Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

On Andronov-Hopf bifurcations of two-dimensional diffeomorphisms with homoclinic tangencies.

Sergey V. Gonchenko[1], Vladimir S. Gonchenko[1]

submitted: February 28, 2000

[1] Institute for Applied Mathematics and Cybernetics,
10 Ulyanova St.
Nizhny Novgorod, 603005
Russia
E-Mail: gonchenko@focus.nnov.ru

Preprint No. 556
Berlin 2000

1991 Mathematics Subject Classification. 58F12, 58F13.

Key words and phrases. homoclinic tangency, invariant curve, Andronov-Hopf bifurcation, strange attractors, Newhouse regions.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax:	+ 49 30 2044975
E-Mail (X.400):	c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet):	preprint@wias-berlin.de
World Wide Web:	http://www.wias-berlin.de/

Abstract

The bifurcation of the birth of a closed invariant curve in the two-parameter unfolding of a two-dimensional diffeomorphism with a homoclinic tangency of invariant manifolds of a hyperbolic fixed point of neutral type (i.e. such that the Jacobian at the fixed point equals to 1) is studied. The existence of periodic orbits with multipliers $e^{\pm i\psi}$ $(0 < \psi < \pi)$ is proved and the first Lyapunov value is computed. It is shown that, generically, the first Lyapunov value is non-zero and its sign coincides with the sign of some separatrix value (i.e. a function of coefficients of the return map near the global piece of the homoclinic orbit).

Introduction.

Homoclinic orbits are one of the most interesting object of study in the theory of dynamical systems, because their presence leads to nontrivial dynamics. Recall that the Poincaré homoclinic orbit is an orbit which is biasymptotic to a saddle periodic orbit, i.e. it is an orbit lying in the intersection of the stable and unstable invariant manifolds of the saddle. If this intersection is transverse, the homoclinic orbit is called rough; otherwise, it is called an orbit of homoclinic tangency.

It is well-known that the set of all orbits lying entirely in a small neighborhood of a rough homoclinic orbit is hyperbolic and has a nontrivial structure which admits a complete description in terms of symbolic dynamics [1]. The situation is drastically different in the case of homoclinic tangency. Here, the complete study is proven [2, 3, 4] to be impossible in any finite-parameter unfolding. However, some main bifurcations of periodic orbits have been studied sufficiently well [5, 6, 7, 4]. It occurred that the basic feature of the bifurcation of homoclinic tangency is the appearance of a large number (even infinitely many) coexisting periodic orbits of different topological types. This is closely connected with the so-called Newhouse phenomenon: systems with homoclinic tangencies are dense in open regions (the Newhouse regions) in the space of smooth dynamical systems [8, 9, 10, 11, 12]. We note that it is the Newhouse regions to which, presumably, the most of known systems with chaotic behavior belong, e.g. systems with quasistochastic and wild-hyperbolic strange attractors [13, 14, 15, 4, 16, 17]. Therefore, the question on which type periodic orbits (and more complicated invariant sets) can appear via homoclinic bifurcations is especially important. For general (codimension one) homoclinic bifurcations this question was solved (including the multidimensional case) in [7, 4, 18] where necessary and sufficient conditions for the birth of periodic orbits of a given topological type were obtained and the appearance of invariant tori and even infinitely many coexisting strange attractors was established in some situations.

In the two-dimensional case, it is known since [5] that, generically, the bifurcation of homoclinic tangency produces, along with saddle periodic orbits, either stable or completely unstable ones. The type of stability depends on wether the saddle value σ (i.e. the absolute value of the product of the multipliers of the saddle fixed point) is greater or less than 1. As a result, in the Newhouse regions close to two-dimensional diffeomorphisms with a homoclinic tangency systems with infinitely many coexisting saddle and stable (if $\sigma < 1$) or saddle and completely unstable ($\sigma > 1$) periodic orbits are dense [19]. It is also known [2, 3, 20] that diffeomorphisms with infinitely many arbitrarily degenerate periodic orbits are dense in these regions. Note that these degenerate periodic orbits have exactly one multiplier equal to $+1$ or -1 with an arbitrarily large number of Lyapunov values (may be all of them) vanishing[1]. When $\sigma \neq 1$ there cannot be other degeneracies [7, 4, 18] and, in particular, no close diffeomorphism can have closed invariant curves or periodic orbits with the multipliers $e^{\pm i\varphi}$.

In the present paper we show that if $\sigma = 1$ at the moment of homoclinic tangency, then, along with such usual for systems with homoclinic tangencies bifurcations as a saddle-node and a period-doubling, the Andronov-Hopf bifurcations connected with the birth of closed invariant curves from periodic orbits with the multipliers $e^{\pm i\psi}$ take place when the homoclinic tangency unfolds.

Note that if both multipliers of the saddle are positive, two different cases are possible depending on the sign of the separatrix value R defined below (formula (5)). Namely, if $R < 0$, we show that asymptotically stable invariant curves are born near the homoclinic tangency, whereas at $R > 0$ we show the birth of unstable invariant curves. In the case of negative multipliers both stable and unstable invariant curves are born at $R \neq 0$.

Let us proceed to detailed formulation of the results (theorems A and B below). Let f_0 be a two-dimensional orientation-preserving C^r-diffeomorphism ($r \geq 4$) satisfying the following conditions.

A) f_0 has a saddle fixed point O with the multipliers λ, γ such that $|\lambda| < 1, |\gamma| > 1$;

B) the saddle value $\sigma = \lambda\gamma$ equals to 1 (i.e. O is a point of neutral type);

C) the stable and unstable manifolds W^s_0 and W^u_0 of the saddle O have a quadratic homoclinic tangency at the points of some homoclinic orbit Γ_0 (Fig.1a).

[1] For instance, diffeomorphisms with periodic orbits for which the first return map is locally identical on the center manifold are dense in the Newhouse regions [3, 20].

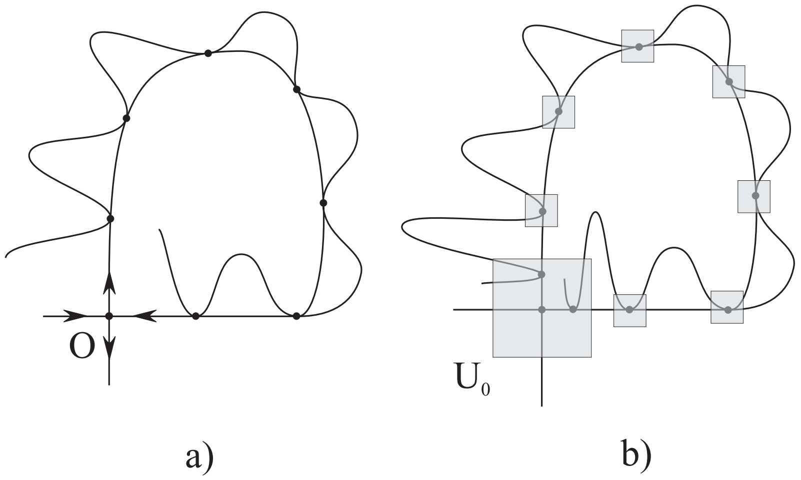

Figure 1

The diffeomorphisms C^r-close to f_0 and satisfying the same conditions A)-C) compose a bifurcational surface $H \in \mathrm{Diff}^r$ of codimension two. Let f_μ, $\mu \equiv (\mu_1, \mu_2)$, be a two-parameter family passing through f_0 at $\mu_1 = \mu_2 = 0$. Assume that

D) the family f_μ is C^r-smooth with respect to all variables and μ and it is transverse to the bifurcational surface H at $\mu = 0$.

It is not hard to understand which should be the nature of the governing parameters μ. One of them, say μ_1, must control the position of the invariant manifolds of O near the points of homoclinic tangency (i.e. we choose μ_1 as the splitting parameter near some homoclinic point). The second parameter μ_2 must control the saddle value at O, i.e.

$$\sigma(\mu) = 1 + \mu_2.$$

Let us take a sufficiently small neighborhood $U = U(O \cup \Gamma_0)$ of the closure of the homoclinic orbit Γ_0. It is the union of a small disc U_0 around O and a finite number of small neighborhoods of those points of Γ_0 which do not lie in U_0 (Fig.1b). We will call as *a p-round periodic orbit* a periodic orbit of f_μ which lies entirely in U and visits every component of $U \backslash U_0$ exactly p times on the period (i.e. every such orbit, before it returns to its initial point, runs U exactly p times).

In the present paper we study the bifurcations of single-round ($p = 1$) periodic orbits in U (for families f_μ satisfying A)–D)). Such orbits correspond to fixed points of the first return map $T_1 T_0^k$ ($k = \bar{k}, \bar{k}+1, \ldots$ for some sufficiently large $\bar{k}$) near the homoclinic orbit. Here T_0 is the so-called *local map* which is the diffeomorphism f_μ restricted to the neighborhood U_0 of its fixed point O_μ. The *global map* T_1 is some power of f_μ acting from a small neighborhood of some point of Γ_0 in $W^u_{loc} \cap U_0$ into a small neighborhood of some point of Γ_0 in $W^s_{loc} \cap U_0$.

The map T_0 is C^r-smooth and it has a saddle fixed point O_μ at all small μ (at $\mu = 0$

it is the original point O). It is shown in [21, 22] that in some C^{r-1}-coordinates (x,y) in U_0 the map T_0 has the following form for all small μ:

$$\bar{x} = \lambda(\mu)x + h(x,y,\mu)x^2y \ , \ \ \bar{y} = \gamma(\mu)y + g(x,y,\mu)xy^2 \ , \tag{1}$$

where $h(x,y,\mu)xy \in C^{r-1}$, $g(x,y,\mu)xy \in C^{r-1}$. Hereafter, we assume that the map T_0 is brought to this form.

In these coordinates, the fixed point O_μ is in the origin and $W^s_{loc}(\mu) \cap U_0$ and $W^u_{loc}(\mu) \cap U_0$ are segments of straight lines $y=0$ and $x=0$, respectively. By assumption C), at $\mu = 0$, the homoclinic orbit Γ_0 have points both in $W^u_{loc} \cap U_0$ and in $W^s_{loc} \cap U_0$. Chose a pair of such points $M^+(x^+,0) \in W^s_{loc} \cap U_0$ and $M^-(0,y^-) \in W^u_{loc} \cap U_0$; without loss of generality we may assume $x^+ > 0$, $y^- > 0$. Let Π_0 and Π_1 be sufficiently small neighborhoods of M^+ and M^-, respectively. We denote the coordinates in Π_0 and Π_1 as, respectively, (x_0,y_0) and (x_1,y_1).

By construction, there exists such positive n_0 that $M^+ = f_0^{n_0}(M^-)$. We define the global map T_1 as follows:

$$T_1 \equiv f_\mu^{n_0} \ : \ \Pi_1 \to \Pi_0. \tag{2}$$

It is defined at all small μ and it is, at least, C^{r-1}-smooth (in the coordinates for which T_0 has the form (1)).

Let us write T_1 in the following form:

$$\bar{x}_0 - x^+ = F(x_1, y_1 - y^-, \mu) \ , \ \ \bar{y}_0 = G(x_1, y_1 - y^-, \mu) \ , \tag{3}$$

where $F(0,0,0) = 0$, $G(0,0,0) = 0$. According to the condition C), the image of the segment $\{x_1 = 0\}$ by T_1 must have a quadratic tangency with $\{y_0 = 0\}$ at $\mu = 0$. Hence,

$$\frac{\partial G(0,0,0)}{\partial y_1} = 0 \ , \ \ \frac{\partial^2 G(0,0,0)}{\partial y_1^2} = 2d \neq 0$$

Thus, one can write

$$\begin{array}{l} F \equiv ax_1 + b(y_1 - y^-) + e_{20}x_1^2 + e_{11}x_1(y_1 - y^-) + e_{02}(y_1 - y^-)^2 + \\ O[(|x_1| + |y_1 - y^-|)^3] \ , \\ G \equiv \mu_1 + cx_1 + d(y_1 - y^-)^2 + f_{20}x_1^2 + f_{11}x_1(y_1 - y^-) + \\ +f_{03}(y_1 - y^-)^3 + f_{30}x_1^3 + f_{21}x_1^2(y_1 - y^-) + f_{12}x_1(y_1 - y^-)^2 + \\ +o[(|x_1| + |y_1 - y^-|)^3] \ , \end{array} \tag{4}$$

where the coefficients $a, b, ..., f_{03}$ (as well as x^+ and y^-) are some functions of μ. Since T_1 preserves orientation,

$$bc < 0.$$

Note that we choose the parameter μ_1 such that it enters (in the main order) the right-hand side of the equation for $\bar{y}_0$ additively. It means that μ_1 is the splitting parameter for the invariant manifolds of O near the homoclinic point M^+.

Let us introduce *the separatrix value*

$$R \equiv 2a - \frac{b}{d}f_{11} - 2\frac{c}{d}e_{02} \tag{5}$$

where the coefficients of the global map T_1 (see (4)) are taken at $\mu = 0$. Note that it is important in this definition that the coordinates are chosen such that the local map T_0 has the form (1)).

Theorem A. *If $R \neq 0$, in the parameter plane (μ_1, μ_2) there exists a sequence of open regions Δ_k, accumulating at $\mu = 0$ as $k \to +\infty$, such that the diffeomorphism f_μ has a closed invariant curve at all $\mu \in \Delta_k$. At $\lambda > 0, \gamma > 0$, the invariant curves are asymptotically stable at $R < 0$ and unstable at $R > 0$. If $\lambda < 0, \gamma < 0$, then the invariant curves at $\mu \in \Delta_k$ are stable or unstable depending on parity of k.*

We prove this theorem by means of the study of bifurcations of single-round periodic orbits or, what is the same, of bifurcations of fixed points of the maps $T_k = T_1 T_0^k$ at all sufficiently large k: $k = \bar{k}, \bar{k}+1, \ldots$. By definition, a single-round periodic orbit has exactly one point in each of the neighborhoods Π_0 and Π_1 . Let $M_0 \in \Pi_0$ and $M_1 \in \Pi_1$ be such points. Then, $M_0 = T_1(M_1)$ and there exists such an integer k that $M_1 = T_0^k(M_0)$. Thus, the point $M_0 \in \Pi_0$ is a fixed point of $T_k \equiv T_1 T_0^k$ (the period of the corresponding orbit of f_μ equals $k + n_0$, see (2)).

Theorem B. 1. *In the plane of parameters (μ_1, μ_2), for every sufficiently large k there exist bifurcational curves L_k^+, L_k^- and L_k^φ, corresponding to single-round periodic orbits (fixed points of T_k) with multipliers $+1$, -1 and $e^{\pm i\psi}$ $(0 < \psi < \pi)$, respectively. The curves L_k^+ and L_k^- accumulate to the line $\mu_1 = 0$ as $k \to +\infty$. The curves L_k^φ connect points B_k^{++} and B_k^{--} on, respectively, L_k^+ and L_k^- and accumulate at the point $\mu_1 = \mu_2 = 0$.*
2. *At μ from the region D_k between the curves L_k^+ and L_k^- the diffeomorphism f_μ has two single-round periodic orbits one of which, Q_k, is saddle and the other, P_k is asymptotically stable at $\mu \in D_k^s$ and completely unstable at $\mu \in D_k^u$ where D_k^s (D_k^u) is the region in D_k to the left (resp., to the right) of L_k^φ. The transitions into the region D_k across the curves L_k^+ (without the point $B_k^{++} = L_k^+ \cap L_k^\varphi$) and L_k^- (without the point $B_k^{--} = L_k^- \cap L_k^\varphi$) correspond, respectively, to generic saddle-node and period-doubling bifurcations of P_k (on L_k^+ the orbits P_k and Q_k merge together). At $\mu = B_k^{++}$ the orbit P_k has two multipliers equal to 1, and both the multipliers are equal to -1 at $\mu = B_k^{--}$.*
3. *If $R\lambda^k < 0$, the boundary L_k^φ of stability of P_k is "safe": the first Lyapunov value is negative, so at the transition across L_k^φ (except for two points for which $\psi = \pi/2$, $2\pi/3$) towards the increase of μ_2 the orbit P_k becomes unstable and a stable invariant curve is born from it.*

If $R\lambda^k > 0$, the boundary L_k^φ is "dangerous": the first Lyapunov value is positive, so at the transition across L_k^φ (except for two points for which $\psi = \pi/2$, $2\pi/3$) towards the decrease of μ_2 an unstable invariant curve is born from P_k.

See figure 2 as an illustration to theorem B.

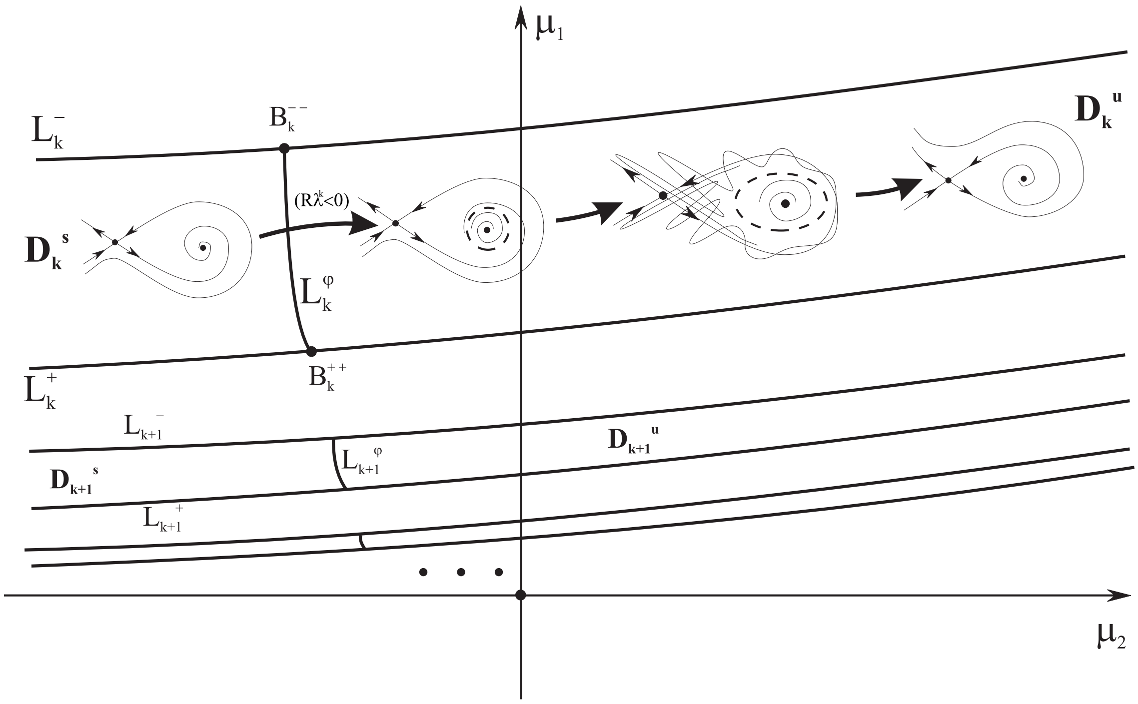

Figure 2
An illustration to theorem B

It is obvious that theorem A follows from theorem B. Here, the region Δ_k is some part of D_k adjoining to L_k^e from the left if $R\lambda^k > 0$ and from the right if $R\lambda^k < 0$, corresponding to the existence of the invariant curve.

The content of the paper is as follows. In section 2 we study properties of iterations of the local map T_0. In section 3 the first return maps T_k are constructed. Here we prove that the map T_k for sufficiently large k may be brought to a certain form close to the quadratic Henon family. Unlike some standard results [23, 2, 4], we take into account also small terms of the order $O(\lambda^k)$. In section 4 bifurcations of fixed points of T_k are studied, and the first Lyapunov value is calculated for the fixed point undergoing the Andronov-Hopf bifurcation. In section 5 theorems A and B are finally proved.

1 Properties of the local map $T_0(\mu)$.

The map T_0 is defined as the restriction of diffeomorphism f_μ of the neighbourhood U_0 , i.e., $T_0(\mu) \equiv f_{\mu|U_0}$. The map T_0 has, at all sufficiently small μ, the fixed saddle point O_μ. It is well-known that in some C^r-coordinates (x, y) on U_0 the map T_0 can be written as

$$\bar{x} = \lambda(\mu)x + h(x, y, \mu) \ , \ \ \bar{y} = \gamma(\mu)y + g(x, y, \mu) \ , \qquad (6)$$

where $h(0,y,\mu) = g(x,0,\mu) \equiv 0$. Here, the axes x and y are eigendirections of the Jacobi matrix of T_0 at O_μ, and the local stable and unstable manifolds of O_μ are straightened.

Note that the form (6) of T_0 is not very convenient from the technical point of view since the right-hand sides of (6) contain too many non-resonant terms. For example, if to write functions h and g in the following extended form

$$h(x,y,\mu) \equiv \varphi_1(x,\mu)x + \varphi_2(y,\mu)x + \tilde{h}(x,y,\mu)x^2y \ ,$$
$$g(x,y,\mu) \equiv \phi_1(y,\mu)y + \phi_2(x,\mu)y + \tilde{g}(x,y,\mu)xy^2 \ ,$$

where $\varphi_\alpha(0,\mu) \equiv 0$, $\phi_\alpha(0,\mu) \equiv 0$, $\alpha = 1,2$, then one can see that functions φ and ϕ contain only nonresonant monomials[2]. It was shown in [21, 22] (see also [24] for the general multidimensional case) that such "always nonresonant monomials" can all be nullified by a sufficiently smooth change of variables. Namely, the following result is valid [21, 22] :

There exist such $\delta_1 > 0$ and $\delta_2 > 0$ that, at $\|(x,y)\| \leq \delta_1$ and $\|\mu\| \leq \delta_2$, the map $T_0(\mu)$ is brought to the form

$$\bar{x} = \lambda(\mu)x + h_1(x,y,\mu)x^2y \ \ , \ \ \bar{y} = \gamma(\mu)y + g_1(x,y,\mu)xy^2 \ \ , \tag{7}$$

where $h_1(x,y,\mu)xy \in C^{r-1}$, $g_1(x,y,\mu)xy \in C^{r-1}$, by means of a C^{r-1}-smooth transformation of coordinates (this transformation is C^{r-2} with respect to parameters).

By (7), the point O_μ is in the origin (at all sufficiently small μ) and $W^s_{loc}(\mu) \cap U_0$ and $W^u_{loc}(\mu) \cap U_0$ have, respectively, equations $y = 0$ and $x = 0$. At $\mu = 0$ let us choose in U_0 a pair of points of the orbit Γ_0 : $M^+(x^+,0)$ and $M^-(0,y^-)$, and take their sufficiently small rectangular neighbourhoods Π_0 and Π_1 . We denote the coordinates (x,y) in Π_0 and Π_1 as (x_0,y_0) and (x_1,y_1), respectively. Without loss of generality, we assume that $x^+ > 0$, $y^- > 0$. The neighborhoods Π_0 and Π_1 are defined as follows

$$\begin{aligned} \Pi_0 &= \{(x_0,y_0) \, \big| |x_0 - x^+| \leq \varepsilon_0 \ , \ |y_0| \leq \varepsilon_0\} \ , \\ \Pi_1 &= \{(x_1,y_1) \, \Big| |x_1| \leq \varepsilon_1 \ , \ |y_1 - y^-| \leq \varepsilon_1\} \ , \end{aligned} \tag{8}$$

where ε_0 and ε_1 are sufficiently small (so $T_0(\Pi_0) \cap \Pi_0 = \emptyset$, $T_0^{-1}(\Pi_1) \cap \Pi_1 = \emptyset$, in particular).

To study the maps $T_k \equiv T_1 T_0^k$ it is necessary, first of all, to have appropriate formulas and estimates for the maps $T_0^k : \Pi_0 \to \Pi_1$ for all sufficiently large k. To this aim, the form (7) of $T_0(\mu)$ is very convenient because the iterations of the map T_0 in form (7) are asymptotically close (as $k \to \infty$) to those in the linear case[3]. Namely, the following lemma holds.

[2]it is a consequence of the inequalities $|\lambda| < 1$ and $|\gamma| > 1$: a monomial $x^m y^n$ in the first equation of (6) is resonant if $\lambda^{m-1}\gamma^n = 1$, and in the second equation it is resonant if $\lambda^m \gamma^{n-1} = 1$

[3]Note, that a smooth linearization is impossible here because of the resonance $\lambda\gamma = 1$ at $\mu = 0$.

Lemma 1 *For any $\varepsilon > 0$ there exists $\bar{k} > 0$ such that for any $k \geq \bar{k}$ and $\|\mu\| \leq \varepsilon$ the map $T_0^k : \Pi_0 \to \Pi_1$ can be represented as follows:*

$$\begin{array}{l} x_1 = \lambda^k(\mu) x_0 (1 + k\hat{\gamma}^{-k} \hat{\xi}_k(x_0, y_1, \mu)) \ , \\ y_0 = \gamma^{-k}(\mu) y_1 (1 + k\hat{\gamma}^{-k} \hat{\eta}_k(x_0, y_1, \mu)) \ , \end{array} \tag{9}$$

where $\hat{\gamma} = \gamma(0)/(1+\varepsilon)$ and functions $\hat{\xi}_k \equiv \tilde{\xi}_k \cdot x_0 y_1$ and $\hat{\eta}_k \equiv \tilde{\eta}_k \cdot x_0 y_1$ and their derivatives (along with derivatives with respect to μ) up to the order $(r-2)$ are bounded, uniformly in k. The derivatives of order $(r-1)$ from the right-hand sides of (9) *tend to zero as $k \to \infty$.*

Proof. The proof of this lemma repeats closely the proof of an analogous statement (lemma 1.2 in [22]). Therefore, we prove here only the boundedness for the functions ξ_k and η_k themselves; the boundedness of derivatives is verified along the same lines (for more detail see [22, 24]).

We will use the method of the boundary-value problem [1, 25] in a modification of [26]. For the sake of simplicity, we write the map $T_0(\mu)$ in the form

$$\bar{x} \ = \ \lambda(\mu) x + \hat{h}(x, y, \mu), \quad \bar{y} \ = \ \gamma(\mu) y + \hat{g}(x, y, \mu) \tag{10}$$

where $\lambda(0)\gamma(0) = 1$ and , by (7), $\hat{h}(x, y, \mu) \equiv x^2 y(\beta_1(\mu) + O(|x| + |y|))$, $\hat{g}(x, y, \mu) \equiv xy^2(\beta_2(\mu) + O(|x| + |y|))$. Also, in the proof, we will use notations $\lambda(\mu) \equiv \lambda$, $\gamma(\mu) \equiv \gamma$.

Let us consider the following operator $\Phi : [(x_j, y_j)]_{j=0}^k \mapsto [(\bar{x}_j, \bar{y}_j)]_{j=0}^k$ where

$$\begin{array}{rcl} \bar{x}_j & = & \lambda^j x_0 + \sum\limits_{s=0}^{j-1} \lambda^{j-s-1} \hat{h}(x_s, y_s, \mu), \\ \\ \bar{y}_j & = & \gamma^{j-k} y_k - \sum\limits_{s=j}^{k-1} \gamma^{j-s-1} \hat{g}(x_s, y_s, \mu), \end{array} \tag{11}$$

$j = 0, 1, \ldots, k$. The operator Φ is defined on the set

$$Z(\delta) \ = \ \{ z = [(x_j, y_j)]_{j=0}^k, \ \|z\| \leq \delta \} \ ,$$

where the norm $\|\cdot\|$ is defined as the maximum of the absolute values of the components x_j, y_j of the vector z. Note that if $z_0 = [(x_j^0, y_j^0)]_{j=0}^k$ is a fixed point of Φ, then

$$(x_0^0, y_0^0) \xrightarrow{T_0} (x_1^0, y_1^0) \xrightarrow{T_0} \ldots \xrightarrow{T_0} (x_k^0, y_k^0),$$

i.e. the fixed point of Φ is the orbit of the map T_0.

For sufficiently small $\delta = \delta_0$ and $\|x_0\| \leq \delta_0/2$, $|y_k| \leq \delta_0/2$ the operator Φ maps the set $Z(\delta_0)$ into itself and is contracting on this set (see the proof in [26]). Thus, map (11) has a unique fixed point $z_0 = [(x_j^0(x_0, y_k), y_j^0(x_0, y_k)]_{j=0}^k$ which is the limit of iterations by Φ of any initial sequence in $Z(\delta_0)$, i.e., the coordinates x_j^0 and y_j^0 can be found by the successive approximations method. As an initial approximation we take the following (the solution of the linear problem)

$$x_j^{0(1)} = \lambda^j x_0 \ , \ y_j^{0(1)} = \gamma^{j-k} y_k$$

As it follows from (11), the second approximation has the form

$$\begin{aligned}
x_j^{0(2)} &= \lambda^j x_0 + \sum_{s=0}^{j-1} \lambda^{j-s-1}\lambda^{2s}\gamma^{s-k} x_0^2 y_k(\beta_1(\mu) + O(|\lambda|^s|x_0| + |\gamma|^{s-k}|y_k|)) = \\
&= \lambda^j x_0 + \lambda^j\gamma^{-k}\sum_{s=0}^{j-1} \lambda^{-1}\lambda^s\gamma^s x_0^2 y_k(\beta_1(\mu) + O(|\lambda|^s|x_0| + |\gamma|^{s-k}|y_k|)) = \\
&= \lambda^j x_0 + \lambda^j\gamma^{-k}\lambda^{-1}\sum_{s=0}^{j-1}(1+\mu_2)^s x_0^2 y_k(\beta_1(\mu) + O(|\lambda|^s|x_0| + |\gamma|^{s-k}|y_k|)),
\end{aligned}$$

$$\begin{aligned}
y_j^{0(2)} &= \gamma^{j-k} y_k + \sum_{s=j}^{k-1} \gamma^{j-s-1}\lambda^s\gamma^{2(s-k)} x_0 y_k^2(\beta_2(\mu) + O(|\lambda|^s|x_0| + |\gamma|^{s-k}|y_k|)) = \\
&= \gamma^{j-k} y_k + \gamma^{j-2k}\sum_{s=j}^{k-1} \gamma^{-1}\lambda^s\gamma^s x_0 y_k^2(\beta_2(\mu) + O(|\lambda|^s|x_0| + |\gamma|^{s-k}|y_k|)) = \\
&= \gamma^{j-k} y_k + \gamma^{j-2k-1}\sum_{s=j}^{k-1}(1+\mu_2)^s x_0 y_k^2(\beta_2(\mu) + O(|\lambda|^s|x_0| + |\gamma|^{s-k}|y_k|))
\end{aligned} \tag{12}$$

(in the last step, in both formulas here, we use the relation $\lambda\gamma = 1 + \mu_2$). Let $|\mu_2| \le \varepsilon$. Then

$$\sum_{s=0}^{j-1}(1+\mu_2)^s \le \sum_{s=0}^{j-1}(1+\varepsilon)^{j-1} = j(1+\varepsilon)^{j-1} \ ,$$

and, analogously,

$$\sum_{s=j}^{k-1}(1+\mu_2)^s \le (k-j)(1+\varepsilon)^{k-1} \ .$$

It follows from (12) that

$$\begin{aligned}
|x_j^{0(2)} - \lambda(\mu)^j x_0| &\le L_1 j\lambda^j\gamma^{-k}(1+\varepsilon)^j \ , \\
|y_j^{0(2)} - \gamma(\mu)^{j-k} y_k| &\le L_2(k-j)\gamma^{j-2k}(1+\varepsilon)^k \ ,
\end{aligned} \tag{13}$$

where L_1 and L_2 are some positive constants independent of j, k and μ. When we plug (13) in (11) as the initial guess for the next approximation, it is not hard to see that inequalities (13) will be valid for the next approximation too, with the same constants L_1 and L_2, etc.. Thus, we obtain the following formula for the coordinates x_j^0 and y_j^0 of the fixed point of Φ

$$\begin{aligned}
x_j^0 &= \lambda(\mu)^j x_0(1 + j\gamma(\mu)^{-k}(1+\varepsilon)^j)F_{jk}(x_0, y_k, \mu) \ , \\
y_j^0 &= \gamma(\mu)^{j-k} y_k(1 + (k-j)\gamma(\mu)^{-k}(1+\varepsilon)^k)G_{jk}(x_0, y_k, \mu) \ ,
\end{aligned} \tag{14}$$

where functions F_{jk} and G_{jk} are uniformly bounded in j and k. Assuming $j = k$ for x_j^0 and $j = 0$ for y_j^0 in (14) and taking into account $\gamma(\mu) = \gamma(0) + O\mu$, we obtain formula (9). □

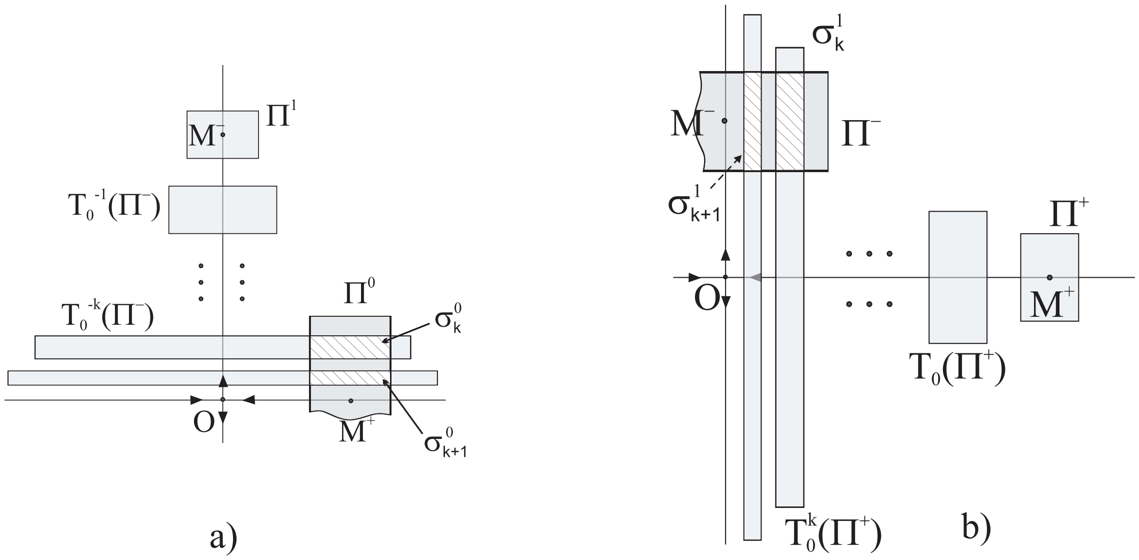

Figure 3

Hereafter, we will write (9) as

$$\begin{aligned} x_1 &= \lambda^k(\mu)x_0(1+\tilde{\gamma}^{-k}\xi_k(x_0,y_1,\mu))\ , \\ y_0 &= \gamma^{-k}(\mu)y_1(1+\tilde{\gamma}^{-k}\eta_k(x_0,y_1,\mu))\ , \end{aligned} \tag{15}$$

where $\tilde{\gamma}$ depends on ε only and $\tilde{\gamma} = \max\{|\lambda(\mu)|^{-1}, |\gamma(\mu)|\}$ for $|||\mu||| \le \varepsilon$.

2 Construction of the first return maps T_k

The domain of the first return map is the set of those points in Π_0 whose iterations by T_0 reach Π_1. One can easily see that it is a union of a countable number of horizontal strips $\sigma_k^0 = \Pi_0 \cap T_0^{-k}\Pi_1$, $k = \bar{k}, \bar{k}+1, \ldots$ which accumulate on the segment $\Pi_0 \cap W^s_{\text{loc}}$ as $k \to \infty$ (Figure 3a). In turn, the images of the strips σ_k^0 by T_0^k are vertical strips $\sigma_k^1 = \Pi_1 \cap T_0^k\Pi_0$ lying in Π_1. These strips accumulate on the segment $\Pi_1 \cap W^u_{\text{loc}}$ as $k \to \infty$ (Figure 3b).

Indeed, if we define Π_0 and Π_1 by formula (8) for some sufficiently small ε_0 and ε_1, then it follows immediately from (15) that

$$\begin{aligned} &\sigma_k^0 = \{(x_0,y_0)|\ |x_0 - x^+| \le \varepsilon_0\ , \\ &\gamma^{-k}(y^- - \varepsilon_1 + O(\gamma^{-k})) \le y_0 \le \gamma^{-k}(y^- + \varepsilon_1 + O(\gamma^{-k}))\}\ , \\ \\ &\sigma_k^1 = \{(x_1,y_1)|\ \lambda^k(x^+ - \varepsilon_0 + O(\gamma^{-k})) \le x_1 \le \lambda^k(x^+ + \varepsilon_0 + O(\gamma^{-k}))\ , \\ &|y_1 - y^-| \le \varepsilon_1\} \end{aligned} \tag{16}$$

The map $T_k \equiv T_1 T_0^k$ is defined on the strip σ_k^0. Using (4) and (15) this map can be represented in the following form

$$\begin{array}{l}\bar{x}_0 = x^+ + a\lambda^k x_0 + b(y_1 - y^-) + e_{02}(y_1 - y^-)^2 + e_{11}\lambda^k x_0(y_1 - y^-) + \\ + E_1(x_0, y_1 - y^-, \mu) \ , \\ \gamma^{-k}\bar{y}_1(1 + \tilde{\gamma}^{-k}\eta_k(\bar{x}_0, \bar{y}_1, \mu)) = \mu_1 + \\ + [c\lambda^k x_0 + f_{11}\lambda^k x_0(y_1 - y^-)](1 + \tilde{\gamma}^{-k}\xi_k(x_0, y_1, \mu)) + d(y_1 - y^-)^2 + \\ + f_{12}\lambda^k x_0(y_1 - y^-) + f_{20}\lambda^{2k}x_0^2 + f_{21}\lambda^{2k}x_0^2(y_1 - y^-) + f_{03}(y_1 - y^-)^3 + \\ + E_2(x_0, y_1 - y^-, \mu),\end{array} \tag{17}$$

where

$$\begin{array}{l}E_1 = O(|\lambda|^k\tilde{\gamma}^{-k}) + O((y - y^-)^3) + |\lambda|^k O((y - y^-)^2) + |\lambda|^{2k}O(y - y^-), \\ E_2 = O(\lambda^{2k}\tilde{\gamma}^{-k})) + o((y - y^-)^3) + |\lambda|^k o((y - y^-)^2) + |\lambda|^{2k}o(y - y^-).\end{array} \tag{18}$$

Hereafter we will adhere the following notation: the symbols $O(\varphi(k))$ and $o(\varphi(k))$ for some function $\varphi(k)$ denote functions (of all variables) which are bounded by a constant (resp., an infinitely small factor) times $\|\varphi(k)\|$ as $k \to +\infty$, and the same asymptotic behavior is shown by all their derivatives up to the order $(r-2)$; the symbol $o(x^m y^n)$ denotes a function vanishing, if $n \neq 0$, at $y = 0$ along with the first n derivatives with respect to y and, if $m \neq 0$, vanishing at $x = 0$ along with the first m derivatives with respect to x; the symbol $O(x^m y^n)$ denotes a function of the form $const \cdot x^m y^n + o(x^m y^n)$; below, we will also write $O(|f| + |g|)$ instead of $O(f) + O(g)$ and $o(|f| + |g|)$ instead of $o(f) + o(g)$, by esthetic reasons.

To study map (17), we use the rescaling technique. Thus, the following result takes place.

Rescaling lemma. *By means of an affine transformation of coordinates and parameters the map T_k can be brought to the form*[4]

$$\begin{array}{l}\bar{X} = Y - \dfrac{1}{bd}(e_{02} + \ldots)\gamma^{-k}Y^2 + O(\tilde{\gamma}^{-2k}) \ , \\ \bar{Y} = M_1 - M_2 X - Y^2 + \left(2a - \dfrac{b}{d}f_{11} + \ldots\right)\lambda^k XY + \\ + \dfrac{1}{d^2}(f_{03} + \ldots)\gamma^{-k}Y^3 + o(\gamma^{-k})o(Y^3) + O(\tilde{\gamma}^{-2k})\end{array} \tag{19}$$

where

$$\begin{array}{l}M_1 = -d\gamma^{2k}[\mu_1 - \gamma^{-k}(y^- + \ldots) + c\lambda^k(x^+ + \ldots)], \\ M_2 = -bc(1 + \mu_2)^k(1 + \ldots);\end{array} \tag{20}$$

the dots stand for the terms, independent of (X, Y), which tend to zero as $k \to +\infty$.

The new coordinates (X, Y), as well as the new parameter M_1, run all finite values at sufficiently large k and M_2 runs all finite positive values[5]*. Namely, there exist positive constants $C_1, ..., C_5$ such that X, Y, M_1, M_2 take values in the following domain*

$$\begin{array}{l}|X| \leq C_1\varepsilon_0|\gamma|^k \ , \ |Y| \leq C_2\varepsilon_1|\gamma|^k \ , \\ |M_1| \leq C_3\varepsilon\gamma^{2k} \ , \ C_4(1 - \varepsilon)^k \leq M_2 \leq C_5(1 + \varepsilon)^k \ ,\end{array} \tag{21}$$

[4] Note that in comparison with the known results on the rescaling near homoclinic [23, 2, 7, 4] or heteroclinic [27, 28] tangencies we compute also terms which are asymptotically small as $k \to +\infty$.

[5] $M_2 > 0$ since f_μ preserves orientation.

where ε_0 and ε_1 are the diameters of Π_0 and Π_1, respectively, and ε is the size of the interval of variation of μ.

Proof. First, we shift the origin to the point (x^+, y^-), i.e. we introduce the coordinates $x = x_0 - x^+$, $y = y_1 - y^-$. The map (17) takes the form

$$\begin{array}{l} \bar{x} = a\lambda^k x + a\lambda^k x^+ + (b + e_{11}\lambda^k x^+)y + e_{11}\lambda^k xy + e_{02}y^2 + \\ +O(|\lambda|^k\tilde{\gamma}^{-k} + (|y| + |\lambda|^k)^3)\ , \\ (\bar{y} + y^-)(1 + \tilde{\gamma}^{-k}\eta_k(\bar{x} + x^+, \bar{y} + y^-, \mu)) = \\ = \mu_1\gamma^k + \lambda^k\gamma^k(x + x^+)[c + f_{11}y][(1 + \tilde{\gamma}^{-k}\xi_k(x + x^+, y + y^-, \mu)] + \\ +d\gamma^k y^2(1 + \frac{f_{12}}{d}\lambda^k(x + x^+))\lambda^{2k}\gamma^k x^2(f_{20} + f_{21}y) + f_{03}\gamma^k y^3 + \\ +O(\lambda^k\tilde{\gamma}^{-k}(1 + \mu_2)^k) + \gamma^k o((|y_1 - y^-| + |\lambda|^k)^3) \end{array} \qquad (22)$$

By an additional shift of coordinates

$$x \to x_{new} + a\lambda^k x^+ - \frac{bf_{11}x^+}{2d}\lambda^k + o(\lambda^k)\ \ ,\ \ y \to y_{new} - \frac{bf_{11}x^+}{2d}\lambda^k + o(\lambda^k)$$

we can bring the map to such form where the right-hand side of the first equation does not contain the constant term and the right-hand side of the second equation does not contain the term linear in y. Then, (22) is rewritten as follows

$$\begin{array}{l} \bar{x} = (a + \ldots)\lambda^k x + by(1 + \nu_1\lambda^k) + +(e_{02} + \ldots)y^2 + \\ +O(|\lambda|^k|x|(|y| + |x|) + |y|^3)\ , \\ \bar{y} + \tilde{\gamma}^{-k}O(|\bar{y}| + |\bar{x}|) = \gamma^k[\mu_1 - \gamma^{-k}(y^- + \ldots) + c\lambda^k(x^+ + \ldots)] + \\ (c + \ldots)\lambda^k\gamma^k x + (f_{11} + \ldots)\lambda^k\gamma^k xy + d\gamma^k y^2(1 + \nu_2\lambda^k) + f_{03}\gamma^k y^3 + \\ +O(|\lambda\gamma|^k|x|y^2 + \lambda^{2k}|\gamma|^k x^2) + \gamma^k o(y^3) \end{array} \qquad (23)$$

where $\nu_1 = e_{11}x^+/b - e_{02}f_{11}x^+/(bd) + \ldots$, $\nu_2 = f_{12}x^+/d - 3f_{11}f_{03}x^+/d + \ldots$; the dots stand for terms asymptotically vanishing as $k \to \infty$ and independent of (x, y).

Let us make the following rescaling of the variables

$$x = -\frac{b(1 + \nu_1\lambda^k)}{d(1 + \nu_2\lambda^k)}\gamma^{-k}X\ ,\ y = -\frac{1}{d(1 + \nu_1\lambda^k)}\gamma^{-k}Y.$$

Then, map (23) is brought to the form

$$\begin{array}{l} \bar{X} = Y + (a + \ldots)\lambda^k X - \frac{1}{bd}(e_{02} + \ldots)\gamma^{-k}Y^2 + O(|\lambda|^k\tilde{\gamma}^{-k}), \\ \bar{Y} + \rho_k^1\bar{Y} + \rho_k^2\bar{X} = \hat{M}_1 + bc(\lambda\gamma)^k X(1 + \ldots) - Y^2 - \frac{b}{d}(f_{11} + \ldots)\lambda^k XY + \\ +(\frac{f_{03}}{d^2} + \ldots)\gamma^{-k}Y^3 + o(\gamma^{-k})o(Y^3) + O(\tilde{\gamma}^{-2k}) \end{array} \qquad (24)$$

where $\rho_k^{1,2}$ are some coefficients of order $\tilde{\gamma}^{-k}$, and

$$\hat{M}_1 = -d\gamma^{2k}[\mu_1 - \gamma^{-k}(y^- + \ldots) + c\lambda^k(x^+ + \ldots)].$$

Substituting expressions for $\bar{X}$ into the left-hand side of the second equation of (24) we obtain

$$\begin{aligned}
&\bar{X} = Y + (a + \ldots)\lambda^k X - \frac{1}{bd}(e_{02} + \ldots)\gamma^{-k}Y^2 + O(\tilde{\gamma}^{-2k}),\\
&\bar{Y} = -\rho_k^2 Y + \hat{M}_1/(1+\rho_k^1) + bc(\lambda\gamma)^k X(1+\ldots) - Y^2(1+\rho_k^1)^{-1} -\\
&-\frac{b}{d}(f_{11} + \ldots)\lambda^k XY + \frac{f_{03+\ldots}}{d^2}\gamma^{-k}Y^3 + o(\gamma^{-k})o(Y^3) + O(\tilde{\gamma}^{-2k}).
\end{aligned} \tag{25}$$

Denote the coefficient of the linear in X term (taken with the sign "minus") in the second equation of (25) as M_2, i.e.

$$M_2 = -bc(\lambda^k\gamma)^k(1+\ldots).$$

Make the new change

$$X = X_{new}(1+\rho_k^1),\ Y = Y_{new}(1+\rho_k^1) - (a+\ldots)\lambda^k X$$

which brings the map to the form

$$\begin{aligned}
&\bar{X} = Y - \frac{1}{bd}(e_{02} + \ldots)\gamma^{-k}Y^2 + O(\tilde{\gamma}^{-2k}),\\
&\bar{Y} = -\rho_k^3 Y + \hat{M}_1/(1+\rho_k^1)^2 - M_2 X - Y^2 + (2a - \frac{b}{d}f_{11} + \ldots)\lambda^k XY +\\
&+\frac{f_{03+\ldots}}{d^2}\gamma^{-k}Y^3 + o(\gamma^{-k})o(Y^3) + O(\tilde{\gamma}^{-2k})
\end{aligned} \tag{26}$$

where the coefficient ρ_k^3 is of order $\tilde{\gamma}^{-k}$. One more shift of coordinates

$$X \to X_{new} + \frac{1}{2}\rho_k^3,\ Y \to Y_{new} + \frac{1}{2}\rho_k^3$$

eliminates the linear in Y term in the second equation and brings, finally, the map to the requried form (19) (with the new constant term M_1 satisfying (20)). This completes the proof of the lemma.

3 Bifurcations of the fixed points of the first return map.

By the rescaling lemma, the map T_k in the rescaled coordinates X, Y is close to the standard quadratic Henon map

$$\bar{X} = Y \quad , \quad \bar{Y} = M_1 - M_2 - Y^2 \tag{27}$$

where the Jacobian M_2 of (27) may take arbitrary finite positive values (see inequalities (20)). The bifurcations of the fixed points of the Henon family are well known. The corresponding bifurcational diagram on the half-plane $M_2 > 0$ is represented in Figure 4. It contains three bifurcational curves: $L^+ \ : M_1 = -\frac{1}{4}(1+M_2)^2$; $L^- \ : M_1 = \frac{3}{4}(1+M_2)^2$ and $L^\varphi \ : \hat{M}_2 = -1, -1 < \hat{M}_1 < 3$. At $(M_1, M_2) \in L^+$

map (27) possesses a fixed point with a multiplier $\nu_1 = +1$; at $M_2 \neq 1$ this point is a saddle-node either with the stable sector at $M_2 < 1$ or with the unstable sector at $M_2 > 1$. At $(M_1, M_2) \in L^-$ map (27) possesses a fixed point with a multiplier $\nu_1 = -1$; this point is stable at $M_2 < 1$ and unstable at $M_2 > 1$. At $M_2 = 1$ the Henon map is area-preserving. It is known that this map has a parabolic fixed point ($\nu_1 = \nu_2 = 1$) of unstable type at $M_1 = -1$; a parabolic fixed point ($\nu_1 = \nu_2 = -1$) of stable type at $M_1 = 3$ and an elliptic fixed point ($\nu_{1,2} = e^{\pm i\psi}$) at $-1 < M_1 < 3$ (i.e., at $(M_1, M_2) \in L^\varphi$). The elliptic fixed point is generic if $\psi \notin \{\pi/2, 2\pi/3, arccos(-1/4)\}$ [29].

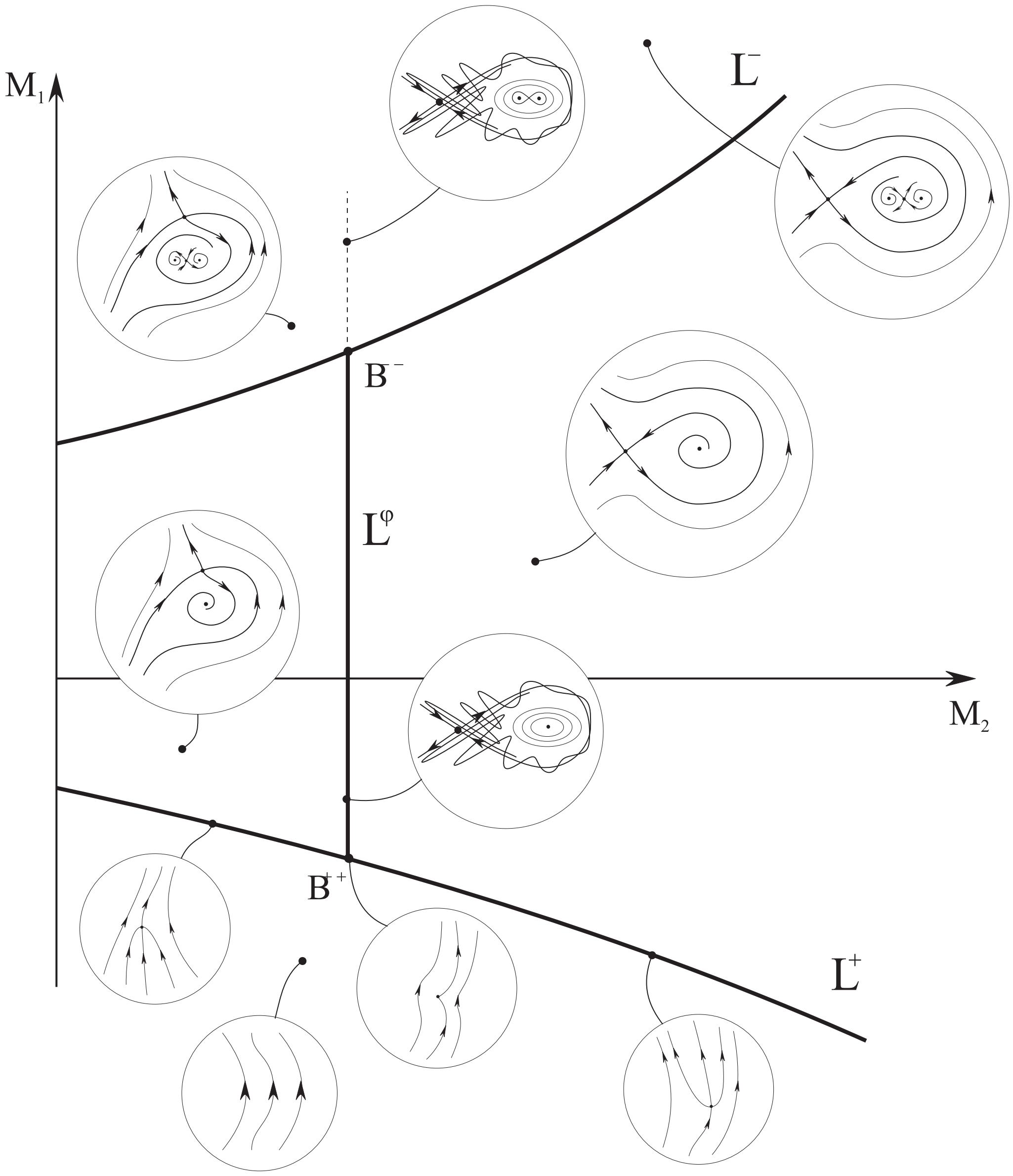

Figure 4

Bifurcation diagram for fixed points in the Henon map for M_2>0

Since the rescaled map T_k given by (19) is C^{r-2}-close to Henon map, it also has in the half-plane (defined by inequalities (20)) three bifurcational curves $L^+_{(k)}, L^-_{(k)}$ and $L^{\varphi}_{(k)}$ close to the curves L^+, L^- and L^{φ}, respectively. Direct calculations give us the following equations for $L^+_{(k)}$ and $L^-_{(k)}$:

$$
\begin{aligned}
L^+_{(k)} : M_1 &= -\frac{(1+M_2)^2}{4}(1+O(|\lambda|^k+|\gamma|^{-k})), \\
L^-_{(k)} : M_1 &= \frac{3(1+M_2)^2}{4}(1+O(|\lambda|^k+|\gamma|^{-k})).
\end{aligned}
\tag{28}
$$

The equation of $L^{\varphi}_{(k)}$ in a parametric form (where the argument ψ of the multiplier is taken as a parameter) can be written as follows

$$
\begin{array}{l}
M_1 = cos^2\psi - 2\cos\psi + O(|\lambda|^k+|\gamma|^{-k}) \ , \\
M_2 = 1+\cos\psi\left[2\gamma^{-k}(e_{02}+\ldots) - (2a-\frac{b}{d}f_{11}+\ldots)\lambda^k\right]
\end{array}
\tag{29}
$$

where $0<\psi<\pi$.

In order to prove that a closed invariant curve is born while crossing the curve $L^{\varphi}_{(k)}$, it is necessary to calculate the first Lyapunov value G_1 at the weak focus (the fixed point of T_k with the multipliers $e^{\pm i\psi}$). We show that the following result takes place

Lemma 2 *The following formula is valid for the first Lyapunov value G_1*

$$
G_1 = \frac{R}{16(1-\cos\psi)}\cdot\lambda^k + o(\lambda^k)
\tag{30}
$$

where R is the separatrix value given by (5).

Proof. The first step of the proof is to write the Taylor expansion for the map T_k given by (19) at the weak focus. Let us fix the values of parameters M_1 and M_2 so that $(M_1,M_2)\in L^{\varphi}_{(k)}$, i.e. T_k has a fixed point P_k with the multipliers $e^{\pm i\psi}$, for some $\psi\in(0,\pi)$. We will denote the corresponding values of M_1 and M_2 as M_1^{ψ} and M_2^{ψ}. By (20), we have $M_2^{\psi} = -bc\lambda^k\gamma^k(1+\ldots)$, so it follows from the second equation of (29) that

$$
\gamma^{-k} = -bc\lambda^k(1+\ldots)
\tag{31}
$$

for values of the parameters at the curve $L^{\varphi}_{(k)}$. Thus, map (19) near the weak focus can be written as follows

$$
\begin{array}{l}
\bar X = Y + h_{02}\lambda^k Y^2 + O(\tilde\gamma^{-2k}) \ , \\
\bar Y = M_1 - M_2 X - Y^2 + s_{11}\lambda^k XY + s_{03}\lambda^k Y^3 + \\
+o(\lambda^k)o(Y^3) + O(\tilde\gamma^{-2k})
\end{array}
\tag{32}
$$

where

$$
h_{02} = \frac{c}{d}e_{02}+\ldots \ , \ s_{11} = 2a - \frac{b}{d}f_{11}+\ldots \ , \ s_{03} = -\frac{bc}{d^2}f_{03}+\ldots;
\tag{33}
$$

the dots stand for the terms, independent of (X, Y), which tend to zero as $k \to +\infty$.

Let us now find coordinates (X^{ψ}, Y^{ψ}) of the fixed point of (32) at $M_1 = M_1^{\psi}, M_2 = M_2^{\psi}$. We use that the trace of the characteristic matrix

$$\begin{pmatrix} \frac{\partial \bar{X}}{\partial X} & \frac{\partial \bar{X}}{\partial Y} \\ \frac{\partial \bar{Y}}{\partial X} & \frac{\partial \bar{Y}}{\partial Y} \end{pmatrix} \tag{34}$$

of T_k at the fixed point is equal to the sum of the multipliers, i.e. at the weak focus it equals to $2\cos\psi$. We have

$$-2Y^{\psi} + s_{11}\lambda^k X^{\psi} + 3s_{03}\lambda^k (Y^{\psi})^2 + o(\lambda^k) = 2\cos\psi \quad ,$$

or, since $X^{\psi} = Y^{\psi} + h_{02}\lambda^k (Y^{\psi})^2 + o(\lambda^k)$,

$$Y^{\psi}(1 - \frac{1}{2}s_{11}\lambda^k - \frac{3}{2}s_{03}\lambda^k Y^{\psi}) = -\cos\psi + o(\lambda^k).$$

We obtain from this that

$$\begin{array}{l} Y^{\psi} = -\cos\psi \cdot (1 + \frac{1}{2}s_{11}\lambda^k + \frac{3}{2}s_{03}\lambda^k \cos\psi) + o(\lambda^k), \\ X^{\psi} = Y^{\psi} + h_{02}\lambda^k \cos^2\psi + o(\lambda^k). \end{array} \tag{35}$$

We shift now the origin to the point (X^{ψ}, Y^{ψ}) and consider map (32) at $M_1 = M_1^{\psi}, M_2 = M_2^{\psi}$. We obtain

$$\begin{array}{l} \bar{X} = Y + h_{02}\lambda^k Y^2 + o(\lambda^k), \\ \bar{Y} = -X + 2Y\cos\psi + s_{11}\lambda^k XY - \\ -Y^2(1 + 2h_{02}\lambda^k \cos\psi + 3s_{03}\lambda^k \cos\psi) + s_{03}\lambda^k Y^3 + o(\lambda^k). \end{array} \tag{36}$$

The following change of variables

$$X = \alpha \cdot X_{new} \quad , \quad Y = \alpha \cdot Y_{new}$$

where $\alpha = (1 + 2h_{02}\lambda^k \cos\psi + 3s_{03}\lambda^k \cos\psi)^{-1}$, brings map (36) to the form

$$\begin{array}{l} \bar{X} = Y + h_{02}\lambda^k Y^2 + o(\lambda^k) \quad , \\ \bar{Y} = -X + 2Y\cos\psi + s_{11}\lambda^k XY - \\ -Y^2 + s_{03}\lambda^k Y^3 + o(\lambda^k). \end{array} \tag{37}$$

This map has a fixed point with multipliers $e^{\pm i\psi}$ at the origin. Evidently, the first Lyapunov value of this point will coincide with that for map (36) up to terms of order $o(\lambda^k)$. It is also obvious that if we omit the $o(\lambda^k)$ terms in the right-hand side of (37), the first Lyapunov value will get only some $o(\lambda^k)$ corrections.

Thus, to prove the lemma it is enough to show that the first Lyapunov value of the fixed point (in the origin) of the map

$$\begin{array}{l} \bar{X} = Y + h_{02}\lambda^k Y^2, \\ \bar{Y} = -X + 2Y\cos\psi - Y^2 + s_{11}\lambda^k XY + s_{03}\lambda^k Y^3. \end{array} \tag{38}$$

satisfies formula (30)

The first Lyapunov value of the weak focus in the map (38) is a polynomial with respect to the coefficients $h_{02}\lambda^k$, $s_{11}\lambda^k$, $s_{03}\lambda^k$. Hence, it has the form

$$G_1 = F_0 + \lambda^k h_{02} \cdot F_1 + \lambda^k s_{11} \cdot F_2 + \lambda^k s_{03} \cdot F_3 + O(\lambda^{2k}) \tag{39}$$

where F_0, F_1, F_2 and F_3 are some coefficients, depending on ψ. We note that if $h_{02} = s_{11} = 0$, then map (37) is conservative, i.e. $G_1 = 0$ in this case. It means that $F_0 = F_3 \equiv 0$ in (39). Hence, the first Lyapunov value is independent of s_{03} in the main order. Therefore, it is sufficient to prove (30) for the first Lyapunov value of the following quadratic map (i.e., at $s_{03} = 0$)

$$\begin{aligned} \bar{X} &= Y + h_{02}\lambda^k Y^2, \\ \bar{Y} &= -X + 2Y\cos\psi - Y^2 + s_{11}\lambda^k XY. \end{aligned} \tag{40}$$

By means of the linear change

$$X = \xi, Y = \cos\psi \cdot \xi - \sin\psi \cdot \eta \tag{41}$$

map (40) is brought to the form where its linear part is the rotation

$$\begin{aligned} \bar{\xi} &= \cos\psi \cdot \xi - \sin\psi \cdot \eta + h_{02}\lambda^k[\cos^2\psi \cdot \xi^2 - \sin 2\psi \cdot \xi\eta + \sin^2\psi \cdot \eta^2], \\ \bar{\eta} &= \sin\psi \cdot \xi + \cos\psi \cdot \eta + \xi^2 \left[\frac{1 + h_{02}\lambda^k \cos\psi}{\sin\psi}\cos^2\psi - s_{11}\lambda^k \frac{\cos\psi}{\sin\psi}\right] + \\ &+ \xi\eta\left[-2\cos\psi \cdot (1 + h_{02}\lambda^k\cos\psi) + s_{11}\lambda^k\right] + \\ &+ \eta^2 \sin\psi \cdot (1 + h_{02}\lambda^k\cos\psi). \end{aligned} \tag{42}$$

In the complex coordinates $z = \xi + i\eta$, $z^* = \xi - i\eta$ map (42) takes the form

$$\bar{z} = e^{i\psi} z + C_{20}z^2 + C_{11}zz^* + C_{02}(z^*)^2 \tag{43}$$

where

$$\begin{aligned} C_{20} &= [(A + B' - C) + i(A' - B - C')]/4 \\ C_{11} &= [2(A + C) + i2(A' + C')]/4 \\ C_{02} &= [(A - B' - C) + i(A' + B - C')]/4 \end{aligned}$$

with A, B, C and A', B', C' denoting the coefficients of the quadratic terms $\xi^2, \xi\eta, \eta^2$ in the right-hand sides of the equations for $\bar{\xi}$ and $\bar{\eta}$, respectively; i.e.

$$A = h_{02}\lambda^k\cos^2\psi\ ,\ B = -h_{02}\lambda^k \sin 2\psi\ ,\ C = h_{02}\lambda^k\sin^2\psi\ ,$$

$$A' = \left[\frac{1 + h_{02}\lambda^k\cos\psi}{\sin\psi}\cos^2\psi - s_{11}\lambda^k\frac{\cos\psi}{\sin\psi}\right]\ ,$$

$$\begin{aligned} B' &= \left[-2\cos\psi \cdot (1 + h_{02}\lambda^k\cos\psi) + s_{11}\lambda^k\right]\ , \\ C' &= \sin\psi \cdot (1 + h_{02}\lambda^k\cos\psi). \end{aligned}$$

Thus, we have

$$
\begin{aligned}
C_{20} &= \tfrac{1}{4}\left[-2\cos\psi - h_{02}\lambda^k + s_{11}\lambda^k\right] + \\
&+\tfrac{i}{4}\left[\frac{\cos 2\psi}{\sin\psi} + \lambda^k\frac{\cos\psi}{\sin\psi}\left(h_{02} - s_{11}\right)\right] \\
C_{11} &= \lambda^k\frac{h_{02}}{2} + \tfrac{i}{2}\left[\frac{1}{\sin\psi} + \lambda^k\frac{\cos\psi}{\sin\psi}\left(h_{02} - s_{11}\right)\right] \\
C_{02} &= \tfrac{1}{4}\left[2\cos\psi + h_{02}\lambda^k\left(3\cos^2\psi - \sin^2\psi\right) - s_{11}\lambda^k\right] + \\
&+\tfrac{i}{4}\left[\frac{\cos 2\psi}{\sin\psi} + h_{02}\lambda^k\frac{\cos\psi}{\sin\psi}\left(\cos^2\psi - 3\sin^2\psi\right) - s_{11}\lambda^k\frac{\cos\psi}{\sin\psi}\right].
\end{aligned}
\tag{44}
$$

Make the following normalizing change of variables in (43)

$$w = z + A_{20}z^2 + A_{11}zz^* + A_{02}(z^*)^2 \tag{45}$$

in order to nullify all quadratic terms in (43). It is not hard to see that the coefficients A_{20}, A_{11} and A_{02} should be taken as follows

$$A_{20} = \frac{C_{20}}{e^{i\psi} - e^{2i\psi}},\ A_{11} = \frac{C_{11}}{e^{i\psi} - 1}\ ,\ A_{02} = \frac{C_{02}}{e^{i\psi} - e^{-2i\psi}}; \tag{46}$$

in this case all quadratic terms will be eliminated indeed, provided $\psi \neq 2\pi/3$ (in the latter case, the term $(z^*)^2$ is resonant and cannot be killed by smooth coordinate changes). Thus, map (43) is brought to the form (if $\psi \neq 2\pi/3$)

$$\bar{w} = e^{i\psi}(w + G_{30}w^3 + G_{21}w^2w^* + G_{12}w(w^*)^2 + G_{03}(w^*)^3 + O(|w|^4) \tag{47}$$

We note that, among the cubic terms, the term w^2w^* is always resonant, the terms w^3 and $w(w^*)^2$ are always nonresonant, while the term $(w^*)^3$ is resonant only in the case where $\psi = \pi/2$. Thus, if $\psi \neq \pi/2, 2\pi/3$, the last three terms may be eliminated by some cubic change of variables. But the coefficient G_{21} is not changed and, hence, the map (47) can be brought to the form

$$\bar{w} = e^{i\psi}(w + G_{21}w^2w^*) + O(|w|^4) \tag{48}$$

In the polar coordinates (ρ, α), where $w = \rho e^{i\alpha}$, the map (48) takes the following form

$$\bar{\rho} = \rho(1 + G_1\cdot\rho^2) + O(\rho^4)\ \ ,\ \ \bar{\alpha} = \alpha + \psi + B\rho^2 + O(\rho^3) \tag{49}$$

where $G_1 = Re(G_{21}), B = Im(G_{21})$. Thus, $Re(G_{21})$ is the first Lyapunov value.

Now we calculate the coefficient G_{21}. Using (43),(45) and (46) we obtain for G_{21} the following expression

$$G_{21} = 2A_{20}C_{11} + A_{11}C_{11}^* + A_{11}C_{20}e^{-2i\psi} + 2A_{02}C_{02}^*e^{-i\psi}$$

By virtue of (46)

$$G_{21} = C_{20}C_{11}\frac{2e^{i\psi}-1}{(1-e^{i\psi}}e^{-2i\psi} - -2\frac{|C_{02}|^2}{(1-e^{3i\psi}} - \frac{|C_{11}|^2}{(1-e^{i\psi}} \tag{50}$$

and, hence,

$$G_1 \equiv Re(G_{21}) = Re\left\{C_{20}C_{11}\frac{2e^{i\psi}-1}{(1-e^{i\psi}}e^{-2i\psi}\right\} - -|C_{02}|^2 - \tfrac{1}{2}|C_{11}|^2, \tag{51}$$

or

$$\begin{aligned} G_1 = {} & Re(C_{20}C_{11})\frac{\cos 3\psi - 3\cos 2\psi + 2\cos\psi}{2(1-\cos\psi)} + \\ & + Im(C_{20}C_{11})\frac{\sin 3\psi - 3\sin 2\psi + 2\sin\psi}{2(1-\cos\psi)} - |C_{02}|^2 - \tfrac{1}{2}|C_{11}|^2 \ . \end{aligned} \tag{52}$$

Finally, (52) can be rewritten as

$$\begin{aligned} G_1 = {} & \tfrac{1}{2}Re(C_{20}C_{11})(3+\cos\psi - 2\cos^2\psi) + \\ & + \tfrac{1}{2}Im(C_{20}C_{11})\left(2\sin\psi - 2\sin 2\psi - \frac{1+\cos\psi}{\sin\psi}\right) - -|C_{02}|^2 - \tfrac{1}{2}|C_{11}|^2 \ , \end{aligned} \tag{53}$$

where C_{20}, C_{11} and C_{02} are given by (44).

In the case where $s_{11} = 0, h_{02} = 0$ one can check directly that $G_1 \equiv 0$. Indeed, it follows from (44) that in this case

$$\begin{aligned} & Re(C_{20}C_{11}) = -\frac{\cos 2\psi}{8\sin^2\psi} \ , \ Im(C_{20}C_{11}) = -\frac{\cos\psi}{4\sin\psi} \ , \\ & |C_{11}|^2 = \frac{1}{4\sin^2\psi} \ , \ |C_{02}|^2 = \frac{1}{4}\cos^2\psi + \frac{\cos^2 2\psi}{16\sin^2\psi} \end{aligned} \tag{54}$$

Then, we obtain from (53) that

$$\begin{aligned} G_1 = {} & -\frac{1}{16}\frac{\cos^2 2\psi}{\sin^2\psi}(3+2\cos\psi - 4\cos^2\psi) - \\ & -\frac{1}{8}\frac{\cos\psi}{\sin\psi}\left(2\sin\psi - 2\sin 2\psi - \frac{1+\cos\psi}{\sin\psi}\right) - \\ & -\frac{1}{4}\cos^2\psi - \frac{1}{16}\frac{\cos^2 2\psi}{\sin^2\psi} - \frac{1}{8\sin^2\psi} \equiv 0. \end{aligned}$$

We are now in the position to compute the coefficients F_1 and F_2 in formula (39) for G_1. To compute F_2, we may assume $h_{02} = 0$. It follows from (44), that in this case

$$\begin{aligned} & C_{20} = \frac{1}{4}\left[-2\cos\psi + s_{11}\lambda^k\right] + \frac{i}{4}\left[\frac{\cos 2\psi}{\sin\psi} - \frac{\cos\psi}{\sin\psi}s_{11}\lambda^k\right] \\ & C_{11} = \frac{i}{2}\left[\frac{1}{\sin\psi} - \frac{\cos\psi}{\sin\psi}s_{11}\lambda^k\right] \\ & C_{02} = \frac{1}{4}\left[2\cos\psi - s_{11}\lambda^k\right] + +\frac{i}{4}\left[\frac{\cos 2\psi}{\sin\psi} - \frac{\cos\psi}{\sin\psi}s_{11}\lambda^k\right]. \end{aligned} \tag{55}$$

We obtain from here

$$
\begin{aligned}
&Re(C_{20}C_{11}) = [Re(C_{20}C_{11})]_{fin} + \frac{(\cos 2\psi + 1)\cos\psi}{8\sin^2\psi} \cdot s_{11}\lambda^k + O(\lambda^{2k}) \ , \\
&Im(C_{20}C_{11}) = [Im(C_{20}C_{11})]_{fin} + \frac{1 + 2\cos^2\psi}{8\sin\psi} \cdot s_{11}\lambda^k + O(\lambda^{2k}) \ , \\
&|C_{11}|^2 = [|C_{11}|^2]_{fin} - \frac{\cos\psi}{2\sin^2\psi} \cdot s_{11}\lambda^k + O(\lambda^{2k}) \ , \\
&|C_{02}|^2 = [|C_{02}|^2]_{fin} - \frac{1}{4}\left(\cos\psi + \frac{\cos 2\psi\cos\psi}{2\sin^2\psi}\right) \cdot s_{11}\lambda^k + O(\lambda^{2k})
\end{aligned}
$$

where $[\cdot]_{fin}$ denotes the finite part of the corresponding coefficients, i.e., its value at $\lambda^k = 0$ according to (54). Substituting these expressions in formula (53) and collecting the terms of order λ^k, we find

$$
\begin{aligned}
16F_2 = &\frac{(\cos 2\psi + 1)\cos\psi}{\sin^2\psi}(3 + 2\cos\psi - 4\cos^2\psi) + \\
&+ \frac{1 + 2\cos^2\psi}{\sin\psi}\left(2\sin\psi - 2\sin 2\psi - \frac{1 + \cos\psi}{\sin\psi}\right) + \\
&+ 4\cos\psi + \frac{2\cos 2\psi \cdot \cos\psi}{\sin^2\psi} + \frac{4\cos\psi}{\sin^2\psi}.
\end{aligned}
$$

It is easy to check that

$$
F_2 = \frac{1}{16(1 - \cos\psi)}
$$

Analogously we compute that

$$
F_1 = -\frac{2}{16(1 - \cos\psi)}
$$

Thus, the following formula

$$
G_1 = \frac{s_{11} - 2h_{02}}{16(1 - \cos\psi)}\lambda^k + O(\lambda^{2k})
$$

defines the first Lyapunov value. Since $R = s_{11} - 2h_{02}$ (see (5) and (33)), this completes the proof.

Now we are able to describe main bifurcations of fixed points of the rescaled first return map T_k. Such a map has three bifurcational curves $L^+_{(k)}$, $L^-_{(k)}$ and $L^{\varphi}_{(k)}$ which divide the half-plane $M_2 > 0$ of parameters M_1 and M_2 into four parts D_0, D_1^s, D_1^u and D_2 (see figure 5). The map T_k has no fixed points for $(M_1, M_2) \in D_0$. Transitions from D_0 into regions D_1^s and D_1^u across the curve $L^+_{(k)}$ corresponds usual saddle-node bifurcations which are the same as in the Henon map (27)[6]. Thus, map T_k has exactly two fixed points Q_k and P_k for values of the parameters from the region from above the curve $L^+_{(k)}$. The point Q_k is a saddle and P_k is a stable point when $(M_1, M_2) \in D_1^s$ and a completely unstable point when $(M_1, M_2) \in D_1^u$; it is a

[6]Except for the point $B^{++}_{(k)}$ where the map T_k has the fixed point with two unit multipliers. It is not hard to check that the "parabolic" fixed point of T_k is not degenerate in the sense that at $R \neq 0$ this point is the so-called Bogdanov-Takens fixed point [31]

weak focus when $(M_1, M_2) \in L^{\varphi}_{(k)}$. Thus, the curve $L^{\varphi}_{(k)}$ is the boundary of stability of P_k. In the case where $R \neq 0$ we can define the type of this boundary, i.e., whether the boundary is "safe" or "dangerous" [32]. Namely, it depends on the sign of the value $R\lambda^k < 0$ as follows.

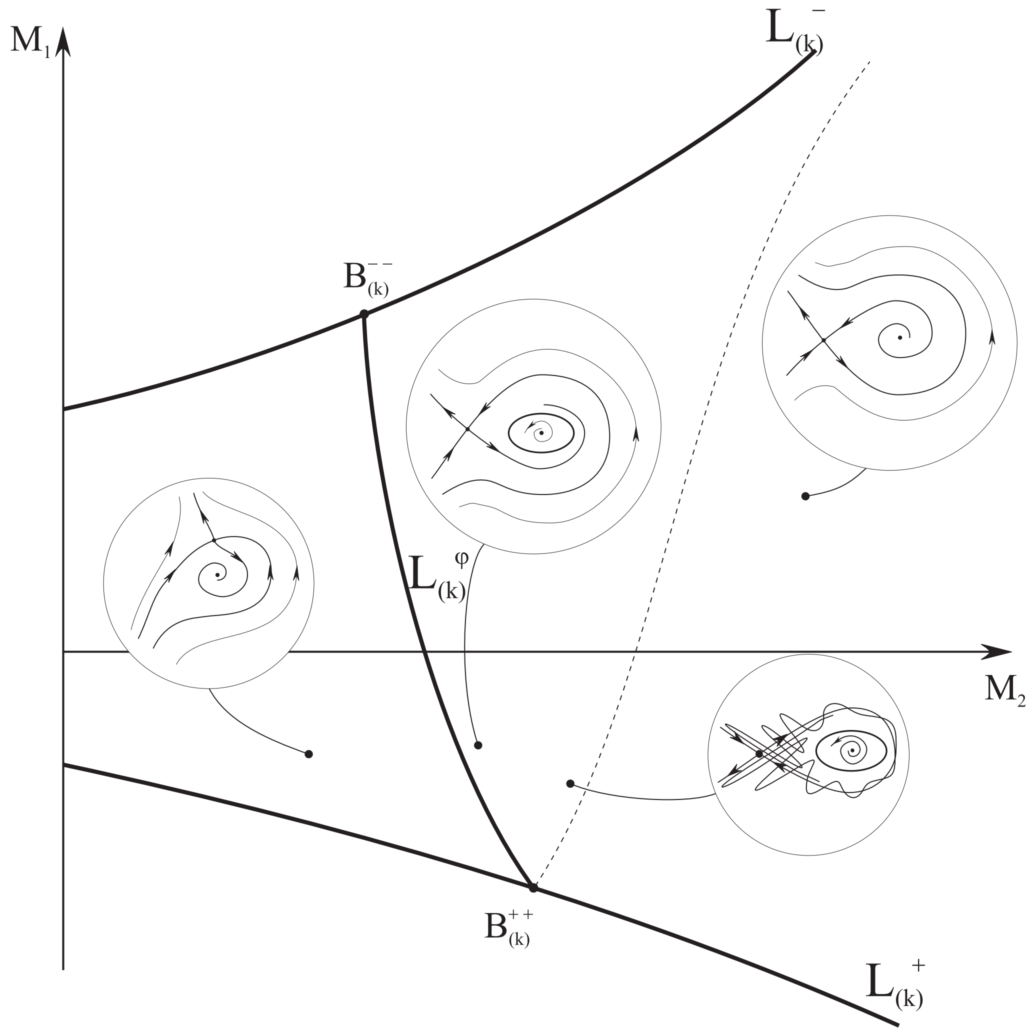

Figure 5

Elements of the bifurcation diagram for the rescalling map in the case $R\lambda^k>0$

If $R\lambda^k < 0$, the boundary $L^{\varphi}_{(k)}$ is "safe": the first Lyapunov value G_1 is negative, so at the transition across $L^{\varphi}_{(k)}$ (except for two points on $L^{\varphi}_{(k)}$ where $\psi = \pi/2, 2\pi/3$, see formula (29)) in the direction from D_1^s to D_1^u (towards the increase of M_2) the orbit P_k becomes unstable and a stable invariant curve is born from it. Moreover, P_k is the stable weak focus at $(M_1, M_2) \in L^{\varphi}_{(k)} \backslash \{\pi/2, 2\pi/3\}$. This boundary is "safe" because, just after the loss of stability, iterations of any initial point close to P_k approach the stable invariant curve and, hence, do not escape a small neighbourhood of the fixed point. The corresponding bifurcation picture is shown in figure 5.

If $R\lambda^k > 0$, the boundary $L^{\varphi}_{(k)}$ is "dangerous": the first Lyapunov value G_1 is positive and ,here, at the transition across $L^{\varphi}_{(k)}$ (except for two points where $\psi = \pi/2, 2\pi/3$) in the direction from D^u_1 to D^s_1 an unstable invariant curve is born from P_k. Moreover, P_k is the unstable weak focus at $(M_1, M_2) \in L^{\varphi}_{(k)} \backslash \{\pi/2, 2\pi/3\}$. This boundary is "dangerous" because the loss of stability is happened when the unstable invariant curve "merges" into the stable point P_k and, as a result, iterations of any initial point close to P_k (except for P_k itself) escape any sufficiently small neighbourhood of the fixed point.

To conclude this section, we note that the importance of the separatrix value R goes beyond the fact that its sign defines the sign of the first Lyapunov value at the weak focus.

Indeed, for example, the Jacobian J of the map

$$\begin{aligned} \bar{X} &= Y + h_{02}\lambda^k Y^2 \quad , \\ \bar{Y} &= M_1 - M_2 X - Y^2 + s_{11}\lambda^k XY + s_{03}\lambda^k Y^3 \end{aligned} \tag{56}$$

calculated in the weak focus (or, which is the same, the Jacobian of map (38) in the origin) is given by

$$J = 1 - R\lambda^k Y + O(\lambda^{2k}) \tag{57}$$

That is, J differs from 1 on a value which is *proportional to $R\lambda^k$ in the main order.*

The second observation is that the shape of the bifurcational curve $L^{\varphi}_{(k)}$ of map (56) depends essentially on the coefficient $R\lambda^k$. Indeed, the equation of this curve has the form (put $\gamma^{-k} = -bc\lambda^k(1 + \ldots)$ in (29)):

$$\begin{aligned} M_1 &= \cos^2\psi - 2\cos\psi + O(\lambda^k) \quad , \\ M_2 &= 1 + \cos\psi \cdot R\lambda^k + O(\lambda^{2k}) \end{aligned} \tag{58}$$

where ψ is the parameter, $0 < \psi < \pi$. We see that the curve $L^{\varphi}_{(k)}$ at $R \neq 0$ is not C^1-close to the curve L^{φ} ($M_2 = 1$) for the Henon map, see figure.

An interesting curve (nonbifurcational) $L^s_{(k)}$ starts with the point $B^{++}_{(k)}$, which corresponds to the existence of a saddle fixed point of (56) of the neutral type, (i.e., the fixed point with multipliers $\lambda_s > 0$ and $\lambda_u > 0$ such that $\lambda_s\lambda_u = 1$). This curve is drawn in figure as the dotted line, its equation is

$$\begin{aligned} M_1 &= \alpha^2 - 2\alpha + O(\lambda^k) \quad , \\ M_2 &= 1 + \alpha \cdot R\lambda^k + O(\lambda^{2k}) \end{aligned} \tag{59}$$

. where $\alpha = (\lambda_s + \lambda_u)/2$ is the parameter, and $\alpha > 1$.[7]

[7] Values $\alpha < -1$ corresponds to that part of the curve $L^s_{(k)}$ where the corresponding neutral type saddle fixed point of (56) has both negative multipliers; values $|\alpha| < 1$ corresponds to the curve $L^{\varphi}_{(k)}$ - in this case $\alpha = \cos\psi$.

We note that the curve $L^s_{(k)}$ may play an important role for the answer to the following principal question: how does the found invariant curve disappear? It is naturally to assume that this curve exists only for values of parameters M_1 and M_2 from some part of the region bounded by the curves $L_{(k)^\varphi}$, $L^s_{(k)}$ and $L^-_{(k)}$ (the dashed region in figure). In any case,it follows from [5, 7] that no invariant closed curve exists for values of the parameters which are sufficiently far from this region. If to assume that our map can be embedded into the flow (this models the situation reasonably near the point $B^{++}_{(k)}$ [31, 24]), then the invariant curve should disappear merging into a homoclinic loop of the saddle. In this case, the homoclinic loop should have the same type of stability as the invariant curve [33, 34]. Thus this phenomenon can occur only for values of the parameters in the dashed region, because the saddle value $\lambda_s\lambda_u$ of the saddle fixed point is less or greater than one for values of the parameters from the left of $L^s_{(k)}$ or from the right of $L^s_{(k)}$, respectively.

In fact, the general mechanism of disappearance of the invariant curve is connected with its break-down [30]: the invariant curve becomes resonant one (in this case it contains alternating saddle and stable (or completely unstable) periodic orbits of the same period) and it is destroyed, typically, by one of the ways given in [30]. In this connection, the questions related to the existence of the resonant zones seem to be very interesting.

Another important question which we will consider in a forthcoming paper addreses the bifurcational phenomena accompanying the transition across the strong resonances $\psi = \pi/2$ and $\psi = 2\pi/3$ (and $\psi = 0, \pi$ also).

All these questions may be studied both in the map (56) itself and in the map

$$\begin{array}{l} \bar{x} = y + A\varepsilon y^2 \ , \\ \bar{y} = M_1 - M_2 x - y^2 + B\varepsilon xy + C\varepsilon y^3 \end{array} \tag{60}$$

where the parameters M_1, M_2, A, B, C are arbitrary and the parameter ε is sufficiently small. Map (60) can be considered as a practically interesting small perturbation of the standard Henon map.

4 The proof of theorems A and B.

We note, first of all, that the rescaling lemma allows to compute immediately the equations of bifurcational curves L^+_k, L^-_k and L^φ_k, $k = \bar{k}, \bar{k}+1, \ldots$, for the first return maps T_k on the plane of the initial parameters (μ_1, μ_2). Namely, using (31) and the relation $\lambda\gamma = 1 + \mu_2$, formulas (28) and (29) are transformed as follows

$$L_k^+ \ : \ \mu_1 = \gamma - k(y^- - (1+\mu_2)^k cx^+) + r_k + \qquad +\gamma^{-2k}\frac{(1+bc(1+\mu_2)^2}{4d}(1+\ldots)$$

$$L_k^- \ : \ \mu_1 = \gamma - k(y^- - (1+\mu_2)^k cx^+) + r_k - \qquad -\gamma^{-2k}\frac{3(1+bc(1+\mu_2)^2}{4d}(1+\ldots)$$

$$\begin{aligned} L_k^\varphi \ : \ & \mu_1 = \gamma - k(y^- + \tfrac{x^+}{b} + \ldots) + r_k - \qquad -\gamma^{-2k}\frac{\cos^2\varphi - 2\cos\varphi}{4d}(1+\ldots) \\ & \mu_2 = -1 + (-bc)^{-1/k}\left(1 - \frac{R\cos\varphi}{bc}\cdot\frac{\lambda^k}{k}(1+\ldots)\right) \end{aligned}$$

where $r_k = o(\gamma^{-k})$ and the dots denote terms tending to zero as $k \to \infty$.

Evidently, curves L_k^+ and L_k^- accumulate on the line $\mu_1 = 0$ corresponding to the diffeomorphisms possessing a (single-round) orbit of homoclinic tangency. Curves L_k^φ connect points B_k^{++} and B_k^{--} on the curves L_k^+ and L_k^-, respectively, and accumulate at the point $\mu_1 = \mu_2 = 0$. The bifurcational part of theorem B (items 2 and 3) follows directly from our analysis of bifurcations of the first return maps (section 4 and 5). Finally, theorem A follows immediately from theorem B: the region Δ_k is some part of D_k adjoining to L_k^φ (on the segment $0 < \psi < \pi/2$, for example) from the left if $R\lambda^k > 0$ and from the right if $R\lambda^k < 0$.

Acknowledgments.

This work was supported by grant INTAS No. 97-804 , by grant of Russian Ministry of Education No. 97-0-1.8-59 and by DFG Schwerpunktprogramm DANSE. We are grateful to L.P.Shilnikov and D.V.Turaev for very useful discussions. The first author is grateful to Professor Klaus Schneider for his warm hospitality during the three month visit to WIAS in 1999-2000.

References

[1] L.P. Shilnikov "On a Poincare-Birkhoff problem".- Math. USSR Sb., 1967, **3**, pp.91-102.

[2] S.V.Gonchenko, D.V.Turaev, L.P.Shilnikov. "On models with a structurally unstable homoclinic curve".- Soviet Math.Dokl., 1992, **44**, No.2, 422-426.

[3] S.V.Gonchenko, D.V.Turaev , L.P.Shilnikov "On models with non-rough Poincare homoclinic curves".- *Physica D*, **62** (1993), Nos.1-4, 1-14.

[4] S.V.Gonchenko, L.P.Shilnikov ,D.V.Turaev "Dynamical phenomena in systems with structurally unstable Poincare homoclinic orbits".- *Interdisc. J. CHAOS*, **6** (1996), No.1, pp. 1-17.

[5] N.K.Gavrilov N.K., L.P.Shilnikov "On three-dimensional dynamical systems close to systems with a structurally unstable homoclinic curve".- Part 1, Math.USSR Sb., 1972, **17**, pp.467-485 ; Part 2, Math.USSR Sb, 1973, **19**, pp.139-156.

[6] S.V.Gonchenko "On stable periodic motions of systems close to a system with structurally unstable homoclinic curve".- *Matematicheskie zametki*, **33** (1983), No.5, pp. 745-755, in Russian.

[7] S.V.Gonchenko, D.V.Turaev , L.P.Shilnikov "Dynamical phenomena in multi-dimensional systems with a structurally unstable homoclinic Poincare curve".- *Russian Acad. Sci. Dokl. Math.*, **47** (1993), No.3, pp.410-415.

[8] S. Newhouse "The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms".- Publ. Math. IHES, 1979. v.50, pp.101–151.

[9] J.Palis, M.Viana "High dimension diffeomorphisms displaying infinitely many periodic attractors".- Annals of Math., 1994, v.140, pp.207-250.

[10] S.V.Gonchenko, D.V.Turaev , L.P.Shilnikov "On the existence of Newhouse regions in a neighbourhood of systems with a structurally unstable homoclinic Poincaré curve (the multidimensional case)".- Russian Acad. Sci. Dokl. Math., 1993. v.47, No.2, pp.268–273.

[11] P.Duarte "Persistent homoclinic tangencies for conservative maps near the identity".- Preprint,6/98, Lisbon, March, 1998.

[12] P.Duarte "Abundance of elliptic isles at conservative bifurcations".- Preprint,7/98, Lisbon, April, 1998.

[13] V.S.Afraimovich, L.P.Shilnikov "Strange attractors and quasiattractors".- in *Nonlinear Dynamics and Turbulence*, 1982, Pitman, Boston, pp.336-339.

[14] L.P.Shilnikov "Mathematical problems of nonlinear dynamics: a tutorial".- Int.J. Bifurcation and Chaos, 1997, v.6, No.6, pp. 969-989.

[15] L.P.Shilnikov "Chua's circuit: rigorous results and future problems".- Int. J. of Bifurcation and Chaos, 1994, v.4, No.3, pp.489-519.

[16] Y.-C.Lai, C.Grebogi, J.A.Yorke, I.Kan "How often are chaotic saddles nonhyperbolic?"- Nonlinearity, 1993, v.6, No.5, pp.779-797.

[17] D.V.Turaev , L.P.Shilnikov "An example of a wild strange attractor".- Sbornik: Mathematics, 1998, v.189, N0.2, pp. 137-160.

[18] D.V.Turaev "On dimension of non-local bifurcational problems".- Bifurcation and Chaos, 1996, v.6.

[19] S.E.Newhouse "Diffeomorphisms with infinitely many sinks".- *Topology*, **13** (1974), pp. 9-18.

[20] S.V.Gonchenko, D.V.Turaev , L.P.Shilnikov "Homoclinic tangencies of arbitrary order in the Newhouse regions".- to appear.

[21] S.V.Gonchenko, L.P.Shilnikov "Invariants of Ω-conjugacy of diffeomorphisms with a structurally unstable homoclinic trajectory".- Ukrainian Math.J., 1990, **42** , No.2, pp.134-140.

[22] S.V.Gonchenko, L.P.Shilnikov "On the moduli of systems with a non-rough Poincare homoclinic curve".- Russian Acad. Sci. Izv. Math., 1993, **41**, No.3, pp.417-445.

[23] L.Tedeshini-Lalli, J.A.Yorke "How often do simple dynamical processes have infinitely many coexisting sinks?"- Comm.Math.Phys., 1995, v.106, pp.635-657.

[24] L.P.Shilnikov, A.L.Shilnikov, D.V.Turaev, L.O.Chua "Methods of qualitative theory in nonlinear dynamics, Part I".- World Scientific, 1998.

[25] L.P.Shilnikov "A contribution to the problem of structure of a neighbourhood of a homoclinic tube of invariant torus".- Soviet Math. Dokl., 1968, **180**, No.2, pp.286-289. (Russian).

[26] V.S.Afraimovich, L.P.Shilnikov "On critical sets of Morse-Smale systems".- Trans. Moscow Math. Soc., 1973, **28**

[27] S.V.Gonchenko, L.P.Shilnikov "On two-dimensional analytic area-preserving diffeomorphisms with infinitely many stable elliptic periodic points".- Regular and Chaotic Dynamics, 1997, v.2,No.3/4, pp.106–123.

[28] E.Colli "Infinitely many coexisting strange attractors".- Ann. Inst. Poincare, 1998, v.15, No.5, pp.539-579.

[29] V.Biragov "Bifurcations in a two-parameter family of conservative mappings that are close to the Henon map".- Selecta Math. Sovietica, 1990, v.9, pp.247-280 [Orig. publ. in *Methods of the Qualitative Theory of Differential Equations*, (E.A.Leontovich-Andronova, ed.), 1987, (Russian)].

[30] V.S.Afraimovich, L.P.Shilnikov "On invariant two-dimensional tori, their desintegration and stochasticity".- in *Methods of the Qualitative Theory of Differential Equations*, (E.A.Leontovich-Andronova, ed.), 1983, pp.3-26. (Russian). [Engl. transl. in: Amer.Math.Soc.Transl., 1991, v.149, pp.201-211].

[31] Kuznetsov Yu.A. "Elements of applied bifurcation theory".- Springer-Verlag, 1995.

[32] N.N.Bautin "Behavior of dynamical systems near the boundaries of stability regions".- OGIZ GOSTEXIZDAT, Leningrad-Moscow, 1949 (Russian).

[33] A.A.Andronov, E.A.Leontovich, I.E.Gordon, A.G.Majer "The theory of dynamical systems on a plane".- Israel Program of Scientific Translations, Jerusalem, 1973.

[34] N.N.Bautin, E.A.Leontovich "Methods and ways (examples) of the qualitative analysis of dynamical systems in a plane".- Moscow, "Nauka", 1976. (Russian)