Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

An equilibrium problem for a thermoelectroconductive body with the Signorini condition on the boundary

Dietmar Hömberg[1], Aleksander M. Khludnev[2]

submitted: 19 April 2000

[1] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstrasse 39
D – 10117 Berlin
Germany
E-Mail: hoemberg@wias-berlin.de

[2] Lavrentyev Institute
of Hydrodynamics
of the Russian
Academy of Sciences
Novosibirsk 630090
Russia
E-Mail: khlud@@hydro.nsc.ru

Preprint No. 574
Berlin 2000

1991 *Mathematics Subject Classification.* 35D05, 35Q72, 73F15.

Key words and phrases. Resistance welding, Signorini condition, thermistor, thermo- viscoelasticity.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We investigate a boundary value problem for a thermoelectroconductive body with the Signorini condition on the boundary, related to resistance welding. The mathematical model consists of an energy balance equation coupled with an elliptic equation for the electric potential and a quasistatic momentum balance with a viscoelastic material law.

We prove existence of a weak solution to the model by using the Schauder fixed point theorem and classical results on pseudomonotone operators.

1 Introduction

In this paper we study a boundary value problem for a thermoelectroconductive body with the Signorini condition on the boundary. The practical application we have in mind is related to the pressure resistance welding of a ring onto a plate. For reasons of symmetry, in this case it is sufficient to consider the 2-dimensional situation depicted in Fig. 1. On the upper boundary a force is applied through the electrode (not visible in Fig. 1). This can be conveniently described by the Signorini boundary condition (cf. (2.6)). After applying electric current to the electrodes, owing to the Joule effect a rise in temperature can be observed. The mathematical model under study in this paper can be viewed as an extension of the classical thermistor problem, where an elliptic equation for the electric potential is coupled with a parabolic heat equation. Since the occuring high temperatures will lead to inelastic behaviour, we couple these equations with a quasi-static momentum balance and a viscoelastic constitutive law corresponding to a linear Maxwell material.

In a previous paper [6], we considered a similar situation but with a different material law and different boundary conditions. There are numerous papers on the mathematical treatment of the classical thermistor problem, see e.g. [3], [4], [12], [13], [14]. For related thermomechanical problems we refer to [1], [2], [7], [8], [9], [11]. Details about the industrial application can be found in [5].

2 Problem formulation and main result

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ; $Q = \Omega \times (0,T)$. In the domain Q, we want to find functions $u = (u_1, u_2), \sigma = \{\sigma_{ij}\}, \varphi, \theta$ such that

$$-\sigma_{ij,j} = f_i, \qquad i = 1, 2, \tag{1}$$

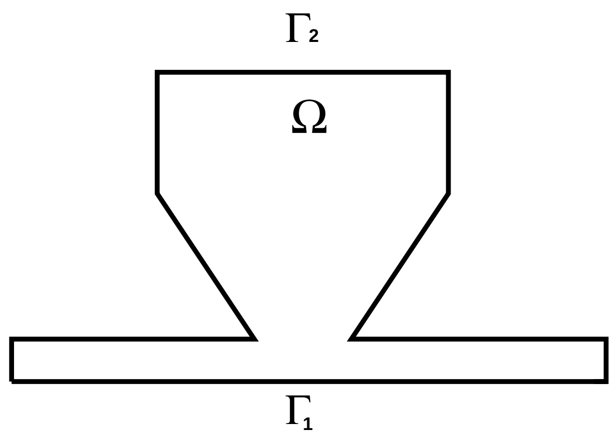

Figure 1: A typical setting related to resistance welding.

$$\varepsilon_{ij}(u) = c_{ijkl}\,\sigma_{kl} + \delta^2\,\beta_{ij}\theta + \int_0^t b(\tau)\,s_{ij}(\tau)d\tau, \qquad i,j = 1,2, \tag{2}$$

$$\theta_t - \Delta\theta + \delta^2\,\frac{\partial}{\partial t}\,\mathrm{div}\,u = \gamma(\theta)|\nabla\varphi|^2, \tag{3}$$

$$\mathrm{div}\,(\gamma(\theta)\nabla\varphi) = 0, \tag{4}$$

$$u = 0 \;\;\text{on}\;\; \Gamma_1 \times (0,T), \tag{5}$$

$$u\cdot n \leq 0,\; \sigma_n \leq 0,\; \sigma_\nu = 0,\; \sigma_n\cdot(u\cdot n) = 0 \quad \text{on} \;\; \Gamma_2 \times (0,T), \tag{6}$$

$$\theta = 0 \;\;\text{on}\;\; \Gamma \times (0,T);\;\; \theta = \theta_0 \text{ for } t = 0, \tag{7}$$

$$\varphi = \varphi_0 \;\;\text{on}\;\; \Gamma \times (0,T)\,. \tag{8}$$

Here $f = (f_1, f_2) \in [L^2(Q)]^2$, $\dot{f} \in [L^2(Q)]^2$; $\delta > 0$ and β_{ij} are constants, $b \in C^1[0,T]$, $b \geq c > 0$; s_{ij} is the deviator of σ_{ij}; c_{ijkl} is the constant elasticity tensor; $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma \cap \Gamma = \emptyset$, meas $\Gamma_1 > 0$; γ is a continuous function such that $0 < \gamma_1 \leq \gamma(\tau) \leq \gamma_2$ for all $\tau \in \mathbb{R}$, γ_i are constants.

Moreover, we assume $\theta_0 \in H_0^1(\Omega)$, $\varphi_0 \in L^\infty(0,T; H^{3/2}(\Gamma))$, and $\{\sigma_{ij}\,n_j\}_{i=1}^2 = \sigma_\nu + \sigma_n n$, where $n = (n_1, n_2)$ is the external unit vector to Γ, $\sigma_n = \sigma_{ij} n_j\, n_i$, and $\nu = (-n_2, n_1)$.

We introduces the spaces

$$\begin{aligned}
\Xi &= \left\{\theta \in L^2(0,T;H_0^1(\Omega)) \,\middle|\, \theta_t \in L^2(Q)\right\},\\
H^1_{\Gamma_1}(\Omega) &= \left\{u = (u_1,u_2) \,\middle|\, u_i \in H^1(\Omega), u_i = 0 \text{ on } \Gamma_1,\, i = 1,2\right\},\\
H &= H^1(0,T;H^1_{\Gamma_1}(\Omega)),\\
\Sigma &= \left\{\sigma = \{\sigma_{ij}\} \,\middle|\, \sigma_{ij}, \dot{\sigma}_{ij} \in L^2(Q),\, i,j = 1,2\right\},
\end{aligned}$$

define the subsets

$$\begin{aligned}
K &= \left\{u \in H^1_{\Gamma_1}(\Omega) | \, u \cdot n \leq 0 \text{ almost everywhere on } \Gamma_2\right\},\\
\mathcal{K} &= \left\{u \in L^2(0,T;H^1_{\Gamma_1}(\Omega)) | \, u(t) \in K \text{ a.e. on } (0,T)\right\}
\end{aligned}$$

and denote $\sigma^b = \sigma \cdot b$, $f^b = f \cdot b$, $c^b_{ijkl} = c_{ijkl} \cdot b^{-1}$.

We will prove the existence of a solution to problem (1) - (8) in the following sense:

Theorem 2.1 *Under the assumptions mentioned above and condition (55) below, for small δ there exist functions θ, u, σ, φ,*

$$\theta \in \Xi,\; u \in \mathcal{K} \cap H,\; \sigma \in \Sigma, \varphi \in L^\infty(0,T;W^1_4(\Omega))$$

satisfying

$$\int_Q \sigma_{ij}\, \varepsilon_{ij}(\bar{u} - u) \geq \int_Q f(\bar{u} - u) \quad \forall \bar{u} \in \mathcal{K}, \tag{9}$$

$$\int_Q \left(c_{ijkl}\, \sigma_{kl} - \varepsilon_{ij}\,(u) + \delta^2 \beta_{ij}\, \theta + \int_0^t b\, s_{ij} \right) \bar{\sigma}_{ij} = 0 \quad \forall \bar{\sigma} \in L^2(Q), \tag{10}$$

$$\int_Q \left(\theta_t + \delta^2 \frac{\partial}{\partial t}\, div\, u - \gamma(\theta)\, |\nabla\varphi|^2 \right) \eta = - \int_Q \nabla\theta \nabla\eta \quad \forall \eta \in L^2(0,T;H_0^1(\Omega)), \tag{11}$$

$$\int_Q \gamma(\theta)\, \nabla\varphi \nabla\psi = 0 \quad \forall \psi \in L^2(0,T;H_0^1(\Omega)) \tag{12}$$

as well as the boundary and initial conditions (7) and (8).

The proof of Theorem 2.1 will be given in the next section. Based on a fixed point argument it is divided into three steps.

STEP 1: We find (u,σ) as a solution of (1), (2), (5), (6) for a given θ and obtain estimates for (u,σ).
Next we find a solution θ of the equation

$$\theta_t - \Delta\theta + \delta^2 \frac{\partial}{\partial t} \operatorname{div} u = h \tag{13}$$

for given $u \in H, h \in L^2(Q)$ with conditions (7), and obtain estimates for θ.

STEP 2: We prove an existence of (θ, u, σ) to the problem (1), (2), (13) with conditions (5), (6), (7) for a given function $h \in L^2(Q)$, and establish estimates for the solution (θ, u, σ).

STEP 3: We solve the problem (1) - (8) by using the Schauder fixed point theorem and the results of the previous two steps.

3 Proof of Theorem 2.1

Step 1: For a given $\theta \in \Xi$ in the domain Q, we want to find $u = (u_1, u_2), \sigma = \{\sigma_{ij}\}$ such that

$$\begin{aligned} -\sigma_{ij,j} &= f_i, \\ \varepsilon_{ij}(u) &= c_{ijkl}\,\sigma_{kl} + \delta^2\,\beta_{ij}\,\theta + \int_0^t b(\tau)\,s_{ij}(\tau), \end{aligned}$$

$$u = 0 \ \text{ on } \ \Gamma_1 \times (0,T),$$

$$u \cdot n \le 0, \ \sigma_n \le 0, \ \sigma_\nu = 0, \ \sigma_n \cdot (u \cdot n) = 0 \ \text{ on } \ \Gamma_2 \times (0,T).$$

To obtain the existence of a solution to this boundary value problem we consider a regularized auxiliary problem. To this end, we introduce $\alpha > 0$ and try to find u, σ^b (depending on α) such that

$$-\alpha\,\Delta u_i - \sigma^b_{ij,j} = f^b_i, \tag{14}$$

$$\varepsilon_{ij}(u) = c^b_{ijkl}\,\sigma^b_{kl} + \delta^2\beta_{ij}\theta + \int_0^t s^b_{ij}, \tag{15}$$

$$u = 0 \ \text{ on } \ \Gamma_1 \times (0,T); \ u \cdot n \le 0, \ \alpha\frac{\partial u_i}{\partial n} n_i + \sigma^b_n \le 0 \ \text{ on } \ \Gamma_2 \times (0,T), \tag{16}$$

$$\left(\alpha\frac{\partial u_i}{\partial n} + \sigma^b_{ij} n_j\right)\nu_i = 0, \ \left(\alpha\frac{\partial u_i}{\partial n} n_i + \sigma^b_n\right)(u \cdot n) = 0 \ \text{ on } \ \Gamma_2 \times (0,T). \tag{17}$$

Here s^b is the deviator of σ^b.

The solution to (14) - (17) is defined as follows:

$$(u, \sigma^b) \in \mathcal{K} \times L^2(Q), \tag{18}$$

$$\alpha \int_\Omega \nabla u_i \, \nabla(\bar{u}_i - u_i) + \int_\Omega \sigma^b_{ij} \, \varepsilon_{ij}(\bar{u} - u) \geq \int_\Omega f^b \, (\bar{u} - u) \quad \forall \bar{u} \in K, \tag{19}$$

$$\int_\Omega \left(c^b_{ijkl} \, \sigma^b_{kl}(t) - \varepsilon_{ij}(u(t)) + \delta^2 \, \beta_{ij}\theta(t) + \int_0^t s^b_{ij} \right) \bar{\sigma}_{ij} = 0 \quad \forall \bar{\sigma}_{ij} \in L^2(\Omega). \tag{20}$$

In order to obtain estimates for u, σ^b we choose a function $\xi = \{\xi_{ij}\} \in L^2(Q)\,, \dot{\xi} \in L^2(Q)$ satisfying the equations

$$-\xi^b_{ij,j} = f^b_i \,, i = 1, 2, \text{ in } Q$$

in the following sense for almost all $t \in (0, T)$,

$$\int_\Omega \xi^b_{ij} \, \varepsilon_{ij}(\bar{u}) = \int_\Omega f^b \bar{u} \qquad \forall \bar{u} \in H^1_{\Gamma_1}(\Omega). \tag{21}$$

Here $\xi^b = \xi \cdot b$. Note that ξ satisfies the identity

$$\int_\Omega \xi_{ij} \, \varepsilon_{ij}(\bar{u}) = \int_\Omega f \bar{u} \qquad \forall \bar{u} \in H^1_{\Gamma_1}(\Omega). \tag{3.21$'$}$$

We take $\bar{u} = 0, \bar{\sigma} = \sigma^b - \xi^b$ in (19), (20) respectively and sum the relations. This gives

$$\alpha \int_\Omega \nabla u_i \, \nabla u_i + \int_\Omega \left\{ \sigma^b_{ij} \, \varepsilon_{ij}\,(u) - f^b u \right\} +$$

$$+ \int_\Omega \left\{ \left(c^b_{ijkl} \, \sigma^b_{kl} - \varepsilon_{ij}(u) + \delta^2 \, \beta_{ij} \, \theta + \Big(\int_0^t s^b_{ij} \Big) \right) \left(\sigma^b_{ij} - \xi^b_{ij} \right) \right\} \;\leq\; 0.$$

Hence, by (21), we derive

$$\alpha \, ||u||^2_{L^2(0,T;H^1_{\Gamma_1}(\Omega))} \, + \, ||\sigma||^2_{L^2(Q)} \leq \, c\delta^2 ||\theta||^2_{L^2(Q)} + c, \tag{22}$$

where the constant is uniform in α, δ, for $\quad \alpha \leq \alpha_0, \;\delta \leq \delta_0$.

Problem (18) - (20) can be written in the form

$$A(u, \sigma^b) \left((\bar{u}, \bar{\sigma}) - (u, \sigma^b) \right) \geq F\left((\bar{u}, \bar{\sigma}) - (u, \sigma^b) \right) \tag{23}$$

for all $(\bar{u},\bar{\sigma}) \in \mathcal{K} \times L^2(Q)\,;\;(u,\sigma^b) \in \mathcal{K} \times L^2(Q)$,

where

$$A : V \to V',$$

$$V = L^2(0,T; H^1_{\Gamma_1}(\Omega)) \times L^2(Q),$$

$F \in V'$ is given by

$$F\,(\bar{u},\bar{\sigma}) = \int\limits_Q f^b \bar{u} - \int\limits_Q \delta^2 \beta_{ij}\,\theta \bar{\sigma}_{ij}$$

and

$$\begin{aligned} A\,(u,\sigma^b)(\bar{u},\bar{\sigma}) \;&=\; \alpha \int\limits_Q \nabla u_i \nabla \bar{u}_i + \int\limits_Q \sigma^b_{ij}\varepsilon_{ij}(\bar{u}) + \\ &\quad + \int\limits_Q \left(c^b_{ijkl}\,\sigma^b_{kl} - \varepsilon_{ij}(u) + \Big(\int\limits_0^t s^b_{ij} \Big) \right) \bar{\sigma}_{ij}\ . \end{aligned}$$

Note that A is a bounded, semicontinuous, monotone operator. The scheme used to obtain (22) allows us to prove the coercivity of A in the sense

$$\frac{A(u,\sigma)(u,\sigma)}{||(u,\sigma)||_V} \to +\infty\,, \quad ||(u,\sigma)||_V \to +\infty$$

and hence, applying Theorem 2.2.2 of [10], the problem (23) has a solution for a given θ.

By (22), from (15) we have

$$||u||^2_{L^2(0,T;H^1_{\Gamma_1}(\Omega))} \leq c\delta^2 ||\theta||^2_{L^2(Q)} + c$$

and consequently we can pass to the limit as $\alpha \to 0$ in (18) - (20) which gives

$$||u||_{L^2(0,T;H^1_{\Gamma_1}(\Omega))} + ||\sigma||_{L^2(Q)} \leq\; c\delta ||\theta||_{L^2(Q)} + c, \tag{24}$$

$$(u,\sigma) \in \mathcal{K} \times L^2(Q);\; \int\limits_Q \sigma_{ij}\varepsilon_{ij}(\bar{u} - u) \geq \int\limits_Q f(\bar{u} - u) \quad \forall \bar{u} \in \mathcal{K}, \tag{25}$$

$$\int\limits_Q \left(c_{ijkl}\,\sigma_{kl} - \varepsilon_{ij}\;(u) \;+\; \delta^2\,\beta_{ij}\,\theta + \int\limits_0^t b\,s_{ij} \right)\; \bar{\sigma}_{ij} = 0 \quad \forall \bar{\sigma} \in L^2(Q). \tag{26}$$

Here we write the limit problem (25) - (26) in terms of (u,σ) instead of (u,σ^b) which is the equivalent formulation.

Now we aim to show that the solution (u,σ) of (25), (26) has an additional regularity in t. Problem (25), (26) can be rewritten in the following equivalent form

$$(u,\sigma) \in \mathcal{K} \times L^2(Q), \tag{27}$$

$$\int_\Omega \sigma_{ij}(t)\,\varepsilon_{ij}(\bar{u}-u(t)) \geq \int_\Omega f(t)\,(\bar{u}-u(t)) \quad \forall \bar{u} \in K, \tag{28}$$

$$\int_\Omega \left(c_{ijkl}\,\sigma_{kl}(t) - \varepsilon_{ij}\;(u(t)) \;+\; \delta^2\,\beta_{ij}\,\theta(t) + \int_0^t b\,s_{ij} \right)\;\bar{\sigma}_{ij} = 0 \quad \forall \bar{\sigma} \in L^2(\Omega). \tag{29}$$

For a given function v introduce the notations

$$v^t = v(t),\; d_\tau v^t = \frac{v^{t+\tau} - v^t}{\tau}\;.$$

Let $\tau > 0$. We take $\bar{u} = u^{t+\tau}$ in (28). Next we consider (28) at the point $t+\tau$ and take $\bar{u} = u^t$. Summing the relations we obtain

$$\int_\Omega d_\tau\,\sigma^t_{ij}\,\varepsilon_{ij}(d_\tau u^t) \leq \int_\Omega d_\tau f^t \cdot d_\tau u^t\;. \tag{30}$$

Now take $\bar{\sigma} = \sigma^{t+\tau} - \sigma^t - \xi^{t+\tau} + \xi^t$ in (29) and evaluate it at times $t+\tau$ and t. Subtracting the equalities obtained we have after division by τ^2

$$\int_\Omega \left\{ c_{ijkl}\,d_\tau\,\sigma^t_{kl} - \varepsilon_{ij}(d_\tau u^t) + \delta^2\,\beta_{ij}\,d_\tau\theta^t + d_\tau \left(\int_0^t b\,s_{ij} \right) \right\} \left(d_\tau\,\sigma^t_{ij} - d_\tau\,\xi^t_{ij} \right) = 0. \tag{31}$$

We add (30),(31), integrate in t from 0 to $T-\tau$, and by $(3.21)'$, (24), we obtain

$$\int_0^{T-\tau} \|d_\tau\,\sigma^t\|_0^2\;dt \leq c\delta^2 \int_0^{T-\tau} ||d_\tau\theta^t||_0^2\,dt + c\delta^2 \int_0^{T-\tau} ||\theta(t)||_0^2\,dt + c. \tag{32}$$

Since $\theta_t \in L^2(Q)$ this inequality implies

$$||\dot{\sigma}||^2_{L^2(Q)} \leq c\delta^2||\dot{\theta}||^2_{L^2(Q)} + c\delta^2||\theta||^2_{L^2(Q)} + c. \tag{33}$$

It follows from (24), (33) that

$$||\sigma||_\Sigma \leq c_1\,\delta||\theta||_\Xi + c_2. \tag{34}$$

In this case equations (15) yield the inequality

$$||u||_H \leq c_3\,\delta||\theta||_\Xi + c_4. \tag{35}$$

Note that the constants c_3, c_4 are independent of δ, $\delta \leq \delta_0$.

Now we consider equation (13) for given $u \in H, h \in L^2(Q)$ and conditions (7). Using standard parabolic theory we can prove the existence of a function $\theta \in \Sigma$ such that $\theta(0) = \theta_0$,

$$\int\limits_Q \left(\theta_t + \delta^2 \frac{\partial}{\partial t}\,\mathrm{div} u - h\right) \eta = \int\limits_0 \nabla\theta\nabla\eta \quad \forall \eta \in L^2(0,T;H_0^1(\Omega)) \tag{36}$$

and moreover,

$$||\theta||_\Xi \leq c_5\,\delta||u||_H + c_6 \tag{37}$$

with constants c_5, c_6 independent of δ, $\delta \leq \delta_0$.

To conclude step 1 we see that problem (1), (2), (5), (6) is uniquely solvable in the sense (25), (26) and that the estimates (34), (35) hold. Equation (13) with conditons (7) is uniquely solvable for given $u \in H$, $h \in L^2(Q)$ with the estimate (37).

Step 2: In the domain Q, consider the problem of finding the functions $u = (u_1, u_2), \sigma = \{\sigma_{ij}\}, \theta$ such that

$$-\sigma^b_{ij,j} = f^b_i, \qquad i = 1,2, \tag{38}$$

$$\varepsilon_{ij}(u) = c^b_{ijkl}\,\sigma^b_{kl} + \delta^2\,\beta_{ij}\theta + \int\limits_0^t s^b_{ij}\,, \qquad i,j = 1,2, \tag{39}$$

$$\theta_t - \Delta\theta + \delta^2 \frac{\partial}{\partial t}\,\mathrm{div}\,u = h, \tag{40}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{41}$$

$$u \cdot n \leq 0,\ \sigma^b_n \leq 0,\ \sigma^b_\nu = 0,\ \sigma^b_n \cdot (u \cdot n) = 0 \qquad \text{on } \Gamma_2 \times (0,T), \tag{42}$$

$$\theta = \theta_0 \text{ for } t = 0;\ \theta = 0 \qquad \text{on } \Gamma \times (0,T). \tag{43}$$

Here we assume that $h \in L^2(Q)$ is a given function.

Define a linear bounded operator

$$L : \mathcal{U} \to \mathcal{U}', \text{ where } \ \mathcal{U} = \Xi \times H,$$

$$\{L(\theta,u)\,,\ (\bar\theta,\bar u)\,\} = \int_Q \left(\theta_t + \delta^2 \frac{\partial}{\partial t}\,\mathrm{div} u\right) \bar\theta + \int_Q \nabla\theta\nabla\bar\theta + \int_Q \sigma^b_{ij}\,\varepsilon_{ij}(\bar u),$$

where $\sigma^b_{ij} = \sigma^b_{ij}(\theta,u)$ are defined from (39), and introduce a convex closed set in $\mathcal{U}$:

$$S = \{(\theta,u) \in \mathcal{U} \mid \theta(0) = \theta_0\,,\ u \in \mathcal{K}\}.$$

Definition 3.1 *An element $(\theta,u,\sigma^b) \in \mathcal{U} \times \Sigma$ is called a solution of the problem (38) - (43) if $(\theta,u) \in S$ for all $(\bar\theta,\bar u) \in S$ satisfies the inequality*

$$\{L(\theta,u),\,(\bar\theta,\bar u) - (\theta,u)\} \geq \int_Q \left(f^b(\bar u - u) + h(\bar\theta - \theta)\right). \tag{44}$$

Lemma 3.1 *For small δ there exists a solution to (44).*

Proof: Introduce two convex closed sets,

$$S_1 = \{\theta \in \Xi \mid \theta(0) = \theta_0\,\}, \quad S_2 = \{u \in H \mid u \in \mathcal{K}\}.$$

Then (44) is equivalent to the following relations

$$\int_Q \left(\theta_t + \delta^2 \frac{\partial}{\partial t}\,\mathrm{div} u - h\right)(\bar\theta - \theta) + \int_Q \nabla\theta(\nabla\bar\theta - \nabla\theta) \geq 0, \quad \forall\bar\theta \in S_1; \quad \theta \in S_1, \tag{45}$$

$$\int_Q \sigma^b_{ij}\,\varepsilon_{ij}(\bar u - u) \geq \int_Q f^b(\bar u - u), \quad \forall\bar u \in S_2;\ u \in S_2, \tag{46}$$

$$\int_Q \left(c^b_{ijkl}\,\sigma^b_{kl} - \varepsilon_{ij}\ (u)\ +\ \delta^2\,\beta_{ij}\,\theta + \int_0^t s^b_{ij}\right)\ \bar\sigma_{ij} = 0, \quad \forall\bar\sigma \in L^2(Q). \tag{47}$$

Notice that (45) can be written equivalently as the identity (36) with intial condition $\theta(0) = \theta_0$. Consider also that relations (46) - (47) and (25) - (26) are equivalent.

Now we write down a variational inequality whose solution exists and prove that this is a solution to (44).

Let $c_* = \max\limits_{3 \leq i \leq 6}\{c_i\}$, where c_i are taken from (35), (37). Assume that δ is so small that

$$m = \frac{c_* + c_*^2 \delta}{1 - (c_* \delta)^2} > 0 \tag{48}$$

we introduce the set,

$$S^0 = \{(\theta, u) \ \in S | \ ||\theta||_\Xi \leq m, \ ||u||_H \leq m\}.$$

Since S^0 is a bounded set in $\mathcal{U}$ and $L : \mathcal{U} \to \mathcal{U}'$ is pseudomonotonous (but not coercive) operator there exists a solution $(\theta, u) \in S^0$ to the problem (see [10], Theorem 2.8.1)

$$\Big\{ L(\theta, u), (\bar{\theta}, \bar{u}) - (\theta, u) \Big\} \geq \int\limits_Q \big(f^b(\bar{u} - u) + h(\bar{\theta} - \theta) \big), \quad \forall (\bar{\theta}, \bar{u}) \in S^0. \tag{49}$$

Next, we introduce two sets

$$S_1^0 = \{\theta \in S_1 \quad | \quad ||\theta||_\Xi \leq m\},$$
$$S_2^0 = \{(u, \sigma) \ \in S_2 \quad | \quad ||u||_H \leq m\}.$$

Then (49) can be rewritten equivalently as the following variational inequalities

$$\int\limits_Q \left(\theta_t + \delta^2 \frac{\partial}{\partial t} \operatorname{div} u - h \right) (\bar{\theta} - \theta) + \int\limits_Q \nabla\theta (\nabla\bar{\theta} - \nabla\theta) \geq 0,$$
$$\forall \bar{\theta} \in S_1^0; \quad \theta \in S_1^0, \tag{50}$$

$$\int\limits_Q \Big\{ \Big(c_{ijkl}^b \, \sigma_{kl}^b + \delta^2 \beta_{ij} \theta - \varepsilon_{ij}(u) + \big(\int\limits_0^t s_{ij}^b \big) \Big) \left(\bar{\sigma}_{ij} - \sigma_{ij}^b \right) +$$
$$+ \sigma_{ij}^b \, \varepsilon_{ij}(\bar{u} - u) \Big\} \geq \int\limits_Q f^b (\bar{u} - u),$$
$$\forall (\bar{u}, \bar{\sigma}) \in S_2^0 \times \Sigma; \quad (u, \sigma^b) \in S_2^0 \times \Sigma. \tag{51}$$

Now we are able to prove the existence of a solution to (44).

Indeed, let (θ, u) be a solution of (49). Then $u \in S_2^0$. Find $\tilde{\theta}$ as a solution of (45) for a given $u \in S_2^0$. We have $\tilde{\theta} \in S_1^0$ by the estimate (37) and (48). But (50) has an unique solution for a given $u \in S_2^0$, hence $\tilde{\theta} = \theta$. On the other hand, $\theta \in S_1^0$. Hence, for a solution $(\tilde{u}, \tilde{\sigma}^b)$ of (46), (47) with the given $\theta \in S_1^0$ we have $\tilde{u} \in S_2^0$ by the estimate (35) and condition (48), whence $(\tilde{u}, \tilde{\sigma}^b)$ is the unique solution of (51), i.e. $(\tilde{u}, \tilde{\sigma}^b) = (u, \sigma^b)$. It follows from these arguments that the solution of (49) is the solution of (45) - (47). Since (45) - (47) is exactly equivalent to (44) the lemma is proved.

Step 3: Now we aim at proving the existence of a solution to (1) - (8). To this end we use the Schauder fixed point theorem.

Let $\bar{\theta} \in L^2(Q)$ be any fixed function. Consider an auxiliary problem for finding φ,

$$\mathrm{div}(\gamma(\bar{\theta})\,\nabla\varphi) = 0 \quad \text{in } \; Q, \tag{52}$$

$$\varphi = \varphi_0 \quad \text{on } \; \Gamma \times (0,T). \tag{53}$$

Similar to [6] we obtain

$$\nabla\,\varphi \in L^\infty(0,T; L^4(\Omega)), \tag{54}$$

provided that

$$\frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}\,\Lambda_4 < 1. \tag{55}$$

Here we use the following well-known regularity result:

Lemma 3.2 *For any given $g = (g_1, g_2, g_3) \in L^p(\Omega)$, $p > 1$, the solution w of the problem*

$$div(\nabla w + g) = 0 \;\; in \;\; \Omega$$

$$w = 0 \;\; on \;\; \Gamma$$

exists and the following estimate holds:

$$\|\nabla w\|_{L^p(\Omega)} \le \Lambda_p \|g\|_{L^p(\Omega)},$$

with the positive constant Λ_p depending on p.

By (54), we have $h \equiv \gamma(\bar{\theta})|\nabla\varphi|^2 \in L^2(Q)$, and consequently, we can solve the following problem with respect to θ, $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$:

$$-\sigma_{ij,j} = f_i, \tag{56}$$

$$\varepsilon_{ij}(u) = c_{ijkl}\sigma_{kl} + \delta^2\beta_{ij}\theta + \int_0^t b\, s_{ij}, \tag{57}$$

$$\theta_t - \Delta\theta + \delta^2\frac{\partial}{\partial t}\mathrm{div} u = \gamma(\bar\theta)|\nabla\varphi|^2, \tag{58}$$

$$\theta(0) = \theta_0, \qquad \theta = 0 \text{ on } \Gamma\times(0,T), \tag{59}$$

$$u = 0 \text{ on } \Gamma_1\times(0,T), \tag{60}$$

$$u\cdot n \le 0,\ \sigma_n \le 0,\ \sigma_\nu = 0,\ \sigma_n\cdot(u\cdot n) = 0 \ \text{ on } \Gamma_2\times(0,T). \tag{61}$$

According to the previous steps, the solution (θ, u, σ) of the problem (56)–(61) exists for the given $h = \gamma(\bar\theta)|\nabla\varphi|^2$ and for δ small enough. Moreover, we have

$$\theta\in\Xi, \quad \sigma\in\Sigma, \quad u\in H$$

and

$$\|\theta\|_\Xi \le c,$$

where c is independent of $\bar\theta$. We see that if

$$\|\bar\theta\|_{L^2(Q)} \le R,$$

then for large R we obtain $\|\theta\|_{L^2(Q)} \le R$, and the imbedding $\Xi\subset L^2(Q)$ is compact. To use the Schauder fixed point theorem it suffices to prove continuity of the operator $B : L^2(Q)\to L^2(Q)$,

$$B : \bar\theta \to \theta.$$

Let $\bar\theta^n \to \bar\theta$ in $L^2(Q)$. We have to prove the convergence

$$B(\bar\theta^n) \to B(\bar\theta) \text{ in } L^2(Q).$$

Consider the equations

$$\theta_t^n - \Delta\theta^n + \delta^2\frac{\partial}{\partial t}\mathrm{div} u^n \;=\; \gamma(\bar\theta^n)|\nabla\varphi^n|^2, \tag{62}$$

$$\theta_t - \Delta\theta + \delta^2\frac{\partial}{\partial t}\mathrm{div} u \;=\; \gamma(\bar\theta)|\nabla\varphi|^2. \tag{63}$$

Here $\theta^n = B(\bar\theta^n), \theta = B(\bar\theta)$, and u^n, u correspond to $\bar\theta^n, \bar\theta$, respectively, and are defined from the boundary-value problem (56)–(61).

Equations (62)–(63) imply

$$\delta^2 \mathrm{div}(u-u^n)(t) = -(\theta-\theta^n)(t) + \int_0^t \Delta(\theta-\theta^n) + \int_0^t \left[\gamma(\bar\theta)|\nabla\varphi|^2 - \gamma(\bar\theta^n)|\nabla\varphi^n|^2\right]. \quad (64)$$

On the other hand, we have the equations

$$-(\sigma_{ij,j} - \sigma^n_{ij,j}) = 0, \quad (65)$$

$$\varepsilon_{ij}(u) - \varepsilon_{ij}(u^n) = c_{ijkl}(\sigma_{kl} - \sigma^n_{kl}) + \delta^2\beta_{ij}(\theta-\theta^n) + \int_0^t b(s_{ij} - s^n_{ij}), \quad (66)$$

which yield the equality

$$\|\sigma-\sigma^n\|^2_{L^2(\Omega)} = -\delta^2\beta_{ij}\int_\Omega (\theta-\theta^n)(\sigma_{ij}-\sigma^n_{ij}) + \int_\Omega \left(\int_0^t b(s_{ij}-s^n_{ij})\right)(\sigma_{ij}-\sigma^n_{ij}),$$

and thereby

$$\|\sigma-\sigma^n\|_{L^2(Q)} \le c\delta\|\theta-\theta^n\|_{L^2(Q)}. \quad (67)$$

By (67), equations (66) imply

$$\|u-u^n\|_{L^2(0,T;H^1_{\Gamma_1}(\Omega))} \le c\delta\|\theta-\theta^n\|_{L^2(Q)}. \quad (68)$$

Next, from (64) it follows that

$$\|\theta-\theta^n\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\int_0^t \nabla(\theta-\theta^n)\|^2_{L^2(\Omega)} = -\delta^2\int_\Omega \mathrm{div}(u-u^n)(\theta-\theta^n) +$$

$$+\int_0^t\int_\Omega \left[\gamma(\bar\theta)|\nabla\varphi|^2 - \gamma(\bar\theta^n)|\nabla\varphi^n|^2\right](\theta-\theta^n).$$

Integrating this relation in t, by (68), we have for small δ,

$$\|\theta-\theta^n\|^2_{L^2(Q)} \le c\int_Q \left[\gamma(\bar\theta)|\nabla\varphi|^2 - \gamma(\bar\theta^n)|\nabla\varphi^n|^2\right]^2. \quad (69)$$

Reasoning similar to the proof of Theorem 3.1 in [6], we derive that the right-hand side of (69) converges to zero, hence $\theta^n \to \theta$ in $L^2(Q)$ and the continuity of the operator B is proved.

By the Schauder fixed point theorem, there exists a $\bar\theta \in L^2(Q)$ such that

$$B(\bar\theta) = \bar\theta$$

which finishes the proof of Theorem 2.1 .

Acknowledgement. A.M.Khludnev was supported by the Weierstrass Institute during his stay in Berlin in January/February 2000 and by the Russian Fund for Basic Research (grant 0001-00842).

References

[1] Ames K.A., Payne L.E. *Uniqueness and continuous dependence of solutions to a multidimensional thermoelastic contact problem*, J. Elasticity, 1994, **34**, N2, 139-148.

[2] Andrews K.T., Shillor M. *A parabolic initial-boundary value problem modelling axially symmetric thermoelastic contact,* Nonlinear Anal., 1994, **22**, 1529-1551.

[3] Antontsev S.N., Chipot M. *The thermistor problem: existence, smoothness, uniqueness, blowup*, SIAM J. Math. Anal., 1994, **25**,N4, 1128-1156.

[4] Chipot M., Cimatti G. *A uniqueness result for the thermistor problem*, Europ. J. Appl. Math., 1991, **2**, 97-103.

[5] Dryer W., Duderstadt F., Hömberg D. *Finite Element Simulation des Widerstands–Impulsschweissens*, Technische Mechanik, 1999, **19**, 331–340.

[6] Hömberg D., Khludnev A.M., Sokolowski J. *On an equilibrium problem for a cracked body with electrothermoconductivity*, Les prepublication de l'Institute Elie Cartan, 1999, N23, 1-16.

[7] Khludnev A.M. *On equilibrium problem for a thermoelastic plate with a crack,* Siberian Math. J., 1996, **37**, N2, 394-404.

[8] Khludnev A.M., Kovtunenko V.A. *Analysis of Cracks in Solids,* WIT Press, Southampton-Boston, 2000.

[9] Khludnev A.M., Sokolowski J. *Modelling and Control in Solid Mechanics*, Birkhauser, Basel-Boston-Berlin, 1997.

[10] Lions J.L. *Quelques methodes de resolution des problemes aux limites nonlineaires,* Dunot, Gauthier-Villars, Paris, 1969.

[11] Shi P., Shillor M. *Existence of a solution to the N- dimensional problem of thermoelastic contact,* Comm.Part. Diff. Eqs, 1992, **17**, N9-10, 1597-1618.

[12] Shi P., Shillor M., Xu X. *Existence of a solution to the Stefan problem with Joule's heating*, J. Diff. Eqs, 1993, **105**, 239-263.

[13] Yuan G. W. *Regularity of solution of the thermistor problem.* Applicable Anal., 1994,**53**, N3-4, 149-155.

[14] Yuan G.W., Liu Z.H. *Existence and uniqueness of the C^α solution for the thermistor problem with mixed boundary value*, SIAM J. Math. Anal., 1994, **25,** N4, 1157-1166.