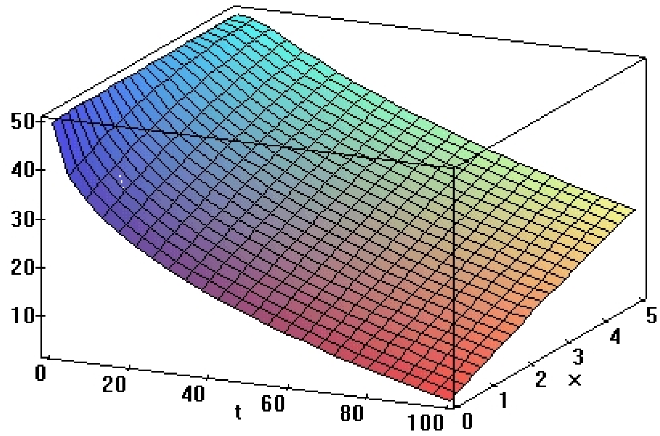



**Figure 3:** *Partial pressure* $p^F$ *as a function of time* $t \equiv \frac{t}{\tau}$ *and depth* $x \equiv \frac{x}{\sqrt{D\tau}}$.

In order to see better the character of this distribution we present below the time behavior of the pressure in the cross-sections $x \equiv \frac{x}{\sqrt{D\tau}} = 0.25, 0.5, 1$. The uppermost curve correspons to $x = 1$.

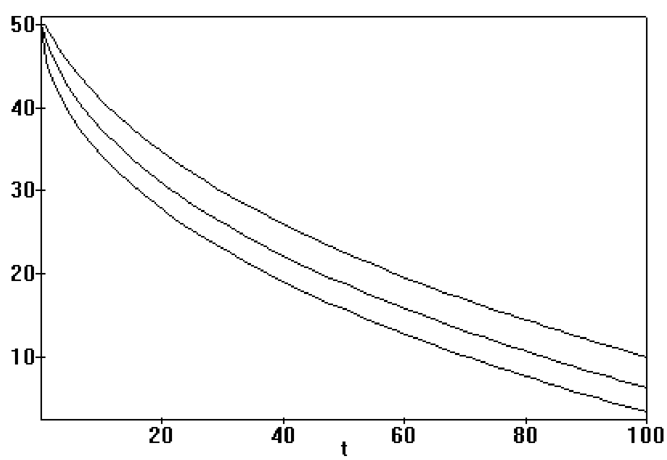



**Figure 4:** *Partial pressure* $p^F$ *as a function of time* $t \equiv \frac{t}{\tau}$ *in cross-sections* $x \equiv \frac{x}{\sqrt{D\tau}} = 1$ *(uppermost),* $\frac{x}{\sqrt{D\tau}} = 0.5$*, and* $\frac{x}{\sqrt{D\tau}} = 0.25$.

In soil mechanics it is customary to work with the so-called hydraulic gradient wich is defined as

$$g := \frac{\partial p^F}{\partial x}. \tag{5.7}$$

Its behavior is shown in the Figure 5.

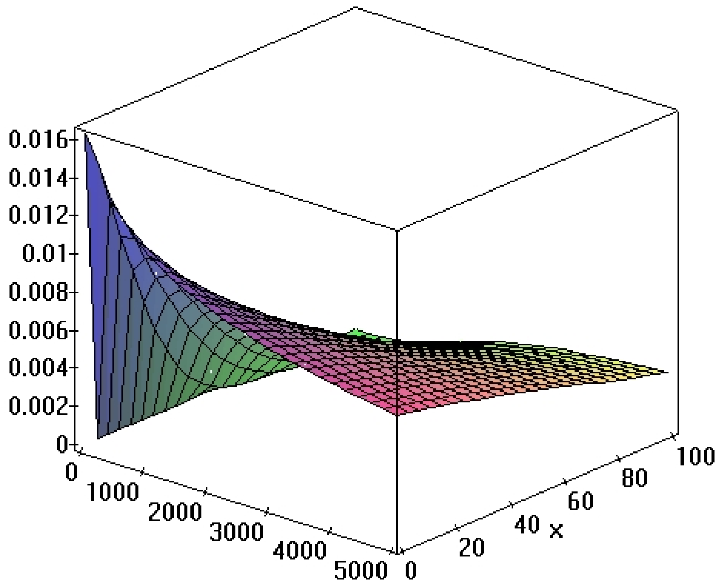
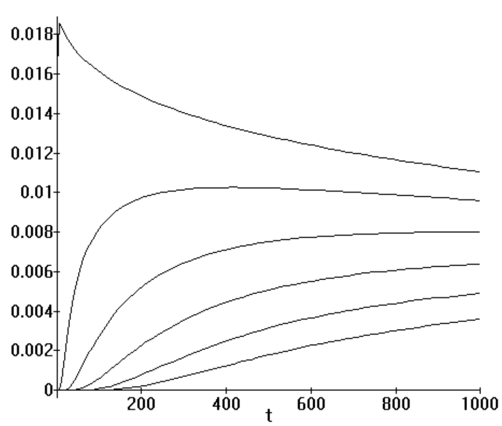



**Figure 5:** *Hydraulic gradient g as a function of time* $t \equiv \frac{t}{\tau}$, *and depth* $x \equiv \frac{x}{\sqrt{D\tau}}$.

Again this corresponds qualitatively with the classical results (e.g. see: [8], Fig.15.7]). It should be mentioned that quantitative differences of the present model and these of classical models of soil mechanics are due to the presence of an additional mechanism of the **flow through the boundary**. In particular cases typical for soil mechanics the present model agrees also quantitatively with the classical results and observations. This yields a particular range of the parameter $\alpha$ representative for phenomena in soils.



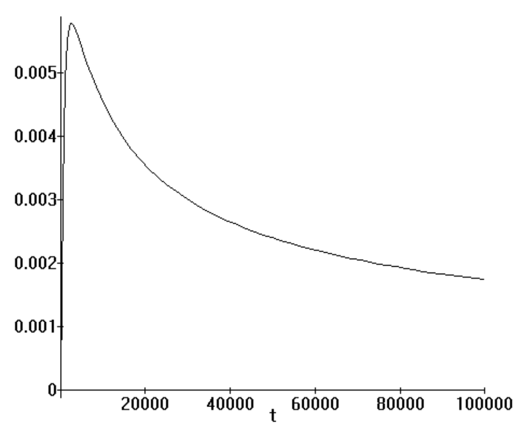




**Figure 6:** *Hydraulic gradient* $g = \frac{\partial p^F}{\partial x}\left(\frac{t}{\tau}\right)$ *for various depth* $x \equiv \frac{x}{\sqrt{D\tau}}$ : 1 *(uppermost),* $10, 20, 30, 40, 50$.

**Figure 7:** *Hydraulic gradient for* $x \equiv \frac{x}{\sqrt{D\tau}} = 40$.

In Figure 7 we demonstrate a chosen hydraulic gradient in order to expose the existence of maximum. This is the property of all curves in Figure 6 but the maximum is shifted to higher times for bigger depth.

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