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# Sound and shock waves in porous and granular materials


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# 1 Introduction

Continuum theories of porous materials are based on models of multicomponent systems. One of the components is a solid matrix - the so-called *skeleton* which is characterized by a complex geometric microstructure. The main feature of this microstructure are randomly distributed interconnected voids. These voids are usually filled with a fluid or a mixture of fluids. Due to interconnections of voids the fluid component may move with its own - one or more - kinematics in addition to the motion of the skeleton. This property of *permeability* of the skeleton leads to diffusion processes. Consequently two main properties describing the microstructure of porous materials are the void fraction, called also the *porosity* and the permeability. Granular materials contain an even larger class of systems as they may appear not only in a compact form when all granulae have contact with neighbours but they may also fluidize, i.e. they may loose contact and behave as a suspension. In the first case their behavior is in many respects similar to porous materials and this is the only case which we consider in this work.

In the paper we present some features of wave motion in poroelastic materials. A porous material is called *poroelastic* if it is an elastic solid in the limit case of zero porosity, i.e. if it becomes in this limit a classical elastic material without voids.

In the next Section a set of governing equations for a two-component poroelastic material under large deformations is presented. Some nonlinear features of this model shall be used in the analysis of nonlinear waves described in the last Section of the paper. Section 3 contains the propagation condition for bulk waves in linear poroelastic materials. It is shown that the model describes a so-called P2-mode discovered by M. Biot. In Section 4 we present some properties of one-dimensional monochromatic waves. In particular we demonstrate the role played in the theory of weak discontinuity waves by two main mechanisms of dissipation: diffusion and dynamic changes of porosity. After a brief presentation of the problem of boundary conditions in Section 5 we discuss in Section 6 main features of surface waves on an impermeable surface of the porous material (contact with vacuum) and on a permeable boundary (drainage). We indicate the existence of three basic types of surface waves: a Stoneley wave, a leaky pseudo-Stoneley wave, and a leaky generalized Rayleigh wave. Finally in Section 7 we report some preliminary results for non-linear waves in porous materials. In particular we show the existence of a soliton-like solution of a Riemann problem for porosity. Such a solution is connected with the dispersion appearing in the field equations for the model of porous materials with the balance equation of porosity. The coupling parameter arising from the existence of dynamical changes of porosity is small and plays the role similar to the parameter $\beta$ in the following Korteweg - de Vries equation

$$\frac{\partial u}{\partial t}+3\frac{\partial u^2}{\partial x}+\beta^2\frac{\partial^3 u}{\partial x^3}=0,\quad x\in\Re^1,\quad t\in\mathcal{T}.$$

It is well-known that limit solutions of this equation as $\beta\to 0$ are entirely different from solutions of the limit equation $\beta=0$ (Burgers equation). For instance a soliton-like solution of this equation converges weakly to the Dirac function as $\beta\to 0$ (e.g. see: V. P. MASLOV, G. A. OMEL'YANOV [1981]). This is also the situation arising in the model of porous materials considered in this work.

# 2 Governing equations

We limit our attention to the case of a two-component system. The thermodynamic construction of a more general model of porous materials can be found, for instance in the papers K. WILMANSKI [1996], [1998].

All processes are assumed to be isothermal.

In the Lagrangian description related to a reference configuration of the skeleton $\mathcal{B}_0$ the unknown fields are

1. mass density of the fluid referred to a unit reference volume: $\rho^F(\mathbf{X},t)$, $\mathbf{X}\in\mathcal{B}_0$, $t\in\mathcal{T}\subset[0,\infty)$,
2. mass density of the skeleton referred to a unit reference volume: $\rho^S(\mathbf{X},t)$, $\mathbf{X}\in\mathcal{B}_0$, $t\in\mathcal{T}$,
3. velocity of the fluid: $\acute{\mathbf{x}}^F(\mathbf{X},t)$, $\mathbf{X}\in\mathcal{B}_0$, $t\in\mathcal{T}$,
4. motion of the skeleton: $\boldsymbol{\chi}^S(\mathbf{X},t)$, $\mathbf{X}\in\mathcal{B}_0$, $t\in\mathcal{T}$,
5. porosity: $n(\mathbf{X},t)$, $\mathbf{X}\in\mathcal{B}_0$, $t\in\mathcal{T}$.

They satisfy the following partial balance equations

$$\begin{aligned}\frac{\partial\rho^F}{\partial t}+\operatorname{Div}\left(\rho^F\acute{\mathbf{X}}^F\right)&=0,\quad\frac{\partial\rho^S}{\partial t}=0,\\ \acute{\mathbf{X}}^F&=\mathbf{F}^{S-1}\left(\acute{\mathbf{x}}^F-\acute{\mathbf{x}}^S\right),\quad J^S:=\det\mathbf{F}^S>0,\end{aligned}$$

$$\rho^F\left(\frac{\partial\acute{\mathbf{x}}^F}{\partial t}+\acute{\mathbf{X}}^F\cdot\operatorname{Grad}\acute{\mathbf{x}}^F\right)=\operatorname{Div}\mathbf{P}^F-\hat{\mathbf{p}},\quad\rho^S\frac{\partial\acute{\mathbf{x}}^S}{\partial t}=\operatorname{Div}\mathbf{P}^S+\hat{\mathbf{p}},\qquad(2.1)$$

$$\frac{\partial n}{\partial t}+\operatorname{Div}\left(\Phi_0\acute{\mathbf{X}}^F\right)=\hat{n},\qquad\mathbf{F}^S:=\operatorname{Grad}\boldsymbol{\chi}^S(\mathbf{X},t),\quad\acute{\mathbf{x}}^S:=\frac{\partial\boldsymbol{\chi}^S}{\partial t}(\mathbf{X},t).$$

These equations become field equations if we specify constitutive relations for partial Piola-Kirchhoff stress tensors $\mathbf{P}^F, \mathbf{P}^S$, the source of momentum $\hat{\mathbf{p}}$, the flux and source of the porosity, $\Phi_0$ and $\hat{n}$, respectively. For processes in poroelastic materials filled with an ideal fluid these relations have the form (e.g. K. WILMANSKI $[1998]_2$)

$$\mathbf{P}^F = -p^F \mathbf{F}^{S-T}, \quad p^F = \rho^F \left( \rho^S \frac{\partial \psi^S}{\partial \rho^F} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} \right) + \beta' \Phi_0 \Delta,$$

$$\mathbf{P}^S = \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} + \beta' \frac{\partial \Phi_0}{\partial \mathbf{F}^S} \Delta, \quad \Delta = n - n_E, \tag{2.2}$$

$$\hat{\mathbf{p}} = \pi \left( \acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S \right), \quad \hat{n} = -\frac{\Delta}{\tau},$$

where the Helmholtz free energy functions $\psi^F, \psi^S$, the flux coefficient $\Phi_0$, and the equilibrium porosity $n_E$ are functions of equilibrium variables $\{\mathbf{F}^S, \rho^S, \rho^F\}$. Obviously the constitutive relations (2.2) do not contain any viscous effects. We have neglected them in order to expose better effects connected with dissipation effects caused by the microstructure itself.

The above system of field equations is assumed to be hyperbolic.

Below we consider some simplified versions of this model. In particular we consider weak discontinuity waves solely for a fully linearized model[1].

# 3 Propagation conditions for acoustic waves

In this Section we consider the propagation of a weak discontinuity wave in a linear poroelastic material with a constant initial porosity $n_E = const.$ In such a case it is convenient to use the Eulerian description in which the mass densities and the velocities are transformed in the following way

$$\begin{aligned} \rho_t^F &= \rho^F J^{S-1} = \rho_t^F (\mathbf{x}, t), \quad \rho_t^S = \rho^S J^{S-1} = \rho_t^S (\mathbf{x}, t), \\ \mathbf{v}^F &= \acute{\mathbf{x}}^F (\mathbf{x}, t), \quad \mathbf{v}^S = \acute{\mathbf{x}}^S (\mathbf{x}, t), \quad \mathbf{x} \in \mathcal{B}_t := \boldsymbol{\chi}^S (\mathcal{B}_0, t), \quad t \in \mathcal{T}. \end{aligned} \tag{3.1}$$

As we use further solely mass densities refering to current configurations we skip the subscript to simplify the notation.

A weak discontinuity wave is defined as a singular surface moving with a normal speed $c$ on which the following conditions are satisfied

$$\left[\left[\rho^F\right]\right] = 0, \quad \left[\left[\mathbf{v}^F\right]\right] = 0, \quad \left[\left[\mathbf{v}^S\right]\right] = 0, \quad \left[\left[\mathbf{e}^S\right]\right] = 0, \quad [[n]] \equiv [[\Delta]] = 0, \tag{3.2}$$

[1] The general nonlinear case of poroelastic materials was consider in the paper K. WILMANSKI $[1995]_2$. As the main results are analogous to those for linear materials we present here solely a linearized case.

where $[[\cdots]] = (\cdots)^{+} - (\cdots)^{-}$ is the difference of limits on both sides of this surface. The derivatives of the fields may possess finite discontinuities and we denote

$$r := \left[\left[\frac{\partial \rho^F}{\partial t}\right]\right], \quad \mathbf{a}^F := \left[\left[\frac{\partial \mathbf{v}^F}{\partial t}\right]\right], \quad \mathbf{a}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t}\right]\right], \quad D := \left[\left[\frac{\partial \Delta}{\partial t}\right]\right]. \tag{3.3}$$

The balance equations appropriate for the linear model (Eulerian description) follow from the equations (2.1) and have the form

$$\begin{aligned} \frac{\partial \rho^F}{\partial t} + \operatorname{div}\left(\rho^F \mathbf{v}^F\right) &= 0, \quad \rho^S = const., \\ \rho^F\left(\frac{\partial \mathbf{v}^F}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} \mathbf{v}^F\right) &= \operatorname{div} \mathbf{T}^F - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right), \\ \rho^S \frac{\partial \mathbf{v}^S}{\partial t} &= \operatorname{div} \mathbf{T}^S + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right), \\ \frac{\partial n}{\partial t} + \operatorname{div}\left[\varphi\left(n_E\right)\left(\mathbf{v}^F - \mathbf{v}^S\right)\right] &= -\frac{\Delta}{\tau}, \quad \Delta := n - n_E, \quad \varphi\left(n_E\right) \approx n_E. \end{aligned} \tag{3.4}$$

The linear model is based on the assumption of small deformations of the skeleton. Namely

$$\sup_{t \in \mathcal{T}} \sup_{\mathbf{x} \in \mathcal{B}_t} \left|\mathbf{e}^S \cdot \mathbf{n} \otimes \mathbf{n}\right| \ll 1 \text{ for all } \mathbf{n}, \quad |\mathbf{n}| = 1,$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad}\left(\mathbf{v}^S\right), \quad J^S \approx 1 + \operatorname{tr} \mathbf{e}^S \approx 1. \tag{3.5}$$

The field equations are constructed by means of the following constitutive relations for partial Cauchy stress tensors

$$\begin{aligned} \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S\left(tr \mathbf{e}^S\right)\mathbf{1} + 2\mu^S \mathbf{e}^S + \beta \Delta \mathbf{1}, \\ \mathbf{T}^F &= -p_0^F \mathbf{1} - \kappa\left(\rho^F - \rho_0^F\right)\mathbf{1} - \beta \Delta \mathbf{1}. \end{aligned} \tag{3.6}$$

These relations follow from (2.2) by linearization. The contribution of the porosity flux to both partial stresses is then identical and its coefficient has been denoted by $\beta := \beta' \Phi_0$. It may depend solely on $n_E$. The above relations are linear not only with respect to the deformation of the skeleton but also with respect to changes of the mass density of the fluid.

Evaluation of jumps of the equations (3.4) on the wave front yields the algebraic homogeneous set of equations for the amplitudes (3.3). As usual one has to make use of kinematic (Hadamard) compatibility conditions which are standard (e.g. K. WILMANSKI [1999]). We obtain the following relations

$$r = \frac{1}{c}\mathbf{a}^F \cdot \mathbf{n}, \quad D = \frac{1}{c}\varphi\left(\mathbf{a}^F - \mathbf{a}^S\right) \cdot \mathbf{n}, \quad \mathbf{a}^F = \mathbf{a}^F \cdot \mathbf{n}\, \mathbf{n}, \tag{3.7}$$

$$\mathbf{a}^S_\perp \neq \mathbf{0} \Longrightarrow \left(c^2 - \frac{\mu^S}{\rho^S}\right) = 0, \quad \mathbf{a}^S_\perp := \mathbf{a}^S - \mathbf{a}^S \cdot \mathbf{n}\,\mathbf{n},$$

$$\mathbf{a}^S \cdot \mathbf{n} \neq \mathbf{0}, \quad \mathbf{a}^\mathbf{F} \cdot \mathbf{n} \neq \mathbf{0} \Longrightarrow$$

$$\Longrightarrow \left(c^2 - \kappa - \frac{\varphi\beta}{\rho^F}\right)\left(c^2 - \frac{\lambda^S + 2\mu^S}{\rho^S} - \frac{\varphi\beta}{\rho^S}\right) - \left(\frac{\varphi\beta}{\rho^F}\right)^2 \frac{\rho^F}{\rho^S} = 0. \tag{3.8}$$

Clearly we obtain three modes of propagation. Two speeds of the so-called P1-, and P2-mode follow from the equation $(3.8)_2$. The slower mode P2 is called also the Biot wave and according to the relation $(3.7)_3$ it is longitudinal with respect to the amplitude of the fluid component $\mathbf{a}^F$. The third mode is transversal with respect to the amplitude $\mathbf{a}^S$ and its speed is identical with the speed of the shear wave in the solid component (formula $(3.8)_1$). The weak discontinuities of the mass density in the fluid and of the porosity are carried by the P1- and P2-modes. These results are confirmed by numerous laboratory experiments on rocks and sintered glass (e.g. T. BOURBIE, O. COUSSY, B. ZINSZNER [1987]). *In situ* experiments are extremally difficult due to a very heavy attenuation of P2-waves. We present this problem in the next Section.

# 4 Monochromatic one-dimensional waves

In order to expose the most important features of bulk waves in poroelastic materials we consider the propagation of monochromatic waves described by the following set of linear field equations

$$\frac{\partial \rho^F}{\partial t} + \rho_0^F \frac{\partial v^F}{\partial x} = 0, \quad \rho_0^F \frac{\partial v^F}{\partial t} = -\frac{\partial p^F}{\partial x} - \pi\left(v^F - v^S\right),$$

$$\rho^S \frac{\partial v^S}{\partial t} = \frac{\partial \sigma^S}{\partial x} + \pi\left(v^F - v^S\right), \quad \frac{\partial \Delta}{\partial t} + n_E \frac{\partial\left(v^F - v^S\right)}{\partial x} = -\frac{\Delta}{\tau},$$

$$\begin{aligned} p^F &= p_0^F + \kappa\left(n_E\right)\left(\rho^F - \rho_0^F\right) + \beta\left(n_E\right)\Delta, \\ \sigma^S &= \sigma_0^S + \left(\lambda^S\left(n_E\right) + 2\mu^S\left(n_E\right)\right)e^S + \beta\left(n_E\right)\Delta, \\ \frac{\partial e^S}{\partial t} &= \frac{\partial v^S}{\partial x}. \end{aligned} \tag{4.1}$$

In these relations $\rho_0^F, \rho^S, n_E, p_0^F, \sigma_0^S$ denote constant reference values of partial mass densities, porosity, partial pressure and normal stress in the skeleton, respectively. Material parameters $\kappa, \lambda^S, \mu^S, \beta, \pi, \tau$ depend parametrically on the initial porosity

$n_E$. Consequently the essential coupling between components is due to changes of porosity $\Delta$ (parameter $\beta$) and the relative motion $v^F - v^S$ (coefficient of permeability $\pi$).

We seek the solution in the following form

$$\begin{aligned} \rho^F &= \rho_0^F + R\exp(kx-\omega t), \\ v^F &= V^F \exp(kx-\omega t), \quad v^S = V^S \exp(kx-\omega t), \\ e^S &= E^S \exp(kx-\omega t), \quad \Delta = D\exp(kx-\omega t), \end{aligned} \tag{4.2}$$

where $R, V^F, V^S, E^S, D$ are constant amplitudes of the wave. As usual we obtain from (4.1) the dispersion relation as condition for existence of bulk waves. It has the following form

$$\left(\omega^2 - U^{S2}k^2 - \frac{i\omega\tau}{1+i\omega\tau}\frac{n_E\beta}{\rho^S}k^2 - i\frac{\pi\omega}{\rho^S}\right)\times \tag{4.3}$$
$$\times\left(\omega^2 - U^{F2}k^2 - \frac{i\omega\tau}{1+i\omega\tau}\frac{n_E\beta}{\rho_0^F}k^2 - i\frac{\pi\omega}{\rho_0^F}\right) + \left(\frac{i\omega\tau}{1+i\omega\tau}\frac{n_E\beta}{\rho_0^F}k^2 + \frac{\pi\omega}{\rho_0^F}\right)^2\frac{\rho_0^F}{\rho^S} = 0.$$

In this relation the speeds $U_\parallel^S, U^F$ are defined as follows

$$U_\parallel^{S2} := \frac{\lambda^S + 2\mu^S}{\rho^S}, \quad U^{F2} := \kappa. \tag{4.4}$$

In the one-dimensional case one mode of propagation - the shear wave cannot appear. Consequently relation (4.3) yields to branches of the dispersion relation connected with the P1- and P2-wave.

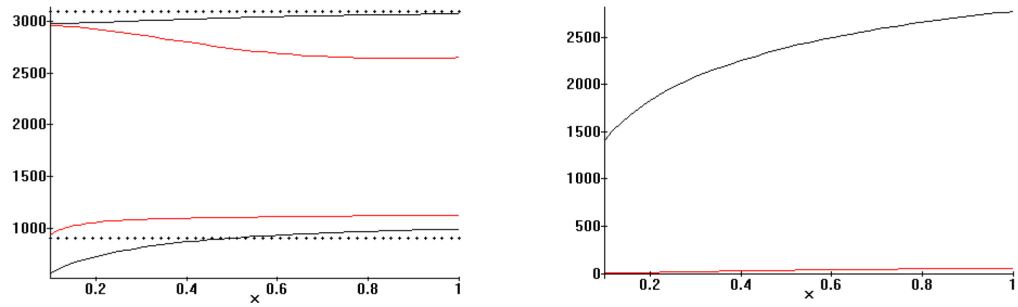
We consider monochromatic waves of the frequency $\omega$. The above dispersion relation yields then the complex wave numbers $k$ which define the phase velocity $c_{ph}$, the group velocity $c_g$, and the attenuation $\gamma$ according to the following relations

$$k = k(\omega), \quad c_{ph} := \frac{\omega}{\operatorname{Re} k}, \quad c_g := \frac{\partial\omega}{\partial \operatorname{Re} k}, \quad \gamma := \operatorname{Im} k. \tag{4.5}$$

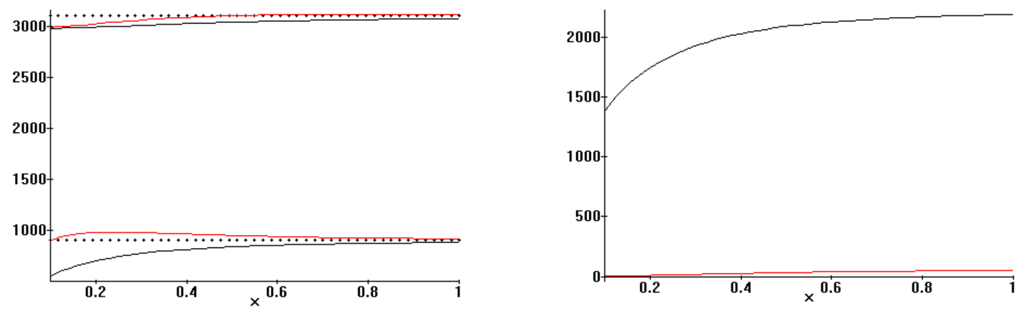
We illustrate the results by an example shown in the Figures below. The data of the parameters are typical for rock materials used in dynamical experiments (see: T. Bourbie, O. Coussy, B. Zinszner [1987], K. Wilmanski [1999]).The first two Figures show the behavior of velocities and attenuations in the case of the full coupling of waves: both coefficients $\beta$ and $\pi$, describing the dissipation in the system are different from zero. Two upper curves of the Figure on the left hand side correspond to the P1-wave. As expected both the phase velocity (upper curve) and the group velocity(lower curve) tend to the speed of the first longitudinal wave discussed in the previous Section of this work: $\lim_{\omega\to\infty} c_{ph} = \lim_{\omega\to\infty} c_g = U_\parallel^S$.

**Figure 1**: *Phase, group velocities and attenuation of monochromatic 1D waves* $(x = \omega\tau)$

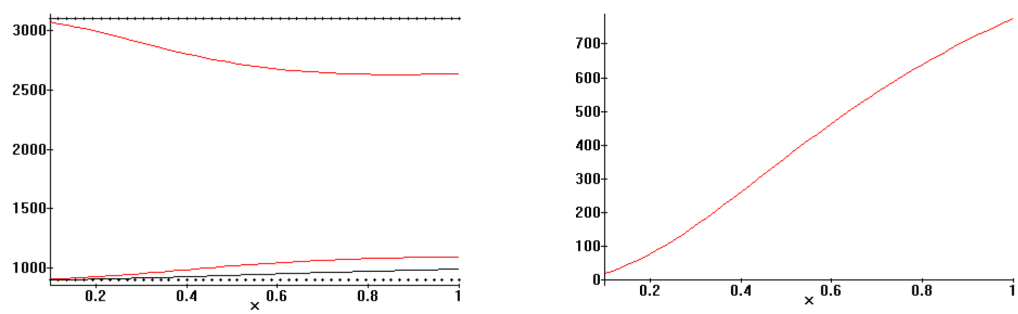
$\beta \neq 0, \quad \pi \neq 0$




$\beta = 0$ :




$\pi = 0$ :




*Numerical data*

$U^S_{\parallel} = 3.1 \frac{km}{s}, \quad U^F = 0.9 \frac{km}{s}, \quad n = 0.23, \quad \rho^S = 2400 \frac{kg}{m^3}, \quad \rho^F = 230 \frac{kg}{m^3},$

$\pi = 2.602 \times 10^7 \frac{kg}{m^3 s}, \quad \beta = 0.313 \times 10^9 \frac{kg}{ms^2}, \quad \tau = 3.7 \times 10^6 s.$

In addition the group velocity has the characteristic minimum value at app. $\omega\tau = 2$ (i.e. app. $0.54\, MHz$, beyond the range of the Figure). Similarily the two lower curves corresponding to the P2-wave approach the limit speed of the second longitudinal wave: $\lim_{\omega\to\infty} c_{ph} = \lim_{\omega\to\infty} c_g = U^F$. The group velocity (upper curve) has the maximum at app. $\omega\tau = 1$ (i.e. app. $0.27\, MHz$). The most characteristic feature of the P2-wave is shown in Figure on the right where we have plotted the attenuation. The attenuation of P1-wave (lower curve) is rather small and reaches the value of app. $50\,\frac{1}{m}$ for $\omega\tau = 1$. On the other hand the attenuation of P2-wave is very large indeed. It exceeds the attenuation of the P1-wave on two orders of magnitude and in our example reaches the value of app. $2500\,\frac{1}{m}$ for $\omega\tau = 1$.

In further Figures we show the influence of both coefficients $\beta$ and $\pi$, determining the dissipation. In the case of lack of coupling due to the changes of porosity $\Delta$ (i.e. $\beta = 0$) the shape of curves for velocities is similar to those discussed above apart from the shift of the extremum values of group velocities to lower frequencies. Attenuation of the P2-waves is reduced in this case on app. 25%. This influence of the porosity on the propagation conditions is quite remarkable if we compare it with an almost total lack of influence of $\beta$ on solutions of steady-state problems (e.g.: B. ALBERS, K. WILMANSKI [1999]). As we see further it is even more important in the case of nonlinear waves.

In the other case in which $\pi = 0$ and $\beta \neq 0$ we see that the phase velocity of P1-waves is practically independent of the frequency and that these waves are not attenuated. There is still a considerable attenuation of P2-waves (the curve in the right Figure) but this is also reduced to app. 25% of the values which it possesses in the general case.

Let us mention in passing that the coupling through the changes of porosity $\Delta$ reminds the couplings appearing in the extended thermodynamics of fluids in which the whole contribution $\beta\Delta$ corresponds to the fourteenth moment of such models (see: I. MÜLLER, T. RUGGERI [1998]).

# 5 Boundary conditions on interfaces, drainage

In spite of the importance of this problem not much has been done to formulate boundary conditions on boundaries of the skeleton and on the contact interfaces of two different porous materials in cases when the boundary is permeable. On the one hand side there are a few papers of G. S. BEAVERS and D. D. JOSEPH (e.g. [1967]) who investigated theoretically and experimentally the problem of discontinuity of the tangential component of relative velocity of the fluid with respect to the boundary. On the other hand conditions for the normal relative displacement were introduced by H. DERESIEWICZ [1962]. They have been modified in the PhD-Thesis by K. RUNESSON [1978]. A new version of this condition has been proposed in some recent papers (K. WILMANSKI [1995], W. KEMPA [1997], B. ALBERS, K. WILMANSKI [1999]). Applications to the theory of surface waves are due to be published (see: I. Ya. EDELMAN, K. WILMANSKI, E. RADKEVICH [1999], I. Ya. EDELMAN, K.

WILMANSKI [1999]) Some of their important aspects shall be presented in the next Section.

The form of this boundary condition follows from the assumption that the drainage is controlled by the pressure jump in the fluid component. For the case of a single ideal fluid component defined by the constitutive relation $(3.6)_2$ it has the form

$$\rho^F \left( \mathbf{v}^F - \mathbf{v}^S \right) \cdot \mathbf{n} = -\alpha' \left[\left[ \frac{p^F}{n} \right]\right] \equiv \alpha \left( p^{F-} - \frac{n^-}{n^+} p^{F+} \right), \quad \alpha := \frac{\alpha'}{n^-}. \tag{5.1}$$

The left hand side can be evaluated on either side of the boundary due to the mass conservation

$$\left[\left[ \rho^F \left( \mathbf{v}^F - \mathbf{v}^S \right) \cdot \mathbf{n} \right]\right] = 0. \tag{5.2}$$

The ratio $\frac{p^F}{n}$ approximates the value of the pore pressure. This approximation seems to be reasonable in the case of small diffusion velocities. According to relation (5.1) the jump of the pore pressure is assumed to be the driving force for the mass transport through a permeable boundary. In the limit case $\alpha = 0$ the boundary is impermeable and the condition (5.1) reduces to the condition for the normal component of velocities $\mathbf{v}^F \cdot \mathbf{n} = \mathbf{v}^S \cdot \mathbf{n}$.

Let us mention that in the case of the interface between a porous material (the negative side of the boundary) and a liquid (the positive side of the boundary) the pore pressure on the positive side must be identified with the external pressure (i.e. with the pressure in the liquid): $\frac{p^{F+}}{n^+} =: p_{ext}$. This pressure is a part of control variables on the boundary.

In the case of ideal fluid components considered in this work the remaining part of the kinematic boundary condition is classical

$$\mathbf{v}^F - \mathbf{v}^F \cdot \mathbf{n}\, \mathbf{n} = \mathbf{v}^S - \mathbf{v}^S \cdot \mathbf{n}\, \mathbf{n}. \tag{5.3}$$

Otherwise one has to use conditions analogous to those proposed by G. S. BEAVERS, D. D. JOSEPH [1967].

The second vectorial boundary condition follows from the dynamical compatibility conditions. Under the assumption of small diffusion velocities we have

$$\left[\left[ \mathbf{T}^S + \mathbf{T}^F \right]\right] \mathbf{n} = 0. \tag{5.4}$$

In the case of contact of a porous material with an external world the above condition contains an external loading vector $\mathbf{t}_{ext}$

$$\left( \mathbf{T}^S + \mathbf{T}^F \right)^- \mathbf{n} = \mathbf{t}_{ext}, \tag{5.5}$$

which is assumed to be the second part of control variables on the boundary.

In the next Section we apply these boundary conditions to the problem of surface waves.

# 6 Surface waves on boundaries of a porous medium with vacuum and with a liquid

We proceed to investigate propagation conditions for waves in linear poroelastic materials with a boundary. In the classical case of an elastic solid two basic bulk waves - longitudinal wave and shear wave - combine on the boundary and in this way yield the existence of surface waves. The most important of them is the so-called Rayleigh wave which propagates along the boundary of a semiinfinite medium with a speed lower than the speed of the shear wave. Its wave vector varies in time in this way that the particles near the boundary possess elliptic trajectories. Such waves are particularly important in seismological applications because they have very large amplitudes and they disperse much weaker than the bulk waves.

Additional bulk modes of propagation in porous materials suggest that the number of surface modes in such systems is also larger than in a single component system. This is indeed the case as some experiments clearly show (e.g. P. B. NAGY [1992], L. ADLER, P. B. NAGY [1994], W. LAURIKS; L. KELDERS, J. F. ALLARD [1998]). Moreover their attenuation is much weaker than this of P2-wave (see: Section 4) which means that they can be much easier applied in various devices.

We present here briefly the procedure of the analytical investigation of such waves. Let us begin with a poroelastic semiinfinite medium whose boundary is impermeable ($\alpha = 0$ in the condition (5.1)). We seek a solution of a two-dimensional problem assuming the existence of potentials for velocity fields, i.e.

$$\begin{aligned} \mathbf{v}^S &= \frac{\partial \mathbf{u}^S}{\partial t}, \quad \mathbf{u}^S = \operatorname{grad} \varphi^S + \operatorname{rot} \boldsymbol{\psi}^S, \\ \mathbf{v}^F &= \frac{\partial \mathbf{u}^F}{\partial t}, \quad \mathbf{u}^F = \operatorname{grad} \varphi^F + \operatorname{rot} \boldsymbol{\psi}^F. \end{aligned} \tag{6.1}$$

The boundary is supposed to coincide with $y = 0$, and we assume solutions of field equations (3.4) to have the form

$$\begin{aligned} \varphi^{S,F} &= A^{S,F}(y) \exp\left[i\left(kx - \omega t\right)\right], \\ \psi^{S,F}_{x,y} &\equiv 0, \quad \psi^{S,F}_z = B^{S,F}(y) \exp\left[i\left(kx - \omega t\right)\right], \\ \rho^{S,F} &= \rho^{S,F}_0 + A^{S,F}_\rho(y) \exp\left[i\left(kx - \omega t\right)\right], \\ n &= n_0 + A^{\Delta}(y) \exp\left[i\left(kx - \omega t\right)\right]. \end{aligned} \tag{6.2}$$

Substitution of these relations in (3.4) yields

$$\begin{aligned} &\left(\frac{d^2}{dy^2} - k^2\right) A^F + k^F\left(k^F + \frac{i\pi}{\rho^F_0 U^F}\right) A^F - \frac{i\pi}{\rho^F_0 U^F} k^F A^S + \\ &+ \frac{\beta}{\rho^F_0 U^F} \frac{n_E k^F}{\frac{i}{\tau} + \omega} \left(\frac{d^2}{dy^2} - k^2\right)\left(A^F - A^S\right) = 0, \end{aligned} \tag{6.3}$$

$$\left(\frac{d^2}{dy^2}-k^2\right)A^S+k^{s2}A^S-\frac{i\pi}{\rho_0^F U_\parallel^S}k_\parallel^S\left(A^F-A^S\right)+$$

$$+\frac{\beta}{\rho_0^S U_\parallel^S}\frac{n_E k_\parallel^S}{\frac{i}{\tau}+\omega}\left(\frac{d^2}{dy^2}-k^2\right)\left(A^F-A^S\right)=0, \qquad (6.4)$$

$$\left(\frac{d^2}{dy^2}-k^2\right)B^S+\left(k_\perp^S-\frac{i\pi}{\rho_0^S U_\perp^S}\frac{\omega\rho_0^F}{\omega\rho_0^F+i\pi}\right)B^S=0, \qquad (6.5)$$

where

$$\begin{aligned} k_\parallel^{S2} &:= \frac{\omega^2}{U_\parallel^{S2}}, \quad U_\parallel^{S2}:=\frac{\lambda^S+2\mu^S}{\rho_0^S}, \quad k_\perp^{S2}:=\frac{\omega^2}{U_\perp^{S2}}, \quad U_\perp^{S2}:=\frac{\mu^S}{\rho_0^S}, \qquad (6.6)\\ k^{F2} &:= \frac{\omega^2}{U^{F2}}, \quad U^{F2}:=\kappa. \end{aligned}$$

Functions $B^F, A^\Delta, A_\rho^{S,F}$ are given by algebraic relations containing $A^S, A^F$ and $B^S$ which we shall not quote in this work (for details, see: I. Ya. EDELMAN, K. WILMANSKI, E. RADKEVICH [1999]). Integration of the above set of ordinary differential equations together with boundary conditions as described in the previous Section yields the solution of the problem. It can be shown that this solution describes two different modes of propagation. In the limit of short waves ($k\to\infty$; these are the fastest waves, i.e. these which define the front of propagation) we obtain

- Stoneley surface waves which propagate almost without attenuation and have the speed

$$U_{St}^2\approx\kappa\left(1-\frac{\rho_0^{F2}}{4\mu^{S2}\left(1-\gamma\right)^2}\kappa^2\right)<U^{F2}, \quad \gamma:=\left(\frac{U_\perp^S}{U_\parallel^S}\right)^2<1; \qquad (6.7)$$

this is the slowest wave of all,

- Rayleigh leaky surface wave with the speed $U_R\in\left(U^F,U_\perp^S\right)$. It is attenuated and as it is faster than the P2-wave it looses its energy to the P2-wave.

The situation is more complicated for permeable boundaries, i.e. for $\alpha\neq 0$. The problem has been solved for the contact of a poroelastic materials with the fluid flowing out of the porous medium. In such a case it can be shown (we refer here to the papers I. Ya. EDELMAN, K. WILMANSKI [1999], I. Ya. EDELMAN [2000] for details) that there exist three modes of surface waves

- Stoneley wave with $U_{St} < U^F$; this wave propagates almost without attenuation,
- leaky pseudo-Stoneley wave with the speed $U_{LSt} \in \left(U^F, U_\perp^S\right)$; the speed $U_{LSt}$ exceeds $U^F$ on a contribution proportional to $\alpha^2$. This wave degenerates into P2-wave for $\alpha \to 0$;
- generalized leaky Rayleigh wave $U_R \in \left(U^F, U_\perp^S\right)$.

All these waves were indeed observed by the authors mentioned at the beginning of this Section.

It is still an open question what kinds of surface waves may exist on interfaces dividing different porous materials saturated by the same fluid. The existence of different modes depends on the relation of material parameters on both sides of the interface. This requires a much more carefull "bookkeeping" than it is the case on a real boundary. This problem is also strongly related to a more general problem of reflection and transmission of acoustic waves between porous materials. No theoretical results are available as yet in such a general case.

# 7 A Riemann problem for poroelastic materials and some nonlinear waves

The problem of shock waves in porous and granular materials has been investigated primarily by means of one-component models. Results were applied in the analysis of underground nuclear explosions, combustion problems for solid fuels etc. Very little has been done for multicomponent systems. A preliminary result that a strong discontinuity wave may indeed grow from a weak discontinuity if the model contains some nonlinearity has been presented in the paper K. WILMANSKI [1998]$_3$. However the main problem of construction of such wave solution is still open. In the forthcoming paper E. RADKEVICH, K. WILMANSKI [2000] the method of an asymptotic analysis has been applied to show some singular solutions for nonlinear poroelastic materials. In the first stage kinematic nonlinearities and a nonlinear dependence on the equilibrium porosity changing according to the constitutive relation

$$n_E = n_E\left(\rho^S, \rho^F\right), \quad \frac{\partial n_E}{\partial \rho^F} > 0, \quad \frac{\partial n_E}{\partial \rho^S} < 0, \tag{7.1}$$

has been investigated. It was shown that, using a dimensionless description in which the coupling parameter $\beta$ is small, one can construct asymptotic solutions for a one-dimensional case in the following dimensionless form

$$\begin{aligned}
v^F &= v_{as}^F + w^F, \quad v^S = v_{as}^S + w^S, \\
\rho^F &= \rho_{as}^F + r^F, \quad \rho^S = \rho_{as}^S + r^S, \quad \frac{\Delta}{\beta} = \Pi_{as} + \varpi,
\end{aligned}$$

$$w^F, w^S, r^F, r^S, \varpi \in W_p^1\left(\Re^1 \times \mathcal{T}\right)$$

$$\begin{aligned}
\Pi_{as} &= \sum_{j=0}^{N} \beta^j \left(\Upsilon_j(x,t) + Y_j^p(\sigma, x, t)\right), \quad \sigma := \frac{x - x(t)}{\beta}, \\
Y_j^p &= \Pi_j(\sigma, t) + H_j^p(x,t)\ z_0(\sigma, t), \quad j \geq 1,
\end{aligned} \tag{7.2}$$

where $\sigma$ is the so-called fast variable, $x(t)$ denotes the position of the wave front, $\Pi_j(\sigma,t)$, $z_0(\sigma,t)$ are smooth. This solution yields shock wave structures for velocity fields with a corresponding soliton-like solution for the porosity. The latter is possible due to the presence of an additional balance equation in the model - the balance of porosity. There are already first indications that such a dynamic structure appears indeed in nature.

# 8 Final remarks

The above presented review shows that it is still rather little what we know about waves in systems with such a complicated microstructure as porosity. On the other hand the propagation of waves - spontaneous (e.g. earthquakes) and artificial (explosions, vibrations etc.) - seems to be an only way to obtain *in situ* data needed in geophysics, soil mechanics, materials sciences. Particularly such problems as transmition and reflection of weak discontinuity waves on interfaces, scattering of such waves on microheterogeneities, strong discontinuity waves in nonlinear materials, relations between propagation properties and material parameters in the bulk and on surfaces of porous materials must be extensively investigated - most likely in many cases by means of modern numerical methods for hyperbolic sets of field equations.

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