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# Dynamical phenomena near homo- and heteroclinic connections involving saddle-foci in a Hamiltonian system

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**Abstract**

The main features of the orbit behavior for a Hamiltonian system in a neighborhood of homoclinic orbit to a saddle-focus equilibrium or a contour made up of two saddle-foci and two heteroclinic orbits to them are presented. These features includes description of hyperbolic subsets and main bifurcations when varying a value of the Hamiltonian. The proofs of results about bifurcations are given.

# 1 Introduction

The goal of this paper is to present results about dynamical behavior, especially bifurcations, in a two-degrees-of-freedom Hamiltonian system in a neighborhood of a homoclinic orbit to saddle-focus equilibrium. Such the study allows one to comprehend deeper the structure of a Hamiltonian system in the large being in the same time more tractable technically. First results in this direction were obtained by Devaney [1] who carried over the impressive and unexpected results by Shilnikov [2] from general systems to Hamiltonian ones that required of a special (symplectic) tool. The main task in [1] was to distinguish a hyperbolic subset in a neighborhood of a transverse (in a level of the saddle-focus $p$) homoclinic orbit to $p$. Namely, it was proved that a hyperbolic subset exists such that on a cross-section to orbits of this set the related Poincaré map was conjugated to the Bernoulli shift of 2 symbols. Since in [1] the system examined in the level $H = H(p)$, a bifurcational nature of the problem was not displayed. Bifurcations in this and similar problems involving homoclinic orbits to equilibria, not to periodic orbits, appear naturally when changing the internal parameter of any Hamiltonian system - the value of its Hamiltonian. In the problem under consideration a rich bifurcational structure was indicated in [3]. Though proofs of these results were absent there, all principal points to carry out the proofs were presented. For an interested reader we point out that the proofs of results concerning hyperbolic behavior and related symbolic dynamics, in particular, the description of homo- and heteroclinic orbits to the saddle-focus and nearby periodic orbits have been given in [4]. Here we present proofs about bifurcations following the lines given in [3]. Namely, we show that as $c \to H(p)$ countably many times parabolic periodic orbits emerge, they break up into elliptic and hyperbolic ones, then the elliptic orbit goes through doubling, giving rise to the beginning of the doubling cascade which ends with the enlargement of the hyperbolic set (the related Bernoulli shift acquires two new states). In addition, we point out the boundary points of intervals in $c$ where bifurcations related with changing the hyperbolic set take place.

Another problem intimately connected with the just mentioned is the structure of a Hamiltonian system near a heteroclinic connection with two saddle-foci $p_1, p_2$ which was studied in [4]. Naturally, such the connection can appear only if saddle-foci belong to the same level of the Hamiltonian. Under a perturbation $p_1, p_2$ generically diverse to different levels and connection breaks up. Thus, such the problem should be studied in at least one-parameter family of Hamiltonian systems. In such the setting the problem contains two parameters, the value of the Hamiltonian $c$ and the external (governing) parameter $\mu$. It has to expect a possibility of more complex degeneracies in such the system. For instance, we have shown in [4] that, in contrast to the case of a transverse homoclinic orbit to a saddle-focus, where all nearby homoclinic orbits are transverse, here two infinite sequences $\mu_k^{(i)}$, $i = 1, 2$, exist such that the system at $\mu = \mu_k^{(i)}$, has a nontransverse homoclinic orbit to $p_i$ with the quadratic tangency.

One more important reason of the interest to such the homoclinic phenomena is a possibility to understand scenaria of appearance, the existence and the structure of localized (pulses and fronts) traveling waves and stationary patterns to parabolic gradient-like 1D PDEs. Such the solutions can be temporally stable [5].

There is one circumstance why Hamiltonian systems with saddle-foci are not too known in mechanics, where usually so-called classical systems are studied. These systems have Hamiltonians which can be written in the form "kinetic energy plus potential energy" with a positive definite quadratic form (depending on a point of a configuration manifold) as the kinetic energy. For such the case the following Lemma holds.

**Lemma 1** *Let $H(p,q) = T(p,q)+V(q)$, $\quad T(p,q) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)p_i p_j = \frac{1}{2}(A(q)p,p)$ with positively defined symmetric matrix $A(q)$ and inner product $(\cdot,\cdot)$. If a point $q_*$ is a critical point $V$, $dV(q_*) = 0$, then complex eigenvalues of a linear system being a linearization of a Hamiltonian system with a Hamilton function $H$ at the point $p = 0, q = q_*$, if they exist, are pure imaginary.*

**Proof.** Indeed, as the matrix $A(q)$ is nondegenerate, then singular points of the system

$$\dot{p} = -H_q = -V'(q) - \frac{1}{2}(A'(q)p,p), \quad \dot{q} = H_p = A(q)p$$

are found from the system $p = 0$, $V'(q) = 0$. If $q = q_*$ is a solution of this system then the linearized system takes the form

$$\dot{\xi} = -B\eta = -V''(q_*)\eta, \quad \dot{\eta} = A(q_*)\xi = A\xi,$$

with positively defined $A$. In particular, there exists a positive definite matrix $A^{1/2}$. Let us change variables $P = A^{1/2}\xi$, $Q = A^{-1/2}\eta$. This transformation is linear symplectic one and reduces the system to the form

$$\dot{P} = -A^{1/2}BA^{1/2}Q, \ \dot{Q} = P$$

with symmetric matrix $\hat{B} = -A^{1/2}BA^{1/2}$. Therefore, squares of eigenvalues of this latter systems are eigenvalues of $\hat{B}$ which are real ones due to symmetricity of $\hat{B}$. Lemma is proved. Thus, systems with saddle-foci can appear in mechanics if related equations contain hyroscopic terms, etc..

The paper is organized as follows. The next section contains the setting up and statements of the main theorems. The necessary technical assertions are given in Sec.3. Sec.4 is devoted to the proof of the theorem on bifurcations. Results of [4] about hyperbolic behavior are essentially used in the paper.

# 2 The set up and main results

Let $(M, \Omega)$ be a smooth (analytic or $C^\infty$) four-dimensional symplectic manifold with symplectic form $\Omega$. Consider a $C^r$-smooth Hamiltonian vector field $X_H$ (necessary smoothness will be specified later on) with Hamiltonian $H$ such that $X_H$ has a singular points $p$ of saddle-focus type. The latter means the spectrum of a linearization operator of $X_H$ at $p$ consists of a quadruple of eigenvalues $\pm\alpha \pm i\beta$, $\alpha\beta \neq 0$. Such the point $p$ has two local smooth submanifolds, stable one $W^s$ and unstable $W^u$, lying both in the level $H = H(p)$. This set, outside of singular points is a smooth 3-dimensional submanifold. In particular, stable and unstable manifolds of the same or different saddle points (if they belong to the same level) generically intersect each other transversely.

## 2.1 Homoclinic connection

**Main assumption**. There is a homoclinic orbit $\Gamma$ to $p$ being transverse intersection of $W^s$, $W^u$ along $\Gamma$ in the level $H = H(p)$. Such the orbit is usually called the transverse homoclinic orbit to $p$ though, strictly speaking, this intersection is not transverse in $M$.

A general problem is to describe the orbit behavior of nearby orbits in some neighborhood $U$ of $\Gamma$. It is worth emphasizing that we consider only those orbits of $X_H$ which lie entirely in $U$ for all $t$. It turns out that even this problem is too hard as it will be clear from our results. Particular results in this direction are presented below.

To describe orbit behavior near the homoclinic connection we need in some notions of the symbolic dynamics (see, for instance, [6]). A symbolic system is constructed by means of a compact topological space called the alphabet $\mathcal{A}$, and some continuous mapping $\mathcal{T} : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$ called a transition matrix. The symbolic system consists of the space $Y$ being a set of all two-sided sequences $(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$ with the fixed zero position (the topology is given by the Tichonov product structure) such that any two symbols $\omega_i, \omega_{i+1}$ can follow one after another iff $\mathcal{T}(\omega_i, \omega_{i+1}) = 1$, and of a continuous map $\sigma : Y \to Y$ being a shift to the left for any symbolic sequence. This symbolic system denotes $(Y,\ \sigma)$. For our case we use the following alphabets and transition matrices. Consider a compact countable space $\mathcal{B}$ consisting of points $\pm n^{-1}, n \in \mathbf{N}\setminus\{0\}$ complemented with the nonseparable two-point space $\{0^+,\ 0^-\}$. This set has the discrete topology everywhere except for $0^+,\ 0^-$, and neighborhoods of the point $0^+$ are the sets $\{n^{-1}, n \geq k > 0\}$ along with the points $0^+,\ 0^-$, the sets $\{-n^{-1}, n \geq k > 0\}$ along with the points $0^-,\ 0^+$ are the neighborhoods for $0^-$.

As an alphabet we take a set $\mathcal{B}$, transitions are described as follows: i) after symbol $0^+$ can follow only $0^+$; ii) after any symbol of $\mathcal{B}$, but not $0^+$, can follow any symbol from $\mathcal{B}$ excepting for $0^-$; iii) only $0^-$ can precede $0^-$. The corresponding symbolic system is denoted $(Y_0, \sigma)$. Another symbolic system we use is $(Y_m, \sigma)$ (Bernoulli shift), here the alphabet is $\{\pm n^{-1}, n = 1, \dots, m\}$ and all transitions are admissible.

The orbit behavior in some neighborhood $U$ of $\Gamma$ is described via the description of orbits of the related Poincaré map on some cross-section $N$ to $\Gamma$. This section near the trace of $\Gamma$ is foliated by levels $H = c$ of the Hamiltonian into two-dimensional symplectic disks $N_c$ with respect to the restriction of 2-form $\Omega$. Thus, one obtains a one-parameter family of symplectic maps $P_c : N_c \to N_c$. The first theorem describes hyperbolic subsets existing in any level $N_c$. Recall that we describe only those orbits which lie in $U$ for $t \in \mathbf{R}$.

**Theorem 1** *1. At $c = 0$ Poincaré map $P_0$ on $N_0$ is conjugated to the symbolic system $(Y_0, \sigma)$. 2. There is $c_0 > 0$ such that for $|c| \leq c_0$ in the level $H = c$ an invariant hyperbolic subset exists for which the related Poincaré map is conjugated to symbolic system $(Y_m, \sigma)$, where $m = 2n(c)$, and function $n(c)$ has the following asymptotics as $|c| \to 0 : n(c) \sim -\frac{\beta}{2\pi\alpha} ln|c| + const.$ 3. In a segment $[-c_0, c_0]$ there is a countable set of accumulating zero disjoint intervals $I_n$, $n \in \{\mathbf{Z}\setminus 0\}$, such that for $c \in I_n$ the set of all orbits lying entirely in $U \cap \{H = c\}$ coincides with the hyperbolic subset of the item 2.*

We call intervals of the item 3 hyperbolicity intervals. In accordance with the construction, periodic orbits of $X_H$ correspond to periodic points of Poincaré map $P$, moreover, fixed points of $P$ give periodic orbits of the field that make one round along $\Gamma$, $n$-periodic points of $P$ give $n$-round periodic orbits of the field. In the same way the notion of $n$-round homoclinic orbits is introduced: these are homoclinic orbits of $X_H$ which are homotopic to $n\Gamma$ in a thin tube near $\Gamma$. The proof of the Theorem 1 is given in [4]. It relies on several auxiliary assertions which are presented below.

The construction of hyperbolic subsets gives the following property of these sets. If one fixes the number $2n$ of states in the Bernoulli shift then the hyperbolic set with this number of states exists for all values of $c$ with $|c| < c_n < c_0$. In particular, for $|c| \leq c_0$ there exists a hyperbolic set with 2 states. This set contains two fixed saddle points, one orientable and one nonorientable. Stable and unstable manifolds of the orientable saddle periodic orbit play an essential role in detecting boundaries of bifurcational intervals in $c$ (see, Subsec. 4.1).

It follows from the Theorem 1 that for any $n \in \mathbf{N}$ there are orbits corresponding to sequences $(\ldots, 0^-, 0^-, a_1, a_2, \ldots, a_n, 0^+, 0^+, \ldots)$. These orbits are homoclinic to $p$ and $n+1$ is their roundness.

**Corollary 1** *In a neighborhood of $\Gamma$ there are countably many homoclinic orbits of any roundness.*

**Remark 1** *It follows from the proof of this theorem that all these homoclinic orbits are transverse as $\Gamma$ itself.*

Further assertions concern with the bifurcational phenomena occurring when $c$ varies near $H = H(p)$. Theorem 1 implies that, as $|c| \to 0$, the number of states in the related Bernoulli shift $(Y_m, \sigma)$ increases, hence, bifurcations have to occur giving rise reconstructions in the orbit structure in levels $H = c$. It turns out that on the segment $[-c_0, c_0]$ in the complementary set to hyperbolicity intervals there are subintervals such that when $c$ runs them bifurcations really take place.

Let us fix $c > 0$ to be definite, and denote $(c'_{n+1}, c''_{n+1})$, $(c'_n, c''_n)$ two neighboring hyperbolic intervals, $c''_{n+1} < c'_n$.

**Theorem 2** *1. In each interval $(c''_{n+1}, c'_n)$ a subinterval $J_n$ exists such that in $J_n$ there are points $d_0 > d_1$ corresponding to the following bifurcations of the Poincaré map $P_c$: i) at $c = d_0$ inside of rectangle $N_c$ a parabolic fixed point appears which breaks up for $c < d_0$ into elliptic and hyperbolic fixed points, both of them persist till $c = d_1$; ii) at $c = d_1$ the elliptic point becomes a degenerate fixed point with double multiplier $-1$, two-dimensional Jordan box of the linearization matrix and nonzero Lyapunov value that leads to its doubling for $c < d_1$ and appearing a period 2 elliptic periodic point, the degenerate fixed point changes into a nonorientable saddle fixed point.*

The same is valid for $c < 0$.

**Remark 2** *The bifurcation occuring at $c = d_1$ is, in fact, the beginning of a doubling cascade leading to the formation of new Smale horseshoe constructed on two saddle fixed points, namely, the orientable saddle (with positive eigenvalues) appearing from the parabolic point after its destruction and nonorientable (Möbius) saddle having appeared from the elliptic point in the process of the first doubling. See [13], where this process is discussed in more details.*

## 2.2 Heteroclinic connection.

Suppose now that $X_{H_0}$ has two singular points $p_1, p_2$ both of saddle-focus type with eigenvalues $\pm\alpha_i \pm i\beta_i$, $\alpha_i\beta_i \neq 0$.

ASSUMPTION 1. For $X_{H_0}$ both points $p_i$, $i = 1, 2$, belong to the same level and there are two heteroclinic orbits $\gamma_1, \gamma_2$ joining respectively, when time increases, $p_1$

with $p_2$ (for $\gamma_1$) and $p_2$ with $p_1$ (for $\gamma_2$). Intersection of $W^s(p_i)$ and $W^s(p_j)$, $i \neq j$, along $\gamma_k$, $k = 1, 2$, is transverse.

We shall call the set made up of $\gamma_i$, together with the points $p_i$, $i = 1, 2$, heteroclinic connection or contour and denote it $\Gamma$. The existence of a heteroclinic contour is a codimension one phenomenon in the space of all Hamiltonians with $C^r$ topology, $r \geq 2$. Thus, the most natural problem is to study orbit structure and its bifurcations in an, at least, one-parameter unfolding $H_\mu$ of $H_0$. The unfolding $H_\mu$ is supposed to be $C^2$-smooth in $\mu$.

As $p_i$ are nondegenerate, $C^2$-functions $p_i : (-\mu_0, \mu_0) \to M$ are defined such that $p_i(\mu)$ is a saddle-focus singular point for the vector field with the Hamiltonian $H_\mu$, $p_i(0) = p_i$. Changing Hamiltonian $H_\mu \to H_\mu - H_\mu(p_1(\mu))$ we may assume $H_\mu(p_1(\mu)) \equiv 0$. Then $f(\mu) = H_\mu(p_2(\mu))$ is a $C^2$-function of $\mu$ and we impose the following genericity condition

ASSUMPTION 2. Function $f(\mu)$ has a simple zero at $\mu = 0$ on $(-\mu_0, \mu_0)$, i.e., $f'(0) \neq 0$.

This condition means that when changing $\mu$ the point $p_2(\mu)$ crosses the level where $p_1$ lies with nonzero velocity .

Let $U$ be some neighborhood of $\Gamma$. We study those orbits of the unfolding which lie entirely in $U$. Their study is carried out by means of investigating of the related Poincare map $P_{\mu,c}$ constructed on some cross-section to $\gamma_1$, where $H_\mu = c$ fixes the level.

For any fixed $\mu$ the section is smoothly foliated by levels $H_\mu = c$ and flow preserves the levels, so every map can be restricted at such a level giving a two-parameter family of related symplectic maps with respect to restriction of the symplectic form $\Omega$ to these levels.

In this case symbolic dynamics is given by the alphabet $\mathcal{A} = \mathcal{B}_1 \cup \mathcal{B}_2$ consisting of two copies of $\mathcal{B}$. The transitions are described as follows: 1) after any symbol from $\mathcal{B}_i$, excepting for $0^+$, can follow only a symbol from $\mathcal{B}_j$, $i \neq j$, moreover, it can be any symbol from $\mathcal{B}_j$ but not $0_j^-$; after symbol $0_1^+$ can follow only $0_1^+$, similarly, after $0_2^+$ can follow only $0_2^+$; 2) only $0_1^-$ can precede $0_1^-$, only $0_2^-$ can precede $0_2^-$.

The set of all admissible sequences obtained is denoted $Y_0$ and the symbolic system $(Y_0,\ \sigma)$. We also will use symbolic systems $(Y_m,\ \sigma)$ and $(Y_{nm},\ \sigma)$ whose alphabets are the sets $\mathcal{B} \cup B_m$ and $B_n \cup B_m$.

The set of all admissible sequencies obtained is denoted $Y_0$, and the symbolic system is $(Y_0, \sigma)$. We aslo use symbolic systems $(Y_m, \sigma)$ and $(Y_{mn}, \sigma)$ whose alphabets are the sets $\mathcal{B} \cup B_m$ and $B_n \cup B_m$, here $B_m$ is a set of numbers $\pm 1$, $\pm 1/2, \ldots, \pm 1/m$. For corresponding symbolic systems transitions are described in the same way with the evident change concerning $\mathcal{B}_m$ since it does not contain symbols $0^-$, $0^+$.

Now we can formulate our results.

**Theorem 3** *There are positive constants $\mu_0$, $c_0$ small enough and a neighborhood $U$*

*of $\Gamma$ such that for $H_\mu$, $|\mu| \le \mu_0$, in $U \cap \{H_\mu = c\}$ there exists an invariant hyperbolic subset $\mathcal{H}_{\mu,c}$ such that the Poincare map $P_{\mu,c}$ for $\mathcal{H}_{\mu,c}$ is conjugated to:*

*1) symbolic system $(Y_0,\ \sigma)$ for $\mu = 0,\ c = 0$;*

*2) symbolic system $(Y_n,\ \sigma)$ for $\mu \ne 0,\ c = 0$ with some $n = n(\mu),\ n(\mu) \sim \frac{\beta_2}{2\pi\alpha_2}\ln(|f(\mu)|) + const$;*

*3) symbolic system $(Y_{nm},\ \sigma)$ for $\mu \ne 0,\ c \ne 0$ with $n \sim \frac{\beta_1}{2\pi\alpha_1}\ln|c| + const$; $m \sim \frac{\beta_2}{2\pi\alpha_2}\ln(|f(\mu)|) + const$;*

*4) symbolic system of the item 2 when $\mu \ne 0,\ c = f(\mu)$ with $n \sim \frac{\beta_1}{2\pi\alpha_1}\ln|c| + const$;*

*For $\mu = c = 0$ the set $\mathcal{H}_{\mu,c}$ exhausts all orbits lying entirely in $U$. There is a countable set of disjoint intervals $I_l \subset [-\mu_0,\ \mu_0]$ accumulating $\mu = 0$ such that for these $\mu$ the set $\mathcal{H}_{\mu,0}$ exhausts all orbits in $U \cap \{H_\mu = 0\}$ lying entirely in $U$.*

*There are two sequences $\{\mu_k^{(1)}\},\ \{\mu_k^{(2)}\},\ \ \mu_k^{(i)} \in [-\mu_0,\ \mu_0]$ such that the vector field for $H_\mu,\ \mu = \mu_k^{(i)}$, possesses a nontransverse homoclinic orbit to $p_i$ with quadratic tangency.*

Symbolis dynamics gives all necessary information concerning homo- and heteroclinic orbits as well as other types of orbits. For a reader convenience we formulate two corollaries about the formers.

**Corollary 2** *For $H_0$ the vector field on the level $H_0 = 0$ has: i) a countable set of transverse heteroclinic contours of any circuitness; ii) each saddle-focus $p_i$ has a countable set of transverse homoclinic orbits of any circuitness; iii) for each integer $N > 0$ there is a countable set of saddle periodic orbits of circuitness $N$, every such orbit $\gamma$ possesses a countable set of heteroclinic orbits with $p_i,\ i = 1, 2,\ \gamma \to p_i$,* and *$p_i \to \gamma$, where the arrow points out the direction of increasing time.*

**Corollary 3** *For $\mu \ne 0,\ c = 0$ the same assertions as in the items 2, 3 of Cor. 1 are true for $p_1$ and periodic orbits in the level $H_\mu = 0$. For $\mu \ne 0,\ c = f(\mu)$ the same is valid for the saddle-focus $p_2(\mu)$.*

Proofs of the Theorem 3 and corollaries are given in [4].

# 3 Auxiliary results

We use the Moser's normal form for a Hamiltonian [9] to represent the local flow. Though it was found for analytic Hamiltonians, it also works in $C^\infty$-case (Lychagin) and sufficiently smooth case [7, 8]. It is sufficient the Hamiltonian to be $C^{12}$, then by means of $C^4$-smooth symplectic transformation it can be brought into the following normal form in a symplectic frame $(x_1, x_2, y_1, y_2), \Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, near a saddle-focus

$$H(x_1, x_2, y_1, y_2) = h(\xi, \eta) = \alpha\xi + \beta\eta + \cdots, \xi = x_1y_1 + x_2y_2,\ \eta = x_1y_2 - x_2y_1 \quad (1)$$

with a polynomial $h$. If $H$ $C^r$-smoothly depends on a parameter $\mu \in (-\mu_0,\ \mu_0)$ then for all $|\mu|$ small enough there are symplectic coordinates $C^r$-smoothly depending on $\mu$ such that $H_\mu$ has the same form (1), only $h$ will depend on $\mu$.

The related equations are the following

$$\dot{x_1} = -h_\xi x_1 + h_\eta x_2,\ \dot{x_2} = -h_\xi x_2 - h_\eta x_1,\ \dot{y_1} = h_\xi y_1 + h_\eta y_2,\ \dot{y_2} = h_\xi y_2 - h_\eta y_1, \quad (2)$$

The functions $\xi, \eta$ are local integrals so equations are immediately integrated

$$\begin{aligned} x_1(t) &= \exp(-t\ \overset{\circ}{h}_\xi)(x_1^0 \cos(t\ \overset{\circ}{h}_\eta) + x_2^0 \sin(\overset{\circ}{h}_\eta)),\\ x_2(t) &= \exp(-t\ \overset{\circ}{h}_\xi)(-x_1^0 \sin(t\ \overset{\circ}{h}_\eta) + x_2^0 \cos(\overset{\circ}{h}_\eta)),\\ y_1(t) &= \exp(t\ \overset{\circ}{h}_\xi)(y_1^0 \cos(t\ \overset{\circ}{h}_\eta) + y_2^0 \sin(\overset{\circ}{h}_\eta)),\\ y_2(t) &= \exp(t\ \overset{\circ}{h}_\xi)(-y_1^0 \sin(t\ \overset{\circ}{h}_\eta) + y_2^0 \cos(\overset{\circ}{h}_\eta)), \end{aligned} \quad (3)$$

here index zero means calculation at an initial point. Without loss of generality one may assume $\alpha$ to be positive that can be achieved by a change of variables preserving the form (1) of the Hamiltonian.

To find the local map $T$ and its properties let us take as a local sections submanifolds $N^s = \{(x_1, x_2, y_1, y_2)|x_1^2 + x_2^2 = \rho_s^2,\ y_1^2 + y_2^2 \le \delta_s^2\}$, $N^u = \{(x_1, x_2, y_1, y_2)|y_1^2 + y_2^2 = \rho_u^2,\ x_1^2 + x_2^2 \le \delta_u^2\}$. These submanifolds are foliated by levels $H = c$ into two-dimensional annulas, $N^s(c),\ N^u(c)$. It is convenient to represent these annuli via coordinates $(\theta, \eta, c), (\varphi, \eta, c)$, respectively. These coordinates are introduced by the following relations

$$x_1 = \rho_s \cos(\theta),\ x_2 = \rho_s \sin(\theta),\ y_1 = \rho_s^{-1}(\xi \cos(\theta) - \eta \sin(\theta)),\ y_2 = \rho_s^{-1}(\xi \sin(\theta) + \eta \cos(\theta))$$

When choosing a neighborhood $U$ one may regard the equation $h(\xi, \eta) = c$ to be uniquely solved with respect to $\xi$ in $U$, $\xi = a_c(\eta) = \alpha^{-1}(c - \beta\eta + \cdots)$. Easy calculation shows that restrictions of the form $\Omega$ on these annuli are given by 2-forms $d\theta \wedge d\eta$ and $d\varphi \wedge d\eta$, respectively, so these coordinates are symplectic on $N^s(c),\ N^u(c)$, respectively. It means, in particular, that local maps $T_1(c),\ T_2(c)$ and global ones, $S_1(c),\ S_2(c)$, are area preserving in these coordinates.

**Remark 3** *Making a shift $\theta \to \theta + \theta_0$ one can regard the origin in $\theta$ at any point on the circle, the same is valid for $\varphi$. For our problem it is convenient to choose the origins at the points of intersection of $\Gamma$ with sections. In the sequel, it is supposed it is the case.*

To write down $T$ one needs to find the time of passage for any orbit from $N^s(c)$ to $N^u(c)$. In coordinates this time is given

$$t_{pas} = (\overset{\circ}{h}_\xi)^{-1} \ln\left(\rho_s\rho_u/\sqrt{\eta^2 + a_c^2(\eta)}\right)$$

Inserting $t_{pas}$ and $a_c(\eta)$ into (3)

The orbit behavior is studied by means of related Poincaré map constructed on some cross-sections $N^s$, $N^u$ to stable and unstable manifolds of $p$. They are foliated by levels $H = c$ into annulae $N^s_c$, $N^u_c$. For the case of homoclinic connection this map is a superposition of two maps, local one near $p$ and global one, near a global piece of $\Gamma$. Using (1) and introducing a function $\xi = a_c(\eta) = \alpha^{-1}(c - \beta\eta + \cdots)$ being a unique solution of the equation $h(\xi, \eta) = c$ with respect to $\xi$ in some neighborhood of $p$, we obtain the following representation of the local map $T_c$ ([1, 3, 4])

$$\varphi = \theta + B_c(\eta)\ (mod 2\pi), \quad \eta = \eta \tag{4}$$

with

$$B_c(\eta) = a'_c(\eta) \ln\left(\rho_s\rho_u/\sqrt{\eta^2 + a_c^2(\eta)}\right) + \Phi_c(\eta) \tag{5}$$

the function $\Phi_c(\eta)$ is defined as the principal branch of $\mathrm{Arctan}(\eta/a_c(\eta))$ with $\Phi_0(+0) = \pi - \arctan(\alpha/\beta)$, $\Phi_0(-0) = -\arctan(\alpha/\beta)$.

The global map $S_c$ is defined in some neighborhoods of traces of $\Gamma$ on the sections, it is given in coordinates $(\theta, \eta, c)$ on the sections $N^s$ and $(\varphi, \eta, c)$ on $N^u$ as

$$\theta_1 = f(\varphi, \eta, c) = f_c(\varphi, \eta), \ \eta_1 = g(\varphi, \eta, c) = g_c(\varphi, \eta) \tag{6}$$

with $D(f,\ g)/D(\varphi,\ \eta) \equiv 1$ (symplecticity), $f(0,0,0) = g(0,0,0) = 0$ (since the trace of $\Gamma$ on $N^u$ is transformed to the trace of $\Gamma$ on $N^s$), $(\partial g/\partial\varphi)(0,0,0) \neq 0$ (transversality condition of $W^s$ and $W^u$).

**Remark 4** *It is worth noting that, in dependence on the sign of the quantity $l = (\partial g/\partial\varphi)(0,0,0)$ two cases can be distinguished, $A:\ l > 0$ and $B:\ l < 0$. In the case $A$ a region $\eta > 0$ by the map (6) transforms to a region to the right of the trace of of unstable manifold on $N^s(0)$, and onto a region to the left of this trace in the case $B$. This disposition determines later on the position of orientable saddle fixed points playing the essential role in dynamics.*

All further considerations are carried out in some neighborhoods of points $\Gamma \cap N^s$, $\Gamma \cap N^u$. These neighborhoods $\Pi^s$, $\Pi^s$ are determined by inequalities

$$\Pi^s = \{(\theta,\ \eta,\ c)|\ |\theta| \leq \delta,\ |\eta| \leq \epsilon,\ |c| \leq c_0\},\ \ \Pi^u = \{(\varphi,\ \eta,\ c)|\ |\varphi| \leq \delta,\ |\eta| \leq \epsilon,\ |c| \leq c_0\}$$

for $\epsilon,\ \delta,\ c_0$ small enough. The levels $V_c = \{H = c\}$ are invariant sets, so we obtain a family of Poincaré maps depending on parameter $c$, given on rectangles $\Pi^s_c = \Pi^s \cap V$, $\Pi^u_c = \Pi^u \cap V_c$.

The properties of the local map $T$ are formulated in the next Lemmas, their proofs are given in [4]. Here and later on we denote $O_k(x)$ a function that is given on a neighborhood of $x = 0$ and such that $O_k(x)/x^k$ is bounded as $x \to 0$, $o(x)$ means that $o(x)/x \to 0$ as $x \to 0$, and $O(x)$ denotes a function which tends to zero as $x \to 0$.

**Lemma 2** *For $|c|$, $|\eta|$ small enough the following holds*

$$B_0'(\eta) = \frac{\beta/\alpha + O(\eta)}{\eta}, \ |B_0'| \leq \frac{\beta}{2\alpha|\eta|}; \tag{7}$$

$$B_c'(\eta) = \left[L(c,\eta) + a_c''R(c,\eta) + O_2(c,\eta)\right]/(\eta^2 + a_c^2(\eta)) \tag{8}$$

*with* $L(c,\eta) = \alpha^{-3}(\alpha^2 - \beta^2)c + \beta(\beta^2+\alpha^2)\eta,\ R(c,\ \eta) = (\eta^2 + a_c^2(\eta))\ln(\eta^2 + a_c^2(\eta))$*;*

$$B_c''(\eta) = \left[q_2(c,\ \eta) + O_3(c,\ \eta)\right]/(\eta^2 + a_c^2(\eta))^2 \tag{9}$$

*with a quadratic form* $q_2(c,\ \eta) = \alpha^{-2}\sigma(3-\sigma^2)c^2 + 2\alpha^{-1}(\sigma^4-1)c\eta - \sigma(1+\sigma^2)^2\eta^2,\ \ \sigma = \beta/\alpha$*, having positive discriminant* $\Delta = \alpha^{-2}(\sigma^2+1)^3$*.*

To formulate next lemma let us consider a standard covering of the annulus $N_c^u$. It is a strip on the plane $(\varphi,\ \eta)$, where $\varphi$ is considered as affine coordinate, $|\eta| \leq \epsilon$. If $\theta = u(\eta)$ is a function given for $|\eta| \leq \epsilon$, then the image of its graph w.r.t. $T_c$ is a curve in the strip $(\varphi,\ \eta)$, being graph of a function (see (4))

$$\varphi = u(\eta) + B_c(\eta). \tag{10}$$

**Lemma 3** *There are positive $\epsilon$, $c_0$ small enough such that in the strip $(\varphi,\ \eta)$, $|\eta| \leq \epsilon$, the image under $T(c)$ of the graph of a $C^2$-function $\theta = u(\eta)$, $|u(\eta)| \leq \delta$, $|u'(\eta)| \leq d_1$, $|u''(\eta)| \leq d_2$, is*

*1. for $c = 0$ the graph of a function $\varphi(\eta)$ for $c = 0$, being $C^2$-smooth everywhere on $|\eta| \leq \epsilon$ except for the point $\eta = 0$ where it has a logarithmic singularity, derivative of $\varphi(\eta)$ satisfies the estimate $|\varphi'(\eta)| \geq \beta/2\alpha|\eta|$;*

*2. for $c \neq 0$, $|c| \leq c_0$, $c_0$ small enough, the graph of a $C^2$-smooth function $\varphi(\eta)$ such that*

*i) $\varphi'(\eta)$ is a monotone function with a unique zero at a minimum point $\eta_c$, $\eta_c = \frac{\beta^2-\alpha^2}{\beta(\beta^2+\alpha^2)}c + o(c)$, $o(c) \to 0$ as $|c| \to 0$;*

*ii) the value $\varphi(\eta_c)$ tends to $-\infty$ when $|c| \to 0$, moreover, the following representation is valid: $\varphi(\eta_c) = (\beta/\alpha)\ln|c| + E(c)$ with a bounded function $E(c)$, and $\frac{d}{dc}\varphi(\eta_c) = (\beta/\alpha + O(c))/c$.*

The following lemma allows one to distinguish the region of hyperbolicity and a band on $\Pi_c^s$ where the creation of parabolic fixed points occurs.

**Lemma 4** *For any $K > 0$ there exists $\gamma > 0$ such that for all $|c| \leq c_0$ there is a region on the segment $|\eta| \leq \epsilon$, where the estimate $|B_c'(\eta)| \geq K$ holds. Furthermore,*

*if $c = 0$ then this region coincide with the segment $|\eta| \leq \epsilon$;*

*if $c \neq 0$ then this region consists of two segments given with inequalities $\eta_c + \gamma c^2 \leq \eta \leq \epsilon$ and $-\epsilon \leq \eta \leq \eta_c - \gamma c^2$.*

Next lemma is used for proofs that tangency is quadratic if stable and unstable manifolds of some periodic orbit in a neighborhood of $\Gamma$ are tangent.

**Lemma 5** *Consider a family of smooth $C^2$-functions of the form $\varphi = v(\eta)$, $|\eta| \leq \epsilon$ with $C^2$-norms bounded with some constant $D$. Then there is a positive $c_1$ small enough such that for all $c$, $|c| \leq c_1$, graphs of any function $\varphi(\eta)$ from lemma 3 and of $v(\eta)$ are quadratically tangent if they have a tangency point.*

Now we are able to describe the domain of the map $T$ and its restrictions $T_c$ (see (4)). Let us denote $\eta = \lambda_{\pm}(\xi)$ two branches of the inverse function for $\xi = B_c(\eta)$. Fix $\epsilon > 0$, $\delta > 0$ such that conclusions of preceding Lemmas 1-4 would hold.

1. $c = 0$. Then for any $\theta$, $|\theta| \leq \delta$, curves $\varphi = \theta + B_0(\eta)$ (here $\theta$ is a parameter marking the curve) monotonically decrease for $\eta < 0$ and increase for $\eta > 0$. Since $B_0(\eta) \to -\infty$ as $|\eta| \to 0$, then graphs of inverse functions $\eta = \lambda_+(\varphi - \theta)$ and $\eta = \lambda_-(\varphi - \theta)$ being projected into the annulus $N_0^u$ are the curves which go round the annulus infinitely many times approaching the circle $\eta = 0$ as $\varphi \to -\infty$. Take two such the curves with $\theta = \pm\delta$. Then, beginning from some $n_0 > 0$ these curves will intersect all segments $\varphi = \delta - 2\pi|n|$, $n \geq n_0$, $|\eta| \leq \epsilon$. Thus, we obtain infinitely many strips

$$\begin{array}{ll} \sigma_n^u = \{(\varphi,\ \eta) | |\varphi| \leq \delta,\ \lambda_+(\varphi - \delta + 2\pi n) \leq \eta \leq \lambda_+(\varphi + \delta + 2\pi n)\}, & \text{if } n > 0 \text{ (i.e., } \eta > 0), \\ \sigma_n^u = \{(\varphi,\ \eta) | |\varphi| \leq \delta,\ \lambda_-(\varphi + \delta - 2\pi n) \leq \eta \leq \lambda_-(\varphi - \delta - 2\pi n)\}, & \text{if } n < 0 \text{ (i.e.., } \eta < 0), \end{array} \tag{11}$$

being the domain of $T_0$.

2. $c \neq 0$. Lemma 3 implies that curves $\varphi = \delta + B_c(\eta)$ for all $|n| \geq n_0$ intersect segments $\varphi = \delta - 2\pi|n|$, $n_0 \leq n \leq n_1(c)$, $|\eta| \leq \epsilon$, where $n_1(c)$ is equal to that maximal $n$ such that $\delta + B_c(\eta) + 2\pi n < -\delta$. For these $n$ equation $\varphi = \theta + B_c(\eta)$ can be solved for $\theta = \pm\delta$ giving for $n > 0$ lower (at $\theta = \delta$) and upper (for $theta = -\delta$) boundaries of the strip $\sigma_n^u$. If $n < 0$ then lower and upper boundaries change places. The difference $n_1(c) - n_0$ gives the upper estimate for the number of strips. This differs from the number of states $n(c)$ in the Bernoulli shift as the latter requires hyperbolicity that can be achieved for lower value of strips. In fact, hyperbolicity can be proved for intervals $|\eta - \eta_c| \geq \gamma c^2$ where $|B_c'| > K$. One can be shown (see [4]) that asymptotically

$$n(c) \sim -\frac{\beta}{2\pi\alpha} \ln |c| + const. \tag{12}$$

We have constructed strips $\sigma_n^u$ making up the range of the local map $T$. The domain of this map, $\sigma_n^s$, are preimages of $\sigma_n^u$, and they are determined with inequalities similar to (11) with $|\lambda_{\pm}(\pm\delta - \theta - 2\pi|n|)$.

## 4 Proof of Theorem 2

We look for fixed points of the Poincaré map as follows. For the global map $S_0$ the inequality $(\partial g_0/\partial\varphi)(0,0) \neq 0$ holds, $f_0(0,0) = 0, \quad g_0(0,0) = 0$, then for some positive small $c_0$ and $|c| < c_0$ the second equation in (6) can be solved w.r.t. $\varphi$, therefore (6) can be rewritten in the "cross" form

$$\theta_1 = Q_c(\eta, \eta_1) = -\frac{\partial F_c}{\partial \eta_1}, \quad \varphi = R_c(\eta, \eta_1) = \frac{\partial F_c}{\partial \eta} \tag{13}$$

for $|\eta|$, $|\eta_1|$, $|\varphi|$, $|\theta|$ small enough, here $F_c$ is a generating function of the symplectic map. Using (4) the equations for finding fixed points take the form

$$Q_c(\eta, \eta) = \theta \ (mod\ 2\pi), \quad R_c(\eta, \eta) = \theta + B_c(\eta) \ (mod\ 2\pi).$$

Eliminating $\theta$ from these equations we come to the equation

$$r_c(\eta) = B_c(\eta) \ (mod\ 2\pi), \quad \text{with } r_c(\eta) = R_c(\eta, \eta) - Q_c(\eta, \eta) = \frac{\partial}{\partial \eta} F_c(\eta, \eta). \tag{14}$$

It is easily seen that the function $\psi = r_c(\eta)$ in l.h.s of this equation is a smooth function in $\eta, c$, which vanishes at $\eta = 0, c = 0$. Considering it as a family of smooth functions of $\eta$ depending smoothly on a parameter $c$ we get that the functions of this family are $C^3$-close to that which corresponds to $c = 0$. The graph of this latter function contains the point $(0,0)$, therefore graphs of all functions of the family lie in the band $|\psi| \leq \delta$ for $\epsilon$ small enough.

On the other hand, due to Lemmas 3, 4 the function $B_c(\eta)$ on the segment $[-\epsilon, \epsilon]$ has a unique minimum that monotonically tends to $-\infty$ as $|c| \to 0$. Considering its graph in the strip $|\eta| \leq \epsilon, -\pi \leq \psi \leq \pi$ one obtains that it consists of finitely many branches with their range the segment $[-\pi, \pi]$ and a middle part in the form of a parabola-like sharp tongue that stretches monotonically till $\psi = -\pi$ when decreasing $|c|$. It implies that these two graphs always have finitely many points of the transverse intersection inside of the band $|\psi| \leq \delta$, these intersection points correspond to fixed points of the hyperbolic set (they also correspond to the fixed points of the Bernoulli shift), and at any passage of the tongue through the band one obtains a point of tangency of these graphs. So, we have countably many such values of $c$ when $|c| \to 0$. That the tangencies are quadratic follows from the representation (9) implying $|B_c''(\eta)| \to \infty$ in the region $|\eta - \eta_c| \leq \gamma c^2$ where tangencies occur.

In order to connect the intersection points of the graphs and fixed points of the map let us apply the idea from [10] which connects fixed points of an area preserving map and critical points of its generating function.

**Lemma 6** *Let $S : (x, y) \to (x_1, y_1) = (f(x, y), g(x, y))$ be a symplectic map and suppose $g_x \neq 0$ in some simply connected region $G$ such that the map can be written in the cross form $x = P(y, y_1)$, $x_1 = Q(y, y_1)$ with a generating function $F(y, y_1)$, that is, $P(y, y_1) = F_y$, $Q(y, y_1) = -F_{y_1}$, and $F(y, y_1)$ is defined in some simply* *connected region $D$, where $F_{yy_1} \neq 0$. Then if $(x_*,\ y_*)$ is any isolated fixed point of $S$ in $G$ then $y_*$ is an isolated critical point of the function $f(y) = F(y,\ y)$. Conversely, if $y_*$ is a critical point of this function such that the point $(x_*,\ y_*),\ x_* = F_y(y_*,\ y_*)$ belongs to $G$ then $(x_*,\ y_*)$ is the fixed point of $S$. Moreover, nondegenerate critical points of the generating function correspond to hyperbolic and elliptic points of the map in dependence of the sign of the second derivative, and vice versa.*

For our case the Poincaré map takes the form

$$\theta_1 = f_c(\theta + B_c(\eta) + 2\pi k,\ \eta),\ \eta_1 = g_c(\theta + B_c(\eta) + 2\pi k,\ \eta). \tag{15}$$

As we have already known (see Sec. 3), the domain of the local map $T_c$ consists of either countably many strips for $c = 0$ or of the finite number of strips (always exist for $c \neq 0$) and, in addition for some $c$, of a middle connected part where $B_c'$ can vanish ("dangerous" zone). For these latter $c$ there are a positive integer $k$ and values of $(\theta, \eta)$ such that the value of the first argument belongs to the interval $(-\delta,\ \delta)$. A generating function $\hat{F}_c(\eta, \eta_1)$ of the map is $F_c(\eta, \eta_1) - 2\pi k\eta - \int^{\eta} B_c(s)ds$, therefore,

$$\frac{\partial \hat{F}_c}{\partial \eta} = \frac{\partial F_c}{\partial \eta} - B_c(\eta) - 2\pi k,\ \ \frac{\partial \hat{F}_c}{\partial \eta_1} = \frac{\partial F_c}{\partial \eta_1}$$

. So, using (13) we get

$$\theta = R_c(\eta, \eta_1) - B_c(\eta) - 2\pi k = \frac{\partial \hat{F}_c}{\partial \eta}, \quad \theta_1 = Q_c(\eta, \eta_1) = -\frac{\partial \hat{F}_c}{\partial \eta_1}, \tag{16}$$

and the equation for searching for critical points is

$$f_c'(\eta) = \left(\frac{\partial \hat{F}_c}{\partial \eta} + \frac{\partial \hat{F}_c}{\partial \eta_1}\right)|_{y=y_1} = R_c(\eta, \eta) - B_c'(\eta) - 2\pi k - Q_c(\eta, \eta) = 0,$$

that is, it precisely coincides with (14). To determine the type of the point appearing at the tangency we use the following assertion [10].

**Lemma 7** *(Parabolicity conditions) Let, under the conditions of the preceding Lemma, the critical point be simplest degenerate, i.e., $f''(y_*) = 0$ but $f'''(y_*) \neq 0$. Then the corresponding fixed point of the map is parabolic, that is, it has 1 as a double multiplier, Jordan form of the related linearization matrix is 2-dimensional box, and the related coefficient (see below) in the normal form of the second order at this point does not vanish. If, in addition, the family of the functions $f$ depends smoothly on a parameter $c$, and at $c = 0$ a simplest degenerate critical point exists at which $f_y'''(y_*,\ 0) \neq 0$ and $\frac{\partial^2}{\partial c \partial y} f(y_*,\ 0) \neq 0$, then, when passing through $c = 0$ the following bifurcation of the map occurs: on the one side of $c = 0$ the related map has not fixed points near $(x_*,\ y_*)$, but on the other side there are two fixed points, elliptic and hyperbolic ones.*

The related normal form of the second order to which any area preserving map near its parabolic point can be transformed is the following

$$x_1 = x + y + Ax^2 + \ldots, \quad y_1 = y + Ax^2 + \ldots.$$

Parabolicity condition (i.e., not more higher degeneracy) is $A \neq 0$. Two-dimensionality of Jordan box follows from the inequality $g_x \neq 0$.

This Lemma implies that a fixed point will be a parabolic if it corresponds to a simplest degenerate critical point of generating function. The Lemma works in our case. Indeed, let us calculate $f_c^{''}(\eta_*)$. As is easily seen, in notations of Lemma 6, this quantity is equal to the value of function $F_{yy} + 2F_{yy_1} + F_{y_1y_1}$ evaluated at the point $y = y_1 = y_*$. Since the trace of Jacobi matrix is $f_x + g_y = P_{y_1}^{-1}(Q_{y_1} - P_y)$, then $f_x + g_y - 2 = P_{y_1}^{-1}(Q_{y_1} - P_y - 2P_{y_1}) = -(F_{y_1y_1} + F_{yy} + 2F_{yy_1})/F_{yy_1}$, the numerator of this fraction at the point $y = y_1 = y_*$ is equal to $f^{''}(y_*)$, this implies that the trace is equal to 2 if and only if $f_c^{''}(\eta_*) = 0$. This calculation shows that vanishing this quantity is equivalent to tangency of curves in (14), and their transversality means that the related critical point is nondegenerate. Since $|B_c'(\eta)|$ is large enough in the region $|\eta - \eta_c| \geq \gamma c^2$, then all intersection points of two graphs over this region are transverse. Thus, we have got countably many points with the double multiplier 1, one needs to verify that they are parabolic.

The condition $f'''(y_*) \neq 0$ reads in our case as nonvanishing the quantity

$$\begin{aligned} f_c^{'''}(\eta_*) = \; & -B_c''(\eta) + \tfrac{\partial^2 R_c}{\partial \eta^2}(\eta,\eta_1) + 2\tfrac{\partial^2 R_c}{\partial \eta \partial \eta_1}(\eta,\eta_1) + \tfrac{\partial^2 R_c}{\partial \eta_1^2}(\eta,\eta_1) - \\ & \tfrac{\partial^2 Q_c}{\partial \eta^2}(\eta,\eta_1) - 2\tfrac{\partial^2 Q_c}{\partial \eta \partial \eta_1}(\eta,\eta_1) - \tfrac{\partial^2 Q_c}{\partial \eta_1^2}(\eta,\eta_1), \end{aligned}$$

where one should set $\eta = \eta_1 = \eta_*$ in the function at the r.h.s.. Similarly, the function $\frac{\partial^2}{\partial c \partial y} f$ takes the form

$$\frac{\partial^2}{\partial c \partial y} f = -\frac{\partial B_c'(\eta)}{\partial c} - \frac{\partial Q_c}{\partial c} + \frac{\partial R_c}{\partial c}.$$

The properties of the function $B_c(\eta)$ (Lemma 2), namely, $B_c^{''}(\eta_*) \to \infty$, and $|\frac{\partial}{\partial c} B_c(\eta_c)| \to \infty$ prove Lemma 7. From this we get a countable set of $c = d_0(n)$ in each semi-interval $c > 0$ and $c < 0$.

The points $c = d_1(n)$ are obtained from another statement connecting the presence of a fixed point with the double multiplier $-1$ with some properties of generating function of the map under consideration.

**Lemma 8** *Let, under the conditions of Lemma 6, a critical point $y_*$ of the function $F(y,\ y)$ be such that the function $F_{yy} - 2F_{yy_1} + F_{y_1y_1}$ evaluated at the point $y = y_1 = y_*$ is equal to zero. Then the related fixed point $(x_*,\ y_*)$ has double multiplier $-1$ with two-dimensional Jordan box. If, in addition, $F$ depends on a parameter $c$ and quantities similar to those in lemma 7 do not vanish, then for $|c|$ small enough the following bifurcation does occur: in the space $(x, y, c)$ near the point $(0, 0, 0)$ there* *is a smooth curve $(x(c), y(c), c)$ through $(0,0,0)$ which consists of fixed points of the map. The fixed points are elliptic for $c < 0$ and they are hyperbolic for $c > 0$, or vice versa. Furthermore, from this family of fixed points a family of period 2 points branches at $c = 0$. This latter family exists only on one side of $c = 0$. In dependence on the sign of some coefficient in the normal form of the second order the bearing family consists of either elliptic period 2 points (then it exist for those $c$ where the main family consists of hyperbolic points) or, for opposite sign of the coefficient, it consists of hyperbolic period 2 points (then it exists for those $c$ where the main family consists of elliptic points).*

**Proof.** Let us calculate the trace of linearization matrix $D(f,g)/D(x,y)$. It is easily verified that $f_x + g_y = P_{y_1}^{-1}(Q_{y_1} - P_y)$. Therefore one gets $\sigma = f_x + g_y + 2 = P_{y_1}^{-1}(Q_{y_1} - P_y + 2P_{y_1}) = -(F_{y_1 y_1} + F_{yy} - 2F_{yy_1})/F_{yy_1}$, it implies that the trace is equal to $-2$ iff the numerator of this expression vanishes. Two-dimensionality of the Jordan box follows, as before, from the inequality $g_x \neq 0$.

In our case the numerator is equal to $-B_c'(\eta) + \frac{\partial R_c}{\partial \eta} + 2\frac{\partial Q_c}{\partial \eta} - \frac{\partial Q_c}{\partial \eta_1}$. So, as above, since fixed points appear in the region $|\eta - \eta_c| \leq \gamma c^2$, we conclude that at the bearing elliptic point the quantity $f_x + g_y$ decreases monotonically, when $|c|$ decreases reaching the value $-2$. It follows from this lemma the existence of the fixed point with double eigenvalue $-1$, two-dimensional Jordan box for linearized map and nonzero Lyapunov value, that leads to doubling bifurcation for the further varying $c$. The theorem 2 is proved.

It is worth emphasizing that the same tool can be used for searching for 1-round periodic orbits and determining their types near a heteroclinic connection. Let us briefly outline related details based on the results of the Sec.3 and 4. Recall that in this case one has two local maps $T_{c,\mu}^{(i)} : (\theta_i, \eta_i) \to (\varphi_i, \eta_i) = (\theta_i + B_{c,\mu}^{(i)}(\eta_i), \eta_i)$, $i = 1, 2$, and two global ones, $S_{c,\mu}^{(1)} : (\varphi_1, \eta_1) \to (\varphi_2, \eta_2) = (f_{c,\mu}^{(1)}(\varphi_1, \eta_1), g_{c,\mu}^{(1)}(\varphi_1, \eta_1))$, and $S_{c,\mu}^{(2)} : (\varphi_2, \eta_2) \to (\bar{\varphi}_1, \bar{\eta}_1) = (f_{c,\mu}^{(2)}(\varphi_2, \eta_2), g_{c,\mu}^{(2)}(\varphi_2, \eta_2))$. Symplecticity of these global maps means, as usually, that $d\varphi_1 \wedge d\eta_1 = d\varphi_2 \wedge d\eta_2$, and $d\varphi_2 \wedge d\eta_2 = d\bar{\varphi}_1 \wedge d\bar{\eta}_1$. Transversality of heteroclinic orbits is expressed in two inequalities $l_i = \partial g_{c,\mu}^{(i)}/\partial \varphi_i \neq 0$, $i = 1, 2$, at the points $(0,0)$ for $(c, \mu) = (0,0)$ (we suppose, as above, the traces of heteroclinic orbits have zero coordinates that always possible to do, see Remark 3). We can also write global maps in the cross form, as above, since $l_i \neq 0$, namely

$$
\begin{aligned}
&\theta_2 = F_{c,\mu}^{(1)}(\eta_1, \eta_2) = -\partial R_{c,\mu}^{(1)}(\eta_1, \eta_2)/\partial \eta_2, \quad \varphi_1 = G_{c,\mu}^{(1)}(\eta_1, \eta_2)) = \partial R_{c,\mu}^{(1)}(\eta_1, \eta_2)/\partial \eta_1, \\
&\bar{\theta}_1 = F_{c,\mu}^{(2)}(\eta_2, \bar{\eta}_1) = -\partial R_{c,\mu}^{(2)}(\eta_2, \bar{\eta}_1)/\partial \bar{\eta}_1, \quad \varphi_2 = G_{c,\mu}^{(2)}(\eta_2, \bar{\eta}_1)) = \partial R_{c,\mu}^{(2)}(\eta_2, \bar{\eta}_1)/\partial \eta_2,
\end{aligned}
$$

where $R_{c,\mu}^{(1)}, R_{c,\mu}^{(2)}$ are corresponding generating fuctions of the symplectic global maps. The existence of a 1-round periodic orbit means that $(\bar{\theta}_1, \bar{\eta}_1) = (\theta_1, \eta_1)$, therefore we get the following relations.

$$
\begin{aligned}
&\theta_2 = F_{c,\mu}^{(1)}(\eta_1, \eta_2) = -\partial R_{c,\mu}^{(1)}(\eta_1, \eta_2)/\partial \eta_2, \quad \theta_1 = F_{c,\mu}^{(2)}(\eta_2, \eta_1) = -\partial R_{c,\mu}^{(2)}(\eta_2, \eta_1)/\partial \eta_1, \\
&\theta_1 + B_{c,\mu}^{(1)}(\eta_1) = G_{c,\mu}^{(2)}(\eta_2, \eta_1)) = \partial R_{c,\mu}^{(2)}(\eta_2, eta_1)/\partial \eta_2, \\
&\theta_2 + B_{c,\mu}^{(2)}(\eta_2) = G_{c,\mu}^{(2)}(\eta_2, \eta_1)) = \partial R_{c,\mu}^{(2)}(\eta_2, \eta_1)/\partial \eta_2.
\end{aligned}
$$

Inserting $\theta_2, \theta_1$ from the first and third relations into the second and fourth relations, respectively, we come to the following system of two equations

$$\partial[R^{(2)}_{c,\mu}(\eta_2,\eta_1) + R^{(1)}_{c,\mu}(\eta_1,\eta_2)]/\partial\eta_1 = B^{(1)}_{c,\mu}(\eta_1) \ \ (mod \ \ 2\pi),$$
$$\partial[R^{(2)}_{c,\mu}(\eta_2,\eta_1) + R^{(1)}_{c,\mu}(\eta_1,\eta_2)]/\partial\eta_2 = B^{(2)}_{c,\mu}(\eta_1) \ \ (mod \ \ 2\pi).$$

Now we can proceed as above. Solutions of this system gives $\eta_i$-coordinates of fixed points. As was said above in this Section, l.h.s. of this equations are functions which are slightly depend on parameters $c, \mu$ as they are obtained from the global maps but the functions $B^{(i)}_{c,\mu}$ depend on $c$ essentially and have the form of very sharp parabola (see Sec.2 for the properties of these functions, they are similar). Solutions of each equation consist of a collection of curves in the plane $(\eta_1, \eta_2)$. Some of them exist for all $c$ small enough, for instance, those that lie outside of regions $|\eta_i - \eta_i(c,\mu)| \leq \gamma c^2$. Such the curves from different collections are transverse, and their intersection gives a saddle periodic orbits described in Theorem 3 by periodic symbolic sequences of the form $(..., b, b, b, ...)$, where block $b = (\omega^1_1, \omega^1_2, \omega^2_1, \omega^2_2)$ made up of symbols $\omega^j_i$ from alphabets $\mathcal{B}_\rangle$ (see Subsection 2.2 for the description of the related symbolic systems). In the "dangerous zone" the curves (when they exist in the corresponding regions of the parameters $(c,\mu)$) have the form of the narrow tongues, the curves from different families can have tangency points of order 2 and 3 which correspond to periodic orbits of parabolic types and more higher degeneracy. We do not dwell on the study of these points that requires of a special investigation.

## 4.1 Bifurcational intervals revisited

Bifurcational intervals, inside of which bifurcations described by theorem 2 occur, can be characterized in more details. Namely, boundary points of these intervals can be explicitly pointed out. To this end, let us enumerate strips lying outside of the region $D_c$=$\{|\eta - \eta_c| \leq \gamma c^2\}$ (see Lemma 4) in such a way that their numeration begins with 1 (for the upper strip) and $-1$ (for lower strip). Consider, for definiteness, the case when $(\partial g/\partial\varphi)(0,0,0) > 0$ in (6). We distinguish a region in $\Pi^s(c)$ bounded with segments of stable and unstable manifolds of the orientable saddle fixed point $O$ lying in the strip $\sigma^s_1$ and corresponding to the sequence $(\ldots, 1, \ 1, \ldots)$. Another saddle fixed point $N$ corresponding to $(\ldots, -1, \ -1, \ldots)$ is nonorientable (Möbius's one). Stable manifold of the point $O$ intersects transversely unstable manifold of $N$, giving a heteroclinic point $q_1$ (in the upper strip), and unstable manifold of $O$ transversely intersects stable manifold of $N$ giving a heteroclinic point $q_2$ (in the lower strip) (see Fig.1). Let us construct a curvilinear rectangle $R_c$ in $\Pi^s_c$, whose boundaries are: the upper one is the stable manifold of the point $O$, the lower one is the preimage under $P_c$ of a local piece through $q_1$ of the stable manifold of point $O$ (this preimage is a smooth curve in $\sigma^s{}_{-1}$ lying beneath the stable manifold of the point $N$, due to nonorientability of $N$, and intersecting $W^u(O)$); from the left it is bounded with unstable manifold of $O$, and from the right – with the image under $P_c$ of that local piece through $q_2$ of the unstable manifold of $O$ which belongs to $\sigma^s{}_{-1}$

(this image is a smooth curve that lies to the right of the unstable manifold of $N$, due to nonorientability of $N$, and intersecting $W^s(O)$).

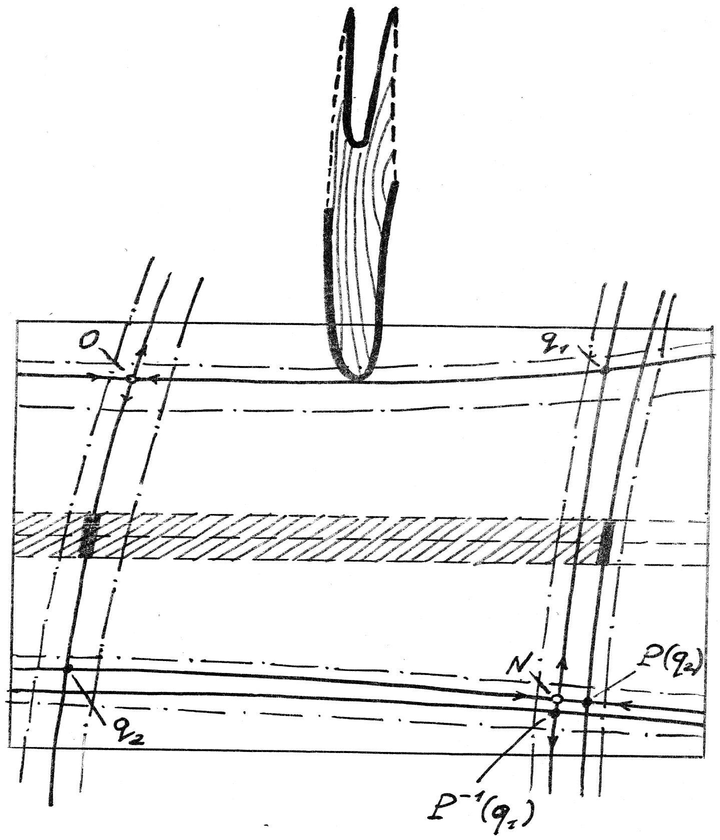


Figure 1: The first tangency of stable and unstable manifolds of the point $O$. The dashed region is the dangerous one, it is determined by the inequalities $|\eta - \eta_c| \leq \gamma c^2$. Only two extreme strips are plotted

It is readily seen, due to the construction, that the region constructed is invariant in the sense that image (and preimage) of any strip $\sigma_j^s(c)$, $|j| \leq n(c)$, belongs to this region, but points from the strip $D_c \cap R_c$ can be transformed outside of $R_c$. When $c$ belongs to a hyperbolic interval, then $P_c(D_c)$ is situated outside of $R_c$ and orbits lying entirely in $U$ cut $\Pi^s(c)$ only in strips. When $|c|$ decreases, $T_c(D_c)$ moves monotonically (see Lemma 3) around the annulus $N_c^u$ and countably many times passes through $\Pi^u(c)$, therefore $P_c(D_c)$ monotonically passes through $R_c$. The first value of $c$, when this intersection is not empty, corresponds exactly to the first quadratic (Lemma 5) tangency point of the stable manifold of $O$ and of the image of that piece of the unstable manifold of $O$ (on the left side of the boundary of $R_c$) which belongs to $D_c$ (see Fig.1 and [14] for the explanation what the first tangency point means). Before this value of $c$ no orbits of $P_c$ exist which begin in $D_c$ and hit $R_c$ one more time. After that, bifurcations take place related with formation of multi period elliptic points in the first strip in a neighborhood of the tangency point [11, 12, 14]. In fact, bifurcational structure, when $P_c(D_c)$ passes through $R_c$, is very complicated, in particular, Newhouse phenomena are expected here [16]. Moreover, during subsequent decreasing $c$ $W^u(O)$ will be tangent with stable manifold of the orientable saddle fixed point in the strip $\sigma_2^s(c)$, etc.. It is also easily to construct here two saddle fixed points for which one pair of stable and unstable separatrices intersect transversely but another pair is tangent, as in [15]. Then a countable set of elliptic periodic points can be expected. To this end, for instance, one should take saddle points $O$ and $N$ and a piece of $W^u(N)$ which belongs to the right boundary of $D_c$. Then, when decreasing $|c|$, this piece under $P_c$ comes from above forming parabola-like smooth curve and cross $W^s(O)$ at a tangency point. Here, $W^u(O)$ (i.e., the left boundary of $R_c$) and a part of $W^s(N)$ belonging to $\sigma^s_{-1}(c)$ remain transverse.

The last point of the bifurcational interval is determined with that value of $c$, when the so-called "last tangency" case of stable and unstable manifolds for the point $O$ arises [14]. Namely, at this value of $c$ the image of the piece of the unstable manifold of $O$ on the right boundary of $R_c$ belonging $D_c$, is tangent to the lower boundary of $R_c$. As follows from the geometry of the map (Fig.1), at this $c$ region $P_c(D_c)$ has a unique common point with $R_c$, its tangency point. At this moment the local structure of invariant set in a neighborhood of this point is hyperbolic excepting the tangency point itself [14]. After that, when $|c|$ further decreases, hyperbolic structure of that larger invariant set of orbits restores and the whole invariant set of orbits lying entirely in $N_c^s$ acquires two new states in the Bernoulli shift.

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