

Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Singularity of super-Brownian local time at a point catalyst

ON 1

D.A. Dawson¹, K. Fleischmann², Y. Li³ and C. Müller³

submitted: 1st July 1993

¹ Carleton University Ottawa ² Institute of Applied Analysis
Department of Mathematics and Stochastics
and Statistics Mohrenstraße 39
Ottawa, Canada K1S 5B6 D - 10117 Berlin
Germany

³ University of Rochester
Department of Mathematics
Rochester, NY 14627
USA

Preprint No. 56
Berlin 1993

1991 Mathematics Subject Classification. Primary 60 J 80, secondary 60 J 65, 60 G 57.
Key words and phrases. Point catalytic medium, critical branching, super-Brownian local time, occupation time, occupation density, measure-valued branching, superprocess.

Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39
D - 10117 Berlin

Fax: + 49 30 2004975
e-Mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e-Mail (Internet): preprint@iaas-berlin.d400.de

SINGULARITY OF SUPER-BROWNIAN LOCAL TIME
AT A POINT CATALYST

BY DONALD A. DAWSON¹, KLAUS FLEISCHMANN², YI LI³ AND CARL MUELLER⁴

¹CARLETON UNIVERSITY OTTAWA,

²INSTITUTE OF APPLIED ANALYSIS AND STOCHASTICS BERLIN,

^{3,4}UNIVERSITY OF ROCHESTER

ABSTRACT. In a one-dimensional single point-catalytic continuous super-Brownian motion studied by Dawson and Fleischmann (1993), the occupation density measure λ^c at the catalyst's position c is shown to be a singular (diffuse) random measure. The source of this qualitative new effect is the irregularity of the varying medium δ_c describing the point catalyst. The proof is based on a probabilistic characterization of the law of the Palm canonical clusters χ appearing in the Lévy-Khinchin representation of λ^c in a historical process setting and the fact that these χ have infinite left upper density (with respect to Lebesgue measure) at the Palm time point.

AMS 1980 subject classifications. Primary 60J80; secondary 60J65, 60G57

Key words and phrases. Point catalytic medium, critical branching, super-Brownian local time, occupation time, occupation density, measure-valued branching, superprocess.

1. INTRODUCTION

1.1. Motivation and Main Result. A *one-dimensional point-catalytic continuous super-Brownian motion* $\mathfrak{X} = \{\mathfrak{X}_t; t \geq 0\}$, in which branching is allowed only at a *single point catalyst*, was discussed in some detail in Dawson and Fleischmann (1993). This critical measure-valued branching process \mathfrak{X} is associated with mild solutions of the nonlinear equation

$$(1.1.1) \quad \frac{\partial}{\partial t} u(t, z) = \kappa \Delta u(t, z) - \delta_c(z) u^2(t, z), \quad t > 0, z \in \mathbb{R}.$$

Here $\kappa > 0$ is a diffusion constant, the (one-dimensional) Laplacian Δ acts on the space variable $z \in \mathbb{R}$, and the *branching rate* degenerates to the Dirac δ -function δ_c describing a *point catalyst* situated at $c \in \mathbb{R}$. In fact, the *Laplace transition functional* which determines this Markov process \mathfrak{X} has the form

$$(1.1.2) \quad \mathbb{E}\left\{ \exp(\mathfrak{X}_t, -\varphi) \mid \mathfrak{X}_s = \mu \right\} = \exp(\mu, -u_\varphi(t-s)),$$

$0 \leq s \leq t$, $\varphi \in \Phi_+$, $\mu \in \mathcal{M}_f$. Here u_φ is the unique mild solution of (1.1.1) with initial condition $u|_{t=0+} = \varphi$, where φ lies in some set Φ_+ of non-negative test functions, \mathcal{M}_f is the space of all finite measures on \mathbb{R} , and (m, f) is an abbreviation for the integral $\int dm f$ of the function f with respect to the measure m .

It was demonstrated that there exists a version of \mathfrak{X} for which the associated *point-catalytic occupation time process*

$$(1.1.3) \quad \mathfrak{Y}_t := \int_0^t dr \mathfrak{X}_r(\cdot), \quad t \geq 0,$$

possesses a *jointly continuous occupation density field* $\eta = \{\eta_t(z); t \geq 0, z \in \mathbb{R}\}$ with probability one. Consequently $\mathfrak{Y}_t(dz) = \eta_t(z) dz$, $t \geq 0, z \in \mathbb{R}$, in spite of the irregularity of the branching rate δ_c (see Theorem 1.2.9 in [2]). The *occupation density measures* $\lambda^z(dr) =: d\eta_r(z)$, $z \in \mathbb{R}$, on \mathbb{R}_+ (*super-Brownian local time measures*) were shown to be absolutely continuous a.s. provided that z is different from c (see (1.2.8) there). Moreover, the super-Brownian local time measure λ^c at the catalyst's location has *carrying Hausdorff-Besicovitch dimension one* (cf. Theorem 1.2.14).

On the other hand, [2] Theorem 1.2.7 says, roughly speaking, that, at a fixed time, the density of mass *vanishes* stochastically as the catalyst's position is approached. Together these facts suggest that λ^c is a *singular* diffuse random measure in contrast to the well-known absolute continuity in the case of one dimensional regular branching (cf. Konno and Shiga (1988), or Reimers (1989)).

The heuristic picture is that density of mass arriving at c normally dies instantaneously due to the infinite branching rate at c , but that additionally bursts of creation of absolutely continuous mass occur on an "exceptional" set of times.

The *main purpose of this note* is to prove that λ^c is in fact singular a.s. Here we restrict our consideration to the case in which \mathfrak{X} starts off at time 0 with a unit mass δ_c at the catalyst's position c :

Theorem 1.1.4 (singularity of λ^c). *Assume that $\mathfrak{X}_0 = \delta_c$. The occupation density measure λ^c at the catalyst's position is with probability one a singular (diffuse) random measure on \mathbb{R}_+ .*

1.2. Methodology. Our approach to this requires the development of some tools which may be of some independent interest. First, we consider an enriched version of \mathfrak{X} , namely, the *historical point-catalytic super-Brownian motion* $\tilde{\mathfrak{X}} := \{\tilde{\mathfrak{X}}_t; t \geq 0\}$ which is necessary for our argument. Here the state $\tilde{\mathfrak{X}}_t$ at time t keeps track of the entire history of the population masses alive at t and their "family relationships" and arises as the diffusion limit of the reduced branching tree structure associated with the approximating branching particle system; cf. for instance, Dynkin (1991b) or Dawson and Perkins (1991).

In this setting, $\lambda^c(dr)$ is replaced by $\tilde{\lambda}^c(d[r, w])$, where $\tilde{\lambda}^c([r_1, r_2] \times B)$ is the contribution to the "occupation density increment" $\lambda^c([r_1, r_2])$ due to paths in the subset B of *c-Brownian bridge paths w on $[0, r]$* which start at time 0 at c and also end up in c at time r , $r_1 \leq r \leq r_2$. These *c-Brownian bridge paths w on $[0, r]$* can be interpreted as the "particles" trajectories which "contributed to" the occupation density increment $\lambda^c([r_1, r_2])$.

Next, the infinite divisibility of the law of the random measure $\tilde{\lambda}^c$ allows us to use the framework of *Lévy-Khinchin representation*. Moreover, we can disintegrate the corresponding Lévy-Khinchin measure Q with respect to its intensity measure to obtain its *Palm distributions* $Q^{r,w}(d\chi)$. Roughly speaking, $Q^{r,w}(d\chi)$ is the law of a *canonical cluster χ* but "given that it contains the pair" $[r, w]$. Given such an r and a fixed *c-Brownian bridge path w on $[0, r]$* we derive a *probabilistic representation* (see Theorem 3.3.9 below) for the Laplace functional of the Palm distribution $Q^{r,w}$ in terms of the (deterministic) *Brownian local time measure $L^c(w, dt)$* at c of this given bridge path w and solutions of an "historical" version of the singular equation

$$(1.2.1) \quad -\frac{\partial}{\partial r}u = \kappa\Delta u + f\delta_c - \delta_c u^2, \quad r > 0,$$

(see (3.1.4)).

Exploiting this probabilistic representation of $Q^{r,w}$, the key step of the proof of Theorem 1.1.4 is then to demonstrate that the random measure $\chi(d[r', w'])$ distributed according to $Q^{r,w}$ has with probability one at the Palm point r an *infinite left upper density* with respect to the Lebesgue measure dr' (Theorem 4.2.2 below). This is then shown to imply that the original super-Brownian local time measure λ^c at c cannot have an absolutely continuous component.

1.3. Outline. In Section 2 the point-catalytic super-Brownian motion \mathfrak{X} , its occupation time process \mathfrak{Y} , and the occupation density measures λ^z are adapted to the framework of historical processes. Based on this, the Palm representation formula is derived in the following section. The singularity proof then follows in the final Section 4.

Our general reference for standard facts on random measures is Kallenberg (1983), Chapters 6 and 10, and Kerstan et al. (1982), Chapters 1,3 and 11; for Brownian motion and in particular for Brownian local time we refer to Revuz and Yor (1991), Chapters 6 and 8.

2. OCCUPATION DENSITY MEASURES IN THE SETTING OF HISTORICAL PROCESSES

The purpose of this section is to sketch how the point-catalytic super-Brownian motion \mathfrak{X} and related objects of interest in the present paper can be fit into the general framework of historical processes. For the latter, we refer, for instance, to Dynkin (1991b) and Dawson and Perkins (1991).

2.1. Preliminaries: Terminology. We start by introducing some terminology we will use. If A, B are sets and $a \mapsto B^a$ is a map of A into the set of all subsets of B , then we often write

$$(2.1.1) \quad A \hat{\times} B^\bullet := \{[a, b]; a \in A, b \in B^a\} = \bigcup_{a \in A} \{a\} \times B^a$$

for the *graph* of this map. Note that $A \hat{\times} B^\bullet \subseteq A \times B$.

We adopt the following conventions. If E is a topological space then subsets of E will always be equipped with the subset topology. Products of topological spaces will be endowed with the product topology. Measures on a topological space E will be defined on the Borel σ -algebra $\mathcal{B}(E)$, (generated by the open subsets of E). A measure m on E with $m(E \setminus E') = 0$ for some $E' \in \mathcal{B}(E)$, that is, if m is concentrated on E' , will also be regarded as a measure on E' (and conversely).

If E_1 is a topological space and E_2 a normed space with norm $\|\cdot\|$, let $\mathbf{B}[E_1, E_2]$ denote the space of all *bounded measurable* maps $f : E_1 \mapsto E_2$ equipped with the supremum norm $\|f\|_\infty := \sup \{\|f(e_1)\|; e_1 \in E_1\}$, $f \in \mathbf{C}[E_1, E_2]$, of *uniform convergence*. Note that $\mathbf{B}[E_1, E_2]$ is a Banach space if E_2 is. By $\mathbf{C}[E_1, E_2]$ we denote the subspace of all *continuous* (bounded) f in $\mathbf{B}[E_1, E_2]$.

Once and for all we fix a closed finite time interval $I := [0, T]$, $0 < T < \infty$. We write \mathbf{C} for the Banach space $\mathbf{C}[I, \mathbb{R}]$ of all real-valued continuous paths on I . The lower index $+$ on the symbol of a space will always refer to the subset of all of its non-negative members.

2.2. Brownian Paths W and Brownian Path Processes \tilde{W} . In this subsection we provide some preliminaries on historical processes starting at the level of a *one-particle motion* (underlying Brownian motion process).

For $t \in I = [0, T]$ and a path $w \in \mathbb{C} = \mathbb{C}[I, \mathbb{R}]$, write $\tilde{w}_t := \{w_{s \wedge t}; s \in I\}$ for the corresponding path in \mathbb{C} stopped at time t , and \mathbb{C}^t for the set of all those *stopped paths*. Note that \mathbb{C}^t is a *closed* subspace of the Banach space \mathbb{C} , that $\mathbb{C}^s \subseteq \mathbb{C}^t$ for $s \leq t$, that $\mathbb{C}^T = \mathbb{C}$, and that \mathbb{C}^0 can be identified with \mathbb{R} . Given a path $w \in \mathbb{C}$, we interpret $\tilde{w} := \{\tilde{w}; t \in I\}$ as a "*path trajectory*". Since, for $s \leq t$,

$$(2.2.1) \quad \|\tilde{w}_t - \tilde{w}_s\|_\infty = \sup_{s \leq r \leq t} |w_r - w_s| \xrightarrow{t-s \rightarrow 0} 0,$$

\tilde{w} belongs to the Banach space $\mathbb{C}[I, \mathbb{C}]$, hence, to its *closed* subspace

$$(2.2.2) \quad \hat{\mathbb{C}}[I, \mathbb{C}] := \{\omega \in \mathbb{C}[I, \mathbb{C}]; \omega_t \in \mathbb{C}^t \ \forall t \in I\}.$$

Moreover,

$$(2.2.3) \quad \|\tilde{w}' - \tilde{w}''\|_\infty = \|w' - w''\|_\infty, \quad w', w'' \in \mathbb{C},$$

implies that $w \mapsto \tilde{w}$ maps \mathbb{C} continuously into $\hat{\mathbb{C}}[I, \mathbb{C}]$. Note, the graph $\{[s, \omega_s]; s \in I\}$ of $\omega \in \hat{\mathbb{C}}[I, \mathbb{C}]$ is a subset of $I \hat{\times} \mathbb{C}^\bullet$ (recall (2.1.1) and that $I' \hat{\times} \mathbb{C}^\bullet$ is a *closed* subset of $I \times \mathbb{C}$ provided that I' is a closed subinterval of I).

For each $z \in \mathbb{R}$, denote by Π_z the law on \mathbb{C} of a *Brownian path* W with diffusion constant κ and starting at $W_0 = z$ (we normalize in such a way that $\kappa \Delta$ is the generator of this strong Markov process). Given the Brownian path W we denote its continuous *Brownian local time measure* at the catalyst's position \mathfrak{c} by $L^\mathfrak{c}(W, dt)$ and note that formally

$$(2.2.4) \quad L^\mathfrak{c}(W, dt) = \delta_\mathfrak{c}(W_t) dt.$$

Using the continuous map $w \mapsto \tilde{w}$ of \mathbb{C} into $\hat{\mathbb{C}}[I, \mathbb{C}]$ introduced above, we arrive at the so-called *Brownian path process* $\tilde{W} = [\tilde{W}, L^\mathfrak{c}, \tilde{\Pi}_{s,w}, s \in I, w \in \mathbb{C}^s]$ on I which is a time-inhomogeneous strong Markov process realized in the Banach space $\hat{\mathbb{C}}[I, \mathbb{C}]$ (defined in (2.2.2)). Note that we adjoined the additive functional $L^\mathfrak{c} = L^\mathfrak{c}(W)$. This makes sense since given the Brownian path process \tilde{W} we can recover the Brownian path W itself by *projection*: $W_t = (\tilde{W}_t)_t, t \in I$.

The *semigroup* of \tilde{W} will be denoted by

$$(2.2.5) \quad \tilde{S}_{s,t} \varphi(w) := \tilde{\Pi}_{s,w} \varphi(\tilde{W}_t), \quad 0 \leq s \leq t \leq T, \quad w \in \mathbb{C}^s, \quad \varphi \in \mathbf{B}[\mathbb{C}, \mathbb{R}],$$

and the related *generator* by $\{\tilde{A}_s; s \in I\}$. (Note that as a rule occurrences of the capital letter W refer to random objects, whereas related lower case letters denote fixed elements in path spaces such as \mathbb{C} or $\mathbb{C}[I, \mathbb{C}]$.)

Below we will use the continuous (non-negative) additive functional $L^\mathfrak{c}(W)$ as a clock to govern the branching of a particle whose motion is described by W . Actually,

L^c will serve as a probabilistic refinement of the rough characteristics of the super-Brownian motion $\tilde{\mathfrak{X}}$ which is provided by the branching rate δ_c . More precisely L^c will be exploited for the description of the historical point-catalytic super-Brownian motion $\tilde{\mathfrak{X}}$ we will deal with in the next subsection (Dynkin's additive functional approach).

2.3. Historical Point-catalytic Super-Brownian Motion $\tilde{\mathfrak{X}}$. Let \mathcal{M}_f^I denote the set of all finite (non-negative) measures μ defined on $\mathbb{C} = \mathbb{C}[I, \mathbb{R}]$, equipped with the topology of *weak convergence*. Write $\|\mu\|$ for the *total mass* $\mu(\mathbb{C})$ of a measure $\mu \in \mathcal{M}_f^I$. Introduce the closed subsets $\mathcal{M}_f^t := \{\mu \in \mathcal{M}_f^I; \mu(\mathbb{C} \setminus \mathbb{C}^t) = 0\}$ of measures on paths stopped at time $t \in I = [0, T]$. Note that $\mathcal{M}_f^T = \mathcal{M}_f^I$.

Now we are in a position to formulate as a proposition the existence and characterization of the *historical point-catalytic super-Brownian motion* $\tilde{\mathfrak{X}}$ related to the super-Brownian motion $\tilde{\mathfrak{X}}$ which was mentioned in the introduction.

Proposition 2.3.1 (historical point-catalytic super-Brownian motion $\tilde{\mathfrak{X}}$). *There exists a time-inhomogeneous right Markov process $\tilde{\mathfrak{X}} = [\tilde{\mathfrak{X}}, \tilde{\mathbb{P}}_{s,\mu}, s \in I, \mu \in \mathcal{M}_f^s]$ with states $\tilde{\mathfrak{X}}_t \in \mathcal{M}_f^t, t \in I$, having the following Laplace transition functional*

$$(2.3.2) \quad \tilde{\mathbb{P}}_{s,\mu} \exp(\tilde{\mathfrak{X}}_t, -\varphi) = \exp(\mu, -u_\varphi(s, \cdot, t)),$$

$0 \leq s \leq t \leq T, \mu \in \mathcal{M}_f^s, \varphi \in \mathbf{B}_+[\mathbb{C}^t, \mathbb{R}]$, where $u_\varphi(\cdot, \cdot, t)$ is the unique $\mathcal{B}([0, t] \times \mathbb{C}^\bullet)$ -measurable (recall (2.1.1)) bounded non-negative solution of the nonlinear integral equation

$$(2.3.3) \quad u_\varphi(s, w, t) = \tilde{\mathbb{P}}_{s,w} \left[\varphi(\tilde{W}_t) - \int_s^t L^c(W, dr) u_\varphi^2(r, \tilde{W}_r, t) \right],$$

$0 \leq s \leq t, w \in \mathbb{C}^s$, or symbolically,

$$(2.3.4) \quad -\frac{\partial}{\partial s} u_\varphi(s, w, t) = \tilde{A}_s u_\varphi(s, w, t) - \delta_c(w_s) u_\varphi^2(s, w, t), \quad 0 \leq s \leq t, w \in \mathbb{C}^s,$$

with "terminal condition" $u_\varphi(t, \cdot, t) = \varphi$.

The following expectation and variance formulas hold:

$$(2.3.5) \quad \tilde{\mathbb{P}}_{s,\mu}(\tilde{\mathfrak{X}}_t, \varphi) = \int \mu(dw) \tilde{\mathbb{P}}_{s,w} \varphi(\tilde{W}_t),$$

$$(2.3.6) \quad \tilde{\text{Var}}_{s,\mu}(\tilde{\mathfrak{X}}_t, \varphi) = 2 \int \mu(dw) \tilde{\mathbb{P}}_{s,w} \int_s^t L^c(W, dr) \left[\tilde{\mathbb{P}}_{r,\tilde{W}_r} \varphi(\tilde{W}_t) \right]^2,$$

$0 \leq s \leq t \leq T, \mu \in \mathbf{B}[\mathbb{C}^t, \mathbb{R}]$.

Consequently, the historical population $\tilde{\mathfrak{X}}_t$ at time t is now a measure on paths stopped at time t which, in contrast to \mathfrak{X}_t , includes information on which routes in space the masses present at time t had followed up to t . Of course, by the *projection*

$\tilde{\mathfrak{X}}_t(\{w \in \mathbb{C}^t; w_t \in \cdot\}) = \mathfrak{X}_t$, from $\tilde{\mathfrak{X}}$ we can deduce the super-Brownian motion \mathfrak{X} (but not its path continuity established in [2]).

Note that (2.3.4) *formally* follows from (2.3.3) by using the representation (2.2.4) of the Brownian local time measure $L^c(W, dr)$, by writing (2.3.3) in terms of the semigroup of the Markov process \tilde{W} (recall (2.2.5)), and then by a formal differentiation to the time variable s ; compare with (1.1.1) where we could use a forward formulation because of time-homogeneity.

Remarks on the Proof of Proposition 2.3.1. The construction of the process is essentially a special case of the general construction of Dynkin (see for example Dynkin (1991a,b)). In our case the Brownian path process \tilde{W} serves as the underlying right Markov process ξ (motion component), the Brownian local time measure $L^c(W, dt)$ at the catalyst's position as the continuous additive functional K of ξ , and the branching mechanism ψ is specialized to

$$\psi(u)(s, w) := u^2(s, w), \quad 0 \leq s \leq T, \quad w \in \mathbb{C}^s, \quad u \in \mathbf{B}[I \times \mathbb{C}, \mathbb{R}_+].$$

The right property (and consequently right continuity and strong Markov property) of $\tilde{\mathfrak{X}}$ follows from the arguments in Dynkin (1992), Theorem 2.1 together with Subsection 1.6. ■

2.4. Historical Point-catalytic Occupation Time Measures $\tilde{\mathfrak{Y}}$. In this subsection we introduce an "historical analog" of the point-catalytic occupation time process \mathfrak{Y} of (1.1.3). For this purpose, for fixed $0 \leq s \leq t \leq T$ let

$$(2.4.1) \quad (\tilde{\mathfrak{Y}}_{s,t}, \psi) := \int_s^t dr (\tilde{\mathfrak{X}}_r, \psi(r, \cdot)), \quad \psi \in \mathbf{B}_+ [[0, t] \hat{\times} \mathbb{C}^\bullet, \mathbb{R}],$$

getting a finite random measure $\tilde{\mathfrak{Y}}_{s,t}$ defined on $[s, t] \hat{\times} \mathbb{C}^\bullet$. In fact, by the Proposition 2.3.1, $\tilde{\mathfrak{X}}$ is right continuous and hence the integral in (2.4.1) makes sense; see also Theorem 3.1 in Dynkin (1992).

We call $\tilde{\mathfrak{Y}} := \{\tilde{\mathfrak{Y}}_{s,t}; 0 \leq s \leq t \leq T\}$ the *historical point-catalytic occupation time process related to $\tilde{\mathfrak{X}}$* . By standard arguments (see also Theorem 1.2 in Dynkin (1991b)) we get:

Proposition 2.4.2 (historical point-catalytic occupation time measures $\tilde{\mathfrak{Y}}$). *The Laplace transition functional of $[\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}}]$ is given by*

$$(2.4.3) \quad \tilde{\mathbb{P}}_{s,\mu} \exp \left[- (\tilde{\mathfrak{X}}_t, \varphi) - (\tilde{\mathfrak{Y}}_{s,t}, \psi) \right] = \exp \left(\mu, -u_{\varphi,\psi}(s, \cdot, t) \right),$$

$0 \leq s \leq t \leq T$, $\mu \in \mathcal{M}_f^s$, $\varphi \in \mathbf{B}_+ [\mathbb{C}^t, \mathbb{R}]$, $\psi \in \mathbf{B}_+ [[0, t] \hat{\times} \mathbb{C}^\bullet, \mathbb{R}]$, where $u_{\varphi,\psi}(\cdot, \cdot, t)$ is the unique $\mathcal{B}([0, t] \hat{\times} \mathbb{C}^\bullet)$ -measurable bounded non-negative solution of

the nonlinear integral equation

$$(2.4.4) \quad u_{\varphi,\psi}(s, w, t) = \tilde{\Pi}_{s,w} \left[\varphi(\tilde{W}_t) + \int_s^t dr \psi(r, \tilde{W}_r) - \int_s^t L^\varepsilon(W, dr) u_{\varphi,\psi}^2(r, \tilde{W}_r, t) \right],$$

$0 \leq s \leq t$, $w \in \mathbb{C}^s$, or, more formally,

$$(2.4.5) \quad -\frac{\partial}{\partial s} u_{\varphi,\psi}(s, w, t) = \tilde{A}_s u_{\varphi,\psi}(s, w, t) + \psi(s, w) - \delta_\varepsilon(w_s) u_{\varphi,\psi}^2(s, w, t),$$

with terminal condition $u_{\varphi,\psi}(t, \cdot, t) = \varphi$.

The following expectation formula holds:

$$(2.4.6) \quad \tilde{\mathbb{P}}_{s,\mu}(\tilde{\mathfrak{Y}}_{s,t}, \psi) = \int \mu(dw) \int_s^t dr \tilde{\Pi}_{s,w} \psi(r, \tilde{W}_r),$$

$0 \leq s \leq t \leq T$, $\mu \in \mathcal{M}_f^s$, $\psi \in \mathbf{B}_+[[0, t] \hat{\times} \mathbb{C}^\bullet, \mathbb{R}]$.

$\tilde{\mathfrak{Y}}_{s,t}(d[r, w])$ measures the increment of the historical process $\tilde{\mathfrak{X}}$ at time r at the path w stopped at time r . Note that again by projection, $\tilde{\mathfrak{Y}}_{0,t}([0, t] \times \{w \in \mathbb{C}^t; w_t \in \cdot\}) = \mathfrak{Y}_t$, from $\tilde{\mathfrak{Y}}$ we can deduce the point-catalytic occupation time process \mathfrak{Y} related to \mathfrak{X} .

2.5. Historical Super-Brownian Local Time Measures. The next step in our program is to turn to the *historical super-Brownian local time measures (historical occupation density measures)* $\tilde{\lambda}_{s,t}^z$ which can loosely be defined by

$$(2.5.1) \quad (\tilde{\lambda}_{s,t}^z, \psi) := \int \tilde{\mathfrak{Y}}_{s,t}(d[r, w]) \psi(r, w) \delta_z(W_r),$$

$z \in \mathbb{R}$, $0 \leq s \leq t \leq T$, $\psi \in \mathbf{B}_+[[s, t] \hat{\times} \mathbb{C}^\bullet, \mathbb{R}]$. Roughly speaking, $\tilde{\lambda}_{s,t}^z(d[r, w])$ measures the increment of the *density* of those paths w concerning the historical process $\tilde{\mathfrak{X}}$ which at time r are stopped at z .

To be more precise, we introduce the set

$$(2.5.2) \quad \mathbb{C}^{t,z} := \{w \in \mathbb{C}^t, w_t = z\}, \quad t \in I, z \in \mathbb{R},$$

of continuous paths on I stopped at time t at z . Note that $I' \hat{\times} \mathbb{C}^{\bullet,z}$ (for any $z \in \mathbb{R}$) is a *closed* subset of $I \times \mathbb{C}$ if I' is a closed subinterval of I . Now we are ready to rigorously define the historical occupation density measures:

Proposition 2.5.3 (*historical super-Brownian local time measures* $\tilde{\lambda}_{s,t}^z$).
Fix $z \in \mathbb{R}$, $0 \leq s \leq t \leq T$, and $\mu \in \mathcal{M}_f^s$. Then there is a finite random measure $\tilde{\lambda}_{s,t}^z$ defined on $[s, t] \hat{\times} \mathbb{C}^{\bullet,z}$ (recall (2.5.2) and (2.1.1)) having Laplace functional

$$(2.5.4) \quad \tilde{\mathbb{P}}_{s,\mu} \exp(\tilde{\lambda}_{s,t}^z, -\psi) = \exp(\mu, -u_{\psi,z}(s, \cdot, t)), \quad \psi \in \mathbf{B}[[0, t] \hat{\times} \mathbb{C}^{\bullet,z}, \mathbb{R}],$$

where $u_{\psi,z}(\cdot, \cdot, t)$ is the unique $\mathcal{B}([0, t] \hat{\times} \mathbb{C}^\bullet)$ -measurable bounded non-negative solution of the nonlinear integral equation

$$(2.5.5) \quad u_{\psi,z}(s, w, t) = \tilde{\Pi}_{s,w} \left[\int_s^t L^z(W, dr) \psi(r, \tilde{W}_r) - \int_s^t L^c(W, dr) u_{\psi,z}^2(r, \tilde{W}_r, t) \right],$$

$0 \leq s \leq t \leq T$, $w \in \mathbb{C}^s$, or, more formally,

$$(2.5.6) \quad -\frac{\partial}{\partial s} u_{\psi,z}(s, w, t) = \tilde{A}_s u_{\psi,z}(s, w, t) + \delta_z(w_s) \psi(s, w) - \delta_c(w_s) u_{\psi,z}^2(s, w, t),$$

with terminal condition $u_{\psi,z}(t, \cdot, t) = 0$.

The random measure $\tilde{\lambda}_{s,t}^z$ is $\tilde{\mathbb{P}}_{s,\mu}$ -a.s. diffuse, i.e. it does not carry mass at any single point set. Finally, the following expectation formula holds:

$$(2.5.7) \quad \tilde{\mathbb{P}}_{s,\mu} \left(\tilde{\lambda}_{s,t}^z, \psi \right) = \int \mu(dw) \tilde{\Pi}_{s,w} \left[\int_s^t L^z(W, dr) \psi(r, \tilde{W}_r) \right],$$

$\psi \in \mathbf{B}_+ \left[[0, t] \hat{\times} \mathbb{C}^\bullet, \mathbb{R} \right]$.

Note that according to this proposition $\tilde{\lambda}_{s,t}^z$ is defined only for fixed z, s, t (and not as a family of random measures on a common probability space). This is sufficient for the purpose of proving Theorem 1.1.4. In fact, the "original" super-Brownian local time measures λ^z restricted to the interval $[s, t]$ and then denoted by $\lambda_{s,t}^z$ coincide in law with the marginal of $\tilde{\lambda}_{s,t}^z$:

$$(2.5.8) \quad \lambda_{s,t}^z(B) \stackrel{\mathcal{L}}{=} \tilde{\lambda}_{s,t}^z(B \times \mathbb{C}), \quad z \in \mathbb{R}, \quad B \in \mathcal{B}(I), \quad 0 \leq s \leq t \leq T.$$

Remarks on the Proof of Proposition 2.5.3. As in the non-historical case, one can derive the Laplace functional of $\tilde{\lambda}_{s,t}^z$ by approximating the δ -function δ_z in the heuristic expression (2.5.1) by the smooth functions $p(\varepsilon, \cdot - z)$, $\varepsilon \rightarrow 0$. Here p denotes the continuous Brownian transition density function (with generator $\kappa\Delta$):

$$(2.5.9) \quad p(r, y) := (4\pi\kappa r)^{-1/2} \exp \left[-y^2/4\kappa r \right], \quad r > 0, \quad y \in \mathbb{R}.$$

Since the original super-Brownian local time measures λ^z are a.s. diffuse (Theorem 1.2.9 in [2]), the a.s. diffuseness of $\tilde{\lambda}_{s,t}^z$ follows from (2.5.8) by contradiction. The derivation of the expectation formula is standard. \blacksquare

Note that the laws of the random measures $\tilde{\lambda}_{s,t}^z$ are *infinitely divisible*; in fact, in the representation (2.5.4) pass from μ to μ/n , $n \geq 1$.

3. A PALM REPRESENTATION FORMULA

The purpose in this section is to derive a *probabilistic representation* of the Laplace functional concerning the Palm canonical clusters in the Lévy-Khinchin representation of the infinitely divisible random measure $\tilde{\lambda}_{s,T}^c$ (see Theorem 3.3.9 below).

3.1. Specialization. *From now on we assume* in this section that, for $s \in I = [0, T]$, the starting measure $\mu \in \mathcal{M}_f^s$ of $\tilde{\mathcal{X}}$ at time s is a unit measure $\delta_{\mathbf{w}}$ concentrated at $\mathbf{w} \in \mathbb{C}^{s,c}$, a path, stopped at time s at the catalyst, recall (2.5.2). (Note that we reserve the boldface letter \mathbf{w} , for a "starting path".) For convenience, we write $\tilde{\mathbb{P}}_{s,\mathbf{w}}$ instead of $\tilde{\mathbb{P}}_{s,\mu}$. Also, let

$$(3.1.1) \quad \mathbb{C}_{s,\mathbf{w}}^{r,c} := \{w \in \mathbb{C}^{r,c}; w \text{ is a continuous extension of } \mathbf{w}\},$$

$0 \leq s \leq t \leq T$, $\mathbf{w} \in \mathbb{C}^{s,c}$, denote the set of all *c-bridge paths* w on $[s, r]$, that is, bridge paths on $[s, r]$ which start and end at the catalyst's position \mathbf{c} . (The path w before time s is determined by \mathbf{w} whereas after time r it is constantly \mathbf{c} , but the behavior of w outside $[s, r]$ is not relevant in most cases.) Note that the subsets $[s, T] \hat{\times} \mathbb{C}_{s,\mathbf{w}}^{r,c}$ of $I \times \mathbb{C}$ are *closed*.

By Proposition 2.5.3, with respect to $\tilde{\mathbb{P}}_{s,\mathbf{w}}$, the finite random measures $\tilde{\lambda}_{s,T}^c$ are concentrated on $[s, T] \hat{\times} \mathbb{C}_{s,\mathbf{w}}^{r,c}$ and have Laplace functionals

$$(3.1.2) \quad \tilde{\mathbb{P}}_{s,\mathbf{w}} \exp \left(\tilde{\lambda}_{s,T}^c, -\psi \right) = \exp \left[-u_{\psi,c}(s, \mathbf{w}, T) \right],$$

$s \in I$, $\mathbf{w} \in \mathbb{C}^{s,c}$, $\psi \in \mathbf{B}_+ [0, T] \hat{\times} \mathbb{C}^{\cdot,c}, \mathbb{R}$. Here $u_{\psi,c}(\cdot, \cdot, T)$ is the unique bounded non-negative solution of the nonlinear integral equation

$$(3.1.3) \quad u_{\psi,c}(s, w, T) = \tilde{\mathbb{P}}_{s,w} \int_s^T L^c(W, dr) \left[\psi(r, \tilde{W}_r) - u_{\psi,c}^2(r, \tilde{W}_r, T) \right],$$

$s \in I$, $w \in \mathbb{C}^{s,c}$, or, as a formal short-hand,

$$(3.1.4) \quad -\frac{\partial}{\partial s} u_{\psi,c}(s, w, T) = \tilde{A}_s u_{\psi,c}(s, w, T) + \delta_c(w_s) \psi(s, w) - \delta_c(w_s) u_{\psi,c}^2(s, w, T),$$

$s \in I$, $w \in \mathbb{C}^{s,c}$, with terminal condition $u_{\psi,c}(T, \cdot, T) = 0$.

3.2. Vanishing Deterministic Part of $\tilde{\lambda}_{s,T}^c$. In this subsection we show that the historical super-Brownian local time measures $\tilde{\lambda}_{s,T}^c$ do *not* contain a deterministic part:

Lemma 3.2.1 (vanishing deterministic component of $\tilde{\lambda}_{s,T}^c$). *Fix $s \in I$ and $w \in \mathbb{C}^{s,c}$. If $\tilde{\mathbb{P}}_{s,\mathbf{w}} \left(\tilde{\lambda}_{s,T}^c \geq \nu \right) = 1$ for some deterministic measure ν on $I \times \mathbb{C}$ then $\nu = 0$.*

Proof. Take ν as in the assumption of the lemma, and suppose that $m := \nu(I \times \mathbb{C}) > 0$. Hence, $\tilde{\lambda}_{s,T}^c(I \times \mathbb{C}) \geq m$ with $\tilde{\mathbb{P}}_{s,\mathbf{w}}$ -probability one. Therefore, by the representation (3.1.2) with $\psi \equiv \theta$, $\theta > 0$,

$$(3.2.2) \quad \exp[-\theta m] \geq \tilde{\mathbb{P}}_{s,\mathbf{w}} \exp[-\theta \tilde{\lambda}_{s,T}^c(I \times \mathbb{C})] = \exp[-u_\theta(T - s)],$$

where, by (3.1.3), $u_\theta \geq 0$ solves the simplified forward equation

$$(3.2.3) \quad u_\theta(t) = \theta \int_0^t dr p(r, 0) - \int_0^t dr p(t - r, 0) u_\theta^2(r), \quad t \geq 0,$$

related to the formal equation

$$(3.2.4) \quad \frac{\partial}{\partial t} u_\theta = \kappa \Delta u_\theta + \theta \delta_c - \delta_c u_\theta^2, \quad t \geq 0, \quad u_\theta \Big|_{t=0+} = 0,$$

(for convenience, by time-homogeneity, we switched to a *forward* setting). In the next lemma, we will show that $u_\theta(t) \sim \sqrt{\theta}$ as $\theta \rightarrow \infty$ holds, for fixed $t > 0$. Then from (3.2.2) we conclude $\theta m \leq u_\theta(T - s) \sim \sqrt{\theta}$ as $\theta \rightarrow \infty$, for fixed $s < T$, which is an obvious contradiction, since $m > 0$ by assumption. On the other hand, if $s = T$ then use $u_\theta(0) = 0$ to again derive a contradiction using (3.2.2). The proof will be finished after verifying the following lemma. \square

Lemma 3.2.5. *The (non-negative) solutions to (3.2.3) satisfy $u_\theta(t) \sim \sqrt{\theta}$ as $\theta \rightarrow \infty$, for each fixed $t > 0$.*

Proof. Setting $v_\theta(s) := \theta^{-1/2} u_\theta(\theta^{-1} s)$, $s > 0$, from (3.2.3) we get

$$v_\theta(s) = \int_0^s dr p(r, 0) - \int_0^s dr p(s - r, 0) v_\theta^2(r).$$

Hence, by uniqueness, $v_\theta = u_1$. Therefore, $u_\theta(t) = \theta^{1/2} u_1(\theta t)$ holds. But by [2], Lemma 4.3.2 (ii) (where $\mathfrak{c} = 0$ without loss of generality), we have $u_1(s) \uparrow 1$ as $s \uparrow \infty$. Then the claim follows since $t > 0$ by assumption. \square

3.3. The Palm Representation Formula. Recall that each historical occupation density measure $\tilde{\lambda}_{s,t}^c$ has an *infinitely divisible distribution*. Therefore, we can apply the so-called *Lévy-Khinchin representation*, which, for convenience we state in the following lemma (where by Lemma 3.2.1 the deterministic component is dropped):

Lemma 3.3.1 (Lévy-Khinchin representation). *Fix $s \in I$ and $\mathbf{w} \in \mathbb{C}^{s,c}$. There is a uniquely determined σ -finite measure $Q_{s,\mathbf{w}}$ defined on the set of all non-vanishing finite measures χ on $[s, T] \hat{\times} \mathbb{C}_{s,\mathbf{w}}^{s,c}$ (recall (3.1.1)) such that*

$$(3.3.2) \quad \tilde{\mathbb{P}}_{s,\mathbf{w}} \exp(\tilde{\lambda}_{s,T}^c, -\psi) = \exp\left[-\int Q_{s,\mathbf{w}}(d\chi) (1 - \exp(\chi, -\psi))\right],$$

$\psi \in \mathbf{B}_+ [[s, T] \hat{\times} \mathbb{C}_{s,\mathbf{w}}^{s,c}, \mathbb{R}]$, where the latter integrals are finite.

Roughly speaking, the *Lévy-Khinchin measure* $Q_{s,w}$ describes canonical clusters χ which by a "Poissonian superposition" can be added up to form $\tilde{\lambda}_{s,T}^c$. In this case a canonical cluster $\chi =: \int \chi(d[r,w]) \delta_{[r,w]}$ is interpreted as a "collection" of weighted pairs $[r,w]$ referring to \mathfrak{c} -bridge paths w on $[s,r]$ (recall (3.1.1)). In particular, the "randomness" of the paths w "given" r concerns the behavior of w *before time* r .

Next we want to determine the *intensity measure* $\bar{Q}_{s,w}$ (first moment measure) of the Lévy-Khinchin measure $Q_{s,w}$ defined by

$$(3.3.3) \quad (\bar{Q}_{s,w}, \psi) = \int Q_{s,w}(d\chi) (\chi, \psi), \quad \psi \in \mathbf{B}_+ [[s, T] \hat{\times} \mathbf{C}_{s,w}^{\bullet,c}, \mathbb{R}].$$

Since $\tilde{\lambda}^c$ does not have a deterministic component, by (2.5.7) we have

$$(3.3.4) \quad (\bar{Q}_{s,w}, \psi) = \tilde{\mathbb{P}}_{s,w}(\tilde{\lambda}_{s,T}^c, \psi) = \tilde{\mathbb{P}}_{s,w} \int_s^T L(W, dr) \psi(r, \tilde{W}_r).$$

For convenience, we introduce the *\mathfrak{c} -Brownian bridge laws*

$$(3.3.5) \quad \tilde{\mathbb{P}}_{s,w}^{r,c}(A) := \mathbb{P}_{s,w}(\tilde{W}_r \in A | W_r = c), \quad 0 \leq s \leq t \leq T, w \in \mathbf{C}^s, A \in \mathcal{B}(\mathbf{C}).$$

Note that $\tilde{\mathbb{P}}_{s,w}^{r,c}$ is a law on the path space \mathbf{C} (and not on $\mathbf{C}[I, \mathbf{C}]$), so we could write $\mathbb{P}_{s,w}^{r,c}$ as well. Using this notation, from (3.3.4) we conclude:

Lemma 3.3.6 (intensity measure $\bar{Q}_{s,w}$). *Fix $s \in I$ and $w \in \mathbf{C}^{s,c}$. The intensity measure $\bar{Q}_{s,w}$ of $Q_{s,w}$ is a finite measure on $[s, T] \hat{\times} \mathbf{C}_{s,w}^{\bullet,c}$ given by*

$$(3.3.7) \quad (\bar{Q}_{s,w}, \psi) = \int_s^T dr p(r-s, 0) \int \tilde{\mathbb{P}}_{s,w}^{r,c}(dw) \psi(r, w),$$

$$\psi \in \mathbf{B}_+ [[s, T] \hat{\times} \mathbf{C}_{s,w}^{\bullet,c}, \mathbb{R}].$$

Roughly speaking, $\bar{Q}_{s,w}(d[r,w])$ selects pairs $[r,w]$ in such a way that r is absolutely continuous "distributed", and "conditioned" on r , the path w is a \mathfrak{c} -bridge on $[s,r]$ with law $\tilde{\mathbb{P}}_{s,w}^{r,c}$.

For almost all w with respect to $\tilde{\mathbb{P}}_{s,w}^{r,c}$, the *Brownian bridge local time measure* $L^c(w, dt)$ makes sense (note that $\tilde{\mathbb{P}}_{s,w}^{r,c}$ describes a semi-martingale). It will be involved in the Palm representation formula below.

For $\bar{Q}_{s,w}$ -almost all $[r,w] \in [s, T] \hat{\times} \mathbf{C}_{s,w}^{\bullet,c}$ we can build the *Palm distributions* $Q_{s,w}^{r,w}$ formally defined by *disintegration*:

$$(3.3.8) \quad Q_{s,w}(d\chi) \chi(dr, dw) = \bar{Q}_{s,w}(dr, dw) Q_{s,w}^{r,w}(d\chi).$$

Roughly speaking, $Q_{s,w}^{r,w}$ describes a canonical cluster χ according to $Q_{s,w}$ but "given" that it contains the pair $[r,w]$. Recall that by (3.3.7) the bridge local time measure $L^c(w)$ makes sense.

Theorem 3.3.9 (Palm representation formula). Fix $s \in I$ and $w \in \mathbb{C}^{s,c}$. For $\bar{Q}_{s,w}$ -almost all $[r, w] \in [s, T] \hat{\times} \mathbb{C}_{s,w}^{s,c}$, the Palm distribution $Q_{s,w}^{r,w}$ has Laplace functional

$$(3.3.10) \quad \int Q_{s,w}^{r,w}(d\chi) \exp(\chi, -\psi) = \exp \left[-2 \int_s^r L^c(w, dt) u_{\psi,c}(t, w_{\wedge t}, T) \right],$$

$\psi \in \mathbf{B}_+ [[0, T] \hat{\times} \mathbb{C}^{s,c}, \mathbb{R}]$, where $u_{\psi,c}(\cdot, \cdot, T) \geq 0$ is the unique bounded solution of equation (3.1.3).

Consequently, "given the Palm time point" r and a corresponding c -bridge path w , the Laplace functional of $Q_{s,w}^{r,w}$ is expressed with the help of the (deterministic) bridge local time measure $L^c(w)$ and solutions of the equation related to $\tilde{\lambda}_{s,T}^c$. The proof of this theorem will immediately follow.

3.4. Proof of Theorem 3.3.9. To get (3.3.10), we must show that, for functions $\varphi, \psi \in \mathbf{B}_+ [[0, t] \hat{\times} \mathbb{C}^{s,c}, \mathbb{R}]$,

$$(3.4.1) \quad \begin{aligned} & \int \bar{Q}_{s,w}(dr, dw) \varphi(r, w) \int Q_{s,w}^{r,w}(d\chi) \exp(\chi, -\psi) \\ &= \int \bar{Q}_{s,w}(dr, dw) \varphi(r, w) \exp \left[-2 \int_s^r L^c(w, dt) u_{\psi,c}(t, w_{\wedge t}, T) \right]. \end{aligned}$$

Using (3.3.8) and (3.3.4), this can be written as

$$(3.4.2) \quad \begin{aligned} & \int Q_{s,w}(d\chi) (\chi, \varphi) \exp(\chi, -\psi) \\ &= \tilde{\Pi}_{s,w} \int_s^T L^c(W, dr) \varphi(r, \tilde{W}_r) \exp \left[-2 \int_s^r L^c(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right]. \end{aligned}$$

Here $u_{\psi,c}(\cdot, \cdot, T) \geq 0$ solves (3.1.3). To prove (3.4.2) we shall first reformulate both sides of (3.4.2) separately in order to show that they satisfy the same equation (3.4.5) below. Start with the l.h.s. It equals

$$(3.4.3) \quad = \left. \frac{\partial}{\partial \varepsilon} \int Q_{s,w}(d\chi) (1 - \exp(\chi, -(\psi + \varepsilon\varphi))) \right|_{\varepsilon=0+} =: v_{\psi,\varphi}(s, w, T)$$

which by (3.3.2), (3.1.2) and (3.1.3) coincides with $\left. \frac{\partial}{\partial \varepsilon} u_\varepsilon(s, w, T) \right|_{\varepsilon=0+}$. Here $u_\varepsilon(\cdot, \cdot, T) \geq 0$ (for given $\psi + \varepsilon\varphi$) is a solution to

$$(3.4.4) \quad u_\varepsilon(s, w, T) = \tilde{\Pi}_{s,w} \int_s^T L^c(W, dr) \left[(\psi + \varepsilon\varphi)(r, \tilde{W}_r) - u_\varepsilon^2(r, \tilde{W}_r, T) \right],$$

$s \in I$, $w \in \mathbb{C}^{s,c}$, that is (3.1.3) with ψ replaced by $\psi + \varepsilon\varphi$. Consequently, $v_{\psi,\varphi}(\cdot, \cdot, T)$ solves

$$(3.4.5) \quad v(s, w, T) = \tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr) \varphi(r, \tilde{W}_r) \\ - 2\tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr) u_{\psi,c}(r, W_r, T) v(r, \tilde{W}_r, T),$$

$s \in I$, $W \in \mathbb{C}^{s,c}$, with $u_{\psi,\varphi}$ taken from (3.1.3). In other words, this equation describes the "Laplace transform" of the *Campbell measure* of $Q_{s,w}$ as written at the left hand side of (3.4.2). By the way, replacing formally φ by $\delta_{r,w}$ we get the equation related to the Laplace transform of the Palm distribution $Q_{s,w}^{r,w}$. We also mention that by a formal differentiation of (3.4.5)

$$(3.4.6) \quad -\frac{\partial}{\partial s} v(s, w, T) = \tilde{A}_s v(s, w, T) + \delta_c(w_s) \varphi(s, w) \\ - 2\delta_c(w_s) u_{\psi,c}(s, w, T) v(s, w, T), \quad s \in I, \quad w \in \mathbb{C}^{s,c},$$

with terminal condition $v(T, \cdot, T) = 0$.

Therefore, the r.h.s of (3.4.2) can formally be thought of as a "*Feynman-Kac solution*" of equation (3.4.5).

To complete the proof, by uniqueness it remains to show that the r.h.s. of (3.4.2) also satisfies (3.4.5). We change the notation from $[s, w]$ to $[r, \tilde{W}_r]$ and put the r.h.s. of (3.4.2) into the second term of the r.h.s. of (3.4.5) in place of $v(r, \tilde{W}_r, T)$. Then, for $s \in I$, $w \in \mathbb{C}^{s,c}$, that second term equals

$$- 2\tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr) u_{\psi,c}(r, \tilde{W}_r, T) \\ \tilde{\Pi}_{r, \tilde{W}_r} \int_r^T L^\zeta(w, dr') \varphi(r', \tilde{W}_r) \exp \left[-2 \int_r^{r'} L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right].$$

Applying the Markov property at time r to the Brownian path process \tilde{W} , this simplifies to

$$- 2\tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr) u_{\psi,c}(r, \tilde{W}_r, T) \\ \int_r^T L^\zeta(w, dr') \varphi(r', \tilde{W}_r) \exp \left[-2 \int_r^{r'} L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right].$$

Change the order of integration to get

$$= -2\tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr') \varphi(r', \tilde{W}_{r'}) \\ \int_s^{r'} L^\zeta(W, dr) u_{\psi,c}(r, \tilde{W}_r, T) \exp \left[-2 \int_r^{r'} L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right].$$

The latter expression compensates the two remaining terms

$$\tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr) \varphi(r, \tilde{W}_r) \\ - \tilde{\Pi}_{s,w} \int_s^T L^\zeta(W, dr') \varphi(r', \tilde{W}_{r'}) \exp \left[-2 \int_s^{r'} L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right]$$

of (3.4.5) where we used the r.h.s. of (3.4.2) instead of $v(s, w, T)$. In fact,

$$\int_s^{r'} L^\zeta(W, dr) 2u_{\psi,c}(r, \tilde{W}_r, T) \exp \left[-2 \int_r^{r'} L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right]$$

equals

$$\int_s^{r'} L^\zeta(W, dr) \frac{d}{L^\zeta(W, dr)} \exp \left[-2 \int_r^{r'} L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right] \\ = 1 - \exp \left[-2 \int_r^s L^\zeta(W, dt) u_{\psi,c}(t, \tilde{W}_t, T) \right].$$

This finishes the proof of Theorem 3.3.9. \square

4. SINGULARITY OF THE CATALYST'S OCCUPATION DENSITY MEASURE

In this final section we shall carry out the following program. The main point will be to show that the Palm canonical clusters χ according to $Q_{s,w}^{r,w}$, more precisely their marginals $\chi^* := \chi(\cdot \times \mathbb{C})$ on $[0, T]$, have an infinite left upper density $\bar{d}(\chi^*)$ at r (see definition (4.2.1) and Theorem 4.2.2 below). This implies the analogous property for the "Palm realizations" of the historical super-Brownian local time measures $\tilde{\lambda}_{s,T}^\zeta$. Therefore $\lambda_{s,T}^\zeta \stackrel{L}{=} \tilde{\lambda}_{s,T}^\zeta(\cdot \times \mathbb{C})$ also has infinite left upper densities at a random exceptional set of times that carries $\lambda_{s,T}^\zeta$ (Corollary 4.3.1 below). Since $\lambda_{s,T}^\zeta$ is diffuse and for an absolutely continuous measure ν the left upper density $\bar{d}(\nu)$ is finite at ν a.e. r , this finally implies that $\lambda_{s,T}^\zeta$ is singular, for any $0 \leq s \leq T$, thus proving Theorem 1.1.4.

4.1. A Simple Equation Estimate. Fix a constant $b > 0$. To realize the program outlined above we need a *lower estimate* of the solution to the following (ordinary) integral equation:

$$(4.1.1) \quad u_\varepsilon(t) = (b\varepsilon)^{-1} \int_0^{t \wedge b\varepsilon} ds p(t-s, 0) - \int_0^t ds p(t-s, 0) u_\varepsilon^2(s),$$

$t \geq 0$, where $\varepsilon > 0$. (This is related to the formal equation

$$\begin{aligned} \frac{\partial}{\partial t} u_\varepsilon(t, z) &= \kappa \Delta u_\varepsilon(t, z) + (b\varepsilon)^{-1} \mathbf{1}_{(0, b\varepsilon)}(t) - \delta_c(z) u_\varepsilon^2(t, z), \quad t \geq 0, \quad z \in \mathbb{R}, \\ u_\varepsilon|_{t=0+} &= 0. \end{aligned}$$

Lemma 4.1.2. *Recall that $b > 0$ is fixed. For each constant $a \in (0, b]$, there is a constant $\delta > 0$ such that for the solution u_ε of equation (4.1.1)*

$$(4.1.3) \quad u_\varepsilon(t) \geq \delta \varepsilon^{-1/2}, \quad t \in [\varepsilon a, \varepsilon b], \quad \varepsilon > 0.$$

Proof. As in the proof of Lemma 3.2.5 we rescale by setting $v(t) := \sqrt{\varepsilon} u_\varepsilon(\varepsilon t)$, $\varepsilon, t > 0$. Then we immediately see that v solves the same equation as u_1 . By uniqueness we conclude that $v = u_1$. Thus we only need to show that u_1 is bounded away from 0 on the interval $[a, b]$. Restricting our attention to $t \leq b$, we can exploit the fact that $u_1(t)$ is related to the super-Brownian local time $\eta_t(\mathbf{c}) = \lambda^c([0, t])$ (recall (2.5.8)):

$$\mathbb{E}_{0, \mathbf{c}} \exp[-b^{-1} \eta_t(\mathbf{c})] = \exp[-u_t(t)], \quad 0 \leq t \leq b,$$

cf. Proposition 2.5.3 or [2] Theorem 1.2.9. Hence u_1 is *monotone* non-decreasing. Therefore it suffices to show that the solution to

$$(4.1.4) \quad v(t) = b^{-1} \int_0^t ds p_1(s, 0) - \int_0^t ds p_1(t-s, 0) v^2(s), \quad 0 < t < b,$$

is strictly positive for sufficiently small $t \in (0, a]$.

The first term on the r.h.s. of (4.1.4) equals $k\sqrt{t}$ with some constant $k > 0$. We use it to bound $v^2(s) \leq k^2 s \leq k^2 t$ in the second term to arrive at

$$v(t) \geq k\sqrt{t} - k^3 \sqrt{t^3}, \quad t < b,$$

which is certainly strictly positive for all $t > 0$ sufficiently small. This finishes the proof. \square

4.2. Infinite Left Upper Densities of Palm Canonical Clusters. Let ν be a fixed measure on $I = [0, T]$. The *left upper density* $\bar{d}_r(\nu)$ at $r \in I$ of ν is defined in the following way:

$$(4.2.1) \quad \bar{d}_r(\nu) := \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \nu((t - \varepsilon, t)).$$

In the following, this notation will be applied to the *marginal* $\chi^* := \chi(\cdot \times \mathbb{C})$ of a measure χ on $I \times \mathbb{C}$. Recall the definition (3.3.3) of the intensity measures $\bar{Q}_{s, w}$

as well as (3.3.8) of the Palm distributions $Q_{s,w}^{r,w}$. Now we show that the left upper densities blow up at Palm time points.

Theorem 4.2.2 (infinite left upper densities at Palm times). Fix $s \in I = [0, T]$ and $w \in \mathbb{C}^{s,c}$. For $\bar{Q}_{s,w}$ -almost all pairs $[r, w] \in (s, T] \hat{\times} \mathbb{C}_{s,w}^{\bullet,c}$,

$$(4.2.3) \quad Q_{s,w}^{r,w}(\bar{d}_r(\chi^*) = +\infty) = 1.$$

Proof. Given $[r, w]$, it suffices to verify that

$$Q_{s,w}^{r,w} \exp \left(\chi, (b\varepsilon)^{-1} \mathbf{1} \{ (r - b\varepsilon, r) \times \mathbb{C}^{\bullet,c} \} \right) \rightarrow 0$$

along some sequence $\varepsilon_n \downarrow 0$ (depending on $[r, w]$), where we set $b = r/2\kappa$ (recall that $\kappa > 0$ is the diffusion constant). However, according to the Palm representation formula (3.3.10), the latter Laplace functional expression equals

$$\exp \left[-2 \int_s^r L^c(w, dt) u_{\psi,c}(t, w_{\cdot \wedge t}, T) \right],$$

where $u_{\psi,c}(\cdot, \cdot, T) \geq 0$ is the unique bounded solution of our equation (3.1.3) with $\psi = (b\varepsilon)^{-1} \mathbf{1} \{ (r - b\varepsilon, r) \times \mathbb{C}^{\bullet,c} \}$. Since this ψ is constant in the second variable (belonging to \mathbb{C}), by uniqueness of solutions we conclude that $u_{\psi,c}(\cdot, \cdot, T) =: v_\varepsilon$ only depends on the time variable and satisfies the (ordinary) equation

$$v_\varepsilon(t) = \int_t^T d\tau p(\tau - t, 0) (b\varepsilon)^{-1} \mathbf{1} \{ r - b\varepsilon < \tau < r \} - \int_t^T d\tau p(\tau - t, 0) v_\varepsilon^2(\tau),$$

$s \leq t \leq T$. Hence $v_\varepsilon(t) = 0$ for $t \geq r$, and the upper limit in the integrals on the right hand side can be replaced by r . Setting now $v_\varepsilon(t) =: u_\varepsilon(r - t)$, $s \leq t \leq r$, we can easily verify that u_ε solves the simplified forward equation (4.1.1) on $[0, r - s]$. Consequently, it suffices to prove that

$$(4.2.4) \quad \lim_{n \rightarrow \infty} \int_s^r L^c(w, dt) u_{\varepsilon_n}(r - t) = \infty$$

for some sequence $\varepsilon_n \downarrow 0$ (depending on $[r, w]$) for $\bar{Q}_{s,w}$ -almost all $[r, w] \in [s, T] \hat{\times} \mathbb{C}_{s,w}^{\bullet,c}$.

Recall that according to (3.3.7), "given" r , such a \mathfrak{c} -bridge path w on $[s, r]$ has the law $\tilde{\Pi}_{s,w}^{r,c}$. In view of homogeneity there is no loss of generality in setting $s = 0$ and $\mathfrak{c} = 0$ in order to prove (4.2.4). By reversibility in law of the 0-Brownian bridge, the statement (4.2.4) will follow if, for $0 < r \leq T$,

$$\limsup_{\varepsilon \downarrow 0} \int_0^r L^0(w, dt) u_\varepsilon(t) = \infty$$

for $\tilde{\Pi}_{0,0}^{r,0}$ -almost all 0-Brownian bridge paths w on $[0, r]$, where u_ε solves (4.1.1). Restrict the domain of integration additionally to $t \in [\varepsilon a, \varepsilon b]$ with $a = r/5\kappa$ (recall that $b = r/2\kappa$). Then by Lemma 4.1.2 it suffices to show that

$$(4.2.5) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1/2} L^0(W^r, [\varepsilon a, \varepsilon b]) = \infty \quad \text{a.s.}$$

where W^r denotes a 0-Brownian bridge on $[0, r]$ distributed according to $\tilde{\Pi}_{0,0}^{r,0}$.

Let $\{B_t; t \geq 0\}$ be a *standard* Brownian motion (i.e. the diffusion constant κ equals 1/2) starting at 0. Then the 0-Brownian bridge $\{W_t^r; 0 \leq t < r\}$ has the same law as $\{(1-t/r)B_{2\kappa t/(r-t)}; 0 \leq t < r\}$ (cf. for instance, [14], Exercise 1.3.10). Hence, for $0 < \varepsilon < \kappa$, in (4.2.5) we may replace the 0-Brownian bridge local time

$$L^0(W^r, [\varepsilon a, \varepsilon b]) = \int_{\varepsilon a}^{\varepsilon b} dt \delta_0(W_t^r)$$

by

$$\int_{\varepsilon a}^{\varepsilon b} dt \delta_0(B_{2\kappa t/(r-t)}).$$

(For convenience, we proceed formally with the δ -functions setting.) Substitute $s = 2\kappa t/(r-t)$ and by recalling that $\varepsilon < \kappa$, the latter integral can be estimated below by

$$\geq 2\kappa r \int_{\varepsilon/2}^{\varepsilon} ds (2\kappa + s)^{-2} \delta_0(B_s) \geq \text{const } L^0(B, [\varepsilon/2, \varepsilon])$$

(recall that r and κ are fixed). Setting $L(t) := L^0(B, [0, t])$, for (4.2.5) now it suffices to show that there exists a sequence $\varepsilon_n \downarrow 0$ of random times such that a.s.

$$(4.2.6) \quad \varepsilon_n^{-1/2} [L(\varepsilon_n) - L(\varepsilon_n/2)] \xrightarrow[n \rightarrow \infty]{} \infty.$$

Recall that the standard Brownian local time (at 0) process $\{L(\varepsilon); \varepsilon \geq 0\}$ has the same law as $\{\max_{0 \leq x \leq 1} B_{x\varepsilon}; \varepsilon \geq 0\}$; see for instance, Revuz and Yor (1991), Theorem 6.2.3.

Strassen's law of the iterated logarithm says that with probability one the set of limit points of $\{[2\varepsilon \log \log(1/\varepsilon)]^{-1/2} B(x\varepsilon); 0 \leq x \leq 1\}$ in \mathbb{C} (now with $T = 1$) as $\varepsilon \downarrow 0$ is given by the set of all absolutely continuous functions g on $[0, 1]$ satisfying $g(0) = 0$ and $\int_0^1 dx [g'(x)]^2 \leq 1$.

In fact this version (for ε tending to 0) can be concluded for instance from Theorem 1 in Mueller (1981) by setting $t := 1/\varepsilon$, $h(t) := \log \log t$, and using the single point set $\mathcal{P}(t) := \{(0, 1/t)\}$. (Note that then $dA_1(t) = t^{-1} dt$.)

Applying Strassen's law for the identity function $g(x) = x$, we can find a sequence $\varepsilon_n \downarrow 0$ of (random) times such that for $0 \leq s \leq 1$,

$$(4.2.7) \quad [2\varepsilon_n \log \log(1/\varepsilon_n)]^{-1/2} \max_{0 \leq x \leq s} B_{x\varepsilon_n} \longrightarrow s \quad \text{as } n \rightarrow \infty.$$

Exploiting this for $s = 1$ and $s = 1/2$, we see that the expressions in (4.2.6) are of order $[\log \log(1/\varepsilon_n)]^{1/2} \xrightarrow{n \rightarrow \infty} \infty$. This finishes the proof. \square

4.3. Completion of the Proof of Theorem 1.1.4. Fix $s \in I = [0, T]$ and $w \in \mathbb{C}^{s,c}$. According to Theorem 4.2.2,

$$\int \bar{Q}_{s,w}(d[r, w]) \int Q_{s,w}^{r,w}(d\chi) \mathbf{1} \{ \bar{d}_r(\chi^*) < \infty \} = 0.$$

Using the fact that the Palm canonical clusters χ distributed by $Q_{s,w}^{r,w}$ related to the Lévy-Khinchin measure $Q_{s,w}$ concerning $\tilde{\lambda}_{s,T}^c$ are *stochastically smaller* than the Palm clusters χ distributed by $P_{s,w}^{r,w}$, say, related to the law of $\tilde{\lambda}_{s,T}^c$ itself (see Lemma 10.6 in Kallenberg (1983)), the previous statement implies (recalling (3.3.4))

$$\tilde{\mathbb{P}}_{s,w} \int \tilde{\lambda}_{s,T}^c(d[r, w]) \int P_{s,w}^{r,w}(d\chi) \mathbf{1} \{ \tilde{d}_r(\chi^*) < \infty \} = 0.$$

From a *disintegration* formula analogous to (3.3.8),

$$\tilde{\mathbb{P}}_{s,w} \int \tilde{\lambda}_{s,T}^c(dr, \mathbb{C}) \mathbf{1} \{ \tilde{d}_r((\tilde{\lambda}_{s,T}^c)^*) < \infty \} = 0, \quad s \in I, \quad w \in \mathbb{C}^{s,c}.$$

Consequently, we derived the following result, writing $[0, c]$ instead of $[s, w]$ in the special case $s = 0$.

Corollary 4.3.1 (infinite left upper densities at exceptional times). *With $\tilde{\mathbb{P}}_{0,c}$ -probability one, $\tilde{\lambda}_{0,T}^c(\cdot \times \mathbb{C})$ has infinite left upper density at $\tilde{\lambda}_{0,T}^c(\cdot \times \mathbb{C})$ -almost every time.*

Now it is very easy to *verify Theorem 1.1.4*: recall (2.5.8) combine the previous corollary with the fact that the original super-Brownian local time measure $\lambda_{0,T}^c$ at the catalyst's location is *diffuse* a.s. (Theorem 1.2.9 in [2]), and take into account that $T > 0$ is arbitrary. \square

REFERENCES

1. Dawson, D.A. (1993). Measure-valued Markov Processes, in: P.L. Hennequin, ed. *École d'Été de Probabilités de Saint Flour XXI-1991*, Lecture Notes in Mathematics, Springer-Verlag.
2. Dawson, D.A. and Fleischmann, K. (1993): A super-Brownian motion with a single point catalyst. *Stoch. Proc. Appl.* (to appear).
3. Dawson, D.A. and Perkins, E.A. (1991). *Historical Processes*, *Memoirs of the American Mathematical Society* 93, no. 454.
4. Dynkin, E.B. (1991a): Branching particle systems and superprocesses. *Ann. Probab.* 19, 1157-1194.
5. Dynkin, E.B. (1991b): Path processes and historical superprocesses. *Probab. Th. Rel. Fields* 90, 1-36.
6. Dynkin, E.B. (1992): On regularity of superprocesses. *Techn. Rep. 92-51*, Cornell Univ. Ithaca.
7. Dynkin, E.B. (1993): Branching measure-valued processes and partial differential equations, CRM, Montréal (in preparation).
8. Fleischmann, K. (1993): Superprocesses in catalytic media, CRM Proceedings and Lecture Notes, Montréal (submitted).
9. Kallenberg, O. (1983): *Random measures*. 3rd. revised and enlarged ed. Akademie-Verlag Berlin.
10. Kerstan, J., Matthes, K. and Mecke, J. (1982): *Infinitely Divisible Point Processes* (in Russian), Nauka, Moscow.
11. Konno, N. and Shiga, T. (1988): Stochastic differential equations for some measure-valued diffusions, *Prob. Th. Rel. Fields* 79, 201-225.
12. Mueller, C. (1981): A unification of Strassen's law and Lévy's modulus of continuity, *Zeitschrift für Wahrsch.* 56, 163-179.
13. Reimers, M. (1989): One dimensional stochastic partial differential equations and the branching measure diffusion, *Probab. Th. Rel. Fields* 81, 319-340.
14. Revuz, D. and Yor, M. (1991): *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin.

DEPARTMENT OF MATHEMATICS AND STATISTICS
CARLETON UNIVERSITY
OTTAWA, CANADA K1S 5B6

donald_dawson@carleton.ca

INSTITUTE OF APPLIED ANALYSIS AND STOCHASTICS
Mohrenstr. 39
D - 10117 BERLIN, GERMANY

fleischmann@iaas-berlin.d400.de

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NY 14627, USA

yili?@troi.cc.rochester.edu
cmlr@troi.cc.rochester.edu

Veröffentlichungen des Instituts für Angewandte Analysis und Stochastik

Preprints 1992

1. D.A. Dawson, J. Gärtner: Multilevel large deviations.
2. H. Gajewski: On uniqueness of solutions to the drift-diffusion-model of semiconductor devices.
3. J. Fuhrmann: On the convergence of algebraically defined multigrid methods.
4. A. Bovier, J.-M. Ghez: Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions.
5. D.A. Dawson, K. Fleischmann: A super-Brownian motion with a single point catalyst.
6. A. Bovier, V. Gayraud: The thermodynamics of the Curie-Weiss model with random couplings.
7. W. Dahmen, S. Pröbldorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations I: stability and convergence.
8. A. Rathsfeld: Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries. Part I: The wedge, Part II: The cube.
9. G. Schmidt: Boundary element discretization of Poincaré-Steklov operators.
10. K. Fleischmann, F.I. Kaj: Large deviation probability for some rescaled superprocesses.
11. P. Mathé: Random approximation of finite sums.
12. C.J. van Duijn, P. Knabner: Flow and reactive transport in porous media induced by well injection: similarity solution.
13. G.B. Di Masi, E. Platen, W.J. Runggaldier: Hedging of options under discrete observation on assets with stochastic volatility.
14. J. Schmeling, R. Siegmund-Schultze: The singularity spectrum of self-affine fractals with a Bernoulli measure.
15. A. Koshelev: About some coercive inequalities for elementary elliptic and parabolic operators.
16. P.E. Kloeden, E. Platen, H. Schurz: Higher order approximate Markov chain filters.

17. H.M. Dietz, Y. Kutoyants: A minimum-distance estimator for diffusion processes with ergodic properties.
18. I. Schmelzer: Quantization and measurability in gauge theory and gravity.
19. A. Bovier, V. Gayrard: Rigorous results on the thermodynamics of the dilute Hopfield model.
20. K. Gröger: Free energy estimates and asymptotic behaviour of reaction-diffusion processes.
21. E. Platen (ed.): Proceedings of the 1st workshop on stochastic numerics.
22. S. Pröbldorf (ed.): International Symposium "Operator Equations and Numerical Analysis" September 28 - October 2, 1992 Gosen (nearby Berlin).
23. K. Fleischmann, A. Greven: Diffusive clustering in an infinite system of hierarchically interacting diffusions.
24. P. Knabner, I. Kögel-Knabner, K.U. Totsche: The modeling of reactive solute transport with sorption to mobile and immobile sorbents.
25. S. Seifarth: The discrete spectrum of the Dirac operators on certain symmetric spaces.
26. J. Schmeling: Hölder continuity of the holonomy maps for hyperbolic basic sets II.
27. P. Mathé: On optimal random nets.
28. W. Wagner: Stochastic systems of particles with weights and approximation of the Boltzmann equation. The Markov process in the spatially homogeneous case.
29. A. Glitzky, K. Gröger, R. Hünlich: Existence and uniqueness results for equations modelling transport of dopants in semiconductors.
30. J. Elschner: The h - p -version of spline approximation methods for Mellin convolution equations.
31. R. Schlundt: Iterative Verfahren für lineare Gleichungssysteme mit schwach besetzten Koeffizientenmatrizen.
32. G. Hebermehl: Zur direkten Lösung linearer Gleichungssysteme auf Shared und Distributed Memory Systemen.
33. G.N. Milstein, E. Platen, H. Schurz: Balanced implicit methods for stiff stochastic systems: An introduction and numerical experiments.
34. M.H. Neumann: Pointwise confidence intervals in nonparametric regression with heteroscedastic error structure.

35. M. Nussbaum: Asymptotic equivalence of density estimation and white noise.

Preprints 1993

36. B. Kleemann, A. Rathsfeld: Nyström's method and iterative solvers for the solution of the double layer potential equation over polyhedral boundaries.
37. W. Dahmen, S. Prössdorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations II: matrix compression and fast solution.
38. N. Hofmann, E. Platen, M. Schweizer: Option pricing under incompleteness and stochastic volatility.
39. N. Hofmann: Stability of numerical schemes for stochastic differential equations with multiplicative noise.
40. E. Platen, R. Rebolledo: On bond price dynamics.
41. E. Platen: An approach to bond pricing.
42. E. Platen, R. Rebolledo: Pricing via anticipative stochastic calculus.
43. P.E. Kloeden, E. Platen: Numerical methods for stochastic differential equations.
44. L. Brehmer, A. Liemant, I. Müller: Ladungstransport und Oberflächenpotentialkinetik in ungeordneten dünnen Schichten.
45. A. Bovier, C. Külske: A rigorous renormalization group method for interfaces in random media.
46. G. Bruckner: On the regularization of the ill-posed logarithmic kernel integral equation of the first kind.
47. H. Schurz: Asymptotical mean stability of numerical solutions with multiplicative noise.
48. J.W. Barrett, P. Knabner: Finite element approximation of transport of reactive solutes in porous media. Part I: Error estimates for non-equilibrium adsorption processes.
49. M. Pulvirenti, W. Wagner, M.B. Zavelani Rossi: Convergence of particle schemes for the Boltzmann equation.
50. J. Schmeling: Most β shifts have bad ergodic properties.
51. J. Schmeling: Self normal numbers.

52. D.A. Dawson, K. Fleischmann: Super-Brownian motions in higher dimensions with absolutely continuous measure states.
53. A. Koshelev: Regularity of solutions for some problems of mathematical physics.
54. J. Elschner, I.G. Graham: An optimal order collocation method for first kind boundary integral equations on polygons.
55. R. Schlundt: Iterative Verfahren für lineare Gleichungssysteme auf Distributed Memory Systemen.