Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Hybrid method and vibrational stability for nonlinear singularly perturbed systems under parametric excitations

Vadim V. Strygin[1]

submitted: March 3, 2000

[1] Department of Applied Mathematics and Mechanics
Voronezh State University
University sq., 1
394693 Voronezh, Russia
E-Mail: strygin@sos.vsu.ru

Preprint No. 561
Berlin 2000

1991 Mathematics Subject Classification. 34E15.

Key words and phrases. Hybrid asymptotical method, averaging, boundary functions, vibrational stability.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

In memory of M.A.Krasnoselski

Abstract

The well-known classic feedback and feedforward techniques are the main tools for investigations of the control problems. Unlike these strategies, the vibrational control technique, introduced by S.M. Meerkov [1], has proven to be a viable alternative to conventional feedback and feedforward strategies in stabilization problems when the outputs, states and disturbances are difficult to access. Mathematical modelling of such systems is closely connected with nonlinear singularly perturbed systems under parametric excitations. In this paper a new asymptotical method based on the periodic solution theory, averaging method and boundary functions method, is presented. Due to it, a vibrational control problem can be investigated. The given example shows the "parasitic" parameters loss in such systems to be extremly dangerous.

1 Introduction

The present paper is at the interface between singular perturbations theory and vibrational control theory. As a rule, for the control engineers, singular perturbations legitimize ad hoc simplifications of dynamic models. One of them is to neglect some of "small" time constants, masses, capacities, and similar "parasitic" parameters which increase the dynamic order of the model. However, the design based on a simplified model may result in a system far from disired performance. In this case, control engineers need a tool which helps to improve the oversimplified design.

The analytic theory of singular perturbations is presented in the monographs [1,2]. Applications of this theory for control system are discussed in the overview of P.V.Kokotovic [3].

Usually, to stabilize a control system the feedback and feedforward principles are used. For these principles to be applied, state coordinates (for feedback) or disturbances (for feedforward) should be measured and an apropriate additive control signal should be introduced. However, for a number of plants the classical methods are not applicable. Vibrational control techique, introduced by S.M. Meerkov [4], has prooved to be a viable alternative to conventional feedback and feedforward strategies in a number of difficult cases. This technique consists in the utilizing appropriately chosen zero mean parametric excitations of a dynamical system to modify the behavior of the original system in the desired manner. Vibrational control has been reported to ensure stabilization of a number of plants which are not easily controllable by feedback.

Further, the Meerkov's approach has been developed by R.E.Bellman, J.Bentsman, S.Meerkov [5,6] for nonlinear dynamic systems.

In recent years the vibrational control principle has beenen extended to systems with time delay [7,8] and to nonlinear parabolic systems [9,10].

In the present paper we consider a family of nonlinear singularly perturbed control systems

$$\frac{dz}{dt} = Z(z, y, \lambda), \quad \frac{dy}{dt} = \Omega Y(z, y, \lambda), \tag{1.1}$$

where $z \in \mathbf{R}^n, y \in \mathbf{R}^m$ are the states, $\lambda \in \mathbf{R}^r$ is a vector parameter, $\Omega \gg 1$ is a large scalar, and t is a dimensionless time. Introducing in (1.1) parametric vibrations according to the law

$$\lambda(t) = \lambda_0 + \varphi(\Omega t), \tag{1.2}$$

where λ_0 is a constant vector and $\varphi(\tau)$ $(\tau = \Omega t)$ is an almost periodic vector function with zero average value. As a result, (1.1) becomes

$$\frac{dz}{dt} = Z(z, y, \lambda_0 + \varphi(\tau)), \quad \frac{dy}{dt} = \Omega Y(z, y, \lambda_0 + \varphi(\tau)). \tag{1.3}$$

Assume that

$$z = P(z, y, \lambda) + \Omega Q(z, \lambda)$$

$$P : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^n, \quad Q : \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^n.$$

First, let us consider the equation

$$\frac{dz}{d\tau} = Q(z, \lambda_0 + \varphi(\tau)). \tag{1.4}$$

We denote the general solution of (1.4) by

$$z = h(c, \tau), \quad h : \mathbf{R}^n \times \mathbf{R} {\to} \mathbf{R}^n, \tag{1.5}$$

where $c \in \mathbf{R}^n$ is a constant uniquely designed for every initial (x_0, τ_0) and propose that (1.5) is almost periodic in $\tau \in [0, \infty)$. Then we make the substitution

$$z = h(x, \tau)$$

which transforms (1.3) into the system

$$\frac{dx}{dt} = f(x, y, \tau), \tag{1.6}$$

$$\frac{dy}{dt} = \Omega F(x, y, \tau), \quad \Omega \gg 1 \tag{1.7}$$

where

$$f = [\frac{\partial h}{\partial x}]^{-1} P(h(x,\tau), y, \lambda_0 + \varphi(\tau)),$$
$$F = Y(h(x,\tau), y, \lambda_0 + \varphi(\tau)).$$

Thus, we have obtained the singularly perturbed system (1.6), (1.7) with high frequency coefficients.

In this paper we suggest the hybrid asymptotical method that allows to construct the solutions of systems (1.6), (1.7) in the interval $[0, T]$ (or even $[0, \infty)$) in the form

$$x_N = u_0(t) + \epsilon[u_1(t) + v_1(t,\tau) + \pi_1(\tau)] + \cdots + \epsilon^N [u_N(t) + v_N(t,\tau) + \pi_N(\tau)] + \epsilon^{N+1}[v_N(t,\tau) + \pi_N(\tau)], \quad (1.8)$$

$$y_N = Y_0(t,\tau) + \Pi_0(\tau) + \epsilon[Y_1(t,\tau) + \Pi_1(\tau)] + \cdots + \epsilon^N [Y_N(t,\tau) + \Pi_N(\tau)] + \epsilon^{N+1} Y_{N+1}(t,\tau), \quad (1.9)$$

where v_i, V_i are the almost periodic functions, and π_i, Π_i are the boundary layer functions. If the functions $f(x,y,\tau), F(x,y,\tau)$ are independent of τ the expansions (1.8), (1.9) coinside with the classical Vasil'eva expansions [2]. If system (1.3) is regular, we get the special averaging expansions [11]. The formulas (1.8), (1.9) highlight a structure of the solutions of system (1.6), (1.7). (1.8), (1.9) are of prime importance for studying of a transient behavior. The analysis in details of (1.8), (1.9) allows to investigate the vibrational stability for nonlinear singularly perturbed systems.

The paper has the following structure. Section 3 discusses preliminaries. Section 4 considers the hybrid asymptotical method. Section 5 presents evaluating of the remainder terms of the expansions (1.8), (1.9). Section 6 discusses two significant special cases. In sections 7, 8 the expansions in an infinite interval $[0, \infty)$ and vibrational stability are studied.

2 Notations

Let $\mathbf{R}^n$ be a space of n- dimensional real vectors with the norm

$$||x|| = \max_{1 \le i \le n} |x_i|,$$

where x_i is the ith coordinate of the vector x and (x, y) is the inner product of vectors x and y.

Let $f : \mathbf{R}^n \Rightarrow \mathbf{R}^n$ be N-times continuously differentiable function in the region $G \subset \mathbf{R}^n$, i.e. $f \in C^N(G, R)$. Let $f^{(k)}(x) = f_x^{(k)}(x)$ denote the kth derivative of f at $x \in G$ The $f^{(k)}$ is a k-multilinear map of $\mathbf{R}^n$ for each $x \in G$; $f^{(k)}(x) \in L(\mathbf{R}^n, \ldots, \mathbf{R}^n)$.

For $\xi_1, \ldots, \xi_k \in \mathbf{R}^n$ and $x \in G$, the value of $f^{(k)}(x)$ at $(\xi_1, \ldots, \xi_k)$ is given by

$$f^{(k)}(x)(\xi_1, \ldots, \xi_k) = \sum \frac{\partial^{(k)} f(x)}{\partial x^{(j1)} \ldots \partial x^{(jk)}} \xi_1^{(j1)} \ldots \xi_k^{(jk)},$$

the summation ranging over $j_1 + \ldots + j_k = k$, $0 \leq j_1, \ldots, j_k \leq n$, and

$$\|f^{(k)}(x)\| = \max_{j_1+\ldots+j_k=k, 0\leq j_1,\ldots,j_k\leq n} \left| \frac{\partial^{(k)} f(x)}{\partial x^{(j1)} \ldots \partial x^{(jk)}} \right|,$$

for all $x \in G$.

In this notation, the Taylor formula for $f \in C^N([0, \varepsilon_0]), \mathbf{R}^n)$ can be written in the form

$$f(\varepsilon) = f(0) + f'(0)\varepsilon + \ldots + \frac{1}{(N-1)!} f^{(N-1)}(0)\varepsilon^{N-1} + \frac{1}{N!} f^{(N)}(\mu)\varepsilon^N,$$

where $\varepsilon \in [0, \varepsilon_0]$, $\mu \in [0, \varepsilon]$.

Now let $F^{(i)}(x, \xi_1, \ldots, \xi_m)(j = 1, \ldots, n)$ be an arbitrary 2π- periodic real analytic function of $\zeta \in \mathbf{R}^m$ for each $x \in G$. Moreover, let $F^{(i)}(x, \xi_1, \ldots, \xi_m)$ be N times continuously differentiable of $x \in G$ for $\zeta \in \mathbf{R}^m$. As $F^N(G, \mathbf{R}^m, \mathbf{R}^n)$ we denote the space $F = (F^{(1)}, \ldots, F^{(N)})$.

Let $\omega \in \mathbf{R}^m$ and $\mathcal{A}_\omega^N(G, \mathbf{R}^n)$ be a class of functions $f(x,t)$ mapping $G \times [0, \infty) \to \mathbf{R}^n$ which satisfy $f(x,t) = F(x, \omega_1 t, \ldots, \omega_m t)$, where F is an arbitrary function of $F^N(G, \mathbf{R}^m, \mathbf{R}^n)$.

Finally, let $M^{m\times m}$ be a space of $(m \times m)$ matrices, and $\mathcal{A}_\omega(M) = \mathcal{A}^N(G, M^{m\times m})$ be a class of $(m \times m)$ real matrices $A(x,t)(x \in G, t \in [0, \infty))$, which are N times continuously differentiable with respect to $x \in G$.

3 Preliminaries

For the following studies we need some preliminary results.

Lemma 3.1 *Let K be a compact set of $\mathbf{R}^n$ and $f \in \mathcal{A}_\omega(K, \mathbf{R}^n)$. Then f is bounded in $K \times [0, \infty)$.*

Lemma 3.2 *If K is a compact set of $\mathbf{R}^n$ and $f, g \in \mathcal{A}_\omega^N(K, \mathbf{R}^n)$, then $f + g, \alpha g \in \mathcal{A}_\omega^N(G, \mathbf{R}^n)$.*

Lemma 3.3 *Let $f \in \mathcal{A}_\omega^j(G, \mathbf{R}^n), \varphi_1, \ldots, \varphi_j \in \mathcal{A}_\omega^0(G, \mathbf{R}^n)$. Then*

$$f^j(x)(\varphi_1, \ldots, \varphi_j) \in \mathcal{A}_\omega^0(G, \mathbf{R}^n).$$

Let $f \in \mathcal{A}^0_\omega(G, \mathbf{R}^n)$ and put

$$M_t[f](x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x,\tau)d\tau,$$

$$K_t[f](x) = f - M_t[f](x),$$

$$I_t[f] = \int_0^t f(x,\tau)d\tau.$$

Lemma 3.4 *The operators*

$$M_t : \mathcal{A}^N_\omega(G, \mathbf{R}^n) \to C^N(G, \mathbf{R}^n),$$

$$K_t : \mathcal{A}^N_\omega(G, \mathbf{R}^n) \to \mathcal{A}^N_\omega(G, \mathbf{R}^n)$$

are defined and bounded.

The following lemma gives a clear to the elements of the space $\mathcal{A}^N_\omega(G, \mathbf{R}^n)$. Let $\mathbf{Z}^m$ be a space of m-dimensional integer vectors.

Lemma 3.5 (13) *Function $f \in \mathcal{A}^N_\omega(G, \mathbf{R}^n)$ if and only if the following conditions are valid:*

$$1. f = \sum_{k\in\mathbf{Z}^m} a_k(x)e^{i(k,\omega)t};$$

$$2. a_k \in C^N(G, \mathbf{R}^n);$$

3. *for all compact set $K \in G$, $j = 0, \ldots, N$ there exist such constants $\alpha > 0, \rho > 0$, that*

$$||a_k^{(j)}(x)|| \leq \alpha\rho^{|k|}(\forall x \in K).$$

Definition 3.1. A vector $\omega \in \mathbf{R}^m$ is called nonresonance if there exist such constants $\gamma, \beta > 0$ that

$$|(k,\omega)| \geq \gamma|k|^{-\beta}$$

for all $k \in \mathbf{Z}^m, k \neq 0$. By $\mathbf{R}^m_{res}$ we denote a set of such vectors [13].

Lemma 3.6 *If $\omega \in \mathbf{R}^m_{res}$ then the operator*

$$I_tK_t : \mathcal{A}^N_\omega(G, \mathbf{R}^n) \to \mathcal{A}^N_\omega(G, \mathbf{R}^n)$$

is defined and bounded.

Finally, consider the linear differential equation

$$\frac{dy}{d\tau} = C(x,\tau)y + h(x,\tau), \tag{3.1}$$

where $h(x,\tau) \in \mathcal{A}^0_\omega(G, \mathbf{R}^n), C(x,\tau) \in \mathcal{A}^N_\omega(G, M^{m\times m})$.

We assume that for any $C > 0$, $\sigma > 0$ the fundamental matrix $V(x,\tau,s)$ of the linear homogeneous system

$$\frac{dy}{d\tau} = C(x,\tau)y$$

satisfies the estimate

$$||V(x,\tau,s)|| \leq Ce^{-\sigma(\tau-s)}, \tag{3.2}$$
$$0 \leq s \leq \tau < \infty,$$

where C, σ do not depend on $x \in G$. By $\mathcal{Y}$ we denote the class of such matrices.

Lemma 3.7 *The equation (3.1) has a unique solution $y(x,\tau) \in \mathcal{A}^0_\omega(G, \mathbf{R}^n)$.*

By $P(\mathbf{R}^m)$ we denote a class of continuous bounded functions $\varphi(\tau)$ with values in $\mathbf{R}^m$, i.e.

$$||\varphi(\tau)|| \leq Ce^{-\kappa\tau} (C > 0, \kappa > 0, 0 \leq \tau \leq \infty)$$

Lemma 3.8 *Let $h(\tau) \in P(\mathbf{R}^m)$ and $C(\tau) \in \mathcal{Y}$. Then the Cauchy problem*

$$\frac{dz}{d\tau} = C(x,\tau)z + h(\tau), z(0) = z_0 \tag{3.3}$$

has a unique solution $z(\tau) \in P(\mathbf{R}^m)$.

If $h = h(x,\tau)$, $C = C(x,\tau)$ are k times continuously differentiable with respect to $x \in K$ (K is a compact), then the function $z(x,\tau)$ has the same smoothness.

To prove this assertion we need to introduce the Banach spaces $X = C^r(K, \mathbf{R}^m)$ and $Y = C^r(K, M^{m\times m})$.

We shall consider $h(x,\tau)$ as a curve

$$h : [0,\infty) \to X$$

in the Banach space X, and denote as $y(x,\tau)$an unknown curve

$$y : [0,\infty) \to X.$$

Then the equation (3.1) can be rewritten as an equation in the Banach space X

$$\frac{dy}{d\tau} = \mathbf{C}(\tau)y + h(\tau), \tag{3.4}$$

where for any fixed $\tau \in [0,\infty)$ the matrix $\mathbf{C} = C(x,\tau)$ is an element of Banach space Y.

Estimate (3.2) yields

Lemma 3.9 *If $C \in \mathcal{A}^r_\omega(K, M^{m\times m})$, $h \in \mathcal{A}^r_\omega(K, \mathbf{R}^m)$, then $y \in \mathcal{A}^r_\omega(K, \mathbf{R}^m)$.*

4 Asymptotics in Finite Interval $[0, T]$

Let $t \in [0,T]$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, G_1 and G_2 be open regions in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively.

Consider the singularly perturbed Cauchy problem

$$\frac{dx}{dt} = f(x,y,\tau), \tag{4.1}$$

$$\varepsilon \frac{dy}{dt} = F(x,y,\tau), \tag{4.2}$$

$$x(0,\varepsilon) = \alpha, \quad y(0,\varepsilon) = \beta, \tag{4.3}$$

where $\tau = t/\varepsilon$ is fast time, $\alpha \in G_1$, $\beta \in G_2$.

We suppose

I. $f \in \mathcal{A}_\omega^{N+2}(G_1 \times G_2, \mathbf{R}^n)$, $F \in \mathcal{A}_\omega^{N+2}(G_1 \times G_2, \mathbf{R}^m)$

II. For every $x \in G_1$ the equation

$$\frac{dy}{dt} = F(x,y,\tau), \tag{4.4}$$

has a unique solution

$$y = \Psi_0(x,\tau) \in \mathcal{A}_\omega^{N+2}(G_1, \mathbf{R}^m), \tag{4.5}$$

moreover, the values $\Psi_0(x,\tau) \in G_2$.

By $C(x,\tau)$ we denote the matrix $F_y(x, \varphi_0(x,\tau), \tau)$.

Let $V(x,\tau,s)$ be the fundamental matrix of the linear homogeneous system

$$\frac{dy}{d\tau} = C(x,\tau)y. \tag{4.6}$$

III. For some $C > 0$, $\sigma > 0$ the uniform estimate

$$||V(x,\tau,s)|| \le Ce^{-\sigma(\tau - s)} (0 \le s \le \tau < \infty) \tag{4.7}$$

is valid, moreover constants C, σ are independent on $x \in G_1$.

We will seek the solution of the problem (1)-(3) in the interval $[0,T]$ in the form

$$x_N = u(t,\varepsilon) + v(t,\tau,\varepsilon) + \pi(\tau,\varepsilon), \tag{4.8}$$

$$y_N = Y(t,\tau,\varepsilon) + \Pi(\tau,\varepsilon), \tag{4.9}$$

where

$$u(\tau,\varepsilon) = u_0(t) + \varepsilon u_1(t) + \ldots + \varepsilon^N u_N(t), 0 \le t \le T$$

is a regular serie ;

$$v(t,\tau,\varepsilon) = \varepsilon v_1(t,\tau) + \ldots + \varepsilon^{N+1} v_{N+1}(t,\tau)$$

is a vibrational serie, $v_i \in \mathcal{A}_\omega^{N+2-i}([0,T],\mathbf{R}^m)$ with $M_t[v_i]=0$ for $i=1,2,\ldots,N+1$;

$$Y(t,\tau,\varepsilon)=Y_0(t,\tau)+\varepsilon Y_1(t,\tau)+\ldots+\varepsilon^{N+1}Y_{N+1}(t,\tau)$$

and $Y_i(t,\tau)\in\mathcal{A}_\omega^{N+2-i}([0,T],\mathbf{R}^m)$; finally

$$\pi(\tau,\varepsilon)=\varepsilon\pi_1(\tau)+\ldots+\varepsilon^N\pi_{N+1}(\tau),$$

$$\Pi(\tau,\varepsilon)=\Pi_0(\tau)+\varepsilon\Pi_1(\tau)+\ldots+\varepsilon^N\Pi_{N+1}(\tau)$$

are bounded functions, i.e. $||\pi_i||, ||\Pi_i||\le Ce^{-\kappa\tau}$ $(i=1,\ldots,N+1,\quad j=0,\ldots,N,\tau\in[0,\infty))$ for some $C>0$, $\kappa>0$.

Further, like in [2], introduce the functions $\overline{f}$, $\overline{F}$, Πf, ΠF putting

$$\overline{f}(t,\tau,\varepsilon)=f\left(u(t,\varepsilon)+v(t,\tau,\varepsilon),Y(t,\tau,\varepsilon),\tau\right),$$

$$\overline{F}(t,\tau,\varepsilon)=F\left(u(t,\varepsilon)+v(t,\tau,\varepsilon),Y(t,\tau,\varepsilon),\tau\right),$$

$$\Pi f(t,\tau,\varepsilon)=f\left(u(t,\varepsilon)+v(t,\tau,\varepsilon)+\pi(\tau,\varepsilon),Y(\varepsilon\tau,\tau,\varepsilon)+\Pi(\tau,\varepsilon),\tau\right)-\overline{f}(\varepsilon\tau,\tau,\varepsilon),$$

$$\Pi F(\tau,\varepsilon)=F\left(u(\varepsilon\tau,\varepsilon)+v(\varepsilon\tau,\tau,\varepsilon)+\pi(\tau,\varepsilon),Y(\varepsilon\tau,\tau,\varepsilon)+\Pi(\tau,\varepsilon),\tau\right)-\overline{F}(\varepsilon\tau,\tau,\varepsilon),$$

Now substituting (4.8), (4.9) into (4.1), (4.2) we obtain

$$\frac{du}{dt}+\frac{\partial v}{\partial t}+\frac{1}{\varepsilon}\frac{\partial v}{\partial\tau}+\frac{1}{\varepsilon}\frac{d\pi}{d\tau} = f(u+v+\pi,Y+\Pi,\tau), \tag{4.10}$$

$$\varepsilon\frac{\partial Y}{\partial t}+\frac{\partial v}{\partial\tau}+\frac{d\Pi}{d\tau} = F(u+v+\pi,Y+\Pi,\tau). \tag{4.11}$$

Rewrite (4.10), (4.11) in the form

$$\frac{du}{dt}+\frac{\partial v}{\partial t}+\frac{1}{\varepsilon}\frac{\partial v}{\partial\tau}+\frac{1}{\varepsilon}\frac{d\pi}{d\tau} = \overline{f}+\Pi F, \tag{4.12}$$

$$\varepsilon\frac{\partial Y}{\partial t}+\frac{\partial v}{\partial\tau}+\frac{d\Pi}{d\tau} = (\overline{F}+\Pi F). \tag{4.13}$$

Expanding $\overline{f}$, $\overline{F}$, Πf, ΠF in power serie of ε, we obtain

$$\begin{aligned}\overline{f} &= \overline{f}_0+\varepsilon\overline{f}_1+\ldots,\\ \overline{F} &= \overline{F}_0+\varepsilon\overline{F}_1+\ldots,\\ \Pi f &= \Pi f_0+\varepsilon\Pi f_1+\ldots,\\ \Pi F &= \Pi F_0+\varepsilon\Pi F_1+\ldots.\end{aligned}$$

Using the operators M_τ, K_τ we separate the regular and the vibrational components of $\overline{f}$

$$\overline{f} = M_\tau[\overline{f}_0] + \varepsilon M_\tau[\overline{f}_1] + \ldots + K_\tau[\overline{f}_0] + \varepsilon K_\tau[\overline{f}_1] + \ldots,$$
$$\overline{F} = M_\tau[\overline{F}_0] + \varepsilon M_\tau[\overline{F}_1] + \ldots + K_\tau[\overline{F}_0] \varepsilon K_\tau[\overline{F}_1] + \ldots.$$

Collecting the terms with the same power of ε in a special manner and, finally, equating the coefficients of the same types to each other, we obtain the equations for u_i, v_i, τ_i, Y_i, Π_i

$$\frac{du_i}{dt} = M_\tau[\overline{f}_i], \tag{4.14}$$
$$\frac{\partial v_i}{\partial \tau} = K_\tau[\overline{f}_i] - \frac{\partial v_i}{\partial t}, \tag{4.15}$$
$$\frac{d\pi_{i+1}}{d\tau} = \Pi_i f, \tag{4.16}$$
$$\frac{\partial Y_i}{\partial \tau} = K_\tau[\overline{F}_i] - \frac{\partial Y_{i-1}}{\partial t}, \tag{4.17}$$
$$\frac{d\Pi_i}{d\tau} = \Pi_i F. \tag{4.18}$$

Let us consider these equations for principal case $i = 0$ in details

$$\frac{du_0}{dt} = M_\tau[f(u_0(t), Y_0(t,\tau), \tau)], \tag{4.19}$$
$$\frac{\partial v_1}{d\tau} = K_\tau[f(u_0(t), Y_0(t,\tau), \tau)], \tag{4.20}$$
$$\frac{d\pi_1}{d\tau} = f(u_0(0), Y_0(0,\tau) + \Pi_0(\tau), \tau) - f(u_0(0), Y_0(0,\tau), \tau). \tag{4.21}$$
$$\frac{\partial Y_0}{\partial \tau} = F(u_0(t), Y_0, \tau) \tag{4.22}$$
$$\frac{d\Pi_0}{d\tau} = F(u_0(0), Y_0(0,\tau) + \Pi_0(\tau), \tau) - F(u_0(0), Y_0(0,\tau), \tau). \tag{4.23}$$

Recalling that $M_\tau = 0$, $\pi_1(\infty) = 0$, $u_0(0) = \alpha$, $Y_0(0,0) + \Pi_0(0) = \beta$ we find u_0, v_0, π_1, Y_0, Π_0 in the following way.

i) From the equation (4.22) and the condition II we have

$$Y_0(t,\tau) = \Psi_0(u_0(t), \tau), \tag{4.24}$$

where $u_0(t)$ is an unknown yet function.

ii) Substituting Y_0 into the equation (4.19), we get the Cauchy proplem for $u_0(t)$

$$\frac{du_0}{dt} = f_0(u_0), u_0(0) = \alpha,$$

where

$$f_0(x) = M_\tau[f(x, \Psi_0(x,\tau), \tau)].$$

IV We assume that this Cauchy problem has a unique solution in the interval $[0, T]$ and $u_0 \in G_1$ for all $t \in [0, T]$. S_0, u_0 and Y_0 are defined.

iii) Recalling $M_\tau[v_0] = 0$, from (4.20) we have

$$v_1 = K_\tau I_\tau K_\tau[f(u_0(t), \Psi_0(t,\tau), \tau)].$$

iv) The equation

$$\frac{dZ}{d\tau} = F(\alpha, \psi_0(\alpha,\tau) + Z, \tau) - F(\alpha, \psi_0(\alpha,\tau), \tau) \tag{4.25}$$

has a zero steady state $Z \equiv 0$. The condition III implies that $Z \equiv 0$ is a stable steady state.

V. We suppose that the point $\beta - \Psi_0(\alpha, 0)$ belongs to the attraction region D of zero steady state.

Now we define $\Pi_0 (0 \le \tau < \infty)$ as the solution of equation (4.25) with the initial condition $Z(0) = \beta - \Psi_0(\alpha, 0)$.

VI. We assume that

$$\Psi_0 + \Pi_0 \in G_2(0 \le t \le T, 0 \le \tau < \infty).$$

v) From (4.21) we have

$$\pi_1 = -\int_\tau^\infty \{f(\alpha, Y_0(\alpha, s) + \Pi_0(s) - f(\alpha, Y_0(\alpha, s), s)\} ds.$$

Thus u_0,v_1,π_1,Y_0,Π_0 are defined completely.

Now let us find the next terms u_1,v_2,π_2,Y_1,Π_1. It is easy to see that for the case $i = 1$ the equation (4.17) is linear with respect to Y_1 and u_1 and has a form

$$\frac{\partial Y_1}{\partial \tau} = C_0(t,\tau) Y_1 + h_1(t,\tau) + H_1(t,\tau) u_1(t).$$

Using the Lemma 3.7 we can determine Y_1 in the form

$$Y_1 = \Phi_1(t,\tau) u_1 + \Psi_1(t,\tau).$$

Substituting Y_1 into the equation (*) for the case $i = 1$ we get the linear Cauchy problem to define u_1

$$\frac{du_1}{dt} = A_1(t) u_1 + b_1(t), u_1(0) = 0.$$

Thus u_1 and Y_1 are completely defined.

Further, v_2 is found due to the operators K_τ, I_τ. It is easy to note that the equation for Π_1 has a form

$$\frac{d\Pi}{d\tau} = C(\tau)\Pi_1 + h(\tau),$$

where $C \in \mathcal{Y}$ and $h \in P(\mathbf{R}^m)$. Hence, $\Pi_1 \in P(\mathbf{R}^m)$. Finally, π_2 is defined like π_1. The terms u_k,v_{k+1},π_{k+1}, Y_k,Π_k $(k > 2)$ can be found in a similar way.

Thus x_N, y_N satisfy the relations

$$\begin{aligned}\frac{dx_N}{dt} &= f(x_N, y_N, \tau) + \varepsilon^{N+1}\mu_N(t,\varepsilon),\\ \varepsilon\frac{dy_N}{dt} &= F(x_N, y_N, \tau) + \varepsilon^{N+1}\nu_N(t,\varepsilon),\end{aligned}$$

and, moreover, for any $\varepsilon_0 > 0$

$$\max_{0\le t\le\tau} ||\mu_N(\tau,\varepsilon)|| \le C, \quad \max_{0\le t\le\tau} ||\nu_N(\tau,\varepsilon)|| \le C,$$

where C is independent on $\varepsilon \in (0, \varepsilon_0]$.

5 Estimating of Remainder Terms by Vasil'eva Technique

To estimate the remainder terms

$$r_N = x - x_N, R_N = y - y_N$$

(see (4.8),(4.9)) we change of variables in the Cauchy problem (4.1)-(4.3):

$$x = r + x_N, y = R + y_N. \tag{5.1}$$

Then for the new variables r, R we get the Cauchy problem

$$\frac{dr}{dt} = f(x_N + r, y_N + R, \tau) - \varepsilon\frac{dx_N}{dt}, \tag{5.2}$$

$$\varepsilon\,\frac{dR}{dt} = F(x_N + r, y_N + R, \tau) - \varepsilon\frac{dy_N}{dt}, \tag{5.3}$$

$$r(o,\varepsilon) = O(\varepsilon^{N+1}), R(o,\varepsilon) = O(\varepsilon^{N+1}) \tag{5.4}$$

Our goal is to prove the existence of such constants $C > 0$, $\varepsilon_0 > 0$ that for all $\varepsilon \in (0, \varepsilon_0]$ the estimate

$$||r(t,\varepsilon)||_{C[0,T]} + ||R(t,\varepsilon)||_{C[0,T]} \le C\varepsilon^{N+1} \tag{5.5}$$

is true.

Introduce the following matrices

$$A(t,\tau) = f_x(u_0(t), Y_0(t,\tau) + \Pi_0(\tau), \tau), \tag{5.6}$$

$$B(t,\tau) = f_y(u_0(t), Y_0(t,\tau) + \Pi_0(\tau), \tau), \tag{5.7}$$

$$C(t,\tau) = F_x(u_0(t), Y_0(t,\tau) + \Pi_0(\tau), \tau), \tag{5.8}$$

$$D(t,\tau) = F_y(u_0(t), Y_0(t,\tau) + \Pi_0(\tau), \tau). \tag{5.9}$$

Linearizing the system (5.2),(5.3) at the point

$$x = u_0(t), y = Y_0(t,\tau) + \Pi_0(\tau)$$

we obtain the following nonlinear system

$$\frac{dr}{dt} = A(t,\tau)r + B(t,\tau)R + G(r,R,t,\tau,\varepsilon), \quad (5.10)$$

$$\varepsilon\frac{dR}{dt} = C(t,\tau)r + D(t,\tau)R + H(r,R,t,\tau,\varepsilon), \quad (5.11)$$

where

$$G = f(x_N + r, y_N + R, \tau) - \frac{dx_N}{dt} - f_x(u_0, Y_0 + \Pi_0, \tau)r - f_y(u_0, Y_0 + \Pi_0, \tau)R, \quad (5.12)$$

$$H = F(x_N + r, y_N + R, \tau) - \varepsilon\frac{dy_N}{dt} - F_x(u_0, Y_0 + \Pi_0, \tau)r - F_y(u_0, Y_0 + \Pi_0, \tau)R. \quad (5.13)$$

Obviously, the nonlinear functions G, H have two principal properties.

1^0. There exist such small $\varepsilon_0 > 0$ and $C > 0$ that for all $\varepsilon \in (0, \varepsilon_0]$

$$||G(0,0,t,\tau,\varepsilon)||_{C[0,T]} + ||H(0,0,t,\tau,\varepsilon)||_{C[0,T]} \le C\varepsilon^{N+1}.$$

2^0. For any small $\Delta > 0$ there exist such numbers $\delta > 0$,$\varepsilon_0 > 0$ that for

$$||u_1||, ||u_2||, ||v_1||, ||v_2|| \le \delta, 0 < \varepsilon \le \varepsilon_0$$

the inequalities

$$||G(u_1,v_1,t,\tau,\varepsilon) - G(u_2,v_2,t,\tau,\varepsilon)||_{C[0,T]} \le \Delta(||u_1 - u_2|| + ||u_1 - u_2||) \quad (5.14)$$

$$||H(u_1,v_1,t,\tau,\varepsilon) - H(u_2,v_2,t,\tau,\varepsilon)||_{C[0,T]} \le \Delta(||u_1 - u_2|| + ||u_1 - u_2||) \quad (5.15)$$

are valid.

The properties 1^0, 2^0 make it possible to prove the contraction mapping principle for the corresponding integral operator.

Without loss of generality we can believe that $r(0,\varepsilon) = 0$, $R(0,\varepsilon) = 0$. Then the Cauchy problem (5.2)-(5.3) is equivalent to integral equations system

$$r = \int_0^t V(t,s,\varepsilon)[B(s,s/\varepsilon)R(s) + G(r(s),R(s),s,s/\varepsilon,\varepsilon)]ds, \quad (5.16)$$

$$R = \frac{1}{\varepsilon}\int_0^t U(t,s,\varepsilon)[C(s,s/\varepsilon)r(s) + G(r(s),R(s),s,s/\varepsilon,\varepsilon)]ds, \quad (5.17)$$

where $V(t,s,\varepsilon)$ and $U(t,s,\varepsilon)$ are fundamental matrices of nonlinear homogeneous systems

$$\frac{dV}{dt} = A(t,\tau)V, \quad V(s,s,\varepsilon) = I, \quad (5.18)$$

$$\frac{dU}{dt} = D(t,\tau)U, \quad U(s,s,\varepsilon) = I \quad (5.19)$$

respectively. Obviously, the matrix $V(t,s,\varepsilon)$ is uniformly bounded. For the matrix $U(t,s,\varepsilon)$ it is possible to prove the estimate

$$||U(t,s,\varepsilon)|| \leq Ce^{-\kappa(t-s)/\varepsilon} \quad (C>0, \kappa>0, 0\leq s\leq t\leq T). \tag{5.20}$$

The constants C, κ are independent on $\varepsilon \in (0,\varepsilon_0]$.

The next two lemmae yield the estimate (5.20).

Lemma 5.1 *Let for all $t\in[0,T]$ the matrix $D_0(t,\tau)$ be a Hurwitz matrix of the class Y, moreover for the fundamental matrix $U_0(t,\tau,s,\varepsilon)$ (t is fixed) of linear homogeneous system*

$$\frac{dU_0}{d\tau} = D_0(t,\tau)U_0, \quad U_0(t,s,s,\varepsilon) = I$$

the estimate

$$||U_0(t,\tau,s,\varepsilon)|| \leq Ce^{-4\sigma(\tau-s)/\varepsilon} \quad (C>1, 0\leq s\leq \tau<\infty)$$

hold. Then for fundamental matrix $W_0(t,s,\varepsilon)$ of the linear homogeneous system

$$\varepsilon\frac{dW_0}{dt} = D(t,\frac{t}{\varepsilon})W_0, \quad 0\leq s\leq t\leq T$$

the estimate

$$||W_0(t,\tau,s,\varepsilon)|| \leq Me^{-\sigma(t-s)/\varepsilon} \quad (M>0, \sigma>0, 0\leq s\leq t\leq T) \tag{5.21}$$

is valid.

Lemma 5.2 *Let $D_0(t,\tau)$ be a matrix from Lemma 5.1 and matrix $D_1(t,\tau)$ satisfies the estimate*

$$||D_1(t,\tau)|| \leq M_1e^{-\sigma_1 t/\varepsilon}(M_1>0, \sigma_1>0, 0\leq t\leq T).$$

Then for fundamental matrix $W(t,s,\varepsilon)$ of the linear homogeneous system

$$\frac{dW}{dt} = (D_0(t,\tau)+D_1(t,\tau))W, \quad W(s,s,\varepsilon) = I$$

the estimate of type (5.21) is true.

The proves of Lemmae 5.1, 5.2 will be given below.

The matrix $D(t,\tau)$ from (5.11) can be presented in the form

$$D = D_0 + D_1,$$

where

$$D_0 = F_x(u_0(t), Y_0(t,\tau)),$$

$$D_1 = F_x(u_0(t), Y_0(t,\tau) + \Pi_0(\tau)) - F_x(u_0(t), Y_0(t,\tau)),$$

moreover, it is obvious that

$$||D_1|| \le M_1 e^{-\kappa_1 t/\varepsilon}.$$

Hence, Lemma 5.1 and Lemma 5.2 imply the estimate (5.20) for the fundamental matrix $U(t,s,\varepsilon)$ of the linear homogeneous system (5.19).

Now reform the integral equation system (5.16), (5.17). Substituting $r(s)$ from (5.16) into the integral

$$\frac{1}{\varepsilon}\int_0^t U(t,s,\varepsilon)C(s,\frac{s}{\varepsilon})u(s)ds,$$

we get new integral equations system for r and R.

$$r = \int_0^t V(t,s,\varepsilon)B(s,\frac{s}{\varepsilon})R(s)ds + \mathcal{Q}(r,R,\varepsilon), \tag{5.22}$$

$$R = \int_0^t \mathcal{K}(t,s,\varepsilon)R(s)ds + \mathcal{P}(r,R,\varepsilon), \tag{5.23}$$

where

$$\begin{aligned}
\mathcal{K}(t,s,\varepsilon) &= \frac{1}{\varepsilon}\int_0^t V(t,p,\varepsilon)C(p,\frac{p}{\varepsilon})V(p,s,\varepsilon)ds + D(s,\frac{s}{\varepsilon}),\\
\mathcal{Q}(r,R,\varepsilon) &= \int_0^t V(t,s,\varepsilon)G(r(s),R(s),\frac{s}{\varepsilon},\varepsilon)ds\\
\mathcal{P}(r,R,\varepsilon) &= \frac{1}{\varepsilon}\int_0^t U(t,s,\varepsilon)G(r(s),R(s),\frac{s}{\varepsilon},\varepsilon)ds+\\
&\quad \frac{1}{\varepsilon}\int_0^t\int_s^t U(t,p,\varepsilon)V(p,s,\varepsilon)C(p,\frac{p}{\varepsilon})H(r(s),R(s),\frac{s}{\varepsilon},\varepsilon)dpds
\end{aligned}$$

The estimate (5.20) implies that the kernel $\mathcal{K}(t,s,\varepsilon)$ of the Volterra operator

$$K\cdot R = \int_0^t \mathcal{K}(t,s,\varepsilon)R(s)ds$$

is bounded, and small nonlinear integral opertors $\mathcal{Q}$ and $\mathcal{P}$ have the properties 1^0 and 2^0, as do the operators G and H.

Now let us reform the Volterra integral equation (5.23). We denote by $\mathcal{R}(t,s,\varepsilon)$ the bounded resolvent of the kernel $\mathcal{K}(t,s,\varepsilon)$. Equation (5.23) can be rewritten in the following equivalent form

$$R = \mathcal{P}^*(r,R,\varepsilon), \tag{5.24}$$

where

$$\mathcal{P}^*(r,R,\varepsilon) = \mathcal{P}(r,R,\varepsilon) + \int_0^t \mathcal{R}(t,s,\varepsilon)\mathcal{P}(r(s),R(s),s,\varepsilon)ds \tag{5.25}$$

Putting

$$\mathcal{Q}^* = \int_0^t V(t,s,\varepsilon)B(s,\frac{s}{\varepsilon})R(s)ds + \mathcal{Q}(r,R,\varepsilon)$$

we obtain for $r \in C([0,T],\mathbf{R}^n)$, $R \in C([0,T],\mathbf{R}^m)$ the system of equations

$$r = \mathcal{Q}^*(r,R,\varepsilon), R = \mathcal{P}^*(r,R,\varepsilon). \tag{5.26}$$

We consider the equations (5.26) in Banach space

$$C_{n\times m}[0,T] = C([0,T],\mathbf{R}^n) \times C([0,T],\mathbf{R}^m)$$

with the vector norm

$$||(r,R)|| = (||r||,||R||)^T \in \mathcal{R}^2$$

We write the equation (5.26) in the form

$$(r,R) = S(r,R,\varepsilon), \tag{5.27}$$

where

$$S(r,R,\varepsilon) = \begin{pmatrix} \mathcal{Q}^*(r,R,\varepsilon) \\ \mathcal{P}^*(r,R,\varepsilon) \end{pmatrix}.$$

Due to 1^0, 2^0 for $\mathcal{Q}^*$, $\mathcal{P}^*$ it is easy to establish

$$||S(r_1,R_1,\varepsilon) - S(r_2,R_2,\varepsilon)|| \le \begin{pmatrix} \Delta_1 & a \\ \Delta_2 & \Delta_3 \end{pmatrix} \begin{pmatrix} ||r_1 - r_2|| \\ ||R_1 - R_2|| \end{pmatrix},$$

where D_1,D_2,D_3 are small numbers if

$$||r_1||, ||r_2||, ||R_1||, ||R_2|| \le \delta$$

and δ is a sufficiently small number.

Obviously, for small Δ_i the spectral radius ρ of the matrix

$$\begin{pmatrix} \Delta_1 & a \\ \Delta_2 & \Delta_3 \end{pmatrix}$$

is less then 1. Therefore, S is a contraction operator in a small ball

$$B(q) = \{(r,R) : ||r||_{C[0,T]} \le q, ||R||_{C[0,T]} \le q\}$$

(q is a sufficiently small number). Simple check shows that for $q = C\varepsilon^{N+1}$

$$S : B(q) \to B(q)$$

and S has a unique fixed point $(r^*,R^*) \in B(q)$ (see [14]).

Thus we prove

Theorem 5.1 *Let for any ω the right hand parts f, F of the system (4.1), (4.2) satisfy the conditions I-VI. Then there exist such constants $C > 0$ and $\varepsilon_0 > 0$ that for all $\varepsilon \in (0,\varepsilon_0]$ the Cauchy problem (4.1) has a unique solution $x \in G_1$, $y \in G_2$, for $t \in [0,T]$ and the inequiality*

$$||x - x_N||_{C[0,T]} + ||y - x_N||_{C[0,T]} \le C\varepsilon^{N+1}$$

is true.

In conclusion let discuss the proof of Lemma 5.1.

Fixing any number $q \in (0,1)$ and take such integer number n that for all $t_1, t_2 \in [0,T]$, $|t_1 - t_2| \leq T/n$ we have

$$||D_0(t_1,\tau) - D_0(t_2,\tau)|| \leq \frac{q\sigma}{2C^2}. \tag{5.28}$$

In the interval $[0,T]$ we choose the uniform grid $\Delta_n : t_i = iT/n, i = 0,1,\ldots,n-1$. By $B_0(t,\tau)$ we denote the piecewise constant with respect to $t \in [0,T]$ matrix

$$B_0(t,\tau) = D_0(t_j,\tau)(t \in [t_j, t_{j+1}), j = 0,1,\ldots,n-1).$$

Let $W_0(t,s,\varepsilon)$ be a fundamental matrix of the auxiliary linear homogeneous system

$$\varepsilon\frac{dW_0}{dt} = B_0(t,\tau)W_0, W_(s,s,\varepsilon) = I.$$

First, we estimate $||W_0(t,s,\varepsilon)||$. Let s be an arbitrary number of $[0,T]$ and let $s \in I_j = [t_j, t_{j+1})$.

a) For $t \in I_j$ we have

$$\varepsilon\frac{dW_0}{dt} = D_0(t,t/\varepsilon)W_0.$$

Hence, $W_0 = U_0(t_j, t/\varepsilon, s/\varepsilon)$, and thus

$$||W_0(t,s,\varepsilon)|| = ||U_0(t_j,t/\varepsilon,s/\varepsilon)|| \leq Ce^{-4\sigma(t-s)/\varepsilon}. \tag{5.29}$$

b) For $t \in I_{j+1}$ we have

$$W_0(t,s,\varepsilon) = U_0(t_{j+1},t/\varepsilon,t_{j+1}/\varepsilon) \cdot U_0(t_{j+1},t_{j+1}/\varepsilon,s/\varepsilon)$$

and, therefore

$$||W_0(t,s,\varepsilon)|| \leq Ce^{-4\sigma(t-s)/\varepsilon}.$$

c) By induction it is easy to show that for $t \in I_{j+k+1}$

$$||W_0(t,s,\varepsilon)|| \leq C^{k+2}e^{\frac{-2k\sigma T}{n\varepsilon}} \cdot e^{-2\sigma(t-s)/\varepsilon} \tag{5.30}$$

Consider the sequence of numbers

$$C^2(C^k e^{\frac{-2k\sigma T}{n\varepsilon}}), k = 1,2,\ldots. \tag{5.31}$$

Obviously, for $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0 = \frac{2\sigma T}{n\ln C}$, $C > 1$, we have

$$Ce^{\frac{-2k\sigma T}{n\varepsilon}} < 1$$

and the sequence (5.31) is bounded by C^2.

Hence,

$$||W_0(t,s,\varepsilon)|| \le C^2 e^{-2\sigma(t-s)/\varepsilon}, \qquad (5.32)$$
$$0 \le s \le t \le T.$$

To complete the proof of Lemma 5.1 we rewrite the equation

$$\varepsilon \frac{dW_0}{dt} = D_0(t,t/\varepsilon)W_0, \quad W(s,s,\varepsilon) = I$$

in the form

$$\varepsilon \frac{dW}{dt} = B_0(t,t/\varepsilon)W + [D_0(t,t/\varepsilon) - B_0(t,t/\varepsilon)]W. \qquad (5.33)$$

For $0 \le s \le t \le T$ from (5.33) we have

$$W(t,s,\varepsilon) = W_0(t,s,\varepsilon) + \frac{1}{\varepsilon}\int_0^t W_0(t,q,\varepsilon)[D_0(q,q/\varepsilon) - B_0(q,q/\varepsilon)]W(q,s,\varepsilon)dq. \quad (5.34)$$

Estimating the left-hand and right-hand parts of (5.22) and (5.23) by norm, we obtain

$$||W_0(t,s,\varepsilon)|| \le C^2 e^{-2\sigma(t-s)/\varepsilon} + \frac{q\sigma}{2\varepsilon}\int_0^t e^{-2\sigma(t-q)/\varepsilon}||W_0(t,q,\varepsilon)||dq. \qquad (5.35)$$

Now introduce the scalar function

$$\omega(t,s,\varepsilon) = e^{\sigma(t-s)/\varepsilon}||W_0(t,s,\varepsilon)||$$

and the number

$$\omega^* = \max_{0\le s\le t\le T} e^{\sigma(t-s)/\varepsilon}||W_0(t,s,\varepsilon)||.$$

Multiplying (5.35) by $e^{\sigma(t-s)/\varepsilon}$, it is easy to get

$$\omega(t,s,\varepsilon) \le C^2 e^{-\sigma(t-s)/\varepsilon} + \frac{q\sigma\omega^*}{2}\int_s^t d(e^{-\sigma(t-q)/\varepsilon}) \le C^2 + q\omega^*.$$

The last relation implies that $\omega^* \le C^2 + q\omega^*$ and hence $\omega^* \le C^2/(1-q)$.

This proves Lemma 5.1.

To prove Lemma 5.2 we denote the fundamental matrix W of the system

$$\frac{dW}{dt} = (D_0(t,\tau) + D_1(t,\tau))W, \quad W(s,s,\varepsilon) = I$$

is a solution of the integral equation

$$W(s,s,\varepsilon) = W_0(s,s,\varepsilon) + \int_s^t \frac{1}{\varepsilon}W(s,p,\varepsilon)D_1(p,p/\varepsilon)W(p,s,\varepsilon)dp,$$

where W_0 is a fundamental matrix of the system

$$\varepsilon\frac{dW_0}{dt} = D_0(t,\tau)W_0, \quad W_0(s,s,\varepsilon) = I.$$

Now the estimate of type (5.21) can be easily obtained by the succesive approximations method.

6 Two Significant Special Cases

6.1 Quasilinear problems

Consider the Cauchy problem

$$\frac{dx}{dt} = f(x, y, \tau), \tag{6.1}$$

$$\varepsilon \frac{dx}{dt} = C(x)y + D(x, \tau) = F(x, y, \tau), \tag{6.2}$$

$$x(0, \varepsilon) = \alpha, \qquad y(0, \varepsilon) = \beta, \tag{6.3}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, ε is a small positive parameter, $t \in [0, T]$, $\tau = t/\varepsilon$, $\alpha \in G_1$, $\beta \in G_2$.

Let ω be a nonresonance vector and

$$f \in \mathcal{A}_\omega^{N+2}(G_1 \times G_2; \mathbf{R}^n),$$

$$C \in C^{N+2}(G_1; M^{m\times m}),$$

$$D \in \mathcal{A}_\omega^{N+2}(G_2; \mathbf{R}^m),$$

moreover, for the eigenvalues $\lambda_i = \lambda_i[C(x)]$ of the matrix $C(x)$ the estimate

$$\sup_{1\le i\le m, x\in G} \Re\lambda_i[C(x)] < 0$$

holds. For the problem (6.1)-(6.3) the solution is convenient to be sought in the form

$$x_N = u(t, \varepsilon) + v(t, \tau, \varepsilon) + \pi(\tau, \varepsilon), \tag{6.4}$$

$$y_N = U(t, \varepsilon) + V(t, \tau, \varepsilon) + \Pi(\tau, \varepsilon), \tag{6.5}$$

where

$$u(t, \varepsilon) = u_0(t) + \varepsilon u_1(t) + \ldots + \varepsilon^N u_N(t), \tag{6.6}$$

$$U(t, \varepsilon) = U_0(t) + \varepsilon U_1(t) + \ldots + \varepsilon^N U_N(t), \tag{6.7}$$

are regular serie;

$$v(t, \tau, \varepsilon) = \varepsilon v_1(t, \tau) + \ldots + \varepsilon^{N+1} v_{N+1}(t, \tau),$$

$$V(t, \tau, \varepsilon) = V_0(t, \tau, \varepsilon) + \varepsilon V_1(t, \tau, \varepsilon) + \ldots + \varepsilon^N V_N(t, \tau, \varepsilon)$$

1 are vibration serie, in addition, $v_i(t, \tau)$, $V_i(t, \tau)$ are quasiperiodic functions with $M_\tau[v_i] = 0$, $M_\tau[V_j] = 0$ for $i = 1, \ldots, N$, $j = 1, \ldots, N+1$;

finally,

$$\pi(\tau, \varepsilon) = \varepsilon\pi_1(\tau) + \ldots + \varepsilon^{N+1}\pi_{N+1}(\tau),$$

$$\Pi(\tau, \varepsilon) = \Pi_0(\tau) + \varepsilon\Pi_1(\tau) + \ldots + \varepsilon^N \Pi_N(\tau)$$

are the boundary layer functions.

It is important to note that if f and F are independent on τ the expansions (6.4), (6.5) do not have terms $v(t,\tau,\varepsilon)$, $V(t,\tau,\varepsilon)$ and the expantions (6.4), (6.5) coinside with the classic Vasil'eva expansions [2]. Repeating arguments of Section 4 for finding u_i, U_i, v_i, V_i, π_i, Π_i we obtain the following relations

$$\begin{aligned}
\frac{du_i}{dt} &= M_\tau[\overline{f}_i], \\
\frac{\partial v_i}{\partial \tau} &= K_\tau[\overline{f}_i] - \frac{\partial v_i}{\partial t}, \\
\frac{d\pi_{i+1}}{d\tau} &= \Pi_i f, \\
M_\tau[\overline{F}_i] &- \frac{dV_{i-1}}{dt} = 0, \\
\frac{\partial V_i}{\partial \tau} &= K_\tau[\overline{F}_i] - \frac{\partial V_{i-1}}{\partial t}, \\
\frac{d\Pi_i}{d\tau} &= \Pi_i F.
\end{aligned}$$

Let us consider carefully these relations for $i = 0$

$$\frac{du_0}{dt} = M_\tau[f(u_0(t)), U_0(t) + V_0(t,\tau),\tau)], \tag{6.8}$$

$$\frac{\partial v_1}{\partial \tau} = K_\tau[f(u_0), U_0(t) + V_0(t,\tau),\tau], \tag{6.9}$$

$$\begin{aligned}
\frac{d\pi_1}{d\tau} = {} & f(\alpha, U_0(0) + V(0,\tau) + \Pi_0(\tau),\tau) - \\
& f(\alpha, U_0(0) + V(0,\tau),\tau),
\end{aligned} \tag{6.10}$$

$$C(u_0(t))U_0(t) + M_\tau[D(u_0(t),\tau)] = 0, \tag{6.11}$$

$$\frac{\partial V_0}{\partial \tau} = C(u_0(t))V_0(t,\tau) + K_\tau[D(u_0(t),\tau)], \tag{6.12}$$

$$\frac{d\Pi_0}{d\tau} = C(\alpha)\Pi_0. \tag{6.13}$$

Recalling that $M_\tau[V_0] = 0$, $M_\tau[V_1] = 0$, $\pi_1(\infty) = 0$, $u_0(0) = \alpha$, $\Pi_0(0) = \beta - U_0(0) - V_0(0)$ we seek u_0, v_1 π_1, V_0, Π_0 in the following way.

i) Put

$$Q(x) = -C^{-1}(x)M_\tau[D(x,\tau)]$$

and define by the Fourier technique the guasiperiodic in τ solution $W(x,\tau)$ of the problem

$$\begin{aligned}
\frac{dW}{d\tau} &= C(x)W + K_\tau[D(x,\tau)], \\
M_\tau[W] &= 0.
\end{aligned}$$

Further we suppose that for all $x \in G_1$, $\tau \in [0, \infty)$ the vector

$$Q(x) + W(x, \tau) \in G_2$$

ii) Setting

$$f_0(x) = M_\tau[f(x, W(x, \tau) + Q(x), \tau), \tau]$$

we define $u_0(t)$ as a solution of the Cauchy problem

$$\frac{du_0}{dt} = f_0(u_0), \quad u_0 = \alpha,$$

and let

$$U_0(t) = Q(u_0(t)), \quad V_0(t, \tau) = W(u_0(t), \tau).$$

We assume $u_0(t) \in G_1$ for all $t \in [0, T]$.

iii) The equation (6.11) and the condition $M_\tau[v_i] = 0$ yield

$$v_1 = K_\tau I_\tau K_\tau[f(u_0(t), U_0(t) + V(t, \tau)), \tau].$$

iv) We denote $\Pi(\tau)$ $(0 \leq \tau < \infty)$ as a solution of the problem

$$\frac{d\Pi_0}{d\tau} = C(\alpha)\Pi_0, \Pi_0(0) = \beta - U_0(0) - V_0(0, 0).$$

v) At last, we have

$$\pi_i = -\int_{-\infty}^{\tau} [f(\alpha, U_0(0) + V_0(0, s) + \Pi_0(s), s) - f(\alpha, U_0(0) + V_0(0, s), s]ds.$$

The functions u_i, v_j, π_k, U_i, V_j, Π_k can be defined in a similar way.

It is easy to verify that if

$$D(x, t) = \sum_{k \neq 0, k \in \mathbf{Z}^m} D_k(x) e^{i(\omega, k)t},$$

then

$$W(x, \tau) = \sum_{k \neq 0, k \in \mathbf{Z}^m} W_k(x) e^{i(\omega, k)t}, \tag{6.14}$$

where

$$W_k(x) = [i(\omega, k)I - C(x)]^{-1} D_k(x).$$

It is clear, that

$$||W_k(x)|| \leq C||D_k(x)||,$$

and, since ω is a nonresonance vector, the series (6.14) is convergent.

6.2 Regular problem

Now consider a Cauchy problem

$$\begin{aligned} \frac{dx}{dt} &= f(x,\tau), && (6.15)\\ x(0,\varepsilon) &= \alpha, && (6.16) \end{aligned}$$

where $f \in \mathcal{A}_\omega^{N+2}(G_1, \mathbf{R}^n)$; $x, \alpha \in G_1$, $\tau = 1/\varepsilon$, $0 \le t \le T$. The equation (6.15) does not have singularities. Therefore we will seek an appropriate solution of the problem (6.15), (6.16) in the form [11,12]

$$x_N = u_0(t) + \varepsilon[u_1(t) + v_1(t,\tau)] + \ldots + \varepsilon^N[u_N(t) + v_N(t,\tau)] + \varepsilon^{N+1}v_{N+1}(t,\tau). \quad (6.17)$$

Substituting the presentation (6.17) into (6.15) we obtain a residual function

$$R_N = \frac{\partial x_N}{\partial t} + \frac{1}{\varepsilon}\frac{\partial x_N}{\partial \tau} - f(x_N(t,\tau,\varepsilon),\tau)$$

and

$$f_N = f(x_N(t,\tau,\varepsilon),\tau).$$

Further we expand R_N in a power series in ε^k and obtain

$$\begin{aligned} R_N = &\left\{\frac{du_0}{dt} + \frac{\partial v_1}{\partial \tau} - f(u_0(t),\tau)\right\} + \varepsilon\left\{\frac{du_1}{dt} + \frac{\partial v_2}{\partial \tau} + \frac{\partial v_1}{\partial t} - f'_{N\varepsilon}(t,\tau,0)\right\} + \ldots +\\ &\varepsilon^N\left\{\frac{du_N}{dt} + \frac{\partial v_{N+1}}{\partial \tau} + \frac{\partial v_N}{\partial \tau} - \frac{1}{N!}f^N_{N\varepsilon}(t,\tau,0)\right\} +\\ &\varepsilon^{N+1}\left\{\frac{du_{N+1}}{dt} - \frac{1}{(N+1)!}f^(N+1)_{N\varepsilon}(t,\tau,\mu)\right\}, \qquad (6.18) \end{aligned}$$

where $\mu \in (0,\varepsilon]$.

Finally, we equate the coefficients of the expantions of R_N at the powers $\varepsilon^k (k = 0, 1, \ldots, N)$ near zero. Then for ε^0 we have

$$\frac{du_0}{dt} + \frac{\partial v_1}{\partial \tau} = M_\tau[f(u_0(t),\tau)] + K_\tau[f(u_0(t),\tau)].$$

Let u_0 be a solution of the Cauchy problem

$$\frac{du_0}{dt} = f_0(u_0), u_0(\alpha), \qquad (6.19)$$

where $f_0(x) = M_\tau[f(x,\tau)]$. Then for v_1 we have

$$\frac{\partial v_1}{\partial \tau} = K_\tau[f(u_0,\tau)],$$

and, hence

$$v_1 = I_\tau K\tau[f(u_0(t),\tau)].$$

From (6.18) for ε^1 we have

$$\frac{du_1}{dt} + \frac{\partial v_2}{\partial \tau} = -\frac{\partial v_1}{\partial t} + f'_{N\varepsilon}(t,\tau,0). \tag{6.20}$$

It makes us sure that

$$f'_{N\varepsilon}(t,\tau,0) - \frac{\partial v_1}{\partial t} = f'_x(u_0(t),\tau)(u_1+v_1) - \frac{\partial v_1}{\partial t} = A_0(t)u_1 + h_1(t,\tau), \tag{6.21}$$

where

$$A_0(t) = M_\tau[f'_x(u_0(t),\tau)] \in C^N([0,\tau], M^{n\times n})$$

and

$$h_1(t,\tau) = f'_x(u_0(t),\tau)(u_1(t)+v_1(t,\tau)) - \frac{\partial v_1}{\partial t} - A_0(t)u_1(t).$$

It is easy to see that $M_\tau[h_1]$ is independent on u_1, and, therefore $M_\tau[h_1]$ is a known function.

Setting

$$\frac{du_1}{dt} = A_0(t)u_1 + M_\tau[h_1(t,\tau)], u_1(0) = 0$$

we find $u_1 \in C^{N+1}([0,T],\mathbf{R}^n)$. Now $h_1 \in \mathcal{A}^N_\omega([0,T],\mathbf{R}^n)$ is a known function of t, τ. Putting

$$\frac{\partial v_2}{\partial \tau} = K_\tau[h_1]$$

we find

$$v_2 = I_\tau K\tau[h_1] \in \mathcal{A}^N_\omega([0,T],\mathbf{R}^n).$$

The other terms of the expansion (6.17) can be found in a similar way.

7 Asymptotics in Infinite Semiaxis $[0,\infty)$

Come back to the problem (4.1)-(4.3). In this section we assume that the averaged system

$$\frac{dx}{dt} = f_0(x)$$

has a stable state $x = x^*$ and, in addition, for the eigenvalues $\lambda_1,\ldots,\lambda_n$ of the matrix $A_0 = f_{0x}(x^*)$

$$\max_i \Re\lambda_i \le -\gamma \quad (\gamma > 0).$$

Let G_0 be the attraction region of the stable steady state x^*. We assume that the initial point $\alpha \in G_0$. Then it is not difficult to prove that for the Cauchy problem

$$\frac{du_0}{dt} = f_0(u_0), \quad u_0 = \alpha$$

the estimate

$$||u_0(t)|| \le Ce^{-\gamma t}(C > 0, 0 \le t < \infty)$$

holds. The analysis of the algorithm, given in Section 4 for the presentation of the approximate solution (4.8), (4.9), shows that this expansions can be considered for $t \in [0,\infty)$, moreover, the following statement is true.

Theorem 7.1 *There exist such constants $C > 0$, $\varepsilon_0 > 0$ that for all $\varepsilon \in (0, \varepsilon_0]$ the Cauchy problem (4.1)-(4.3) has a unique solution $x(t,\varepsilon)$, $x(t,\varepsilon)$ $t \in [0,\infty)$ and the estimate*

$$||x(t,\varepsilon) - x_N(t,\varepsilon)||_{C[0,\infty)} + ||y(t,\varepsilon) - y_N(t,\varepsilon)||_{C[0,\infty)} \le C\varepsilon^{N+1},$$

holds, where C, ε_0 are independent on $\varepsilon \in (0, \varepsilon_0]$.

8 Vibrational Stability

8.1 General case

Let G_1, G_2, G_3 be open regions in $\mathbf{R}^n$, $\mathbf{R}^m$, $\mathbf{R}^r$ respectively. Consider a family of nonlinear singularly perturbed systems

$$\frac{dz}{dt} = P(z,y,\lambda) + \Omega Q(z,\lambda), \tag{8.1}$$

$$\frac{dy}{dt} = \Omega Y(z,y,\lambda), \tag{8.2}$$

$$P : G_1 \times G_2 \times G_3 \to \mathbf{R}^n, Q : G_1 \times G_3 \to \mathbf{R}^n,$$

$$Y : G_1 \times G_2 \times G_3 \to \mathbf{R}^m, 1 \ll \Omega.$$

Let $0 \in G_1 \times G_2$ and $P(0,0,\lambda) = 0$, $Q(0,\lambda) = 0$, and $Y(0,0,\lambda) = 0$. So the steady state $z = 0$, $y = 0$ is an equilibrium state for the system (8.1), (8.2). We assume

$$P \in C^{N+3}(G_1 \times G_2 \times G_3, \mathbf{R}^n),$$

$$Q \in C^{N+3}(G_1 \times G_3, \mathbf{R}^n),$$

$$Y \in C^{N+3}(G_1 \times G_2 \times G_3, \mathbf{R}^m).$$

Let, further, ω be a nonresonance vector. Introduce in (8.1), (8.2) the parametric vibrations with according to the law

$$\lambda(t) = \lambda_0 + \varphi(\tau), \quad \tau = \Omega t \tag{8.3}$$

where $\lambda_0 \in \mathbf{R}^r$ is a constant vector and $\varphi \in \mathcal{A}_\omega^{N+3}([0,\infty), R^r)$, $M_\tau[\varphi] = 0$.

Definition 8.1. An equilibrium state $z = 0, y = 0$ is said to be vibrationally stable (v-stable) if for any $\Delta > 0$ there exist constants $\delta > 0$, $\Omega_0 > 0$ such that for all initial conditions (z_0, y_0) $(||z_0|| < \delta, ||y_0|| < \delta)$ for the solutions $z(t, z_0, y_0)$, $y(t, z_0, y_0)$ of the system

$$\frac{dz}{dt} = P(z, y, \lambda_0 + \varphi(\tau)) + \Omega Q(z, \lambda_0 + \varphi(\tau)), \tag{8.4}$$

$$\frac{dy}{dt} = \Omega Y(z, y, \lambda_0 + \varphi(\tau)), \tag{8.5}$$

the estimate

$$||z(t, z_0, y_0)|| < \Delta, ||y(t, z_0, y_0)|| < \Delta \tag{8.6}$$

holds for $t \in [0, \infty)$ and $\Omega \geq \Omega_0$.

To investigate v-stability of the equilibrium point $z = 0$, $y = 0$ let us consider the equation

$$\frac{dz}{d\tau} = Q(z, \lambda_0 + \varphi(\tau)). \tag{8.7}$$

Denote the general solution of (8.7) as

$$z = h(c, \tau), \quad h(0, \tau) = 0,$$

where $c \in G_1 \subset \mathbf{R}^n$, $h \in \mathcal{A}_\omega^{N+3}(G_1, \mathbf{R}^n)$. We propose that h is one-to-one map,

$$h : G_1 \times [0, \infty) \to G_1 \times [0, \infty),$$

$\partial h/\partial c$ is an invertable matrix and

$$\frac{\partial h}{\partial c} \in \mathcal{A}_\omega^{N+2}(G_1 \times [0, \infty), M^{m \times m}).$$

Due to the replacement

$$z = h(x, \tau)$$

we get the system

$$\frac{dx}{dt} = f(x, y, \tau), \tag{8.8}$$

$$\varepsilon \frac{dy}{dt} = F(x, y, \tau), \tag{8.9}$$

where $\varepsilon = \Omega^{-1}$,

$$\begin{aligned} f &= [\partial h/\partial x]^{-1} P(h(x, \tau), y, \lambda_0 + \varphi(\tau)), \\ F &= Y(h(x, \tau), y, \lambda_0 + \varphi(\tau)). \end{aligned}$$

Obviously, $f(0, 0, \tau) = 0$, $F(0, 0, \tau) = 0$, and Theorem 7.1 is applicable to the system (8.8).

Define the main terms of the asymptotic expansions of x_N, y_N for the system (8.8), (8.9) with initial conditions

$$x(0, \Omega) = \alpha \in G_{10}, \quad y(0, \Omega) = \beta \in G_2.$$

For every fixed $x \in G_{10}$ we consider the equation

$$\frac{dy}{d\tau} = Y(h(x,\tau), y, \lambda_0 + \varphi(\tau)) \tag{8.10}$$

and suppose that this equation has a unique solution

$$y = \psi_0(x,\tau) \in \mathcal{A}_\omega^{N+2}(G_{10}, \mathbf{R}^m), \tag{8.11}$$

moreover, the values $\psi_0(x,\tau) \in G_2$. Obviously, $\psi_0(x,\tau) \to 0$ as $x \to 0$.

By $C(x,\tau)$ we denote the matrix

$$Y_y(h(x,\tau), y, \lambda_0 + \varphi(\tau)).$$

Let for the fundamental matrix $V(x,\tau,s)$ of the linear homogeneous system

$$\frac{dy}{d\tau} = C(x,\tau)y$$

the uniform estimate (4.7) is valid, in addition, the constants C, σ are independent on $x \in G_{10}$. Substituting $y = \psi_0(x,\tau)$ into the equation (8.8) we immediately get equation

$$\frac{du_0}{dt} = f_0(u_0), \quad u_0(0) = \alpha, \tag{8.12}$$

where

$$f_0(x) = M_\tau[(\partial h/\partial z)^{-1} P(h, \psi_0, \lambda_0 + \varphi_0)].$$

Since $f_0(0) = 0$, the equation (8.11)has a zero solution.

Let the matrix

$$A = f_{0x} = M_\tau \left[\frac{\partial}{\partial x} \left[(\frac{\partial h}{\partial x})^{-1} P(h, \psi_0, \lambda_0 + \varphi) \right]_{x=0} \right] \tag{8.13}$$

be Hurwitz . Then zero steady state of system (8.12) is asymptotically stable, and for any $\delta > 0$ and for the solutions $u_0(t,\alpha)$ ($||\alpha|| \le \delta$) the estimate

$$||u_0(t,\alpha)|| \le Ce^{-\kappa t}(0 \le t \le \infty)$$

is true. Thus, the solution of the Cauchy problem

$$\frac{du_0}{dt} = f(u_0), \quad u_0(0) = \alpha, \quad ||\alpha|| \le \delta$$

is defined for $t \in [0, \infty)$.

We assume

$$u_0(t) \in G_{10}$$

for all $t \in [0, \infty)$. S_0, u_0, $Y_0 = \psi_0(u_0(t), \tau)$ are defined completely.

Finally, $\Pi_0(\tau)$ is a solution of the Cauchy problem

$$\frac{dZ}{d\tau} = Y(h(\alpha,\tau), \psi_0(\alpha,\tau) + Z, \lambda + \varphi(\tau)) - Y(h(\alpha,\tau), \psi_0(\alpha,\tau) + Z, \lambda + \varphi(\tau)),$$

$$Z(0) = \beta - \psi(\alpha, 0), \quad ||\beta|| < \delta.$$

Thus, we get the main term of the asymptotical expansions

$$x = u_0(t), \quad y = \psi_0(t,\tau) + \Pi_0(\tau). \tag{8.14}$$

Returning to old variables x, z we obtain

$$z = h(u_0(t), \tau), \quad y = \psi_0(t,\tau) + \Pi_0(\tau) \tag{8.15}$$

Folmulae (8.15) describe the transient behavior of original system (8.4)-(8.5).

Theorem 7.1 implies

Theorem 8.1 *Provided the above conditions the zero steady state of system (8.4), (8.5) is v-stable.*

8.2 Linear Multiplicative Vibrations $Q(z, \lambda_0 + \varphi) = L(\tau)z$

Let us assume that the system

$$\frac{dz}{d\tau} = L(\tau)z$$

has an almost periodic general solution

$$z = \Phi(\tau)c, \quad c \in \mathbf{R}^n, \quad \Phi \in \mathcal{A}^0_\omega(M^{n\times n}).$$

In this case the equation (8.10) has the form

$$\frac{dy}{dt} = Y(\Phi(\tau)x, y, \lambda_0 + \varphi(\tau)), \tag{8.16}$$

and let $y = \Psi(x,\tau)$ be a unique almost periodic solution of equation (8.16) from $\mathcal{A}^{N+2}_\omega(G_2, \mathbf{R}^m)$.

Therefore, in (8.13)

$$A = M_\tau \left[\Phi^{-1}\left\{P_x(0,0,\lambda_0 + \phi(\tau))\Psi_x(0,\tau)\right\}\right]. \tag{8.17}$$

If A is a Gurwitz matrix then zero solution of system (8.4), (8.5) is v-stable.

Now, consider the special case when system (8.4), (8.5) has the form

$$\frac{dz}{dt} = A_0(\tau)z + B_0(\tau)y + \Omega L(\tau)z, \tag{8.18}$$

$$\frac{dy}{dt} = \Omega[C_0(\tau)z + D_0(\tau)y]. \tag{8.19}$$

Let U_0 be the monodromy matrix of the linear homogeneous system

$$\frac{dy}{d\tau} = D_0(\tau)y. \tag{8.20}$$

Let $\lambda_1, \ldots, \lambda_m$ be the characteristic roots of U_0, and

$$|\lambda_i| < 1 (i = 1, 2, \ldots, m). \tag{8.21}$$

Then the system

$$\frac{dy}{d\tau} = D_0(\tau)y + f(\tau), f \in \mathcal{A}^0_\omega(\mathbf{R}^n) \tag{8.22}$$

has a unique solution

$$y = \Psi_0(\tau) \in \mathcal{A}^0_\omega(\mathbf{R}^n).$$

Function Ψ_0 can be presented in the form

$$y = \int_0^\infty G_0(\tau, s) f(s) ds,$$

where $G_0(\tau, s)$ is the Green matrix of the problem (8.22).

In the considered case the matrix (8.17) has the form

$$A_0 = M_\tau \left[\Phi^{-1}(0, \tau) A_0(\tau) \Phi(0, \tau) + B_0(\tau) \int_0^\infty G_0(\tau, s) C_0(s) ds \right]. \tag{8.23}$$

If A_0 is a Hurwitz matrix, zero solution of system (8.18), (8.19) is v-stable.

8.3 Cautionary Example.

Consider a stiff second order system

$$\frac{dz}{dt} = (-0.27 + \lambda(\Omega + 1))z + (0.1 + \lambda)y, \tag{8.24}$$

$$\Omega^{-1}\frac{dy}{dt} = (0.1 + \lambda)z - y \tag{8.25}$$

with a small positive parameter $\varepsilon = \Omega^{-1}$. Under parametric excitation $\lambda = \cos\tau$, $\tau = \Omega t$ we get a new stiff system

$$\frac{dz}{dt} = \Omega \cos\tau z + (-0.27 + \cos\tau)z + (0.1 + \cos\tau)y,$$

$$\varepsilon\frac{dy}{dt} = (0.1 + \cos\tau)z - y.$$

Changing the variable

$$z = e^{\sin\tau} x.$$

We get

$$\frac{dx}{dt} = (-0.27+\cos\tau)x + e^{-\sin\tau}(0.1+\cos\tau)y, \tag{8.26}$$

$$\varepsilon\frac{dy}{dt} = e^{\sin\tau}(0.1+\cos\tau)x - y. \tag{8.27}$$

To investigate this system for small parameter ε we use the proposed above algorithm. First, note that the periodic solution of the equation

$$\frac{dy}{d\tau} = -y + F(\tau) \tag{8.28}$$

can be presented in the form

$$y = (y_1+y_2)x, \tag{8.29}$$

$$\kappa = \frac{e^{-2\pi}}{1-e^{-2\pi}}, \tag{8.30}$$

where

$$y_1 = 0.1e^{-\tau}\left\{\kappa\int_0^{2\pi} e^{s+\sin s}ds + \int_0^{\tau} e^{s+\sin sds}\right\}, \tag{8.31}$$

$$y_2 = e^{-\tau}\left\{\kappa\int_0^{2\pi}\cos se^{s+\sin s}ds + \int_0^{\tau}\cos se^{s+\sin sds}\right\}. \tag{8.32}$$

Substituting (8.29) into (8.26) we obtain

$$\frac{dx}{d\tau} = a(t,\varepsilon)x, \tag{8.33}$$

where

$$a(\tau) = (-0.27+\cos\tau) + e^{-\tau-\sin\tau}[0.1(0.1+\cos\tau)(\kappa\int_0^{2\pi} e^{s+\sin s}ds+$$

$$+\int_0^{\tau} e^{s+\sin s}ds)+$$

$$(\kappa\int_0^{2\pi}\cos se^{s+\sin s}ds + \int_0^{\tau}\cos se^{s+\sin s}ds)].$$

It is to easy compute $\phi = M_\tau[a] < 0$. According to Theorem 8.1 the zero solution of the stiff system (8.26), (8.27) is v-stable .

On the other hand, if we neglect "small parasitic parameter" $\varepsilon = 1/\Omega$ in equation (8.25) and disregard the term $\Omega^{-1}dy/dt$, we get the equation

$$y = (0.1+\lambda)z. \tag{8.34}$$

Substituting (8.34) into (8.24) we obtain the following equation to find z

$$\frac{dz}{dt} = [(-0.27+(\Omega+1)\lambda+(0.1+\lambda)^2]z. \tag{8.35}$$

After the parametric excitation $\lambda = \cos \Omega t$ we have the equation

$$\frac{dz}{dt} = b(t, \Omega) z, \qquad (8.36)$$

where

$$b = -0.27 + (\Omega + 1) \cos \Omega t + (0.1 + \cos \Omega t)^2.$$

Since $M_\tau[b] = 0.24 > 0$ the zero solution of (8.36) is unstable.

Thus, the disregarding of a small "parasitic" perameter Ω^{-1} in (8.25) leads to a mistaken result.

9 Conclusions and Discussions

The material presented in the paper develops the asymptotical theory for nonliner singularly perturbed systems with high frequency coefficients. The expansions obtained allow to study a transient behavior, vibrational control. We give the simple criteria to define v-stability.

It is important to note for singularly perturbed problems under parametric excitation the disregarding of a small parameter at higher derivative may deduce to wrong results. The algorithm proposed in Section 8 is a tool for control engineers to study complicated nonlinear systems.

Acknowledgement

The author sincerely thank the support and hospitality of the Weierstrass Institute during his visits in 1998 and 2000 when this paper was completed.

References

[1] O'Malley R. E.,Jr.(1974), Introduction to Singular Perturbations, Academic Press, New York.

[2] Vasil'eva A. B, Butuzov V. F., Kalachev L. V., Boundary Function Method for Singular Perturbation Problems. Studies in Applied Mathematics, 14. SIAM, 1995.

[3] Kokotovic P. V., Applications of Singular Perturbation Techniques to control problems, SIAM Review, Vol. 26, No 4, October 1984.

[4] Meerkov S. M., Principle of Vibrational Control: Theory and Applications, IEEE Trans. Automat. Contr., Vol. AC-25, pp.755-762, August 1980.

[5] Bellman R. E., Bentsman J., Meerkov S. M., Vibrational Control of Nonlinear Systems: Vibrational Stability, IEEE Trans. Automat. Contr., Vol. AC-31, No 8, pp.710-716, August 1986.

[6] Bellman R. E., Bentsman J., Meerkov S. M., Vibrational Control of Nonlinear Systems: Vibrational Controllability and Tansient Behavior,IEEE Trans. Automat. Contr., Vol. AC-31, No 8, pp.717-724, August 1986.

[7] Lehmann B., Bentsman J., Lunel S.V., Verriest E.I.,Vibrational Control of Nonlinear Time Lag Systems with Bounded Delay: Averaging Theory, Stabilizability, and Transient Behavior, IEEE Trans. Automat. Contr., Vol. 39, No 5, pp. 898-912, May 1994.

[8] Bentsman J., Hong K. S., Fakhfakh J., Vibrational Control of Nonlinear Time Lag Systems: Vibrational Stabilization and Transient Behavior, Automatica, Vol.27, No 3,pp. 491-500, 1991.

[9] Bentsman J., Meerkov S. M., Shu X., Vibrational Stabilizability of a Class of Distributed Parameter Systems, Proc. IV IFAC Symp. on Control of Distributed Parameter Systems, IFAC Proc. Series, Vol 3., pp. 453-457, Pergamon Press, 1987.

[10] Bentsman J., Meerkov S. M., Shu X., Vibrational Control of Nonlinear Parabolic Systems, (1987) IFAC World Congress, Vol 9., pp. 285-288, Munich, Germany, 1987.

[11] Strygin V. V., Esippenko D. G., Hybrid Acymptotical Method of Mechanical Systems Motion Integration Described by Singularly Perturbed Equations with Quasiperiodic Qiuckly Coefficients, Proceedings of the Second International Conference. Asymptotics in Mechanics, pp.253-260, 1997.

[12] Strygin V. V., Esippenko D. G., Method of Separation of Movement and Higher Order Averaging for Nonlinear Systems with Quasiperiodic Coefficients, Nonlinear World (1996), No 4, pp. 807-834.

[13] Moser J., A New Technique for Construction of Solution of Nonlinear Differential Equations. Proc. Nat. Acad. Sci. U.S.A., Vol.47, No 11, 1961, pp.1824-1831.

[14] Daletski Yu. L., Crain M. G., Stability of Equations in Banach Space, Moscow, 1973.