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Homoclinic orbits: since Puancaré till today.

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Abstract

The history and the contemporary results in homoclinic orbits are reported.

In 1885, the Swedish King Oscar II decided to call for an international competition on the best mathematical study of an important scientific problem. The price was supposed to be awarded at the day of the 60-th anniversary of the King, January 21, 1989. The competition had to be organized by the Swedish mathematical journal "Acta Mathematica". The evaluating committee included the following three: Mittag-Leffler, the editor of "Acta", and K.Weierstrass and Ch.Hermit, respected European mathematicians. The committee propose four themes for the competition. It is interesting to note here that only three themes were purely mathematical, whereas the first theme (proposed by K.Weierstrass!) was the problem of Celestial Mechanics. Namely, it was the question on the possibility to represent the solutions of the n body problem in the form of series in some known functions of time, converging uniformly on the whole real axis. It was also added by Weierstrass: "If it would occur impossible to solve the proposed problem to the prescribed date, the prize could be awarded to the work where some other problem of mechanics would be considered in this way and solved completely."

Altogether, eleven memoirs from different countries were submitted (anonymously, in order to make the competition absolutely fair). Two works were awarded by the prize: it was the memoir of Poincaré "On the three body problem and on the equations of dynamics" and a work by Appel "On integrals with weights and their application to the expansion of Abelian functions in trigonometric series." Somewhat later both these papers were published in the 13-th volume of "Acta" (1890) along with the report of Hermit on the paper by Appel. The report of Weierstrass on the paper of Poincaré was not published at that time[1]. Of course, the original report which Weierstrass sent to Mittag-Leffler is available now. Weierstrass wrote: "This paper cannot, in fact, be considered as the solution of the problem announced for the competition, but it is so significant, that with its publication, by my opinion, a new epoch in Celestial Mechanics will start." It is interesting that among many merits of this paper he mentions that this work has absolutely no value for the practical astronomy, moreover, it shows that many methods used by astronomers are wrong. We will not analyse the entire Weierstrass report and quote only that part which is related to biasymptotic solutions, the main subject of the present paper: "... even in the case when the mutually attracting, by the Newton law or by some other law,

[1]The reason was that the German mathematical community was very angry that the prize was given to French scientists.

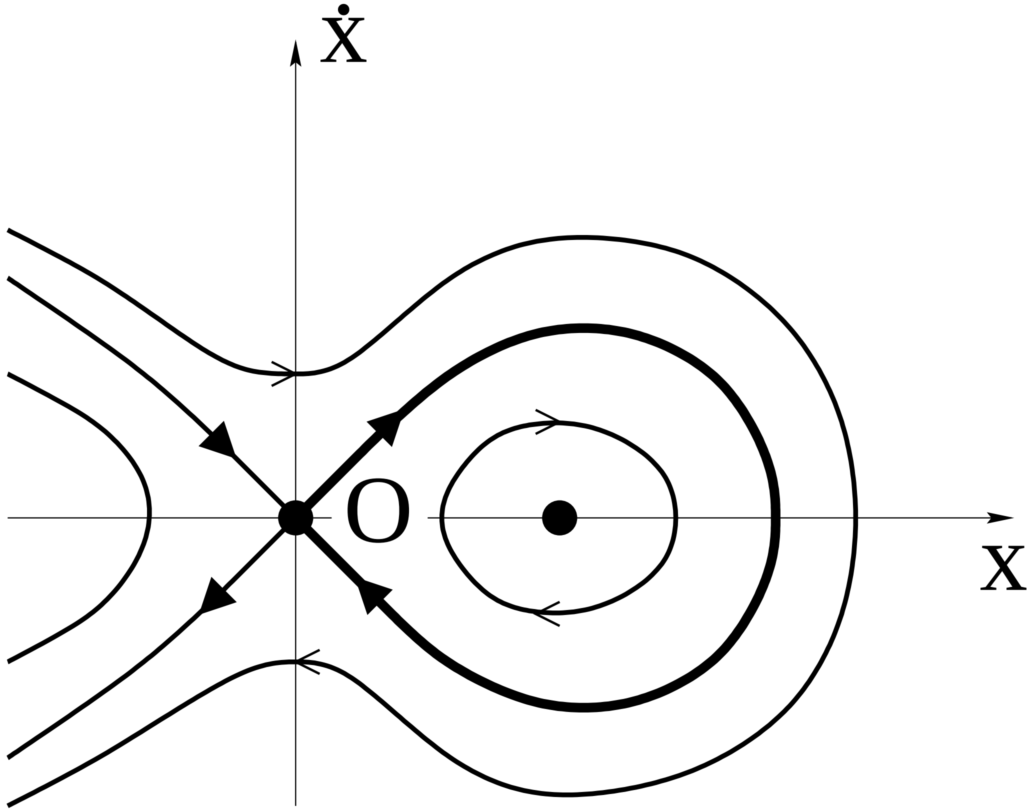

Figure 1:

bodies, in the number greater than two, are moving so that the distances between any two of them stay always in finite bounds, even then there exist the forms of motion which we, before, could hardly suspect about, and for which we do not know an appropriate (valid from $t=-\infty$ to $t=+\infty$) analytic expression; the only thing which is established is that they cannot be given by trigonometric series."

What are these new motions? To begin with, let us consider the equation

$$\ddot{x} - x + x^2 = 0.$$

It is integrable and its phase portrait is shown in Fig.1. The origin is a saddle equilibrium state, and the point $(1,0)$ is a center. One of the outgoing separatrices of the saddle $O(0,0)$ returns to it as $t \to \infty$, forming a loop. Since this orbit tends to O both as $t \to infty$ and $t \to +\infty$, the separatrix loop corresponds to a biasymptotic motion. All this is well known to Weierstrass, since he is, in essence, the author of the geometric method of drawing phase portraits for the equations of the form

$$\ddot{x} + f(x) = 0.$$

Analogously, one can consider the motions which are biasymptotic to saddle periodic orbits. Thus, consider the system

$$\begin{aligned} \ddot{x} - x + x^2 &= 0 \\ \dot{\theta} &= 1, \end{aligned} \tag{1}$$

where the variable θ is cyclic. The phase space of such system is $R^2 \times S^1$ where S^1 is a circle. Since we identify $\theta = 0$ and $\theta = 2\pi$, the study of such system reduces to the study of the map $T : \theta = 0 \ \rightarrow \ \theta = 2\pi$ defined by the orbits of the system. The phase portrait of this map is the same as shown in Fig.1, with the difference that $O(0,0)$ is now a saddle fixed point with the multipliers $e^{2\pi}$ and $e^{-2\pi}$, having one-dimensional stable and unstable manifolds W^s and W^u, halves of which coincide. Note that the set of orbits which are biasymptotic to $O(0,0)$ has now the cardinality of continuum.

The same situation may take place in an integrable system with two degrees of freedom when in some level of a first integral there exists a saddle periodic orbit whose stable and unstable manifolds coincide (entirely, or halves of them). Naturally, even before the given paper by Poincaré, the possibility of existence of such kind of asymptotic motions in integrable Hamiltonian systems was well known. What was shown by Poincaré is that in the nonintegrable cases the stable and unstable manifolds of saddle periodic orbits may intersect not coinciding.

It is the situation which appears, for example, when one considers the following equation

$$\ddot{x} - x + x^2 = \mu A \sin t,$$

which may also be written in the form of the system

$$\begin{aligned} \ddot{x} - x + x^2 &= \mu A \sin \theta \\ \dot{\theta} &= 1. \end{aligned}$$

This system is a small perturbation of (1) as $0 < \mu \ll 1$. The map $T_\mu : \theta = 0 \ \rightarrow \ \theta = 2\pi$ still has a saddle fixed point O_μ, which tends to $O(0,0)$ as $\mu \rightarrow 0$. In turn, the invariant manifolds W^s_μ and W^u_μ will be close to W^s_0 and W^u_0 (on any compact piece). However, they may have transverse intersections now, and the phase portrait at $\mu \neq 0$ will now look as shown in Fig.2.

The possibility of such behavior of the orbits in the three body problem was the subject of one of the chapters of the memoir by Poincaré. Later, in the third volume of "New methods of Celestial Mechanics", when describing the behavior of invariant manifolds in this case, Puancaré exclaims: "If we try to imagine the figure of these two curves and their repeated intersections, each of which corresponds to a biasymptotic solution, then these intersections form something like a grid, a net with infinitely tight loops. None of these loops should intersect itself but it should wind along itself in a very complicated way, so that to intersect all the loops in the net. The complexity of this figure which I would not even try to

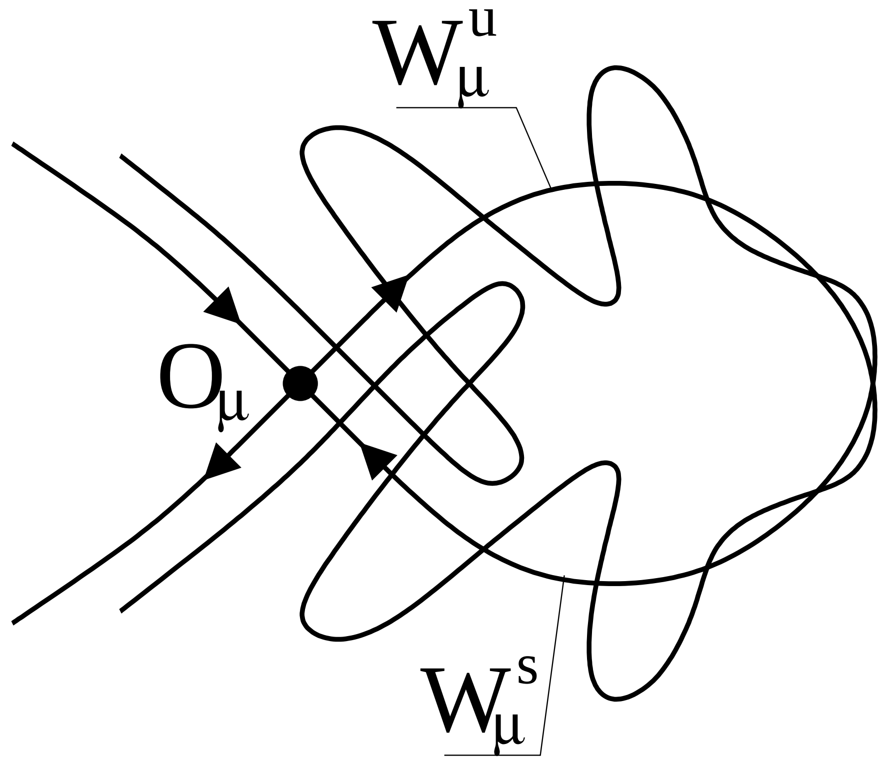

Figure 2:

draw is astounded. Nothing gives us a better impression on the complexity of the three body problem and, in general, all the problems of dynamics where the single-valued integral does not exist.” Now, Poincaré gives the name *homoclinic* to such biasymptotic motions. To those orbits which are asymptotic to two different periodic motions he proposes the name *heteroclinic* by natural reasons. The genie is released from the bottle. In the second half of the XX-th century practically all researchers in the qualitative theory of differential equations and nonlinear dynamics will speak in the language of these notions. As a whole, ”New methods” of Poincaré, which are an extensive version of the prize awarded memoir, became a program monument which determined the development of the qualitative and ergodic theories of the XX-th century for many years ahead. This includes the method of a small parameter for the search of periodic motions in near-integrable systems, the theory of integral invariants, Poisson-stable orbits and recurrence theorems, asymptotic series and many other topics.

As for our main subject, the homoclinic orbits, there is, formally speaking, only one general result belonging to Poincaré: if a two-dimensional map has a homoclinic orbit which corresponds to a transverse intersection of the stable and unstable manifolds of a saddle fixed point, then there exists infinitely many other homoclinic orbits. After that Poincaré had never returned to the study of systems with homoclinic orbits. Naturally, the question appears: why? To some extent, the answer is that Poincaré, being mostly interested in the actual problems of dynamics, gave a special value to stable periodic orbits. Thus, in the first volume of the ”New methods” (1891) he wrote about such solutions that "they give us a unique insight to the region which was considered earlier as inapproachable." Moreover, when he presented the method of small parameter he conjectured that in non-integrable analytic Hamiltonian systems stable periodic orbits are dense in compact levels of the Hamiltonian. For the further understanding, we should remember the following events.

One year before the third volume of "New methods" appeared, i.e. in 1898, a paper "On geodesics on the surfaces of negative curvature" was published by Hadamard. In the case of negative curvature all the geodesics are unstable. Indeed, the equation describing the mutual divergence of geodesics is written, in the linear approximation, in the following form

$$\frac{d^2y}{ds^2} + ky = 0,$$

where k is the curvature of the surface. Since $k < 0$, the behavior near every geodesics is of saddle type. All two-dimensional surfaces, except for the sphere and the torus, admit such metrics for which the curvature is constant and negative. The instability of all geodesics on such surfaces immediately implies that they must behave in a very complicated, messy way. Notably, Hadamard finishes his paper by the question: "Does there exist anything similar in the problems of dynamics and, in particular, in Celestial Mechanics? If it does, then all the setting of the problem on the stability of planetary systems has to be radically reconsidered."

In this connection, the opinion of Poincaré is interesting which can be found in his paper "On geodesic lines on convex surfaces" (1905). He notes that the paper by Hadamard is very important but he believes that the trajectories in the three body problem are not similar to geodesic lines on the surfaces of negative curvature; they, on the contrary, can be compared with the geodesics on the convex surface and only the latter can be interesting to the problem of dynamics! The paper by Hadamard is indeed very interesting because it was in this paper for the first time where the analysis of the dynamics was done by means of the method of symbolic description. It followed from this description that, in particular, closed geodesics are dense and all of them have homoclinic orbits, so in any neighborhood of any periodic orbit and any of orbits homoclinic to it there exist infinitely many other periodic orbits. Naturally, Poincaré could notice this. Since all periodic orbits here are unstable, it was, probably, the reason why he considered such structures sufficiently inessential for the problems of dynamics[2]. It also seems to me that Poincaré must know that at least single-round periodic orbits near a homoclinic orbit are saddle in the general (Hamiltonian, of course) case as well.

The further development of ideas of Poincaré on the study of homoclinic structures is due to Birkhoff. Here, we must recall, first, his paper [1] known as "Pope memoir" (it was submitted to a competition in honour of Pope Pius XI). In this paper Birkhoff shows that a two-dimensional analytic area-preserving diffeomorphism T

[2]Note that it was quite well known at that time that the trajectories of the ideal gas behave in an unstable way. Essentially, Boltzmann used this for the explanation (not rigorous enough, may be) of the irreversibility of the laws of macroscopic behavior. The furious polemic on this matter, in which Poincaré took part, must be well known to the reader.

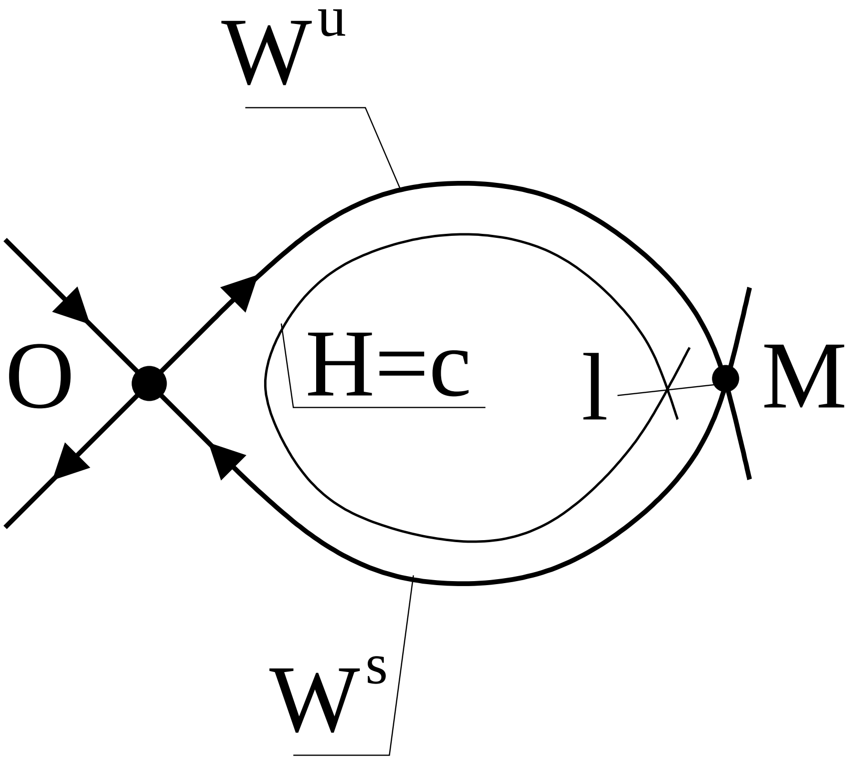

Figure 3:

possessing a saddle fixed point O with a homoclinic orbit Γ which corresponds to a transverse intersection of the stable and unstable manifolds of O has infinitely many single-round periodic orbits (of all periods starting with a sufficiently large one) in any neighborhood of $\Gamma \cup O$. The main idea of the proof can easily be recovered if one assumes that the map T has a smooth first integral[3] $H(x, y)$ in a small neighborhood of the saddle O. Let $H(x, y) = 0$ corresponds to the local stable (W^s) and local unstable (W^u) manifolds. Then the neighborhood of the saddle is foliated by hyperbola-like curves $H(x, y) = C$ with C sufficiently small. Naturally, iterating these invariant curves forward and backward by the maps T and T^{-1}, we can continue them along W^u and W^s, respectively. Since W^s and W^u intersect transversely, the closed invariant curves will have self-intersection points, as shown in Fig.3.

Namely, the set of the self-intersection points is a curve l emanating from the homoclinic point M. The orbit of any point P on l must stay on the invariant curve defined by the corresponding value of C. Therefore, if such orbit returns, after one round, to l, it must return to the same point P, i.e. this orbit is periodic. It is obvious that the number of iterations of a point $P_C \in l$ which are necessary to return again into a neighborhood of M tends to infinity as $C \to 0$. Now, it immediately follows by continuity that there must exist a sequence $\{C_n\}_{n=n_0}^{\infty}$, where $C_n \to 0$ as $n \to \infty$, such that $T^n P_{C_n} \in l$, hence $P_{C_n} = T^n P_{C_n}$. This means indeed that infinitely many periodic points lies in any neighborhood of the homoclinic point M.

[3]The existence of such integral was proved by Moser almost twenty years later.

In the same paper Birkhoff makes an important conjecture on the possibility of a complete description of all the orbits in a small neighborhood of the homoclinic orbit in the language of symbolic dynamics, based on the analogy with the geodesic flows on the surfaces of negative curvature. Note that he stresses the necessity of the use of infinitely many symbols.

In general, we should mention in this connection a great contribution of the Birkhoff school and, in particular, Morse and Hadlund, to the development of symbolic dynamics as an important part of the theory of dynamical systems. Note, however, that the area of their application of symbolic dynamics remained limited by geodesic flows alone.

Explicitly, the problems connected with the study of nonconservative systems was set up by Andronov. We should especially mention that his approach came from the problems of the theory of nonlinear oscillations, which was closely associated with the theoretical radioengineering in those years. It had become clear very soon that in those cases where the problems admit modelling by two-dimensional systems a ready mathematical apparatus had, in fact, existed already in the form of the Poincaré theory of limit cycles and Lyapunov theory of stability. This allowed Andronov to develop an important thesis that the adequate mathematical image of self-oscillations is given by rough stable limit cycles. The next step in this direction was made in the paper by Andronov and Pontryagin "Rough systems" [2]. Here, a rigorous definition of roughness of dynamical systems was given (a system is rough if every C^1-close system is topologically equivalent to it, i.e. if there exist a homeomorphism which maps the trajectories of one system to the trajectories of the other system; an additional requirement was that this homeomorphism must be sufficiently close to identity[4]). Moreover, necessary and sufficient conditions of roughness of systems on a plane were obtained. Thus the idea of roughness as the stability with respect to small smooth perturbations was introduced in the theory as the basis for the study of dynamical systems. In fact, this paper allows us to start to speak about the theory of smooth dynamical systems as a separate mathematical discipline, because such things as the subject of the study and an appropriate equivalence relation were quite explicitly formulated here. Of course, this paper dealt specifically with the systems on a plane, but the significance of the notion of roughness of the system for the multidimensional case was understood quite well. So, the question on the development of the theory of rough systems for the general case arose naturally. Thus, in the introduction to the famous book by Andronov, Vitt and Khaikin "The theory of oscillations" (1937)[5] it was directly announced that in the further volume the authors suppose to develop the multidimensional theory, including the case of distributed systems (described by PDE's). As Evgenija Leontovich told me later "we were going to study multidimensional systems."

However, it did not happen that time due to different circumstances. Andronov

[4]In the later version by Peixoto the roughness was called structural stability and the requirement of the closeness of the homeomorphism to identity was abandoned.

[5]due to the sad circumstances of that time, Vitt was excluded from the authors in th first edition of this book

suddenly switched to the study of nonlinear problems of the theory of automatic control. E.A.Leontovich, after her work on bifurcations of limit cycles from the separatrix loop, devoted herself completely to the writing of the books on the qualitative theory of dynamical systems and their bifurcations. Bautin, after he proved that quadratic two-dimensional systems can have three limit cycles, started to develop the theory of clocks. So, among the four main people of the "Gorky school", only Mayer dealt closely with the problem of roughness of multidimensional systems. However, he did not achieve positive results. Moreover, an opinion of nonroughness of the systems with homoclinic orbits appeared, at least in Gorky[6]. Only systems with simple dynamics (later called Morse-Smale systems) were considered as rough. It is interesting that Smale also adhere to this point of view when he wrote his "Morse inequalities for dynamical systems."

We note that unsuccessful, though comprehensive, analysis of systems with homoclinic orbits led Mayer to the solution of the Birkhoff problem on the ordinal number of the center of dynamical systems. Note that the main object in the construction of Mayer was the geodesic flow on a surface of negative curvature. It is quite probable that Andronov and Mayer could understand that such geodesic flows are rough. But Mayer died in 1951 and Andronov died in the next year. An explicit setting to the problem of roughness of multidimensional systems was given by E.A.Leontovich in her talk on the III Soviet Mathematical congress in 1956: "One should not think that he notion of roughness is extended trivially onto both these cases. Not speaking about the problem that in the case of a nonautonomous system of the second order (depending periodically on time) this question is closely connected with the question about special and regular orbits, which is not clear, here we have a number of principal difficulties. Analogous difficulties appear in the case of an autonomous system in the three-dimensional space. I cannot speak about this in detail. I just can briefly say that the root of all difficulties is connected to a homoclinic point of a transformation of a plane."

To summarize these not so well known events, we limit ourselves by the following general reflection. If one cannot show the transversality of the intersection of the stable and unstable manifolds of saddle periodic orbits, then the probability is sufficiently high that in the system under consideration homoclinic tangencies (the tangencies between W^s and W^u) can appear. It is sufficiently obvious that when there exists one homoclinic tangencies, then arbitrarily small smooth perturbation of the system can be found such that the perturbed system will have new homoclinic tangencies, etc.. Moreover, this can be achieved by generic one-parameter unfoldings of the original tangency. So, the homoclinic tangencies behave in a persistent way in this sense. Undoubtedly, this kind of picture had already been observed by Poincaré when he gave his geometric proof of the existence of infinitely many homoclinic orbits. Indeed, in a one-parameter family of diffeomorphisms T_μ, which includes a diffeomorphism generated by an integrable system with a separatrix loop, even when the single-round homoclinic orbits are rough (transverse) at $\mu \neq 0$ (Fig.4)

[6] In the beginning of 70-s (!) N.F.Otrokov, a former participant of the Andronov seminar, expressed it to me in the following way: "But we know that such systems are nonrough."

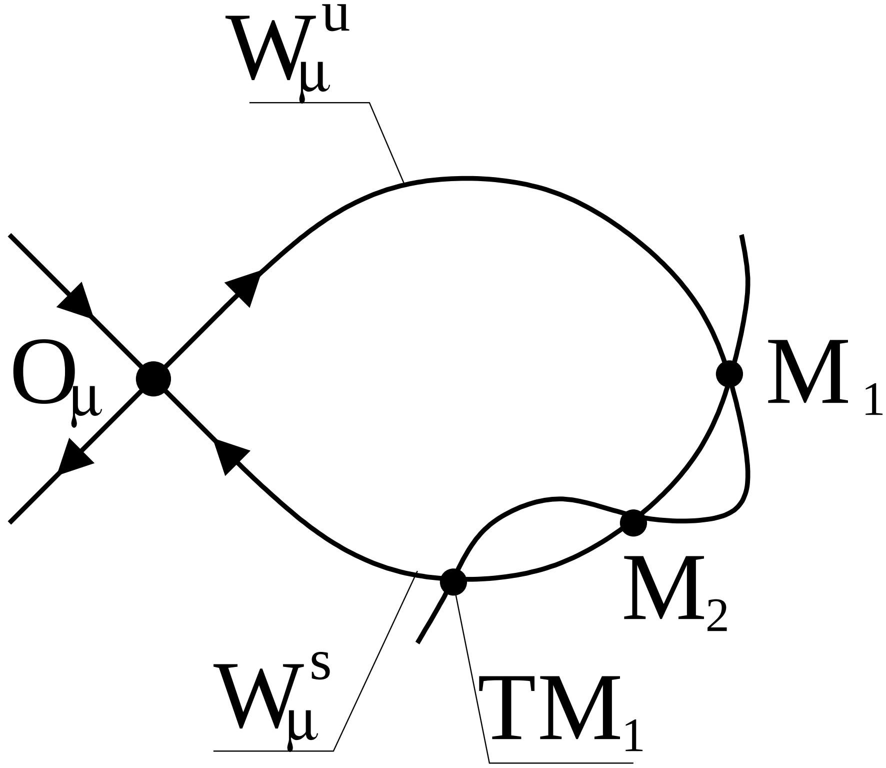

Figure 4:

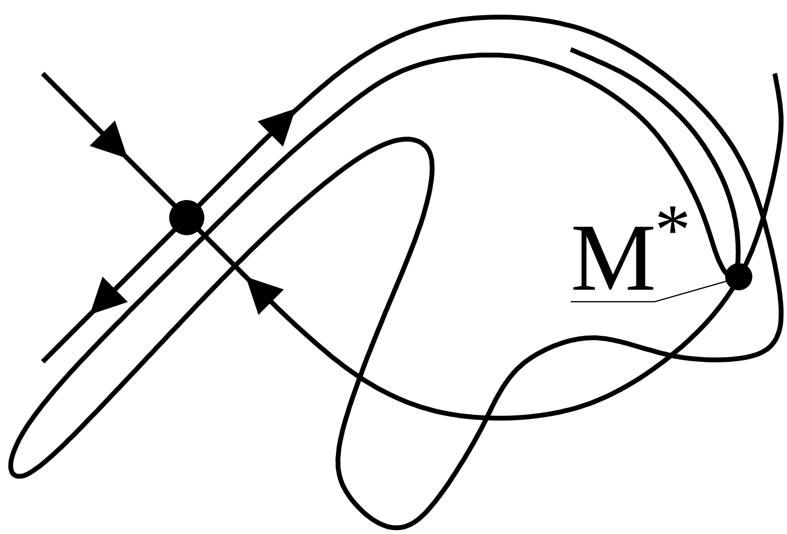

Figure 5:

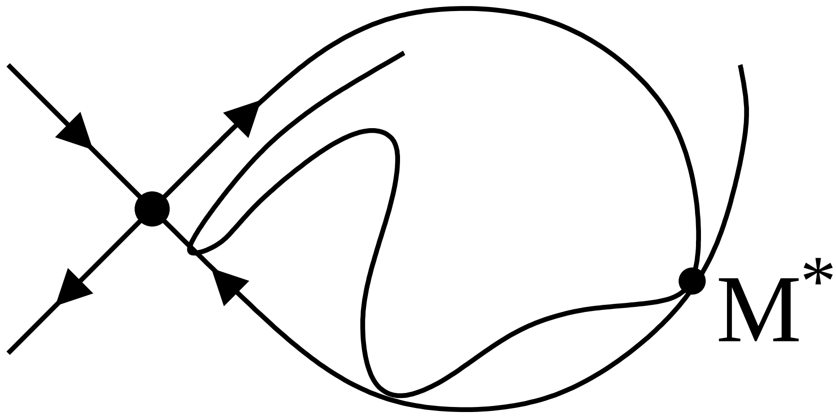

Figure 6:

there exist arbitrarily small values of μ for which the diffeomorphism has a double-round homoclinic orbit at the points of which the manifolds W^s and W^u have a tangency (Fig.5).

For the case of nonconservative systems, like the following equation

$$\ddot{x} + \mu h \dot{x} - x + x^2 = \mu A \sin t,$$

where $\mu \ll 1$, even single-round homoclinic orbits can correspond to the tangency of the stable and unstable manifolds for some relation between h and A (Fig.6). Here, the increase in h leads to the disappearance of the homoclinic orbits, whereas a pair of transverse homoclinic orbits is born when h decreases. Then, double-round orbits of homoclinic tangency can appear and multi-round ones, as well. Therefore, an idea of the nonroughness of the systems with homoclinic orbits arises naturally.

In 50-s, the main source on homoclinic orbits and related dynamical structures was the book by Nemytsky and Stepanov "Qualitative theory of differential equations" (1949). The above-mentioned Birkhoff theorem was presented in this book, along with a number of other statements on the structure of the nonwandering set in a neighborhood of the homoclinic point. Since the presentation of this topic preserved completely the original Birkhoff's style, its understanding was quite difficult. When I turned to the original "Pope memoir", it occured that the statements on homoclinic orbits given in the book by Nemytsky and Stepanov are based solely on the Birkhoff conjecture explained above.

The interest to homoclinic orbits and to the behavior of stable and unstable manifolds of saddle periodic orbits increased in the end of 50-s among physicists. The problem was that the structure of magnetic fields within toroidal accelerators like "Tokamak" us, in general, non-integrable. Taking into account the deviation from integrability shows that due to the appearance of homoclinic intersections and splits between stable and unstable manifolds a charged particle may hit the inner surface of the "Tokamak." As a result of such analysis, a paper [3] by V.K.Melnikov appeared where an estimate (known now as Melnikov formula) for the splitting of the manifolds was given.

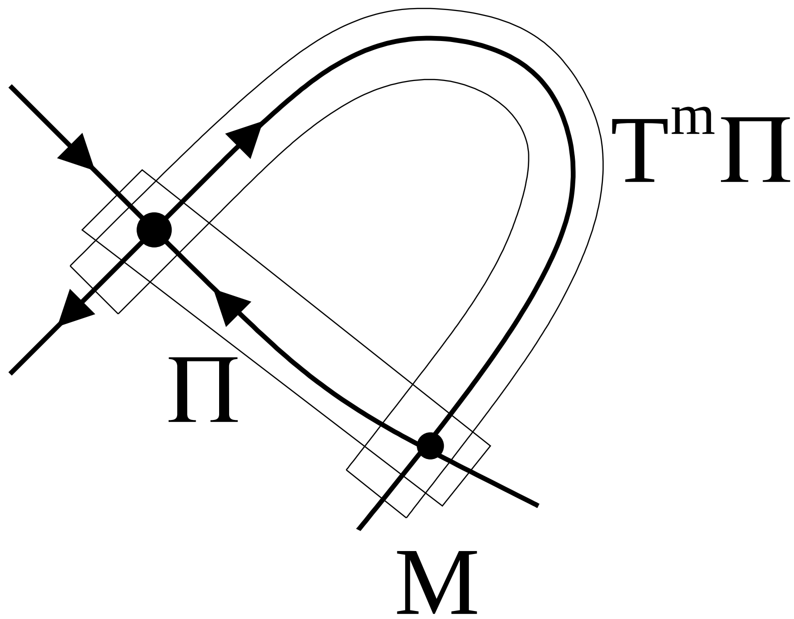

Figure 7:

The progress came at 60-s. In 1961, in Kiev, on a conference on nonlinear oscillations, Smale gives an example (known now as the "Smale horseshoe") of a diffeomorphism of a plane which behaves on the non-wandering set as the Bernoulli scheme of two symbols [4]. As a result, it follows that periodic and homoclinic orbits are dense in the non-wandering set in this example. Although the proof of the roughness of this diffeomorphism which was constructed piecewise linear on the non-wandering set was not proven by Smale, the validity of this fact did not cause any doubt. Little a bit later Anosov proved explicitly [5, 6] the roughness of the so-called U-systems (called Anosov systems now) which include the geodesic flows on compact manifolds of negative curvature and hyperbolic diffeomorphisms of a torus.

The idea of the "horseshoe" was applied by Smale to his proof [7] of the theorem on complicated behavior of orbits in a small neighborhood of a transverse homoclinic point of multidimensional diffeomorphisms. Taking as an initial strip Π a neighborhood of the saddle fixed point O which contains a piece of the stable manifold along with some homoclinic point M, for some integer m we obtain that the m-th iteration of the diffeomorphism, T^m, maps Π to a horseshoe (Fig.7). From this picture it followed that T^m has an invariant set in Π such that T^m is conjugate on this set to the Bernoulli shift of two symbols.

Smale obtained this result in the assumption that T is reduced to a linear form near the saddle point O. It meant, for example, that the theorem could not be applied to the case of Hamiltonian systems and symplectic maps. Moreover, and this is indeed important, the method of "horseshoe" did not give a complete description of all trajectories lying in a neighborhood of the closure of the homoclinic orbit, therefore it did not solve the problem of Birkhoff. The complete solution was published by me in 1967. But this was preceded by the discovery of another situation with

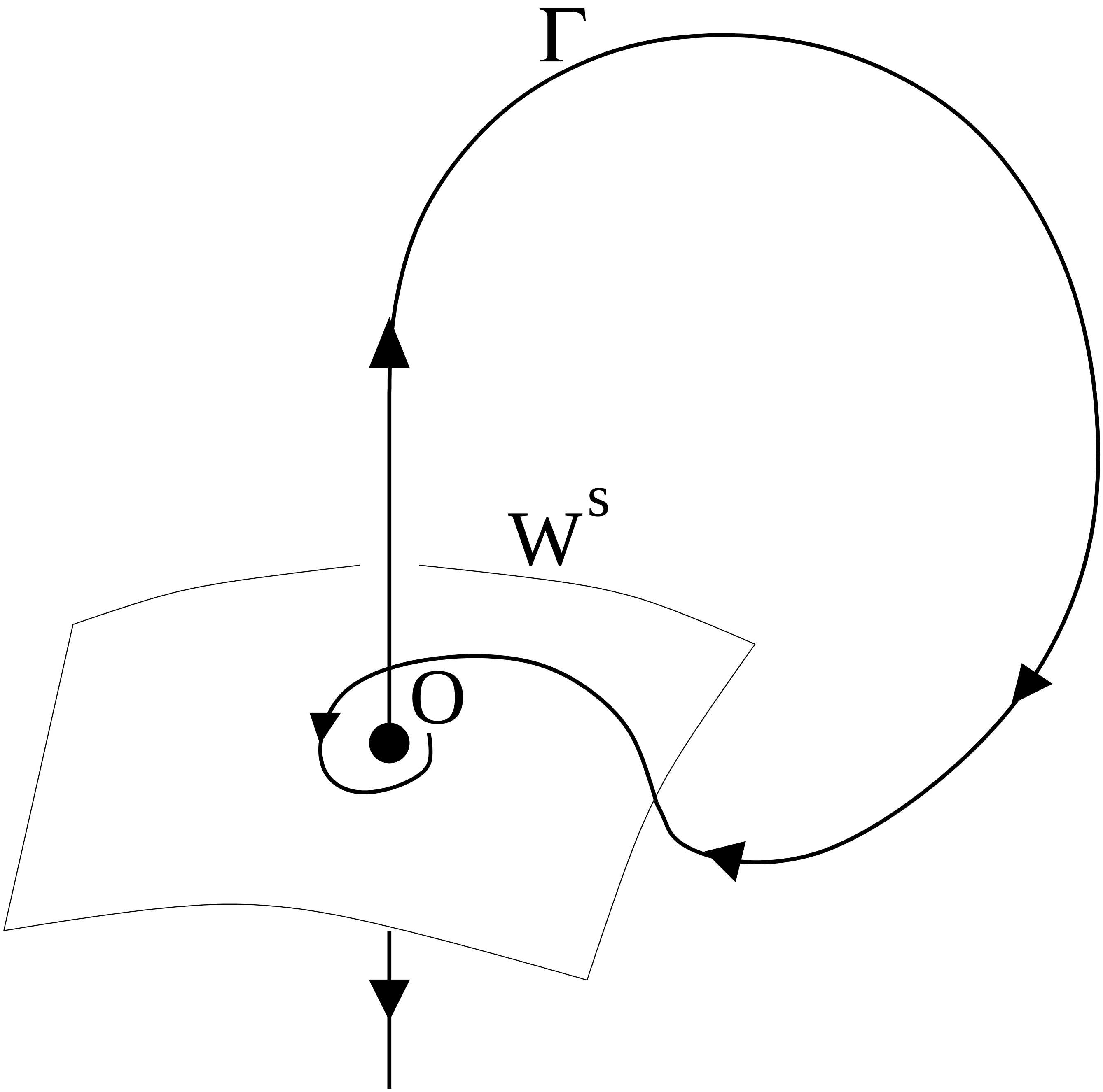

Figure 8:

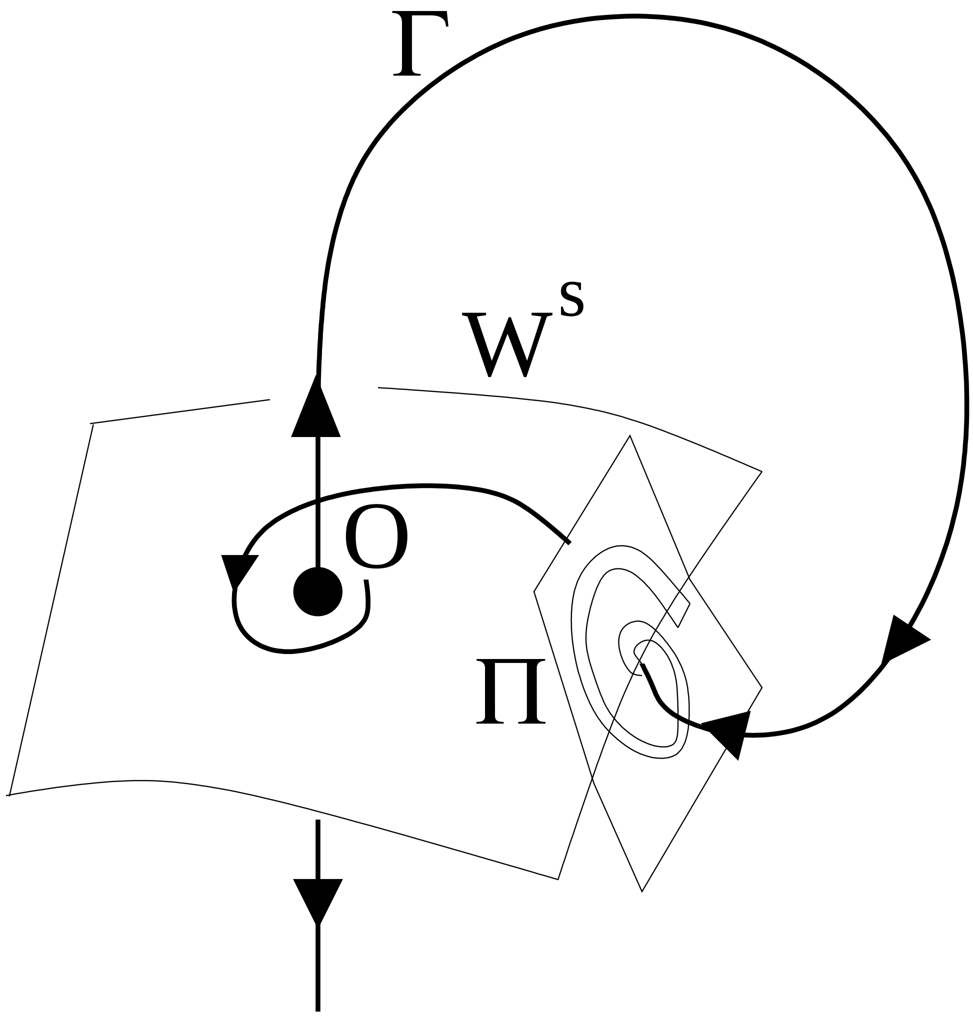

Figure 9:

complicated dynamics.

I started to study multidimensional systems in the end of 50-s. The first problem was to generalize the global bifurcations by Andronov and Leontovich from the two-dimensional case onto the multidimensional one. In principle, the solution of these problems for the case when only one, moreover stable, periodic orbit is born from the separatrix loop could be achieved by the methods known to that time. After the study of these cases [9], I turned to the following problem: let a three-dimensional system have an equilibrium state O of saddle-focus type, i.e. a pair of the roots of characteristic equation is complex and lies in the left half-plane: $\rho \pm i\omega$, where $\rho < 0$, $\omega \neq 0$, and the third root is real and positive: $\lambda > 0$. Assume that one of the orbits coming out of O at $t = -\infty$, tends to O again as $t \to +\infty$, i.e. a homoclinic loop Γ is formed (Fig.8).

The case where the saddle value

$$\sigma = \rho + \lambda$$

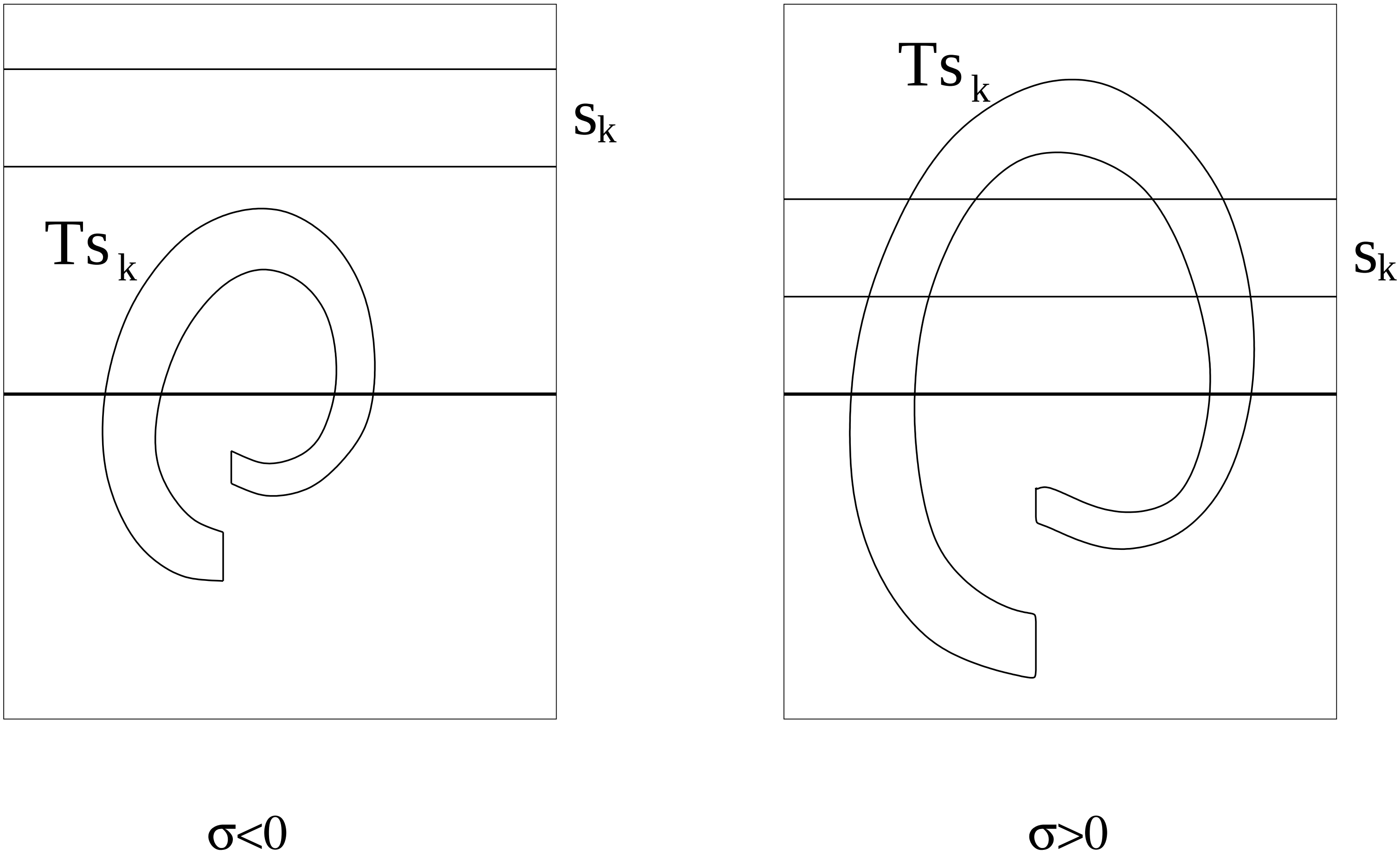

Figure 10:

is less than zero led to the problem of the birth of a stable limit cycle which had already been solved in the general multidimensional setting. On the contrary, the case $\sigma > 0$ required a separate consideration. Both in the cases $\sigma < 0$ and $\sigma > 0$ the image of the upper half of a small cross-section Π under the map T by the trajectories close to the loop Γ is a spiral-shaped region (Fig.9). So, Π can be divided into a sequence of strips s_k, $k = k_0, \ldots, \infty$, such that the image of each strip is exactly one curl of the spiral, Ts_k. In the case $\sigma < 0$ we have $s_k \cap s_k = \emptyset$, but at $\sigma > 0$ the map T acts on s_k as the Smale horseshoe (Fig.10). Thus, it occured that at $\sigma > 0$ the Poincaré map has infinitely many Smale horseshoes, hence - infinitely many saddle periodic orbits [10].

Naturally, the first whom I told about this was E.A.Leontovich. Her reaction, as she said me somewhat later, was: "I wanted to say that it cannot be."

Immediately, the understanding of the dynamics for the case of the transverse homoclinic orbit came. The most convenient way (at least to the author) was to consider the system in the form of flow. Usually the cross-section is chosen transversely to the periodic orbit, but I chose it as a transversal to the stable manifold in a neighborhood of the homoclinic point. In this case it is also possible to construct

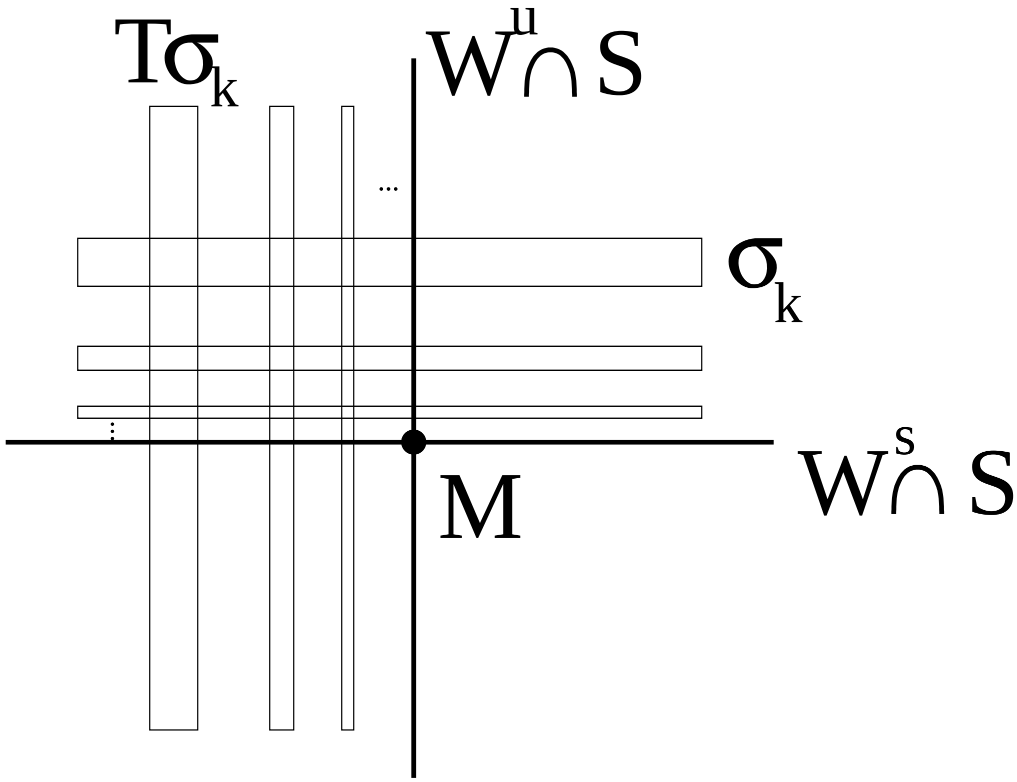

Figure 11:

the Poincaré map T, whose domain of definition is an infinite sequence of disjoint strips σ_k, $k = k_0, k_0+1....$ Here, $T\sigma_k$ intersects all the strips for any k (Fig.11). In essence, this is the picture which gives a complete description identical to that which was conjectured by Birkhoff in the "Pope memoir". Note that, unlike the case of a homoclinic loop of a saddle-focus which is a non-rough structure, the transverse homoclinic orbit by Poincaré is rough. Hence, it can admit a symbolic description with a finite number of symbols. In order to understand this, let us consider different possible codings for the orbits in a neighborhood of a homoclinic orbit of a smooth flow.

Let a system have a periodic motion L of saddle type, i.e. its multipliers does not lie on the unit circle and part of them lies inside this circle and the rest of the multipliers lies outside the unit circle. Then L will have a stable and unstable manifolds W^s and W^u. Suppose that they have a common orbit Γ, different from L (Fig.12).

Take a small neighborhood U of the set $L \cup \Gamma$, it is a solid torus with a handle (Fig.12). Any orbit lying entirely in U will be coded as follows: one round in the solid torus will correspond to the symbol O, and passing along the handle will be coded by the symbol $\hat{1}$. By this rule, the periodic motion L will be coded by the infinite sequence of zeros:

$$(...0, ..., 0...),$$

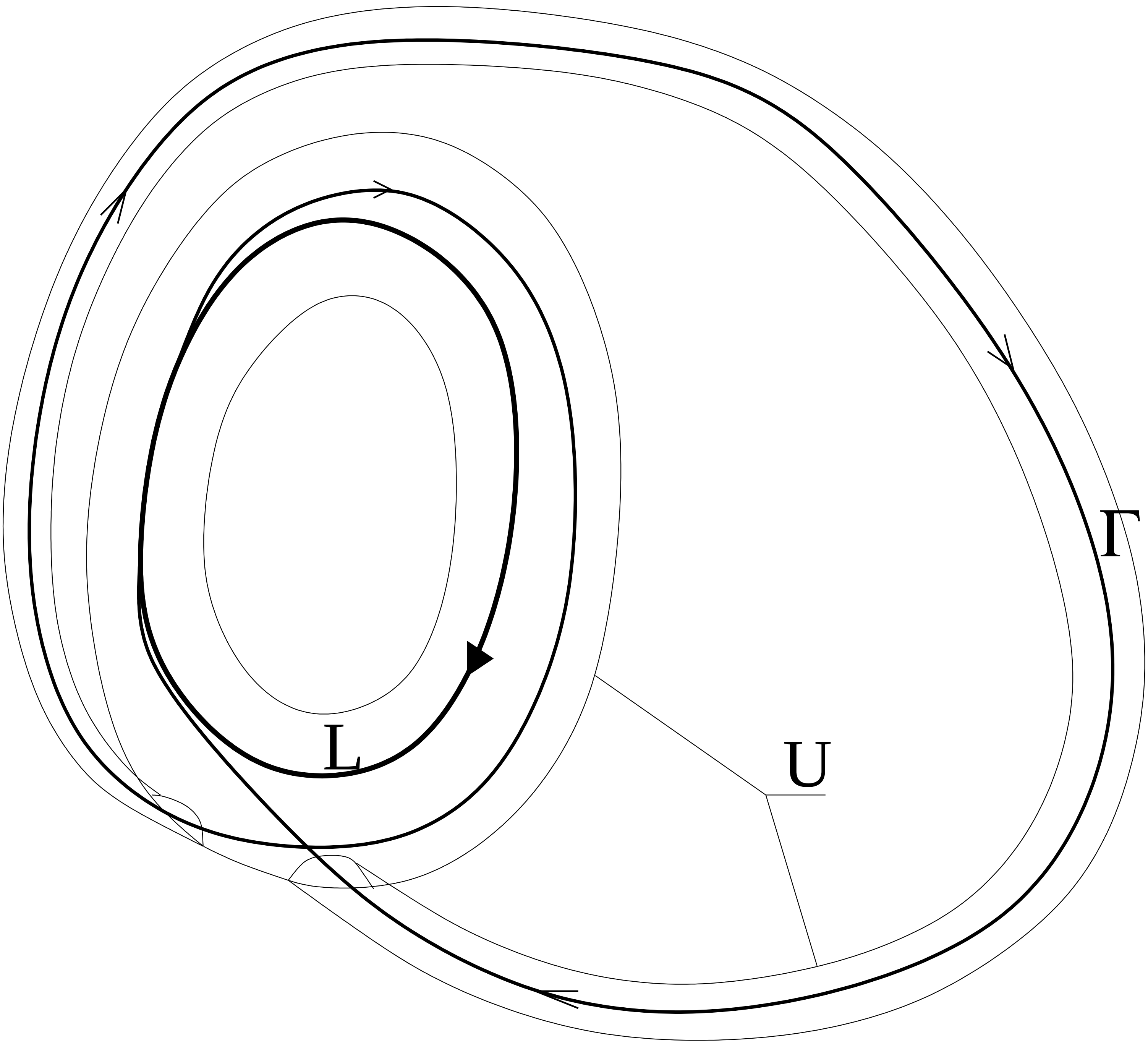

Figure 12:

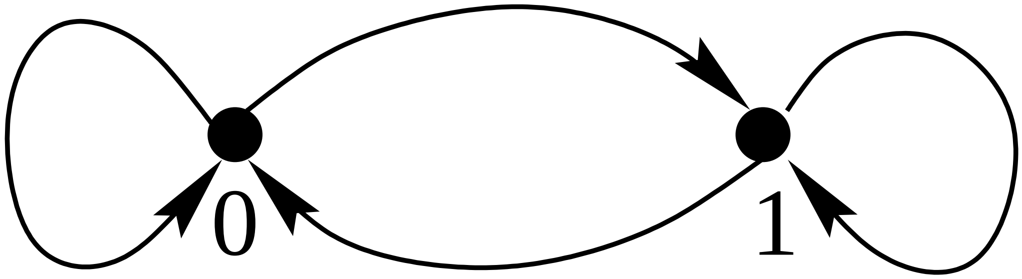

Figure 13:

and the coding of the homoclinic orbit Γ will be

$$(...0, \hat{1}, 0...).$$

. Thus, any orbit in U will be coded by some sequence

$$(..., i_{-k}, ..., i_0, i_1, ...),$$

where the symbol i_m is either 0 or $\hat{1}$. Here, the symbol $\hat{1}$ must be necessarily followed by a sufficiently long string of zeros. The minimal number $\bar{n}$ of zero symbols following $\hat{1}$ depends on the choice of the neighborhood: the less the size of the neighborhood, the larger $\bar{n}$. One cam make the following recoding: denote

$$1 = \{\hat{1}, \underbrace{0 \ldots 0}_{\bar{n}}\}.$$

Then the orbits will be coded by sequences

$$(..., i_{-k}, ..., i_0, i_1, ...),$$

where the symbol 1 may be followed by either of the symbols 0 or 1. In other words, the codings here are the orbits of the Bernoulli shift of two symbols whose graph is shown in Fig.13.

In [11], I used another code. Since the symbol $\hat{1}$ is followed by zeros, any orbit in U which is not asymptotic to L may be coded by the infinite sequence

$$(..., n_{-k}, ..., n_0, n_1, ..., n_k, ...)$$

where n_k is the number of zeros following the corresponding symbol $\hat{1}$; an orbit which is asymptotic to L on one end (let as $t \to -\infty$) is coded by the infinite to one end sequence

$$n_0, n_1, ..., n_k, ...,$$

and an orbit homoclinic to L is coded by the finite sequence

$$(n_0, n_1, ..., n_k)$$

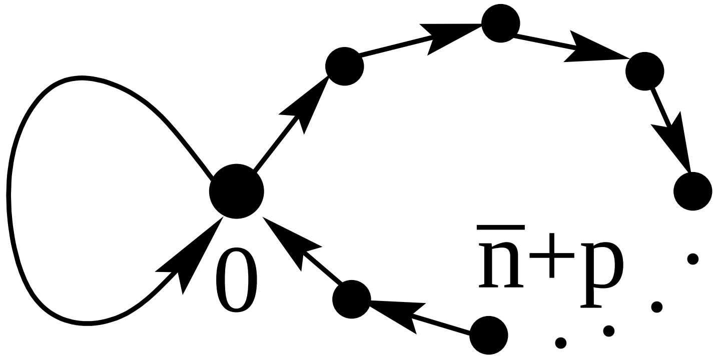

Figure 14:

(here the number of passes along the handle equals to $k+1$). We always have here $n_k \geq \bar{n}$ where $\bar{n}$ is an integer depending on the size of U. In fact, it is the code which appears when we use the numbers n of the strips σ_n as the symbols.

The claim that for every such symbolic sequence there exists in U a unique (and saddle) orbit with the given coding is in the heart of the problem here. It was the claim which I managed to prove in [11] under the assumption that W^s and W^u intersect transversely at the points of the homoclinic orbit Γ.

Rigorously speaking, one should invoke the notion of the suspension here. It is not that important, we just note here that the flow under consideration, when restricted onto the set of the orbits lying entirely in a special small neighborhood of $L \cup \Gamma$, is topologically equivalent to the suspension over the Bernoulli shift of two symbols, independently on the dimension of the system. In the case of diffeomorphism the symbolic description in a small neighborhood of a rough homoclinic orbit Γ is given by the Markov chain shown in Fig.14, where the meaning of $\bar{n}$ is the same as above and p is the number of iterations necessary for the orbit of a homoclinic point M lying on W^u_{loc} in a small neighborhood of O to get to this neighborhood again. Clearly, the chains with different values of $\bar{n}+p$ are not topologically conjugate (because the corresponding values of the topological entropy will be different). It means that the answer depends, in the case of diffeomorphism, on the choice of the neighborhood[7].

The problems related to the study of the behavior in a neighborhood of homoclinic orbits of multidimensional systems required the development of a new technique. One of the elements of it was the construction of the local maps in a neighborhood of saddle periodic orbits and equilibrium states in the so-called cross-form, which means the solution of a special boundary value problem instead of the Cauchy problem[8]. Effective criteria of the existence of aperiodic trajectories was also given (in terms of

[7]Of course, the suspensions over such Markov chains are equivalent.

[8]The inconvenience of the study of these maps in the direct form by means of the solution of the Cauchy problem is due to that the derivatives with respect to the initial values tend to infinity with the increase of the number of iterations, whereas all the derivatives of the map in the cross-form are uniformly bounded (tend to zero, in fact).

theorems on saddle and stable fixed points of operators acting in countable products of Banach spaces).

Along with the problems described above, this technique allowed the author to solve in 60-s an analog of the Poincaré-Birkhoff problem for the case of a homoclinic manifold to a saddle invariant torus [12], as well as a principally new bifurcational problem - the birth of a nontrivial hyperbolic set from a bunch of homoclinic loops to a nonhyperbolic equilibrium state of saddle-saddle type [13]. Later the technique of cross maps was applied effectively to the solution of the Poincaré-Birkhoff problem for the maps in Banach spaces [14], including the case where the unstable manifold of the saddle fixed point is infinite-dimensional, and for non-autonomous systems with an arbitrary aperiodic dependence on time (see [15]).

Now, the natural development of the research led to the necessity of the study of homoclinic tangencies. The systematic study of this problem was begun by N.K.Gavrilov and the author in [16] in the beginning of 70-s. As a subject of the research, we took three-dimensional flows with a saddle periodic orbit L whose stable and unstable manifolds are quadratically tangent along some homoclinic orbit Γ. Let λ and γ be the multipliers of L, and $|\lambda| < 1$, $|\gamma| > 1$. Suppose that the saddle value $\sigma = |\lambda\gamma| \neq 1$; without loss of generality we may assume $|\sigma| < 1$. Let U be a small neighborhood of the closure $\Gamma \cup L$ of the homoclinic orbit and let N be the set of all orbits lying in U. Depending on the signs of multipliers and some coefficients which characterize how the stable and unstable manifolds adjoin to Γ, systems with homoclinic tangencies were grouped in [16] into three classes. It was also established that
1) for the systems of the first class the set N is trivial: $N = \{L, \Gamma\}$
2) for the systems of the second class, N is a nontrivial hyperbolic set which admits a complete description in the language of Bernoulli shift of three symbols,
3) for the systems of the third class, N still contains nontrivial hyperbolic subsets which, however, do not exhaust the whole set N in general; moreover, on the bifurcational surfaces composed of systems of the third class, dense structural instability takes place.

Specifically, it follows from [16] that in any one-parameter family of systems for which the original homoclinic tangency of the third class does not split and for which the quantity

$$\theta = -\frac{\ln|\lambda|}{\ln|\gamma|}$$

changes monotonically, systems with nonhyperbolic periodic orbits are dense. Later, it was shown in [17] that in such one-parameter families systems with infinitely many stable[9] periodic orbits are dense, as well as systems with secondary homoclinic tangencies.

The reason is that for the systems of the third class the structure of the set N depends essentially on the value of θ. Thus, consider the case $\lambda > 0$, $\gamma > 0$, for instance. In this case, the system belongs to the third class when the stable and

[9] if $\sigma < 1$; if $\sigma > 1$, then systems with infinitely many completely unstable

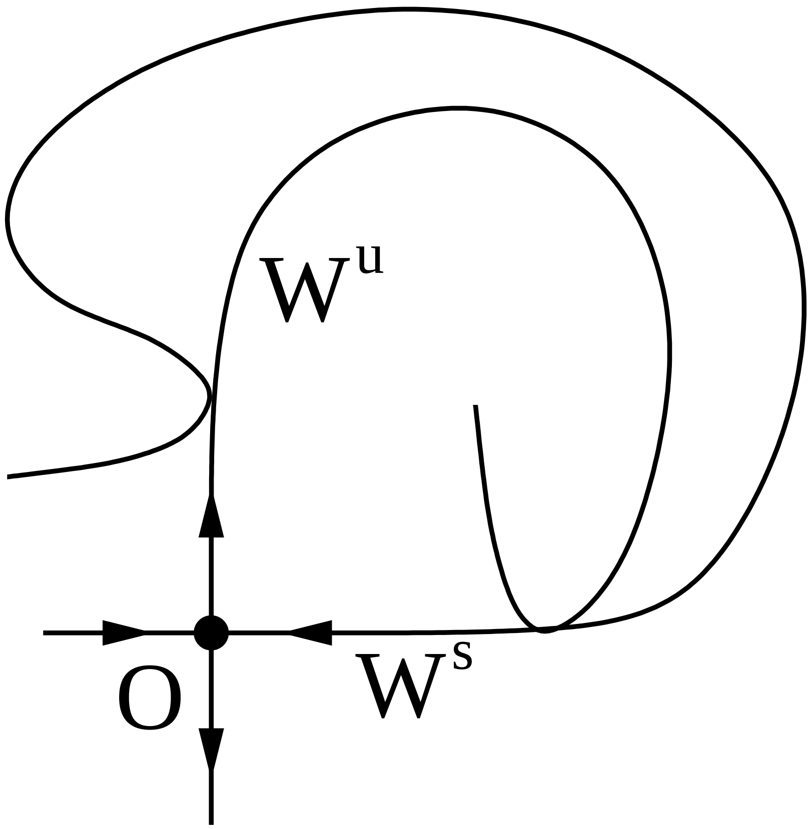

Figure 15:

unstable manifolds behave like it is shown in Fig.15: on a two-dimensional cross-section S to L the manifold W^u is tangent to $W^s_{\rm loc}$ near the point $O = S \cap L$ from above, and W^s is tangent to $W^u_{\rm loc}$ near O from the left side. Like in the case of a rough homoclinic, every orbit from N (except for O and Γ) is coded by a sequence of integers

$$(..., n_{-k}, ..., n_0, n_1, ..., n_k, ...),$$

infinite for the orbits not lying in W^s or W^u, and finite (to one end or to both) for the orbits which are asymptotic to L. It was shown in [16] that for any sequence of sufficiently large integers n_k for which $n_{k+1} < n_k\theta'$ at every k, in U there exists a continuum of orbits with this coding; and vice versa, if $n_{k+1} > n_k\theta''$ at least for one k, then no orbit in U has such coding. Here θ' Í θ'' are some numbers such that $1 < \theta' < \theta < \theta''$; moreover, θ' and θ'' can be taken arbitrarily close to θ if the size of the neighborhood U is sufficiently small. A more precise description was obtained in [18, 17]. In any case, it is immediately clear that if the value of θ changes, the structure of the set N must change permanently.

Later, it was shown in [19] explicitly that θ is an *invariant of the Ω-equivalence* (i.e. topological equivalence on the set of nonwandering points) for systems of the third class. In other words, the systems with different values of θ cannot be Ω-equivalent, which means that *arbitrarily small changes of θ must inevitably cause bifurcations in the nonwandering set*, i.e. the mentioned above bifurcations of periodic and homoclinic orbits, etc..

Systems with quadratic homoclinic tangencies compose bifurcational surfaces of codimension one. It is natural, therefore, that one should consider first what happen in the one-parameter unfolding, when the tangency of invariant manifolds is split. Let μ be the bifurcational parameter responsible for the splitting of the separatrices and let X_μ be a family for which μ changes monotonically. Thus, X_μ intersects transversely the bifurcational surface of systems with a homoclinic tangency at $\mu = 0$. The following fact is of principal value here:

in any transverse one-parameter family X_μ there exists a sequence of intervals accumulating at $\mu = 0$ in which the values of the parameter μ corresponding to new quadratic homoclinic tangencies are dense (and X_μ is transverse to the corresponding bifurcational surfaces).

This remarkable result was proven by Newhouse for nonconservative two-dimensional diffeomorphisms[10] in [24]. This means, roughly speaking, that although every individual homoclinic tangency can be removed by a small perturbation of the system, it is however impossible to prevent the appearance of new homoclinic tangencies.

[10] It was extended onto the general multidimensional case in [20]; the multidimensional case was also done in [21] under condition that the unstable manifold of the saddle periodic orbit is one-dimensional. For the area-preserving maps, and in particular, for the case of small periodic perturbations of Hamiltonian systems on the plane with a separatrix loop (like system (1)) the analogous result has been proved only recently in [22, 23].

The regions of structural instability in the space of C^r-smooth ($r \geq 2$) dynamical systems where systems with homoclinic tangencies are dense are called the Newhouse regions (the above mentioned intervals of the parameter values at which the transverse family X_μ intersects the Newhouse regions are called Newhouse intervals).

The most popular result (discovered in [25]) about dynamics of the two-dimensional maps in the Newhouse regions is that when the saddle value $\sigma = |\lambda\gamma|$ is less than 1,

systems with infinitely many stable[11] *periodic orbits are dense there.*

This statement follows almost immediately from the denseness of the parameter values corresponding to homoclinic tangencies and an earlier result of [16] that if $\sigma < 1$, then for any transverse family there exists a sequence (accumulating at $\mu = 0$) of intervals of μ corresponding to the existence of a stable periodic orbit.

The chain of amazing phenomena in systems with homoclinic tangencies does not end here, as a series of papers by S.V.Gonchenko, D.V.Turaev and the author has shown. As we noticed, it was known that for every system with a homoclinic tangency of the third class, by an arbitrarily small change in θ a one more orbit of homoclinic tangency can be created (and the original homoclinic tangency does not disappear). It occured that this face has quite severe consequences. Namely, if we can produce a new homoclinic tangency without destroying the original one, then we can repeat this procedure and obtain, the third tangency, fourth, etc.. Thus, using localized small smooth perturbations, we were able to prove [28, 29] that

in the set of systems with homoclinic tangencies of the third class systems are dense[12] *each of which has infinitely many saddle periodic orbits with an orbit of homoclinic tangency each, and all these tangencies are of the third class.*

Note that the latter means that such systems have *infinitely many independent continuous invariants (moduli) of the Ω-equivalence* (because for each of these homoclinic tangencies the corresponding θ is such a modulus; we do not claim that the set of all these quantities is a complete invariant - other invariants are also possible, as the value τ from [19], for instance).

The construction with an infinite chain of the orbits of homoclinic tangency was the main element in the proof of the following statement [28, 29, 30].

[11] If $|\sigma| > 1$, we have infinitely many completely unstable periodic orbits. For the multidimensional case the general property of systems in the Newhouse regions is the simultaneous existence of infinitely many periodic orbits with different dimensions of stable manifolds, i.e. with different numbers of positive/negative Lyapunov exponents, see [26, 27]; in the same papers criteria for the existence of infinitely many stable periodic orbits were given for the multidimensional case, see also [21] where a partial case was considered.

[12] We mean here the denseness in C^r-topology, for an arbitrary finite r. When we consider C^∞-smooth systems then the denseness in C^r-topology for every fixed r means, by definition, the denseness in C^∞.

In the set of systems with a quadratic homoclinic tangency of the third class, systems having infinitely many homoclinic tangencies of all, arbitrarily high, orders and infinitely many nonhyperbolic periodic orbits of all orders of degeneracy are dense..

The latter are the periodic orbits with one multiplier equal to 1 or -1 and with arbitrarily large number of zero Lyapunov values (do not miss them with Lyapunov exponents) which are the consecutive coefficients of the nonlinear terms in the normal form of the Poincaré map on the center manifold. Thus, in the case of one multiplier equal to 1, for the periodic orbit of the k-th order of degeneracy the Poincaré map on the center manifold has the form

$$\bar{x} = x + l_{k+1}x^{k+1} + o(x^{k+1}).$$

A complete bifurcational diagram for such periodic orbit requires exactly k parameters. Since we can obtain periodic orbits of any order of degeneracy k, it means that a complete description of the dynamics of systems with homoclinic tangencies *can never be possible*, in any finite-parameter family.

This discouraging result is even more important due to the denseness of the systems with homoclinic tangencies of the third class in the Newhouse regions, that is we have the whole regions in the space of smooth dynamical systems for which a complete description of dynamics (in particular, a complete description of bifurcations of periodic orbits) can never be achieved. When we realized all this I remembered the words of E.A.Leontovich concerning the discovery of chaos near a homoclinic loop to a saddle-focus: "It just cannot be!"

Soon we generalized these results to the multidimensional case [31, 20, 27, 26, 32]: an analogous classification of systems with a quadratic homoclinic tangency was given, the main invariants (moduli) of Ω-equivalence were found (thus, in the case of complex leading multipliers $\lambda e^{i\varphi}$, $\gamma e^{i\psi}$ the arguments φ and ψ occured to be Ω-moduli), the results of Newhouse were generalized and the denseness of multidimensional systems with infinitely degenerate homoclinic and periodic orbits was established in the Newhouse regions. Thus, the conclusion on the principal impossibility of complete description holds in the multidimensional case as well.

All this has a direct relation to the study of specific dynamical models, because homoclinic tangencies (hence, Newhouse regions) are found in practically every known families of systems with complicated dynamics, from the small periodic perturbations of integrable systems discussed above to such popular models as Hénon map, Rössler system and Chua circuit, as well as at the transition to chaos after the breakdown of quasiperiodic regimes and after a period-doubling cascade.

Quite popular among the systems with complicated dynamics are systems with the so-called "spiral chaos", i.e. systems with a homoclinic loop to a saddle-focus. Such systems form, in general, bifurcational surfaces of codimension one in the space of dynamical systems, and in many instances they are analogous to the systems of the third class [33, 34, 35]. Thus, on these surfaces, systems with homoclinic tangencies are dense. Hence, close systems must have Newhouse regions. Therefore, here we also have arbitrarily high degeneracies.

Moreover, it was recently shown by Turaev and the author that there can exist genuine strange attractors[13] containing a saddle-focus (we called these spiral attractors wild). Moreover, the property of instability of all orbits in these attractors is preserved under small perturbations. The fact that these wild spiral attractors contain an equilibrium state makes them partly similar to Lorenz-like attractors. However, if Lorenz-like attractors are two-dimensional (topologically), the wild spiral attractors may have dimension three. We construct our wild attractors in one-parameter families of systems in R^n where $n \geq 4$ and we show [37] that the range of variation of parameters contains Newhouse intervals, with all implications of that. And such attractors must be quite natural objects for nonlinear dynamics.

In his report to the memoir of Poincaré, Weierstrass wrote that the results of that paper kill many illusions of the theory of Hamiltonian dynamics. In essence, this was a starting point for the development of qualitative methods which represent now the essence of nonlinear dynamics. Today, one century later, we see that the illusion of the possibility of a complete qualitative description should be abandoned, in turn. And in both cases, the cause of the crisis was the Poincaré homoclinic curve.

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[13] i.e. all orbits in such attractors are proven to be unstable, whereas many of numerically or experimentally observed examples of chaotic behavior should be regarded as "quasiattractors" due to the presence of practically invisible stable periodic orbits within; see discussion in [36, 26].

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