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Surface Waves at an Interface Separating Two Saturated Porous Media

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Abstract

Surface waves at an interface between two saturated porous media of different structure are investigated. Existence and peculiarities of surface wave propagation are revealed. Four types of surface waves are proven to be possible: true Stoneley surface wave propagating almost without dispersion, leaky slow pseudo-Stoneley wave, leaky generalized Rayleigh wave, and one more new leaky mode. True Stoneley, leaky pseudo-Stoneley, and generalized Rayleigh waves are similar to those waves, which exist at an interface between a saturated porous medium and a liquid. Existence of generalized Rayleigh wave or new mode depends crucially on the parameters of the skeletons.

1 Introduction

In this paper we proceed to study the effect of fluid-filled porous media on the velocity and attenuation of surface waves, which propagate along an interface separating two porous media of different structure.

Comparing the results on classical surface waves [1-3] with those concerning a porous medium bounded by the vacuum [4] and by a liquid [5], in the case under research one might expect an existence of at least three or may be four types of surface waves. First of all this happens because of the presence of a second longitudinal wave in a fluid-saturated porous solid. Since any surface wave is a combination of all bulk waves in both media, existence and distinctive properties of a second longitudinal wave result in new properties of the surface modes. Another significant fact which may lead to the specific properties of the surface waves at an interface of porous media is a relation between parameters of the skeletons. Let us recall that for example at an interface separating two elastic solids depending on the parameters of the media, namely their densities, one or two surface waves may exist [2].

For two porous media of different structure we prove an existance of four different types of surface waves. However only three different modes may exist simultaneously. We proceed to investigate these waves.

It should be reminded that in our approach [see also 4,5] the wave number k is choosen to be real while frequency $\omega = \omega(k)$ is sought as a solution of corresponding dispersion equation and can be complex. As it was proven [5], this approach allows one to consider isolated surface modes without interaction with the bulk waves. Most likely, this approach also allows one to investigate the stability of isolated surface waves and, consequently, to prove the existence of surface waves for nonlinear problems.

2 Mathematical Model and Boundary Conditions

Consider two semi-infinite spaces Ω^I and Ω^{II} having a common interface Γ. Let both regions Ω^I and Ω^{II} are occupied by saturated porous media of different structure. In what follows we consider two typical cases: 1) when porous media are characterized by the different porosities while the Lame constants of the skeletons as well as parameters of saturating liquids are the same; 2) when porous media are characterized by the different porosities and by the different Lame constants of the skeletons while the parameters of saturating liquids are the same. In dimensionless variables the set of field equations describing the porous medium in region Ω^I has the form ($x \in R^3, t \in [0,T]$) [4-6]:

Mass conservation equations

$$\frac{\partial}{\partial t}\varrho_f + \operatorname{div}(\varrho_f \mathbf{v}_f) = 0,$$

$$\frac{\partial}{\partial t}\varrho_s + \operatorname{div}(\varrho_s \mathbf{v}_s) = 0. \tag{2.1}$$

Here ϱ is the mass density, $\mathbf{v}$ is the velocity vector and indices f and s indicate fluid or solid phases, respectively.

Momentum conservation equations

$$\varrho_f \left[\frac{\partial}{\partial t} + (v_{fj}, \frac{\partial}{\partial x_j})\right] v_{fi} - \frac{\partial}{\partial x_j} T^f_{ij} + \pi(v_{fi} - v_{si}) = 0,$$

$$\varrho_s \left[\frac{\partial}{\partial t} + (v_{sj}, \frac{\partial}{\partial x_j})\right] v_{si} - \frac{\partial}{\partial x_j} T^s_{ij} - \pi(v_{fi} - v_{si}) = 0. \tag{2.2}$$

Here T^f_{ij} and T^s_{ij} are the stress tensors, π is a positive constant. The stress tensor in the fluid is assumed to be given by the following linear law:

$$T^f_{ij} = -p_f \delta_{ij} - \beta \Delta_m \delta_{ij}, \quad p_f = p_{f0} + \kappa(\varrho_f - \varrho_{f0}), \tag{2.3}$$

where p_f is the partial fluid pressure. p_{f0} and ϱ_{f0} are the initial values of this pressure and fluid mass density, respectively. κ is the constant compressibility coefficient of the fluid depending only on equilibrium value of the porosity m_E. $\Delta_m = m - m_E$ is the change of the porosity. β denotes the coupling coefficient of the components.

The stress tensor in skeleton has the following form:

$$T^s_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} + \beta \Delta_m \delta_{ij}, \tag{2.4}$$

where λ and μ are the Lame constants of the skeleton and e_{ij} is the strain tensor of small deformations.

Equation for the change of porosity

$$\frac{\partial}{\partial t}\Delta_m + (v_{si}, \frac{\partial}{\partial x_i})\Delta_m + m_E \mathrm{div}(\mathbf{v}_f - \mathbf{v}_s) = -\frac{\Delta_m}{\tau}, \tag{2.5}$$

where τ is the relaxation time of porosity.

For the strain tensor one has:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_{si}}{\partial x_j} + \frac{\partial u_{sj}}{\partial x_i}\right), \tag{2.6}$$

where $\mathbf{u}_s$ is the displacement vector for the solid phase with $\mathbf{v}_s = \partial \mathbf{u}_s/\partial t$.

The same set of equations with corresponding parameters holds true for the porous medium occupying region Ω^{II} (in what follows upper index "$-$" indicates Ω^{II}).

Equation system (2.1)-(2.6) is linearized about some equilibrium state with the following constant values: $\varrho_f = \varrho_{f0}$, $\varrho_s = \varrho_{s0}$, $\mathbf{v}_f = 0$, $\mathbf{v}_s = 0$, and $\Delta_m = 0$. After the introduction of displacement vector for the fluid phase $\mathbf{u}_f$ and linearization, the system (2.1)-(2.6) takes the following form:

$$\frac{\partial}{\partial t}\varrho_f + \varrho_{f0}\frac{\partial}{\partial x_i}\left(\frac{\partial u_{fi}}{\partial t}\right) = 0, \tag{2.7}$$

$$\frac{\partial}{\partial t}\varrho_s + \varrho_{f0}\frac{\partial}{\partial x_i}\left(\frac{\partial u_{si}}{\partial t}\right) = 0, \tag{2.8}$$

$$\varrho_{f0}\frac{\partial^2 u_{fi}}{\partial t^2} + \frac{\partial}{\partial x_j}p_f\delta_{ij} + \frac{\partial}{\partial x_j}\beta\Delta_m\delta_{ij} + \pi\frac{\partial}{\partial t}(u_{fi} - u_{si}) = 0, \tag{2.9}$$

$$\varrho_{s0}\frac{\partial^2 u_{si}}{\partial t^2} - \mu\Delta u_{si} - (\lambda+\mu)\nabla \mathrm{div} u_s - \frac{\partial}{\partial x_j}\beta\Delta_m\delta_{ij} - \pi\frac{\partial}{\partial t}(u_{fi} - u_{si}) = 0, \tag{2.10}$$

$$\frac{\partial}{\partial t}\Delta_m + m_E \mathrm{div}\frac{\partial}{\partial t}(\mathbf{u}_f - \mathbf{u}_s) = -\frac{\Delta_m}{\tau}. \tag{2.11}$$

From now on we consider 2D-problem of propagation of surface waves along the interface $y = 0$, which separates two porous media: one of them occupies semi-infinite space $y > 0$ and another one occupies semi-infinite space $y < 0$.

On the interface $y = 0$ the following linearized boundary conditions, which are the consequence of the general conditions [4], have to be satisfied:

1) the continuity of total stresses:

$$\left(T^s_{ij} + T^f_{ij}\right)n_j\Big|^{I} = \left(T^s_{ij} + T^f_{ij}\right)n_j\Big|^{II}, \tag{2.12}$$

i.e.

$$\mu\left(\frac{\partial u_{s1}}{\partial y}+\frac{\partial u_{s2}}{\partial x}\right)\Big|_{y=0}=\mu^{-}\left(\frac{\partial u_{s1}^{-}}{\partial y}+\frac{\partial u_{s2}^{-}}{\partial x}\right)\Big|_{y=0} \tag{2.13}$$

and

$$\lambda\text{div } \mathbf{u}_s+2\mu\frac{\partial u_{s2}}{\partial y}-\kappa\big(\varrho_f-\varrho_{f0}\big)\Big|_{y=0}=\lambda^{-}\text{div } \mathbf{u}_s^{-}+2\mu^{-}\frac{\partial u_{s2}^{-}}{\partial y}-\kappa^{-}\big(\varrho_f^{-}-\varrho_{f0}^{-}\big)\Big|_{y=0} \tag{2.14}$$

2) the continuity of the displacements of the solid phases (i.e. the boundary Γ is material with respect to the skeleton)

$$\mathbf{u}_s\ |^{I}=\mathbf{u}_s\ |^{II} \tag{2.15}$$

3) the continuity of mass flux across the interface

$$\varrho_{f0}\frac{\partial}{\partial t}\big(u_{f_2}-u_{s2}\big)\Big|_{y=0}=\varrho_{f0}^{-}\frac{\partial}{\partial t}\big(u_{f_2}^{-}-u_{s2}\big)\Big|_{y=0} \tag{2.16}$$

4) proportionality between discontinuity in pressure and relative velocity of the fluid with respect to solid phase

$$-\varrho_{f0}\big(v_{f_2}-v_{s2}\big)\Big|_{y=0}=\alpha\big(p_f-\frac{m_E}{m_E^{-}}p_f^{-}\big)\Big|_{y=0} \tag{2.17}$$

Our goal is to prove that boundary value problem (2.7)-(2.11), (2.13)-(2.17) (let us call it SWP) has solutions as surface waves, i.e. solutions which decrease as $|y|\to\infty$. For this purpose we will investigate the propagation of harmonic wave whose frequency is ω, wave number is k, and amplitude depends on y. It should be noted here that as in [4,5] we consider the solutions of (2.1)-(2.6) in the absence of external forces which are defined uniquely by the Cauchy data. In this case it is natural to derive ω as a function with respect to real wave number $k\in R^1$. Thus, $\mathrm{Re}\omega/k$ defines the phase velocity of the waves, while $\mathrm{Im}\omega$ defines attenuation.

3 Construction of Solution

As for the cases considered in [4,5], solution of SWP is sought in the following form:

$$\mathbf{u}_f=\nabla\varphi_f+\Big((\psi_f)_y,-(\psi_f)_x\Big),\qquad \mathbf{u}_s=\nabla\varphi_s+\Big((\psi_s)_y,-(\psi_s)_x\Big),$$

$$
\begin{aligned}
\varphi_f = A_f(y)\exp\Big(i(kx-\omega t)\Big), &\qquad \varphi_s = A_s(y)\exp\Big(i(kx-\omega t)\Big),\\
\psi_f = B_f(y)\exp\Big(i(kx-\omega t)\Big), &\qquad \psi_s = B_s(y)\exp\Big(i(kx-\omega t)\Big),
\end{aligned}
$$

$$
\begin{gathered}
\varrho_f - \varrho_{f0} = A_{\varrho,f}(y)\exp\Big(i(kx-\omega t)\Big),\\
\varrho_s - \varrho_{s0} = A_{\varrho,s}(y)\exp\Big(i(kx-\omega t)\Big),\\
\Delta_m = A_{\Delta m}(y)\exp\Big(i(kx-\omega t)\Big).
\end{gathered}
\tag{3.1}
$$

Consequently, the relations (3.16)-(3.19) [4] as well as bounded solution (3.32) [4] remain to be valid, namely:

$$
\begin{pmatrix} A_f \\ A_s \end{pmatrix} = C_1(0)\begin{pmatrix} R_{f1} \\ R_{s1} \end{pmatrix}\exp(-\gamma_1\, y) + C_2(0)\begin{pmatrix} R_{f2} \\ R_{s2} \end{pmatrix}\exp(-\gamma_2\, y),
$$

$$
B_s = C_s(0)\exp(-\mu_s\, y), \tag{3.2}
$$

where vectors (R_{f1}, R_{s1}), (R_{f2}, R_{s2}) and radicals μ_s, γ_1, and γ_2 are defined in [4] for the general case. For the case $|k| \gg 1$ which is considered below one has [4,5]: $(R_{f1}, R_{s1}) = (1,0)$, $(R_{f2}, R_{s2}) = (0,1)$, and

$$
\gamma_1 = |k|\sqrt{1-\frac{\tilde\omega^2}{\kappa}}, \qquad \gamma_2 = |k|\sqrt{1-\frac{\tilde\omega^2}{a_{s1}^2}},
$$

$$
\mu_s = |k|\sqrt{1-\frac{\tilde\omega^2}{a_{s2}^2}}, \tag{3.3}
$$

where $\tilde\omega = \omega/k$.

As it has been noted already, we investigate two cases: 1) when porous media have different porosities, and 2) when porous media have different porosities and different Lame constants of the skeletons. Consequently for the first case one gets $\mu^- = \mu$ and $\lambda^- = \lambda$. For the second case we assume (in order to simplify the construction of asymptotic solution) that velocities of longitudinal and shear waves are the same in both media, i.e. $a_{s1}^- = a_{s1}$ and $a_{s2}^- = a_{s2}$. The latter results in relations:

$$
\frac{\mu^-}{\mu} = \frac{\lambda^-}{\lambda} = \frac{\varrho_{s0}^-}{\varrho_{s0}}. \tag{3.4}
$$

Thus, in both cases solution for the porous medium, occupying region $y < 0$, has the form:

$$
\begin{pmatrix} A_f^- \\ A_s^- \end{pmatrix} = C_1^-(0)\begin{pmatrix} R_{f1} \\ R_{s1} \end{pmatrix}\exp(\gamma_1\, y) + C_2^-(0)\begin{pmatrix} R_{f2} \\ R_{s2} \end{pmatrix}\exp(\gamma_2\, y),
$$

$$B_s^- = C_s^-(0)\exp(\mu_s\, y), \tag{3.5}$$

where radicals μ_s, γ_1, and γ_2 are defined above.

In order to derive dispersion relation and define the frequencies of the surface waves, one should substitute solutions (3.2), (3.5) into boundary conditions (2.13)-(2.17). We proceed to do so.

4 Dispersion Relation

Substituting the solution into boundary conditions for the case $\beta = 0$ and $|k| \gg 1$ one gets the following system of equations with respect to unknown constants $C_1(0)$, $C_2(0)$, $C_s(0)$ and $C_1^-(0), C_2^-(0)$, $C_s^-(0)$:

$$-2i\tilde{\gamma}_2 C_2 + \left(\tilde{\mu}_s^2 + 1\right) C_s = 2i\frac{\mu^-}{\mu}\tilde{\gamma}_2 C_2^- + \frac{\mu^-}{\mu}\left(\tilde{\mu}_s^2 + 1\right) C_s^-, \tag{4.1}$$

$$(\lambda + 2\mu)(\tilde{\gamma}_2^2 - 1)C_2 + 2\mu C_2 + 2\mu i\tilde{\mu}_s C_s - \tilde{\omega}^2 \varrho_{f0} C_1$$

$$= (\lambda^- + 2\mu^-)(\tilde{\gamma}_2^2 - 1)C_2^- + 2\mu^- C_2^- - 2\mu^- i\tilde{\mu}_s C_s^- - \tilde{\omega}^2 \varrho_{f0}^- C_1^-, \tag{4.2}$$

$$iC_2 - \tilde{\mu}_s C_s = iC_2^- + \tilde{\mu}_s C_s^- \tag{4.3}$$

$$-\tilde{\gamma}_2 C_2 - iC_s = \tilde{\gamma}_2 C_2^- - iC_s^- \tag{4.4}$$

$$-\tilde{\gamma}_1 C_1 + \tilde{\gamma}_2 C_2 + iC_s = \frac{\varrho_{f0}^-}{\varrho_{f0}}\left(\tilde{\gamma}_1 C_1^- + \tilde{\gamma}_2 C_2 + iC_s\right), \tag{4.5}$$

$$\tilde{\gamma}_1 C_1 - \tilde{\gamma}_2 C_2 - iC_s = i\alpha\tilde{\omega}\left(C_1 - \frac{\varrho_{f0}^-}{\varrho_{f0}}\frac{m_E}{m_E^-} C_1^-\right), \tag{4.6}$$

where

$$\tilde{\gamma}_1 = \sqrt{1 - \frac{\tilde{\omega}^2}{\kappa}}, \qquad \tilde{\gamma}_2 = \sqrt{1 - \frac{\tilde{\omega}^2}{a_{s1}^2}},$$

$$\tilde{\mu}_s = \sqrt{1 - \frac{\tilde{\omega}^2}{a_{s2}^2}}. \tag{4.7}$$

The condition that determinant of the system (4.1)-(4.6) must vanish yields the dispersion equation for the definition of frequencies of the surface waves. First let us consider the case when porous media are characterized by the different porosities only. In this case dispersion equation takes the form:

$$\Bigg(-\frac{1}{2}\mu \mathcal{P}_v - \tilde{\gamma}_1 \Big((\lambda + 2\mu)(\tilde{\gamma}_2^2 - 1) + 2\mu \Big) \Big(\tilde{\mu}_s \tilde{\gamma}_2 - 1 + \frac{\tilde{\omega}^2}{2a_{s2}^2} \Big)$$

$$+\frac{1}{2}\varrho_{f0}\tilde{\omega}^2\tilde{\gamma}_2 \Big(1 - \frac{\bar{\varrho}_{f0}}{\varrho_{f0}}\Big) \Big(\tilde{\mu}_s \tilde{\gamma}_2 - 1\Big) + \varrho_{f0}\tilde{\omega}^2\tilde{\gamma}_2 \frac{\tilde{\omega}^2}{2a_{s2}^2} \Bigg) \Bigg(\tilde{\gamma}_1 - i\alpha\tilde{\omega} \Big(1 + \frac{m_E}{\bar{m}_E} \Big) \Bigg)$$

$$+\bar{\varrho}_{f0}\tilde{\omega}^2\tilde{\gamma}_2 \Big(\tilde{\mu}_s \tilde{\gamma}_2 - 1\Big) \Bigg(\tilde{\gamma}_1 - i\alpha\tilde{\omega} \Big(1 - \frac{\varrho_{f0}}{\bar{\varrho}_{f0}} \Big) \Bigg) = 0 \tag{4.8}$$

Here $\mathcal{P}_v$ is the dispersion relation, corresponding to the case of surface waves at a free interface of a porous medium [4]:

$$\mathcal{P}_v = \tilde{\gamma}_1 \mathcal{P}_R + \tilde{\gamma}_2 \frac{\varrho_{f0}}{\varrho_{s0}} \frac{\tilde{\omega}^4}{a_{s2}^4}, \tag{4.9}$$

whereas $\mathcal{P}_R$ is a classical Rayleigh equation [4]:

$$\mathcal{P}_R = \left(2 - \frac{\tilde{\omega}^2}{a_{s2}^2} \right)^2 - 4\tilde{\gamma}_2 \tilde{\mu}_s. \tag{4.10}$$

Substituting (4.9), (4.10) into (4.8), one can rewrite dispersion equation (4.8) in a simplified form:

$$\Bigg(2\tilde{\gamma}_1 \tilde{\mu}_s \varrho_{s0} + \Big(\varrho_{f0} + \bar{\varrho}_{f0} \Big) \Big(\tilde{\mu}_s \tilde{\gamma}_2 - 1 \Big) \Bigg) \Bigg(\tilde{\gamma}_1 - i\alpha\tilde{\omega} \Big(1 + \frac{m_E}{\bar{m}_E} \Big) \Bigg)$$

$$+2\bar{\varrho}_{f0} \Big(\tilde{\mu}_s \tilde{\gamma}_2 - 1 \Big) \Bigg(\tilde{\gamma}_1 - i\alpha\tilde{\omega} \Big(1 - \frac{\varrho_{f0}}{\bar{\varrho}_{f0}} \Big) \Bigg) = 0 \tag{4.11}$$

For the second case under research, when porous media are characterized by the different porosities and by the different Lame constants of the skeletons, dispersion equation has the following form:

$$\tilde{\gamma}_1\left(\frac{1}{2}\Big(1-\frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\Big)^2\Big(1-\tilde{\mu}_s\tilde{\gamma}_2\Big)\mathcal{P}_R-2\frac{\tilde{\omega}^4}{a_{s2}^4}\frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\tilde{\mu}_s\tilde{\gamma}_2\right)\Big(\tilde{\gamma}_1-i\alpha\tilde{\omega}\Big(1+\frac{m_E}{\bar{m}_E}\Big)\Big)+$$

$$+\tilde{\gamma}_2\frac{\tilde{\omega}^4}{2a_{s2}^4}\Big(1+\frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\Big)(1-\tilde{\mu}_s\tilde{\gamma}_2)\frac{\varrho_{f0}}{\varrho_{s0}}\left(\tilde{\gamma}_1\Big(1+\frac{\bar{\varrho}_{f0}}{\varrho_{f0}}\Big)+i\alpha\tilde{\omega}\Big(1-\frac{m_E}{\bar{m}_E}\Big)\Big(1-\frac{\bar{\varrho}_{f0}}{\varrho_{f0}}\Big)\right)=0. \quad (4.12)$$

Obviously, (4.11) and (4.12) include radicals $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\mu}_s$, which are multi-valued functions. In order to make these function single-valued, consider Riemann surface of $\tilde{\omega}$ with the cuts outgoing from the points $\pm\kappa, \pm a_{s2}, \pm a_{s1}$. Later on we consider this Riemann surface, where the signs at radicals are defined uniquely (depending on the strip of the Riemann surface [5]) in such a way that on the real axis radiation condition [1] is satisfied.

Let one of the following conditions holds true:

Condition 1

$$1 > \max Re\left(\frac{\tilde{\omega}^2}{\kappa}, \frac{\tilde{\omega}^2}{a_{s2}^2}, \frac{\tilde{\omega}^2}{a_{s1}^2}\right) \quad (4.13)$$

and, consequently, $\tilde{\gamma}_1, \tilde{\gamma}_2$ and $\tilde{\mu}_s$ are defined as in (4.7).

Condition 2

$$Re\frac{\tilde{\omega}^2}{\kappa} > 1 > \max Re\left(\frac{\tilde{\omega}^2}{a_{s2}^2}, \frac{\tilde{\omega}^2}{a_{s1}^2}\right) \quad (4.14)$$

and, consequently, $\tilde{\gamma}_2$ and $\tilde{\mu}_s$ are defined as above. However

$$\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa}-1} \quad (4.15)$$

on the first strip of the Riemann surface [5].

Condition 3

$$\max Re\left(\frac{\tilde{\omega}^2}{\kappa}, \frac{\tilde{\omega}^2}{a_{s2}^2}\right) > 1 > Re\frac{\tilde{\omega}^2}{a_{s1}^2}. \quad (4.16)$$

Then $\tilde{\gamma}_2$ and $\tilde{\gamma}_1$ are defined as in (4.7) and (4.15) respectively. However

$$\tilde{\mu}_s = -i\sqrt{\frac{\tilde{\omega}^2}{a_{s2}^2}-1} \quad (4.17)$$

on the upper (second) strip of the Riemann surface [5].

Next we will show that dispersion equation (4.11) has three roots satisfying either (4.13) or (4.14) or (4.16). Dispersion equation (4.12) also has three roots. Depending on the parameters of porous media two situations are proven to be possible: 1) the roots satisfy either (4.13) or (4.14) and 2) the roots satisfy either (4.13) or (4.14) or (4.16). Next we investigate the dependence of the roots of (4.12) on parameters ϱ_{s0} and ϱ_{s0}^{-} and for both dispersion equations we construct the asymptotic expansions of the roots.

5 Asymptotics of the Roots

5.1 Porous Media with Different Porosities

First let us consider dispersion equation (4.11). One can prove that (4.11) has three roots. One of them $\tilde{\omega}_1$ satisfies condition (4.13), i.e. $\mathrm{Re}\tilde{\omega}_1 \in [0, \sqrt{\kappa})$. Another one $\tilde{\omega}_2$ satisfies condition (4.14), i.e. $\mathrm{Re}\tilde{\omega}_2 \in (\sqrt{\kappa}, a_{s2})$ and the last one $\tilde{\omega}_3$ satisfies condition (4.16), i.e. $\mathrm{Re}\tilde{\omega}_3 \in (a_{s2}, a_{s1})$.

It should be noted that we construct outer expansion $\alpha \sim 1$ of the roots with respect to κ.

The asymptotic expansion of $\tilde{\omega}_1$ is sought in the following form [4,5]:

$$\tilde{\omega} = \sqrt{\kappa}(1 - c_1\kappa^2 + \cdots). \tag{5.1}$$

Substituting (5.1) into (4.11) from the lowest appoximation $\mathrm{mod}O(\kappa^{3/2})$ one defines:

$$\sqrt{2c_1} = \frac{(m_E - m_E^{-})(\varrho_{f0} - \varrho_{f0}^{-})}{4(m_E + m_E^{-})\varrho_{s0}} \left(\frac{1}{a_{s2}^2} + \frac{1}{a_{s_1}^2} \right). \tag{5.2}$$

It is obvious that both $\mathrm{Re}c_1$ and $\mathrm{Re}\sqrt{2c_1}$ should be positive. By virtue of physical reasons,

$$(m_E - m_E^{-})(\varrho_{f0} - \varrho_{f0}^{-}) > 0, \tag{5.3}$$

and, consequently, $c_1 > 0$. Therefore

$$\mathrm{Re}\tilde{\omega}_1 = \sqrt{\kappa}\Big(1 - c_1\kappa^2 + O(\kappa^3)\Big) \in [0, \sqrt{\kappa}). \tag{5.4}$$

Similarly to the cases of a free interface of a porous medium and of an interface separating porous and liquid half-spaces [4,5], this phase velocity corresponds to very slow surface wave (true Stoneley wave), propagating almost without dispersion. Its speed is less than the velocities of all bulk waves in the porous media and has order $O(\sqrt{\kappa})$.

Next we will show that dispersion equation (4.11) has also two complex roots $\tilde{\omega}_2$ and $\tilde{\omega}_3$, satisfying conditions (4.14) and (4.16) respectively. These roots correspond to the localized with respect to y surface waves.

Asymptotic expansion of $\tilde{\omega}_2$ has the following form [5]:

$$\tilde{\omega} = \sqrt{\kappa}(1 + c_2\kappa + c_3\kappa^{3/2} + \cdots). \tag{5.5}$$

Substitution of (5.5) into (4.11) allows one to define the coefficients c_2 and c_3. Namely from the lowest $O(\kappa)$ approximation one gets:

$$\sqrt{2c_2} = \alpha\Big(1 + \frac{m_E}{m_E^-}\Big) > 0 \tag{5.6}$$

and, consequently,

$$c_2 = \frac{\alpha^2}{2}\Big(1 + \frac{m_E}{m_E^-}\Big)^2 > 0. \tag{5.7}$$

From the next $O(\kappa^{3/2})$ approximation one has:

$$c_3 = -i\frac{\alpha}{2}\frac{\varrho_{f0}}{\varrho_{s0}}\Big(1 + \frac{m_E}{m_E^-}\frac{\varrho_{f0}^-}{\varrho_{f0}}\Big)\Big(\frac{1}{a_{s2}^2} + \frac{1}{a_{s1}^2}\Big) \tag{5.8}$$

Finally, one gets the following expansion for the second root of dispersion relation (4.11):

$$\tilde{\omega}_2 = \sqrt{\kappa}\Big(1 + c_2\kappa + c_3\kappa^{3/2} + O(\kappa^2)\Big), \tag{5.9}$$

where coefficients c_2 and c_3 are defined above. This root defines slightly dispersive surface wave (pseudo-Stoneley wave), whose phase velocity is close but somewhat more than $\sqrt{\kappa}$. This is a leaky wave, thus reradiation of energy occurs across the interface. Phase velocities of both true and pseudo Stoneley waves are defined primarily by the compressibility coefficient of a liquid. For the pseudo-Stoneley wave it depends additionally on surface permeability α (see (2.17)). As it was proven [5], if $\alpha \to 0$, i.e. surface pores are closed, this wave is degenerated into the slow longitudinal wave.

As it was mentioned already, dispersion equation (4.11) has one more complex root, satisfying condition (4.16). It corresponds to a new surface mode, which is not observed either for the cases of a free interface of a porous medium nor for an interface separating porous and liquid half-spaces. Its phase velocity is close but somewhat more than velocity a_{s2} of a shear wave in unbounded medium. Let us remind that in accordance with condition (4.16) here we have to choose the following branches of the radicals:

$$\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1}, \quad \tilde{\mu}_s = -i\sqrt{\frac{\tilde{\omega}^2}{a_{s2}^2} - 1} \tag{5.10}$$

and, additionally, one can expand

$$\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1} \approx \frac{\tilde{\omega}}{\sqrt{\kappa}}. \tag{5.11}$$

Similar to $\tilde{\omega}_2$, an asymptotic expansion of this root is sought in the following form:

$$\tilde{\omega} = a_{s2}(1 + c_4\kappa + c_5\kappa^{3/2} + \cdots). \tag{5.12}$$

It is easy to get from the lowest approximation $O(\kappa^{-1/2})$ that

$$\sqrt{2c_4} = \frac{\varrho_{f0} + \varrho_{\bar{f}0}}{2\varrho_{s0}a_{s2}} > 0 \tag{5.13}$$

and, consequently,

$$c_4 = \frac{(\varrho_{f0} + \varrho_{\bar{f}0})^2}{2\varrho_{s0}^2 a_{s2}^2} > 0. \tag{5.14}$$

From the next approximation $O(\kappa^0)$ it follows that

$$c_5 = 2\alpha\varrho_{f0}\Big(1 + \frac{m_E}{m_{\bar{E}}}\frac{\varrho_{\bar{f}0}}{\varrho_{f0}}\Big)\frac{\varrho_{f0} + \varrho_{\bar{f}0}}{(2\varrho_{s0}a_{s2})^2} + i\frac{(\varrho_{f0} + \varrho_{\bar{f}0})^3}{(2\varrho_{s0}a_{s2})^3}\sqrt{1 - \frac{a_{s2}^2}{a_{s1}^2}} \tag{5.15}$$

Finally, one has:

$$\tilde{\omega}_3 = a_{s2}\Big(1 + c_4\kappa + c_5\kappa^{3/2} + O(\kappa^2)\Big), \tag{5.16}$$

where coefficients c_4 and c_5 are given by (5.14) and (5.15). This root defines dispersive leaky surface wave, whose phase velocity, similar to the generalized Rayleigh wave, is very close to the velocity a_{s2} of a shear wave. However phase velocity of this surface wave is somewhat more than a_{s2} while phase velocity of the generalized Rayleigh wave is somewhat less than a_{s2}. Moreover, it is not difficult to estimate using (5.16) and (5.10) that some part of the energy of this wave is absorbed by the slow longitudinal wave. In contrast to the generalized Rayleigh wave which is transformed to the classical Rayleigh wave as $\varrho_{f0},\ \varrho_{\bar{f}0} \to 0$ [4,5], this surface mode is degenerated into shear bulk wave if $\varrho_{f0},\ \varrho_{\bar{f}0} \to 0$. It should be noted also, that an amplitude of this mode is slowly growing with respect to t. The latter means that this surface mode exists as isolated surface wave during short time interval [7].

5.2 Porous Media with Different Porosities and Different Lame Parameters of the Skeletons

Next we investigate dispersion equation (4.12), which corresponds to the more complicated case of porous media with different characteristic parameters. One can prove that this dispersion equation also has three roots. Two of them are very similar to the roots $\tilde{\omega}_1$ and $\tilde{\omega}_2$ of dispersion equation (4.11). They satisfy conditions (4.13) and (4.14) respectively. The third root, depending on ratio of ϱ_{s0} and $\varrho_{\bar{s}0}$ may satisfy either condition (4.14) or condition (4.16).

First, let us prove that there exists a root $\tilde{\omega}_1$ of (4.12) satisfying (4.13), i.e. $\mathrm{Re}\tilde{\omega}_1 \in [0, \sqrt{\kappa})$. As above the asymptotic expansion of $\tilde{\omega}_1$ is sought in the following form:

$$\tilde{\omega} = \sqrt{\kappa}(1 - c_1\kappa^2 + \cdots). \tag{5.17}$$

Substituting (5.17) into (4.12) from the lowest appoximation mod$O(\kappa^{7/2})$ one can define coefficient c_1. Depending on relation between ϱ_{s0} and $\varrho_{\bar{s}0}$ the expression for $\sqrt{2c_1}$ is given either by

$$\sqrt{2c_1} = \frac{(m_E - m_{\bar{E}})(\varrho_{f0} - \varrho_{\bar{f}0})}{2(m_E + m_{\bar{E}})\varrho_{s0}a_{s2}^4} \frac{1}{\frac{1}{a_{s2}^2} - \frac{1}{a_{s_1}^2}} > 0 \tag{5.18}$$

if $\frac{\varrho_{\bar{s}0}}{\varrho_{s0}} = O(\sqrt{\kappa})$, or by

$$\sqrt{2c_1} = \frac{(m_E - m_{\bar{E}})(\varrho_{f0} - \varrho_{\bar{f}0})}{4(m_E + m_{\bar{E}})\varrho_{s0}} \left(\frac{1}{a_{s2}^2} + \frac{1}{a_{s_1}^2}\right) > 0 \tag{5.19}$$

if $\frac{\varrho_{\bar{s}0}}{\varrho_{s0}} = O(1)$. Let us note that (5.19) coincides with (5.2).

Taking into account (5.3), one can easily see that in both cases $\sqrt{2c_1}$ and c_1 are positive. Therefore

$$\mathrm{Re}\tilde{\omega}_1 = \sqrt{\kappa}\Big(1 - c_1\kappa^2 + O(\kappa^3)\Big) \in [0, \sqrt{\kappa}) \tag{5.20}$$

and, as before, this phase velocity corresponds to the slowest true Stoneley wave.

Next we will show that dispersion equation (4.12) has a root, satisfying (4.14) and corresponding to the pseudo-Stoneley wave. An asymptotic expansion has the following structure:

$$\tilde{\omega} = \sqrt{\kappa}(1 + c_2\kappa + c_3\kappa^{3/2} + \cdots). \tag{5.21}$$

Substituting this expansion into (4.12) and taking into account that $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1}$, one obtains from the lowest approximation:

$$\sqrt{2c_2} = \alpha\left(1 + \frac{m_E}{m_{\bar{E}}}\right) > 0, \tag{5.22}$$

i.e.

$$c_2 = \frac{\alpha^2}{2}\left(1 + \frac{m_E}{m_{\bar{E}}}\right)^2 > 0. \tag{5.23}$$

From the next appoximation one gets:

$$c_3 = -i\alpha \frac{\frac{\varrho_{f0}}{\varrho_{s0}}\left(1 + \frac{\varrho_{\bar{f}0}}{\varrho_{f0}}\frac{m_E}{m_{\bar{E}}}\right)}{a_{s2}^4\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)} \tag{5.24}$$

if $\frac{\varrho_{\bar{s}0}}{\varrho_{s0}} = O(\sqrt{\kappa})$, or, as in (5.8),

$$c_3 = -i\frac{\alpha}{2}\frac{\varrho_{f0}}{\varrho_{s0}}\left(1 + \frac{m_E}{m_{\bar{E}}}\frac{\varrho_{\bar{f}0}}{\varrho_{f0}}\right)\left(\frac{1}{a_{s2}^2} + \frac{1}{a_{s1}^2}\right) \tag{5.25}$$

if $\frac{\varrho_{\bar{s}0}}{\varrho_{s0}} = O(1)$. In both cases it is easy to see that $\mathrm{Im} c_3 < 0$.
Finally, one gets:

$$\tilde{\omega}_2 = \sqrt{\kappa}\left(1 + c_2\kappa + c_3\kappa^{3/2} + O(\kappa^2)\right), \tag{5.26}$$

where coefficients c_2 and c_3 are given above. This root defines slightly dispersive pseudo-Stoneley wave.

Now let us investigate the third root of dispersion equation (4.12) which satisfies either (4.14) or (4.16) depending on relation between ϱ_{s0} and $\varrho_{\bar{s}0}$. First we consider the case when densities of porous media differ significantly, i.e.

$$\frac{\varrho_{\bar{s}0}}{\varrho_{s0}} = O(\sqrt{\kappa}) = R_0\sqrt{\kappa} + \cdots, \tag{5.27}$$

where $R_0 \sim 1$. Below we prove that if relation (5.27) holds true, dispersion equation (4.12) has a root $\tilde{\omega}_{R'}$ satisfying condition (4.14), namely $\mathrm{Re}\tilde{\omega}_{R'} \in (\sqrt{\kappa}, a_{s2})$. This root similar to [4,5] corresponds to the generalized Rayleigh wave with phase velocity close to the velocity a_{s2} of a shear wave. Asymptotic expansion of $\tilde{\omega}_{R'}$ is sought in the following form:

$$\tilde{\omega} = \Omega_0 + \sqrt{\kappa}\Omega_1 + \cdots \tag{5.28}$$

Because of condition (4.14), the branch of $\tilde{\gamma}_1$ is taken as $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1}$. Additionally, one can assume that

$$\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1} \approx i\frac{\tilde{\omega}}{\sqrt{\kappa}}. \tag{5.29}$$

Using (5.29) and rewritting (4.12) as

$$\gamma_1^2\Big(\frac{1}{2}\Big(1 - \frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\Big)^2\Big(1 - \mu_s\gamma_2\Big)\mathcal{P}_R - 2\frac{\omega^4}{a_{s2}^4}\frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\mu_s\gamma_2\Big)$$

$$+\gamma_1\Big(- i\alpha\omega\Big(1 + \frac{m_E}{\bar{m}_E}\Big)\Big(\frac{1}{2}\Big(1 - \frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\Big)^2\Big(1 - \mu_s\gamma_2\Big)\mathcal{P}_R - 2\frac{\omega^4}{a_{s2}^4}\frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\mu_s\gamma_2\Big) +$$

$$+\gamma_2\frac{\omega^4}{2a_{s2}^4}\Big(1 + \frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\Big)(1 - \mu_s\gamma_2)\frac{\varrho_{f0}}{\varrho_{s0}}\Big(1 + \frac{\bar{\varrho}_{f0}}{\varrho_{f0}}\Big)\Big)$$

$$+i\alpha\omega\gamma_2\frac{\omega^4}{2a_{s2}^4}\Big(1 + \frac{\bar{\varrho}_{s0}}{\varrho_{s0}}\Big)(1 - \mu_s\gamma_2)\frac{\varrho_{f0}}{\varrho_{s0}}\Big(1 - \frac{m_E}{\bar{m}_E}\Big)\Big(1 - \frac{\bar{\varrho}_{f0}}{\varrho_{f0}}\Big) = 0 \tag{5.30}$$

one can easily see that the leading part Ω_0 of expansion (5.28) is defined from $O\Big(\frac{1}{\kappa}\Big)$ approximation and it satisfies the Rayleigh equation: $\mathcal{P}_R(\Omega_0) = 0$, i.e. $\Omega_0 = c_R$, where c_R is a phase velocity of the classical Rayleigh wave [1]. For the next term Ω_1 in $O\Big(\frac{1}{\sqrt{\kappa}}\Big)$ approximation one gets the following equation:

$$\left[\frac{4}{a_{s2}^4}\Omega_0^3 - \frac{8}{a_{s2}^2}\Omega_0 - 4\frac{d}{d\tilde{\omega}}\Big(\sqrt{1 - \frac{\tilde{\omega}^2}{a_{s1}^2}}\sqrt{1 - \frac{\tilde{\omega}^2}{a_{s2}^2}}\Big)\Big|_{\tilde{\omega}=\Omega_0}\right]\Omega_1$$

$$= i\sqrt{1 - \frac{\Omega_0^2}{a_{s1}^2}}\frac{\bar{\varrho}_{f0} + \varrho_{f0}}{\varrho_{s0}}\frac{\Omega_0^3}{a_{s2}^4} - 4R_0\frac{\Omega_0^5}{a_{s2}^4}\frac{\sqrt{1 - \frac{\Omega_0^2}{a_{s1}^2}}\sqrt{1 - \frac{\Omega_0^2}{a_{s2}^2}}}{1 - \sqrt{1 - \frac{\Omega_0^2}{a_{s1}^2}}\sqrt{1 - \frac{\Omega_0^2}{a_{s2}^2}}} \tag{5.31}$$

Finally, one has:

$$\tilde{\omega}_{R'} = c_R + \sqrt{\kappa}\Omega_1 + O(\kappa), \tag{5.32}$$

where Ω_1 is determined by (5.32). Ω_1 is complex, thus the generalized Rayleigh wave is a leaky wave. Similar to the cases investigated in [4,5], $\mathrm{Im}\Omega_1 > 0$ and, consequently, one can prove, estimating the amplitudes of the bulk waves, that part of the energy of this surface wave is absorbed by the slow compressional wave. Let us emphasize that attenuations of the generalized Rayleigh wave at an interface of two porous media and at an interface separating a porous medium and a liquid [5] occur in a similar way (imaginary parts of Ω_1 in (5.31) and in (5.16) [5] coincide). However, in contrast to [4,5], where Ω_1 is pure imaginary, here Ω_1 has additionally the real part which implies its correction to the phase velocity of the generalized Rayleigh wave at an interface of porous media. Namely, its phase velocity is smaller than phase velocity of analogous wave considered in [4,5]. Another distinctive property of this surface mode in comparison for example with [5] concerns its behaviour if $\varrho_{f0},\ \varrho_{\bar{f}0} \to 0$. In [5] it was proven that $\mathrm{Re}\tilde{\omega}_{R'} \to c_R$ if $\varrho_{f0},\ \varrho_{\bar{f}0} \to 0$. However here phase velocity of the generalized Rayleigh wave is somewhat less than phase velocity of the classical Rayleigh wave as $\varrho_{f0},\ \varrho_{\bar{f}0} \to 0$. Also it can be proven that if the ratio $\frac{\varrho_{\bar{0}}}{\varrho_{s0}}$ increases, i.e. $\left(1-\frac{\varrho_{\bar{0}}}{\varrho_{s0}}\right)^2 \to 0$ (see (5.30)), then the generalized Rayleigh wave, similar to the Stoneley wave at an interface of two elastic solids, disappears and the corresponding root of dispersion equation (4.12) tends to a_{s2} (see Appendix for more details).

On the other hand if $\frac{\varrho_{\bar{0}}}{\varrho_{s0}} = O(1)$ then another surface mode appear.

Next we investigate the case when densities of porous media are almost identical, i.e.

$$\frac{\varrho_{\bar{s}0}}{\varrho_{s0}} = O(1) = 1 - \sqrt{R_0}\sqrt{\kappa} + \cdots \quad (\text{or}\left(1-\frac{\varrho_{\bar{s}0}}{\varrho_{s0}}\right)^2 = R_0\kappa + \cdots), \tag{5.33}$$

where $R_0 \sim 1$. One can prove that in this case dispersion equation (4.12) has a root $\tilde{\omega}_3$ satisfying condition (4.16) with $\mathrm{Re}\tilde{\omega}_3 \in (a_{s2}, a_{s1})$. It corresponds to the surface mode whose phase velocity, in contrast to the generalized Rayleigh wave, is somewhat more than phase velocity a_{s2} of a shear wave. Asymptotic expansion of this root has the same as in (5.12) structure:

$$\tilde{\omega} = a_{s2}(1 + c_4\kappa + c_5\kappa^{3/2} + \cdots). \tag{5.34}$$

In accordance with (4.16) one has $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1} \approx i\frac{a_{s2}}{\sqrt{\kappa}}$ and $\tilde{\mu}_s = i\sqrt{\frac{\tilde{\omega}^2}{a_{s2}^2} - 1}$.

From the lowest approximation $O(\kappa^{-1/2})$ one gets:

$$\sqrt{2c_4} = \frac{\varrho_{f0} + \varrho_{\bar{f}0}}{2\varrho_{s0}a_{s2}} > 0 \tag{5.35}$$

and, consequently, exactly as in (5.14),

$$c_4 = \frac{(\varrho_{f0} + \varrho_{\bar{f}0})^2}{2\varrho_{s0}^2 a_{s2}^2} > 0. \tag{5.36}$$

From the next approximation $O(\kappa^0)$ it follows that similar to (5.15)

$$\begin{aligned} c_5 = 2\alpha\varrho_{f0}\left(1 + \frac{m_E}{m_{\bar{E}}}\frac{\varrho_{\bar{f}0}}{\varrho_{f0}}\right)\frac{\varrho_{f0} + \varrho_{\bar{f}0}}{(2\varrho_{s0}a_{s2})^2} + i\frac{(\varrho_{f0} + \varrho_{\bar{f}0})^3}{(2\varrho_{s0}a_{s2})^3}\sqrt{1 - \frac{a_{s2}^2}{a_{s1}^2}} \\ + iR_0\frac{\varrho_{f0} + \varrho_{\bar{f}0}}{8\varrho_{s0}a_{s2}}\frac{1}{\sqrt{1 - \frac{a_{s2}^2}{a_{s1}^2}}} \end{aligned} \tag{5.37}$$

Finally, one has:

$$\tilde{\omega}_3 = a_{s2}\Big(1 + c_4\kappa + c_5\kappa^{3/2} + O(\kappa^2)\Big), \tag{5.38}$$

where coefficients c_4 and c_5 are given by (5.36) and (5.37). This root defines dispersive leaky surface wave, whose phase velocity is somewhat more than a_{s2}. Since $\mathrm{Im}c_5 > 0$, one can conclude estimating $\tilde{\gamma}_1$ and $\tilde{\mu}_s$ that some part of the energy of this wave is absorbed by the slow longitudinal wave. It is easy to see also that this surface mode is degenerated into shear bulk wave if $\varrho_{f0},\ \varrho_{f0}^{-} \to 0$.

6 Conclusions

The results presented in the paper concern surface waves which propagate on an interface separating two saturated porous media of different structure. The present research reveals new features of surface waves in porous media in comparison with those which appear at an interface between two elastic solids. In contrast to the classical case, where one or two surface waves may exist, depending on parameters of the solids, in porous materials three surface modes exist simultaneously. Moreover in porous media four different types of surface waves are proven to be possible. They are due to the combination of all bulk waves.

Two slowest modes, namely true Stoneley and pseudo-Stoneley waves, with phase velocities somewhat less and somewhat more respectively than velocity $\sqrt{\kappa}$ of a slow longitudinal wave always exist. They are very similar to corresponding waves at an interface between a porous medium and a liquid [5]. Their phase velocities are defined primarily by compressibility coefficient of a liquid phase. Phase velocity of leaky pseudo-Stoneley wave is influenced additionally by surface permeability α.

Existence of the other two modes with phase velocities somewhat less and somewhat more than phase velocity a_{s2} of a shear wave is stipulated by the relation between the densities of the skeletons. Namely, if the densities of the solid phases differ then there exists the generalized Rayleigh surface wave with phase velocity less than a_{s2}. Similar to [4,5] it is a leaky wave. It attenuates during the propagation and part of its energy is absorbed by the slow compressional waves. In comparison with analogous waves investigated in [4,5] this generalized Rayleigh wave propagates with smaller phase velocity. However this surface mode and that one considered in [5] have identical attenuations.

At an interafce separating two porous media with almost equal densities generalized Rayleigh wave does not exist. At the same time a new surface mode appears whose phase velocity is slightly more than velocity of a shear bulk wave. It is also a leaky dispersive surface wave and part of its energy is reradiated into the slow compressional waves. In contrast to generalized Rayleigh wave this surface mode is transformed into the bulk shear wave if $\varrho_{f0},\ \varrho_{f0}^{-} \to 0$.

Pseudo-Stoneley, generalized Rayleigh and new surface waves are transitional modes between surface and bulk waves. Due to energy reradiation into interior of the media they exist only in the limited domains, i.e. they are localized waves.

Appendix

Proof of Existence of the Generalized Rayleigh Wave

Let us consider the leading part of dispersion equation (5.30). It can be rewritten as

$$\mathcal{P}_0 = \Big(1 - \mu_s\gamma_2\Big)\mathcal{P}_R - 4\mu_s\gamma_2\frac{\omega^4}{a_{s2}^4}\frac{\varrho_{s0}^-}{\varrho_{s0}}\frac{1}{\Big(1-\frac{\varrho_{s0}^-}{\varrho_{s0}}\Big)^2}. \qquad (A.1)$$

After the changes $\tilde{\omega} = a_{s2}Z$ and $Y = Z^2$ and substitution of (4.10), (A.1) takes the form:

$$\mathcal{P}_0 = \Big(1 - \sqrt{1-Y}\sqrt{1-\nu Y}\Big)\Big((2-Y)^2 - 4\sqrt{1-Y}\sqrt{1-\nu Y}\Big)$$
$$-4\sqrt{1-Y}\sqrt{1-\nu Y}\,Y^2\,\Lambda, \qquad (A.2)$$

where $\nu = \frac{a_{s2}^2}{a_{s1}^2}$ and $\Lambda = \frac{\varrho_{s0}^-\varrho_{s0}}{\Big(\varrho_{s0}^- - \varrho_{s0}\Big)^2}$.

Obviously, in order to prove that there exists a root of dispersion equation (5.30) satisfying condition (4.14), one has to show that equation $\mathcal{P}_0 = 0$ has a root $Y \in (0,1)$. It is not difficult to calculate that:

$$\mathcal{P}_0\Big|_{Y=0} = \frac{d}{dY}\mathcal{P}_0\Big|_{Y=0} = 0,$$
$$\mathcal{P}_0\Big|_{Y=1} = 1 > 0, \qquad (A.3)$$

and

$$\frac{d^2}{dY^2}\mathcal{P}_0\Big|_{Y=0} = -2(1-\nu^2) - 8\Lambda < 0. \qquad (A.4)$$

Thus, there exists a root $Y_0 \in (0,1)$. Let us analyze the behaviour of this root depending on parameter Λ. If the ratio $\frac{\varrho_{s0}^-}{\varrho_{s0}}$ is relatively small and, consequently, Λ is relatively small as well, then the root Y_0 of equation $\mathcal{P}_0 = 0$ is somewhat less than 1 and it defines the generalized Rayleigh wave whose phase velocity is close to a_{s2} (see Fig.1: here $\Lambda = 0.44$ $(\frac{\varrho_{s0}^-}{\varrho_{s0}} = 0.25)$, $\nu = 0.75$).

If $\frac{\varrho_{s0}^-}{\varrho_{s0}} \to 1$, i.e. $\Lambda \to \infty$, then the root Y_0 almost coincides with 1 (see Fig.2: here $\Lambda = 56.0$ $(\frac{\varrho_{s0}^-}{\varrho_{s0}} = 0.87)$, $\nu = 0.75$). The latter means that corresponding root of

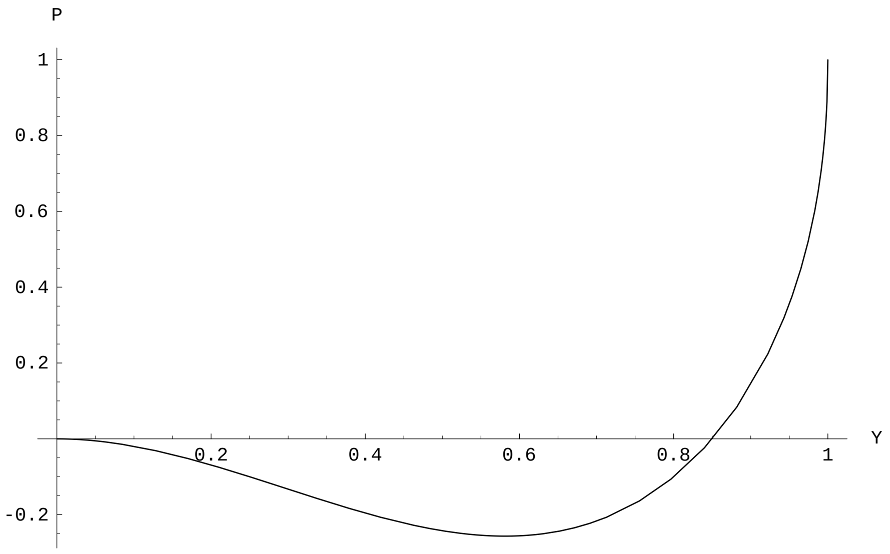

Fig.1

dispersion equation (5.30) tends to a_{s2} and the generalized Rayleigh wave disappears. Moreover, for $\frac{\varrho_{s0}^{-}}{\varrho_{s0}} = O(1)$ one should consider another leading part of (5.30) which will define the root $\tilde{\omega}_3$ with $\mathrm{Re}\tilde{\omega}_3 > a_{s2}$.

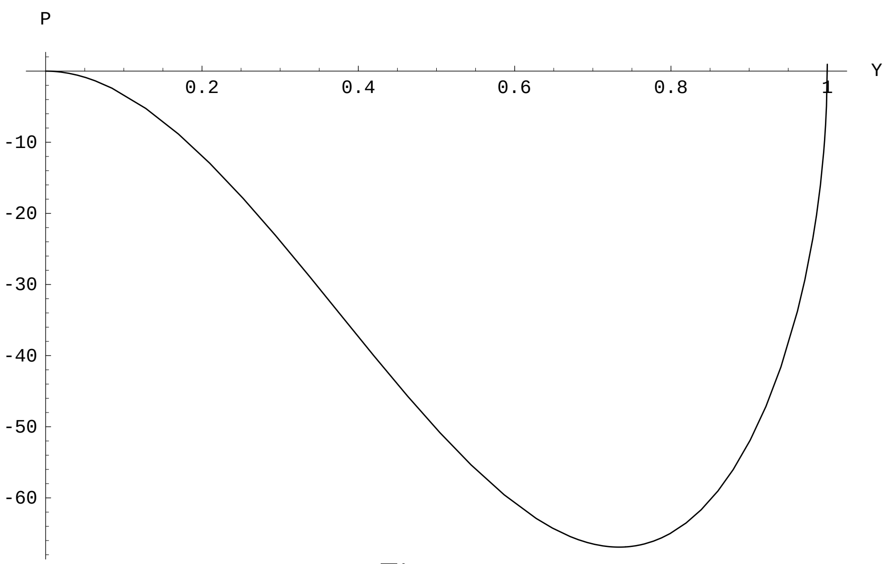

Fig.2

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