THE LONG-TIME BEHAVIOUR OF THE THERMOCONVECTIVE FLOW IN A POROUS MEDIUM.

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ABSTRACT. For the Boussinesq approximation of the equations of coupled heat and fluid flow in a porous medium we show that the corresponding system of partial differential equations posesses a global attractor. We give lower and upper bounds of the Hausdorff dimension of the attractor depending on a physical parameter of the system, namely the Rayleigh number of the flow. Numerical experiments confirm the theoretical findings and raise new questions on the structure of the solutions of the system.

INTRODUCTION.

We consider the equations of coupled heat and fluid transport in a porous medium in the Oberbeck-Boussinesq approximation (see [NB92]). These equations can be used to describe geothermal reservoirs, thermal convection in sediment layers and other processes with bouyancy induced convection. In dimensionless variables, they have the following form:

$$\begin{cases} \nabla_x \cdot v = 0; \quad v = \nabla_x P - \mu\gamma T; \quad P\big|_{\partial\Omega} = P_0 \\ \partial_t T - \nabla_x \cdot (\nabla_x T - Tv) = 0; \quad T\big|_{\partial\Omega} = T_0 \\ T\big|_{t=0} = T(0) \end{cases} \tag{0.1}$$

where $x \in \Omega$, $\Omega \subset\subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary, $(T(t,x), P(t,x))$ – are unknown functions (temperature and pressure respectively), $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a given constant vector ($|\gamma| = 1$) and $\mu > 0$ – is a given scalar parameter called the Rayleigh number, which is combined from physical parameters like characteristic lengths ,heights, temperature differences, temperature expansion coefficients etc. (see [NB92]).

It is observed (see [NB92]) that with growing μ the complexity of the flow patterns grows as well. Our numerical experiments confirm these observations.

This growing complexity of the flow regime leads to the suggestion that the Hausdorff dimension of the attractor of (0.1) should increase with growing Rayleigh number. This suggestion is confirmed by the analysis carried out in this paper. We study this flow from the dynamical point of view. We prove that the semigroup generated by the system (0.1) possesses a global attractor in the corresponding Sobolev space and try to study the complexity of this attractor in dependence of the Rayleigh number μ. Particularly, we present upper and lower bounds for the dimension of the attractor in terms of Rayleigh number μ which in turn shows growing complexity of dynamics when μ is growing. Further, we present results of numerical experiments which are intended to support the intuition of the reader. It is remarkable that these experiments in turn pose questions for further research.

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The paper is organized as follows.

A number of a priori estimates for the solutions of the problem (0.1) is obtained in Section 1. In particular these estimates show that every solution of (0.1) remains bounded when $t \to \infty$ in the appropriate phase space which will be introduced below.

Using these estimates we prove in Section 2 the existence of a global solution of this problem and it's uniqueness. Moreover we prove here the existence of a global attractor $\mathcal{A}_\mu$ for the problem (0.1).

The upper and lower bounds for the Hausdorff dimension of the attractor are obtained in Sections 3 and 4 correspondingly.

Numerical results which show the different types of the long time behaviour of solutions (0.1) in dependence of the Rayleigh number and the discretization parameters are presented in Section 5. While the theoretical results are confirmed, questions for further research in this problem area are raised.

1. A PRIORY ESTIMATES

In this Section we obtain a number of a priori estimates for the solutions of the problem (1.1). Note that we will use these estimates only in order to prove the existence theorems. The sharp estimates for the solutions belonging to the attractor will be obtained below by using the maximum principle. That's why we do not give the explicit expression for the constants and monotonic functions Q which appear in these estimates. Moreover, different in general constants and functions may be denoted by the same symbols (it should not lead to misunerstanding).

A solution of (0.1) is understood to be the pair of functions

$$(T,P) \in W_p^{(1,2)}([0,t_0]\times\Omega) \times W_p^{(1,3)}([0,t_0]\times\Omega) \tag{1.1}$$

which satisfy (0.1) in the sense of distributions.

Here and below

$$W_p^{(l_1,l_2)}([0,t_0]\times\Omega) = W_p^{l_1}([0,t_0],L_p(\Omega)) \cap L_p([0,t_0],W_p^{l_2}(\Omega)) \tag{1.2}$$

(i.e., $u \in W_p^{(l_1,l_2)}$ means by definition that $\partial_t^l u, \partial_x^m \in L_p([0,t_0]\times\Omega)$ if $l \le l_1$ and $|m| \le l_2$). We assume also that the exponent p in (1.2) is chosen in such a way that

$$W_p^{(1,2)}([0,t_0]\times\Omega) \subset C([0,t_0],C^{3/2}(\Omega)) \tag{1.3}$$

It is known (see [BIN96]) that (1.3) holds if $p > 4+2n$ and this embedding is compact.

It is naturally to assume that the initial date $T(0)$ belongs to the trace space $V_0(\Omega)$ at $t=0$ of the class $W_p^{(1,2)}([0,t_0]\times\Omega)$ and the boundary conditions T_0 – to the space $W_0' = W_p^{2-1/p}(\partial\Omega)$ and P_0 – to the space $W_0 = W_p^{3-1/p}(\partial\Omega)$. (The explicit description of the space V_0 is given in [LSU67]: $V_0 = W_p^{2(1-1/p)}(\Omega)$.)

It is assumed moreover that the 1st compatibility condition holds

$$T(0)\big|_{\partial\Omega} = T_0 \tag{1.4}$$

The main result of this Section is the following theorem.

Theorem 1.1. *Let the above assumptions hold and let (T,P) – be a solution of the problem (0.1). Then the following estimate is valid:*

$$\begin{aligned} &\|T\|_{W_p^{(1,2)}([t,t+1]\times\Omega)} + \|P\|_{W_p^{(1,3)}([t,t+1]\times\Omega)} \le \\ &\qquad \le Q(\|T(0)\|_{V_0})e^{-\alpha t} + Q(\|T_0\|_{W_0'} + \|P_0\|_{W_0}) \end{aligned} \tag{1.5}$$

where $\alpha > 0$ and the monotonic function Q are independent of $T(0)$.

Proof. For simplicity we give the proof of this estimate only for the case where

$P_0 = 0$. The general case $P_0 \neq 0$ is completely analogous. This proof will be divided on a number of Lemmata.

Lemma 1.1. *Let the above assumptions be valid. Then for every fixed $t \geq 0$ and for every $1 < r < \infty$ and $l \geq 0$*

$$\|P(t)\|_{l+1,r} \leq C\mu\|T(t)\|_{l,r} \tag{1.6}$$

(Here and below $\|\cdot\|_{l,p} \equiv \|\cdots\|_{W^l_p(\Omega)}$.)

Indeed,

$$\Delta_x P(t) = \mu\gamma\nabla_x T(t) \in W^{l-1,r}(\Omega), \quad P\big|_{\partial\Omega} = 0 \tag{1.7}$$

Applying the elliptic regularity theorem (see [Tem80]), we obtain (1.6).

Corollary 1.2. *The following estimate holds*

$$\|P\|_{W_p^{(1,3)}([t,t+1]\times\Omega)} \leq C\|T\|_{W_p^{(1,2)}([t,t+1]\times\Omega)} \tag{1.8}$$

Thus, it is sufficient to prove (1.5) only for temperature component of the solution (T, P).

Lemma 1.2 (L^2-estimate). *Let (T, P) be the solution of (0.1). Then*

$$\|T(t)\|^2_{0,2} + \int_t^{t+1} \|\nabla_x T(s)\|^2_{0,2}\, ds \leq C\|T(0)\|^2_{0,2} e^{-\alpha t} + \\ + Q(\|T_0\|_{W_0^l})\left(1 + \int_0^t e^{-\alpha(t-s)}\|\nabla_x T(s)\|_{L^1(\partial\Omega)} ds\right) \tag{1.9}$$

where the function Q, and the constant $\alpha > 0$ are independent of $T(0)$.

Proof. Let us multiply the T-equation of (0.1) by T, integrate over $x \in \Omega$ and use Green's formula and the fact that $\nabla_x v \equiv 0$. Then we obtain after simple computations and estimations that

$$\frac{1}{2}\partial_t\|T(t)\|^2_{0,2} + \|\nabla_x T(t)\|^2_{0,2} \leq \\ \leq \|T_0\|_{0,\infty}\int_{\partial\Omega} |\partial_n T(t)| dS + \|T_0\|^2_{0,\infty}\int_{\partial\Omega} |v(t)| dS \tag{1.10}$$

Taking into the accordance that $T\big|_{\partial\Omega} = T_0$ and using Friedrichs inequality we obtain that

$$\|T(t)\|^2_{0,2} \leq \alpha\|\nabla_x T(t)\|^2_{0,2} + C\|T_0\|^2_{W^{1/2,2}(\partial\Omega)} \leq \alpha\|\nabla_x T(t)\|^2_{0,2} + C_1\|T_0\|^2_{W_0^l} \tag{1.11}$$

Using the estimate (1.6) and Sobolev's embedding theorem we will have

$$\|v(t)\|_{L^1(\partial\Omega)} \leq C\|v(t)\|_{1,2} \leq C_1\|T(t)\|_{1,2} \tag{1.12}$$

Inserting these estimates into (1.10) we deduce that

$$\partial_t\|T(t)\|^2_{0,2} + \alpha\|T(t)\|^2_{0,2} + \|T(t)\|^2_{1,2} \leq \\ \leq C\|T_0\|^2_{W_0^l} + \\ + \|T_0\|_{0,\infty}\int_{\partial\Omega} |\partial_n T(t)| dS + C\|T_0\|^2_{0,\infty}(1 + \|T(t)\|_{1,2}) \tag{1.13}$$

Applying the inequality $C\|T_0\|^2_{0,\infty}\|T(t)\|_{1,2} \leq C_1\|T_0\|^4_{0,\infty} + 1/2\|T(t)\|^2_{1,2}$ to the right-hand side of (1.13) we obtain finally that

$$\partial_t\|T(t)\|^2_{0,2} + \alpha\|T(t)\|^2_{0,2} + 1/2\|T(t)\|^2_{1,2} \leq \\ \leq \|T_0\|_{0,\infty}\int_{\partial\Omega} |\partial_n T(t)| dS + C(\|T_0\|^4_{0,\infty} + \|T_0\|^2_{W_0^l}) \tag{1.14}$$

To complete the proof of Lemma 1.2 it remains to apply Gronwall inequality to the estimate (1.14) and use the embedding $W_0^l \subset C(\partial\Omega)$. Lemma 1.2 is proved. $\square$

Lemma 1.3 (L^q-estimate). *Let $q > 2$ and the assumptions of Lemma 1.2 hold. Then*

$$\|T(t)\|_{0,q}^q \le C\|T(0)\|_{0,q}^q e^{-\alpha t} + Q(\|T_0\|_{W_0^l})\left(1 + \int_0^t e^{-\alpha(t-s)}\|\nabla_x T(s)\|_{L^1(\partial\Omega)}ds\right) \tag{1.15}$$

where Q is a certain monotonic function independent of $T(0)$.

Proof. Let us multiply the T-equation of (1.1) by $T|T|^{q-2}$, integrate over $x \in \Omega$ and use Green's formula.

$$\begin{aligned}\frac{1}{q}\partial_t\|T(t)\|_{0,q}^q + \frac{4(q-1)}{q^2}\|\nabla_x(|T(t)|^{q/2})\|_{0,2}^2 \le \\ \le \|T_0\|_{0,\infty}^{q-1}\int_{\partial\Omega}\|\partial_n T(t)\|dS + \frac{1}{q}\|T_0\|_{0,\infty}^q\int_{\partial\Omega}|v(t)|dS\end{aligned} \tag{1.16}$$

Estimating now the last integral in the right-hand side of (1.16) using (1.12) and arguing as in the proof of Lemma 1.2 we obtain that

$$\begin{aligned}\partial_t\|T(t)\|_{0,q}^q + \alpha\|T(t)\|_{0,q}^q \le \\ \le C\left(1 + \|T_0\|_{W_0^l}^{2q}\right)\left(1 + \|\nabla_x T(t)\|_{L^1(\partial\Omega)}\right) + \|T(t)\|_{1,2}^2\end{aligned} \tag{1.17}$$

Applying Gronwall inequality to (1.17) and using the estimate (1.9) for estimating the integral from the last term into the right-hand side of (1.17) we obtain (1.15) with $Q(z) = C_1(1 + z^{2q})$. Lemma 1.3 is proved. $\square$

Now we are in position to complete the proof of Theorem 1.1. To this end we rewrite the equation for the temperature component in the following form:

$$\partial_t T - \Delta_x T = h(t) \equiv -\nabla_x T(t)v(t); \quad T\big|_{\partial\Omega} = T_0 \tag{1.18}$$

and applying the parabolic L_p-regularity theorem (see, e.g., [LSU67]) to the equation (1.18). According to this theorem,

$$\begin{aligned}\|T\|_{W_p^{(1,2)}([t,t-1]\times\Omega)}^p \le C\|T(0)\|_{V_0}^p e^{-\alpha t} + \\ + C\left(\|T_0\|_{W_0^l}^p + \int_0^t e^{-\alpha(t-s)}\|h(s)\|_{0,p}^p\right)\end{aligned} \tag{1.19}$$

with $\alpha > 0$ and $[t, t-1] \equiv [t, 0]$ if $t < 1$.

Let us estimate the integral in the right-hand side in (1.19) by Hölder inequality, the interpolation one and the estimate (1.6):

$$\begin{aligned}\|h(s)\|_{0,p}^p \le \|T(s)\|_{1,2p}^p\|v(s)\|_{0,2p}^p \le C\|T(s)\|_{3/2,2p}^{3p/4}\|T(s)\|_{0,2p}\|^{p/4}\|T(s)\|_{0,2p}^p \le \\ \le C_1\|T(s)\|_{3/2,\infty}^{3p/4}\|T(s)\|_{0,2p}^{5p/4} \le \mu\|T(s)\|_{3/2,\infty}^p + C_\mu\|T(s)\|_{0,2p}^{5p}\end{aligned} \tag{1.20}$$

So, it remains to estimate the last term in the right-hand side of (1.20). According to Lemma 1.3 with $q = 2p$

$$
\begin{aligned}
(1.21)\quad \|T(s)\|_{0,2p}^{5p} &\le C\|T(0)\|_{0,2p}^{5p}e^{-\alpha s} + \\
&+ Q(\|T_0\|_{W_0'})\left(1 + \int_0^s e^{-\lambda(s-s_1)}\|\nabla_x T(s_1)\|_{L^1(\partial\Omega)}^{5/2}ds_1\right) \le \\
&\le C\|T(0)\|_{0,2p}^{5p}e^{-\alpha s} + Q(\|T_0\|_{W_0'})\left(1 + \int_0^s e^{-\lambda(s-s_1)}\|T(s_1)\|_{3/2,\infty}^{5/2}ds_1\right) \le \\
&\le C_\mu\left(\|T(0)\|_{V_0}^{5p}e^{-\alpha s} + Q(\|T_0\|_{W_0'})\right) + \mu\int_0^s e^{-\alpha(s-s_1)}\|T(s_1)\|_{3/2,\infty}^{p}ds_1
\end{aligned}
$$

for a certain monotonic function Q and $\alpha > 0$. Here we have used the fact that $p > 5/2$.

The estimates (1.20) and (1.21) after simple calculations imply that

$$
\begin{aligned}
(1.22)\quad \int_0^t e^{-\alpha(t-s)}\|h(s)\|_{0,p}^p ds &\le C_\mu\left(Q_1(\|T(0)\|_{V_0})e^{-\alpha t} + Q(\|T_0\|_{W_0'})\right) + \\
&+ \mu\int_0^t e^{-\alpha(t-s)}\|T(s)\|_{3/2,\infty}^p ds
\end{aligned}
$$

and this estimate holds for every $\mu > 0$.

Inserting this estimate into the estimate (1.19) and taking into account the embedding (1.4) we deduce that

$$
\begin{aligned}
(1.23)\quad \|T(t)\|_{3/2,\infty}^p &\le C_\mu\left(Q_2(\|T(0)\|_{V_0})e^{-\beta t} + Q_2(\|T_0\|_{W_0'})\right) + \\
&+ \mu\int_0^t e^{-\alpha(t-s)}\|T(s)\|_{3/2,\infty}^p ds
\end{aligned}
$$

Applying Gronwall inequality to (1.23) and fixing μ small enough we obtain that

$$
(1.24)\qquad \|T(t)\|_{3/2,\infty}^p \le Q(\|T(0)\|_{V_0})e^{-\beta t} + Q(\|T_0\|_{W_0'})
$$

for a certain function Q and $\beta > 0$.

Inserting this estimate to the right-hand side of (1.19) we obtain the T-part of the estimate (1.5). The P-part of it follows immediately now from Corollary 1.1. Theorem 1.1 is proved. □

Corollary 1.2. *Let (T, P) be a solution of (0.1). Then*

(1.25)
$$
\|T(t)\|_{2(1-1/p),p} + \|P(t)\|_{3-2/p,p} \le Q(\|T(0)\|_{V_0})e^{-\alpha t} + Q(\|T_0\|_{W_0'} + \|P_0\|_{W_0})
$$

Indeed, the estimate (1.25) for the temperature component follows from (1.5) and from the evident estimate

$$
(1.26)\qquad \|T(t)\|_{2(1-1/p),p} \equiv \|T(t)\|_{V_0} \le C\|T\|_{W_p^{(1,2)}([t,t+1]\times\Omega)}
$$

the estimate for pressure component can be deduced now using the result of Lemma 1.1.

2. Existence of solutions. Uniqueness. The attractor.

The estimates, proved in previous section, allow us to obtain the existence of the global attractor for the problem (1.1) by using the standard arguments. For completeness we give below the sketch of the attractor's construction.

Theorem 2.1. *Let the assumptions of previous Section hold. Then the problem (0.1) possesses at least one solution which satisfies (1.1).*

Sketch of the proof. For simplicity, we again restrict ourselves by considering only the case where $P_0 = 0$. The proof in general case is completely analogous. The existence of a solution for our problem can be derived by using the Leray–Schauder fix point principle (see for instance [HP80]) and based on the a priori estimate, obtained in previous Section.

Leray-Schauder principle. *Let D be a bounded open set in a Banach space W and let $F\colon \overline{D} \to W$ be a compact and continuous operator. Further let the point $h \in D$ be such that*

$$w + sF(w) \neq h \ \textit{for all}\ w \in \partial D,\ s \in [0,1]. \tag{2.1}$$

Then the equation

$$w + F(w) = h \tag{2.2}$$

has at least one solution in D.

In order to apply this principle to our problem we introduce the solving linear operator for the P-equation

$$v(t) = \mathcal{L}T(t) \equiv -\mu\left(\gamma + \nabla_x(-\Delta_x)^{-1}\gamma\nabla_x\right)T(t) \tag{2.3}$$

(note that $\mathcal{L}$ is 0-order PDO and the inverse operator is taken with homogeneous Dirichlet boundary conditions) and rewrite our problem as one equation

$$\partial_t T - \Delta_x T = -\nabla_x T \cdot \mathcal{L}T,\ T\big|_{t=0} = T(0),\ \ T\big|_{\partial\Omega} = T_0 \tag{2.4}$$

Note that if we solve (2.4) then the function P would be uniquely determined from the P-equation.

Now we define a function $\tilde{T}(t)$ as the solution of linear problem

$$\partial_t\tilde{T} - \Delta_x\tilde{T} = 0,\ \tilde{T}\big|_{t=0} = T(0),\ \ \tilde{T}|_{\partial\Omega} = T_0 \tag{2.5}$$

It follows from the parabolic regularity theorem that

$$\tilde{T} \in \mathbb{W} = W_p^{(1,2)}([0,t_0]\times\Omega) \tag{2.6}$$

We introduce the function $u = T - \tilde{T}$. Then $u\big|_{t=0} = u\big|_{\partial\Omega} = 0$ and

$$\partial_t u - \Delta_x u = -\nabla_x(u+\tilde{T})\mathcal{L}(u+\tilde{T}) \tag{2.7}$$

Applying the inverse parabolic operator $\mathcal{P}$ (with homogeneous boundary and initial conditions) to (2.7) we obtain

$$u + F(u) = 0 \tag{2.8}$$

with $F(u) = \mathcal{P}\left(\nabla_x(u+\tilde{T})\cdot\mathcal{L}(u+\tilde{T})\right)$.

The fact that F is compact as an operator from W to W can be easily obtained from the compactness of embedding (1.3) and from the parabolic regularity theorem which implies that $\mathcal{P} : L_p([0,t_0]\times\Omega) \to W$.

The condition (2.1) can be verified for D coincides with a sufficiently large ball in $\mathbb{W}$ by using a priori estimates of previous Section. Indeed, assume that (2.1) is violated, i.e. there exists $s \in [0,1]$ and $u = u_s$, $||u_s||_{\mathbb{W}} = R$ such that

$$u_s + sF(u_s) = 0 \tag{2.9}$$

Denoting $T_s = u_s + \tilde{T}$ and making the inverse transformations with the equation (2.9) we deduce that T_s satisfies the equation

$$\begin{cases} \nabla_x\cdot v_s = 0;\ \ v_s = \nabla_x P_s - s\mu\gamma T_s;\ \ P_s\big|_{\partial\Omega} = 0 \\ \partial_t T_s - \nabla_x\cdot(\nabla_x T_s - T_s v_s) = 0;\ \ T_s\big|_{\partial\Omega} = T_0 \\ T_s\big|_{t=0} = T(0) \end{cases} \tag{2.10}$$

which coincides with the problem (0.1) with μ replaced by $s\mu$. It is not difficult to verify that all estimates of previous section are in fact uniform with respect to $s \in [0,1]$, consequently, Theorem 1.1 implies that $\|T_s\|_{\mathbb{W}} \leq K$ for every solution of (2.10) and therefore

$$\|u_s\|_{\mathbb{W}} \leq K_1 \tag{2.11}$$

for every solution of (2.9) and every $s \in [0,1]$. Thus, if the radius R of the ball D is large then K_1 then the assumption (2.1) is valid and consequenttly the equation (2.8) has a solution. Theorem 2.1 is proved. □

Theorem 2.2. *Under the assumptions of previous Section the solution (T,p) of (0.1), constructed in Theorem 2.1, is unique.*

Proof. Indeed, let (T_1,P_1) and (T_2,P_2) be two solutions of (1.1) and $u(t) = T_1(t) - T_2(t)$. Then

$$\begin{cases} \partial_t u - \Delta_x u + \nabla_x T_1(t)\cdot \mathcal{L}u + \nabla_x u \cdot \mathcal{L}T_2 = 0 \\ u\big|_{\partial\Omega} = u\big|_{t=0} = 0 \end{cases} \tag{2.12}$$

Note that according to Theorem 1.1 and the embedding (1.3) $\|\nabla_x T_1(t)\|_{0,\infty} \leq C$ and $\|v_2(t)\|_{0,\infty} \leq C$, hence multiplying (2.12) by u and integrating over $x \in \Omega$ we obtain that

$$\begin{gathered}\frac{1}{2}\partial_t\|u(t)\|_{0,2}^2 + \|\nabla_x u(t)\|_{0,2}^2 \leq \\ \leq C\|\nabla_x u(t)\|_{0,2}\|u(t)\|_{0,2} + C\|\mathcal{L}u\|_{0,2}\|u\|_{0,2} \leq C_1\|u(t)\|_{0,2}^2 + \frac{1}{2}\|\nabla_x u(t)\|_{0,2}^2\end{gathered} \tag{2.13}$$

Here we have used the fact that $\|\mathcal{L}u\|_{0,2} \leq C\|u\|_{0,2}$.

Applying the Gronwall inequality to (2.13) we prove that $u \equiv 0$. Theorem 2.2 is proved. □

Corollary 2.1. *For every fixed $T_0 \in W_0'$ the equation (0.1) generates a semigroup*

$$S_t : V_0 \to V_0 \text{ by formula } T(t) = S_t T(0) \tag{2.14}$$

Moreover, arguing as in the proof of Theorem 2.2 we obtain that

$$\begin{gathered}\|S_tT_1(0) - S_tT_2(0)\|_{0,2}^2 + \int_t^{t+1} \|\nabla_x S_t T_1(0) - \nabla_x S_t T_2(0)\|_{0,2}^2 dt \leq \\ \leq Ce^{Kt}\|T_1(0) - T_2(0)\|_{0,2}^2\end{gathered} \tag{2.15}$$

Remark 1. Since the value $P(t)$ for a given t can be calculated if we know $T(t)$ for the same t then we will construct the attractor only for the T-component of the solution (T,P) of the problem (0.1).

Now we are in position to prove the existence of the attractor for the semigroup S_t, defined by (2.14). To this end we recall first the definition of the attractor and some sufficient conditions for it's existence (see [BV89] for details).

Definition 2.1. *The set $\mathcal{A}$ is called the attractor for a semigroup S_t acting in Banach space V_0 if the following conditions hold:*

- *$\mathcal{A}$ is compact in V_0;*
- *$\mathcal{A}$ is strictly invariant, i.e., $S_t\mathcal{A} = \mathcal{A}$ for $t \geq 0$;*
- *$\mathcal{A}$ is the attracting set for the semigroup S_t in V_0, i.e., for every bounded $\mathcal{B} \subset V_0$ and for every neighborhood $\mathcal{O}(\mathcal{A})$ of the attractor $\mathcal{A}$ there exists $T = T(\mathcal{B})$ such that*

$$S_t\mathcal{B} \subset \mathcal{O}(\mathcal{A}),\ \textit{for}\ t \geq T \tag{2.16}$$

Proposition 2.1 (see [BV89]). *Let the operator $S_t\colon V_0 \to V_0$ for every fixed $t \geq 0$ have a closed graph, i.e. the set $\{u, S_t u, u \in V_0\}$ is closed in $V_0 \times V_0$. Further, assume that there exists a compact attracting set $K \subset\subset V_0$ of S_t in V_0. Then the semigroup S_t possesses an attractor $\mathcal{A} \subset K$. Moreover, it has the following structure:*

$$\begin{aligned}\mathcal{A} = \big\{\xi \in V_0\colon\ &\exists u(s),\ s \in \mathbb{R},\ \textit{such that}\ u(0) = \xi,\\ &\sup\nolimits_{s\in\mathbb{R}} \|u(s)\|_{V_0} < \infty,\ \textit{and}\ S_t u(s) = u(t+s),\ t \in \mathbb{R}_+,\ s \in \mathbb{R}\big\}.\end{aligned} \tag{2.17}$$

Theorem 2.3. *Let the assumptions of Section 1 hold. Then the semigroup S_t, defined by (2.14), possesses the attractor $\mathcal{A}$ in the space $V_0 = W^{2(1-1/p),p}(\Omega) \cap \{T\big|_{\partial\Omega} = T_0\}$.*

Proof. According to Proposition 2.1 it is sufficient to verify the graph's closeness and the existence of compact attracting set . The closeness of the graph for S_t follows immediately from the fact that S_t is continuous for every fixed t (according to Corollary 2.1). Hence we should check only the existence of compact attracting set.

Note for the first, that according to (1.25) the set

$$\mathcal{B}_0 = \{u \in V_0 : \|u\|_{V_0} \leq Q(\|T_0\|_{W_0'} + \|P_0\|_{W_0})\} \tag{2.18}$$

is an absorbing (and consequently attracting) set for the semigroup S_t. But this set is not compact hence we consider a new set $\mathcal{B}_1 = S_1\mathcal{B}_0$.
Lemma 2.1. *The set $\mathcal{B}_1$ is precompact in V_0.*

Proof. Let $u_0(x)$ be the solution of the Laplace equation

$$\Delta_x u_0 = 0,\quad u_0\big|_{\partial\Omega} = T_0 \subset W^{2-1/p,p}(\partial\Omega) \tag{2.19}$$

Then, according to elliptic regularity theorem (see [Tem80]), $u_0 \in W^{2,p}(\Omega)$. Let us introduce also the function $u(t) = t(T(t) - u_0)$. Then

$$\begin{cases}\partial_t u(t) - \Delta_x u(t) = h(t) \equiv S_t T(0) - t\nabla_x S_t T(0) \cdot \mathcal{L} S_t T(0) - u_0\\ u\big|_{t=0} = 0,\quad u\big|_{\partial\Omega} = 0\end{cases} \tag{2.20}$$

Let $T(0) \in \mathcal{B}_0$. Then due to (1.5) and (1.3)

$$\|h(t)\|_{0,\infty} \leq Q_1(\|T_0\|_{W_0'} + \|P_0\|_{W_0}) \tag{2.21}$$

Applying now the parabolic L^q-regularity theorem to the problem (2.20) with $q > p$ we obtain that $u \in W_q^{(1,2)}([0,1] \times \Omega)$ and

$$\|u\|_{W_q^{(1,2)}([0,1]\times\Omega)} \leq Q_2(\|T_0\|_{W_0'} + \|P_0\|_{W_0}) \tag{2.22}$$

and consequently, the set $S_1\mathcal{B}_0 - u_0$ is bounded in $W^{2(1-1/q),q}(\Omega)$. Since $u_0 \in W^{2,p}(\Omega)$ and $q > p$ then $\mathcal{B}_1 = S_1\mathcal{B}_0$ is precompact in V_0. Lemma 2.1 is proved. □
Thus, we have constructed the absorbing set $\mathcal{B}_1$ for the semigroup S_t which is precompact in V_0. Taking the closure of this set in V_0 we obtain the compact absorbing (and consequently attracting) set for S_t. Theorem 2.3 is proved. □

3. The dimension of the attractor: upper bounds.

In this ection we prove that the attractor $\mathcal{A}$ constructed in previous Section has finite Hausdorff dimension. For the reader's convenience we recall shortly the definition of the Hausdorff dimension and some simple properties of it.

Definition 3.1. *Let X be a compact set in metric space $\mathcal{H}$. Then for any $\varepsilon > 0$, $d \geq 0$ Hausdorff (d, ε)–measure is defined to be the following number:*

$$\mu_H(X, d, \varepsilon) = \inf\{\sum_{i=1}^{\infty} r_i^d : X \subset \cup_{i=1}^{\infty} B_{x_i}^{r_i}, \ |r_i| < \varepsilon\} \tag{3.1}$$

$B_{x_i}^{r_i}$ means a ball of radius r_i centered in $x_i \in \mathcal{H}$ and the infinum is taken over all coverings of the set X.

The Hausdorff d-measure $\mu_H(X, d)$ of X and the Hausdorff dimension $\dim_H(X)$ is defined to be the following numbers:

$$\begin{cases} \mu_H(X, d) = \sup_{\varepsilon > 0} \mu_H(d, \varepsilon) \in [0, \infty] \\ \dim_H(X) = \inf\{d : \mu_H(X, d) = 0\} \in [0, \infty] \end{cases} \tag{3.2}$$

A detailed study of the concept of Hausdorff dimension is given for instance in [Tem80].

Proposition 3.1. *The following properties of Hausdorff dimension can be easily reduced from Definition 3.1:*

1. Let $X_1, X_2 \subset \mathcal{H}$ and let $X_1 \subset X_2$. Then

$$\dim_H(X_1) \leq \dim_H(X_2) \tag{3.3}$$

2. Let X be a Lipschitz manifold in H with dimension N. Then

$$\dim_H(X) = N \tag{3.4}$$

3. Let $L : \mathcal{H} \to \mathcal{H}_1$ be a Lipschitz mapping ($\mathcal{H}, \mathcal{H}_1$ are metric spaces). Then

$$\dim_H(L(X)) \leq \dim_H(X) \tag{3.5}$$

It is not difficult to prove (using (3.3)) that the dimension $\mathcal{A}$ in V_0 coincide with it's dimension in $L^2(\Omega)$

$$\dim_H(\mathcal{A}, V_0) = \dim_H(\mathcal{A}, L^2(\Omega)) \tag{3.6}$$

So, we estimate below the dimension of the attractor in a more simple space $L^2(\Omega)$. To this end we need the following definition

Definition 3.2. *A map $S : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is a subset of certain Banach space X is called uniformly quasidifferentiable on $\mathcal{A}$ if for any $x \in X$ there exists a linear operator $S'(x) : X \to X$ (quasidifferential) such that*

$$||S(x + v) - S(x) - S'(x)v||_X = \overline{\overline{o}}(||v||_X) \tag{3.7}$$

holds uniformly with respect to $x \in X$, $x + v \in X$.

The estimation of the dimension of the attractor $\mathcal{A}$ is based on the following proposition.

Proposition 3.1 [Tem80]**.** *Let S_t be a semigroup in a certain Hilbert space H and let $\mathcal{A} \subset H$ be a compact strictly invariant set of this semigroup ($S_t\mathcal{A} = \mathcal{A}$). Let us suppose also that S_t is uniformly quasidifferentiable on $\mathcal{A}$ for any fixed t and the following inequality holds for some $T > 0$*

$$\omega_d(\mathcal{A}) = \sup_{x \in \mathcal{A}} \omega_d\left(S'_T(x)\right) < 1 \tag{3.8}$$

where $\omega_d(L) \equiv \|\Lambda^d L\|_{\Lambda^d H}$ *is the norm of d-th exterior power of the operator* L *in Hilbert space* $\Lambda^d H$ *(see* [Tem80]*). Then the Hausdorff dimension of the set* $\mathcal{A}$ *is finite in* H*. Moreover,*

$$\dim_H(\mathcal{A}, H) \le d \tag{3.9}$$

Lemma 3.1. *Let the assumptions of Section 1 hold. Then the semigroup* S_t*, defined by (2.11) is uniformly quasidifferentiable on the attractor* $\mathcal{A}$ *and it's quasidifferential* $\theta(t) = (S'(t)T(0))\xi$ *is a solution of the equation of variations for the problem (2.4)*

$$\begin{cases} \partial_t\theta(t) = \Delta_x\theta(t) - \nabla_x T(t)\cdot\mathcal{L}\theta(t) - \nabla_x\theta(t)\cdot\mathcal{L}T(t) \equiv \mathbb{L}_{T(t)}\theta(t) \\ \theta\big|_{t=0} = \xi, \quad \theta\big|_{\partial\Omega} = 0 \end{cases} \tag{3.10}$$

The proof of Lemma 3.1 can be obtained by standard reasonings (see for instance [EZ99] for this proof in much more complicated situation).

Thus, to estimate the dimension of the attractor it remains to estimate d-th exterior powers of the solving operator for the problem (3.10).

Proposition 3.2. *Let the assumptions of Section 1 hold. Then*

$$\omega_d(S_t'(T(0))) \le e^{\int_0^t \mathrm{Tr}_d\{\mathbb{L}_{T(s)}\}\,ds} \tag{3.11}$$

where $T(t)$ *is a solution of (1.1) with* $T(0)\mathcal{A}$*,* $\mathbb{L}_T$ *is defined in (3.10) and* Tr_d *means a d-dimensional trace of the upper semibounded linear operator* $\mathbb{L}$*, i.e.*

$$\mathrm{Tr}_d(\mathbb{L}) \equiv \sup\{\sum_{i=1}^d(\mathbb{L}v_i, v_i) \ : \ \|v_i\|_{0,2} = 1\,,\ i = 1...d;\ (v_i, v_j) = 0 \ \mathit{for}\ i \ne j\} \tag{3.12}$$

The proof of this Proposition can be found for instance in [Tem80].

Thus, Proposition 3.2 reduces the problem of estimating the dimension of $\mathcal{A}$ to the estimating the traces $\mathrm{Tr}_d(\mathbb{L}_\psi)$ for all $\psi \in \mathcal{A}$. To this end we need the following Lemma.

Lemma 3.2. *Let* $\psi(x) \in \mathcal{A}$*. Then*

$$\|\psi\|_{0,\infty} \le \|T_0\|_{0,\infty} \tag{3.13}$$

Indeed, according to (2.17), there exists a solution $T(t)$ defined for all $t \in \mathbb{R}$ of the equation (2.4) such that $T(0) = \psi$ Applying the maximum principle to this equation now we obtain the estimate (3.13).

Remark 3.1. Note that the upper bounds of the C-norm of the attractor $\mathcal{A}_\mu$, given by (3.13) depends only on $\|T_0\|_{0,\infty}$ and independent of P_0 and μ.

Assume that $\psi \in \mathcal{A}$ and $u \in W_0^{1,2}(\Omega)$ then

$$\begin{aligned}(\mathbb{L}_\psi u, u) &= -\|\nabla_x u\|_{0,2}^2 - (\nabla_x\psi\cdot\mathcal{L}u, u) - \\ &- (\nabla_x u\cdot\mathcal{L}\psi, u) = -\|\nabla_x u\|_{0,2}^2 + (\psi, \mathcal{L}u\cdot\nabla_x u) = \\ &= -\|\nabla_x u\|_{0,2}^2 + (\psi, \nabla_x P_u.\nabla_x u) - (\psi, \mu\gamma u, \nabla_x u) \le \\ &\le -\|\nabla_x u\|_{0,2}^2 + \|T_0\|_{0,\infty}\|\nabla_x u\|_{0,2}\,(\|\nabla_x P_u\|_{0,2} + \mu\|u\|_{0,2})\end{aligned} \tag{3.14}$$

Here we have used the fact that $\nabla_x\mathcal{L} \equiv 0$ and the estimate (3.13). Recall that the function P_u satisfies the equation

$$\Delta_x P_u = \mu\gamma\cdot\nabla_x u, \quad P_u\big|_{\partial\Omega} = 0 \tag{3.15}$$

Multiplying (3.15) by P_u and integrating over $x \in \Omega$ we obtain that

$$\|\nabla_x P_u\|_{0,2}^2 = \mu(u, \gamma \cdot \nabla_x P_u) \tag{3.16}$$

Since $|\gamma| = 1$ then (3.16) implies that $\|\nabla_x P_u\|_{0,2} \leq \mu \|u\|_{0,2}$. Inserting this estimate into (3.14) we obtain finally

$$\begin{aligned}(\mathbb{L}_\psi u, u) \leq -\|\nabla_x u\|_{0,2}^2 + 2\mu\|T_0\|_{0,\infty}\|u\|_{1,2}\|u\|_{0,2} \leq \\ \leq -\frac{1}{2}\|\nabla_x u\|_{0,2}^2 + 2\left(\mu\|T_0\|_{0,\infty}\right)^2 \|u\|_{0,2}^2\end{aligned} \tag{3.17}$$

Now we are in position to formulate and prove the main Theorem of this Section.
Theorem 3.1. *Let the assumptions of Section 1 hold and let $d \in \mathbb{N}$ such that*

$$\frac{1}{2}\sum_{i=1}^{d} \lambda_i > 2d\left(\mu\|T_0\|_{0,\infty}\right)^2 \tag{3.18}$$

where λ_i -eigenvalues of the Laplace operator with Dirichlet boundary conditions. Then

$$\dim_H(\mathcal{A}, L^2(\Omega)) < d \tag{3.19}$$

Proof. Indeed, let $\{u_i\}_{i=1}^d$ be the orthonormal system in L^2, $\psi \in \mathcal{A}$. Then, according to (3.17) and min-max principle

$$\begin{aligned}\sum_{i=1}^{d}(\mathbb{L}_\psi u_i, u_i) \leq -\frac{1}{2}\sum_{i=1}^{d}\|\nabla_x u_i\|_{0,2}^2 + 2d\left(\mu\|T_0\|_{0,\infty}\right)^2 \leq \\ \leq -\frac{1}{2}\sum_{i=1}^{d}\lambda_i + 2d\left(\mu\|T_0\|_{0,\infty}\right)^2 < 0\end{aligned} \tag{3.20}$$

Thus, due to Propositions 3.1 and 3.2

$$\dim_H(\mathcal{A}, L^2(\Omega)) < d \tag{3.21}$$

Theorem 3.1 is proved. □

Corollary 3.1. *As known (see* [RS82]*)*

$$C_1 k^{2/n} \leq \lambda_k \leq C_2 k^{2/n} \tag{3.22}$$

and consequently

$$\dim_H(\mathcal{A}, L^2(\Omega)) \leq C\left(\mu\|T_0\|_{0,\infty}\right)^n \tag{3.23}$$

Remark 3.2. Making the varibale changing $T \to T - \delta T_0/2$, where $\delta T_0 = \max T_0 - \min T_0$ and arguing as before that the estimates (3.18) and (3.23) remains valid with δT_0 instead of T_0. Note also that our upper bound for the dimension of the attractor is independent on P_0.

4. The dimension of the attractor: lower bounds.

In this Section we present the lower bound for the dimension of the attractor, constructed in Section 2. For simplicity we restrict ourselves to the case $n=2$. The general case $n>2$ can be considered analogously. As usual, the lower esiamates for this dimension are based on considering the appropriate equilibria, estimating it's instability index, and using the following abstract theorem (see e.g. [BV89], [Tem80], [Hal87])

Theorem 4.1. *Let $S_t : V_0 \to V_0$ be an abstract C^1-semigroup in a Banach space V_0 (i.e. $S_t \in C^1(V_0, V_0)$ for every $\in \mathbb{R}_+$) which posesses an attractor $\mathcal{A}$ in V_0. Assume also that there exists an equilibria z_0 of S_t such that the spectrum of the derivative $D_{v_0}S_1(z_0)$ is discrete (at least outside of the unitary cirle) and the instability index $N(z_0)$ is finite. Then*

$$\dim_H(\mathcal{A}) \geq N(z_0) \tag{4.1}$$

We are going to apply this theorem for our semigroup S_t, which corresponds to the problem (0.1). Note that the differentiability of S_t can be proved in a standard way (see e.g. [BV89] or [EZ99]), hence it remains to find the appropriate equilibria and compute it's instability index.

We realize this program only for a particular choice of the domain $\Omega = [0,1] \times [0,L]$ with periodic boundary conditions with respect to x_2 (i.e. we will consider the problem (0.1) on surface of a cylinder with Dirichlet boundary conditions on the origins of the cylinder).

It is not difficult to verify that

$$\tilde{P}_0(x) = \frac{\mu}{2}x_1^2; \quad \tilde{T}_0(x) = x_1 \tag{4.2}$$

is a homogeneous with respect to x_2 equilibria point of the problem (0.1) with

$$\gamma = (1,0), \ \ T\big|_{x_1=0} = 0, \ \ T\big|_{x_1=1} = 1, \ \ P\big|_{x_1=0} = 0, \ \ P\big|_{x_1=1} = \mu/2 \tag{4.3}$$

Note also that the equilibria point (4.2) satisies

$$\tilde{v}_0(x) = \nabla_x \tilde{P}_0 - \mu\gamma\nabla_x \tilde{T}_0 \equiv 0 \tag{4.4}$$

Let us consider now the linearization of the problem near this equilibria point and let Ψ, w and θ be the pertrubations of $\tilde{T}_0$, $\tilde{P}_0$ and $\tilde{v}_0$ correspondingly. Then the linearized system has the form

$$\begin{cases} \partial_t\Psi - \nabla_x(\nabla_x\Psi) + \tilde{T}_0'(x_1)\theta_1 = 0; \\ \Delta_x w = \mu\partial_{x_1}\Psi; \quad \theta_1 = \partial_{x_1}w - \mu\Psi \\ \Psi\big|_{x_1=0} = \Psi\big|_{x_1=1} = T\big|_{x_1=0} = T\big|_{x_1=1} = 0 \end{cases} \tag{4.5}$$

Solving the second equation of (4.5) we obtain that

$$w = -\mu(-\Delta_x)^{-1}\partial_{x_1}\Psi \tag{4.6}$$

where $(-\Delta_x)^{-1}$ means the inverse to the Laplacian with homogeneous Dirichlet conditions on the origins of the cylinder Ω. Consequently

$$\theta_1 = -\mu\left(1 + \partial_{x_1}(-\Delta_x)^{-1}\partial_{x_1}\right)\Psi \tag{4.7}$$

and the problem (4.5) can be rewritten in the following form

$$\partial_t\Psi = L_\mu\Psi \tag{4.8}$$

where

$$L_\mu = \Delta_x + \mu\left(1 + \partial_{x_1}(-\Delta_x)^{-1}\partial_{x_1}\right) \tag{4.9}$$

Thus, our main task now is to estimate the instability index for the operator (4.9) (i.e. to estimate the number of eigenvalues of it which belong to the right halfplane). The following simple lemma is of fundamental significance in this connection.

Lemma 4.1. *The operator (4.9) with the domain* $D(L_\mu) := W_2^2(\Omega) \cap \{\Psi\big|_{x_1=0,1} = 0\}$ *is selfadjoint in* $L^2(\Omega)$.

The assertion of Lemma 4.1 can be verified in a standard way.

Since L_μ is selfadjoint then it's spectrum is real and we can apply the min-max principle in order to estimate it's instabiltity index.

Lemma 4.2. *Assume that we found functions* $\{\phi_i\}_{i=1}^N$ *in such a way that*

$$(L_\mu\phi_i, \phi_j) = 0, \quad \text{if } i \neq j \text{ and } (L_\mu\phi_i, \phi_i) > 0 \tag{4.10}$$

Then the instable index $N(\mu) \equiv N(L_\mu) \geq N$.

The assertion of this Lemma is a straightforward corollary of the min-max principle. Indeed if we consider the space V_N spanned by functions ϕ_i then (4.10) implies that

$$(L_\mu\phi, \phi) > 0 \text{ for every } \phi \in V_N,\ \phi \neq 0 \tag{4.11}$$

and consequently $N(\mu) \geq N$.

Theorem 4.2. *The unstable index of* L_μ *possesses the following estimate:*

$$N(\mu) \geq C_1\mu - C_2, \quad C_i > 0 \tag{4.12}$$

Proof. According to Lemma 4.2 it is sufficient to find the system which satisfies (4.10) (with a sufficiently large number of functions).

We will seek them in the following form:

$$\phi(x) = \phi_n(x_1)e^{i\frac{2\pi n}{L}x_2} \tag{4.13}$$

Lemma 4.3. *Let* ϕ *be defined by (4.13) and let* $n \neq 0$ $n \in \mathbb{Z}$. *Then*

$$L_\mu\phi = \left(L_\mu^{(n)}\phi_n\right)e^{ip_nx_2} \tag{4.14}$$

where $p_n = \frac{2\pi n}{L}$ *and*

$$L_\mu^{(n)} = \frac{d^2}{dx_1^2} - p_n^2 + \mu\left(1 + \frac{d}{dx_1}(p_n^2 - \frac{d^2}{dx_1^2})^{-1}\frac{d}{dx_1}\right) \tag{4.15}$$

The assertion of this Lemma is a result of direct computations.

Since the functions $e^{ip_nx_2}$ are orthogonal for a different n, then it is sufficient to find systems of functions whcih satisfies (4.10) only for the one dimensional operators (4.15).

Lemma 4.4. *Let* $\phi_{n,k}(x_1) = \phi_k(x_1) = \sin(4\pi kx_1)$ *and* $n \neq 0$, $k \in \mathbb{Z}$ *Then,*

$$L_\mu^{(n)}\phi_k = \lambda_{n,k}\phi_k - \frac{4\pi k}{(4\pi k)^2 + p_n^2}R_n'(x_1) \tag{4.16}$$

where

$$\lambda_{n,k} = -\left((4\pi k)^2 + p_n^2\right) + \mu\frac{p_n^2}{(4\pi k)^2 + p_n^2} \tag{4.17}$$

and the function $R_n(x_1)$ *is a solution of the following boundary problem:*

$$\begin{cases} (p_n^2 - \frac{d^2}{dx_1^2})R_n = 0 \\ R_n(0) = R_n(1) = 1 \end{cases} \tag{4.18}$$

The assertion of this lemma is also a result of direct computations.

Note now that the function $R_n(x_1)$ is a symmetric with respect to $x_1 = 1/2$, consequently R_n' is antisymmetric with respect to $x_1 = 1/2$. Moreover the functions $\phi_k(x_1)$ are also symmetric with respect to $x = 1/2$. Consequently,

$$(R_n, \phi_k) = 0 \tag{4.19}$$

for every $n, k \in \mathbb{Z}$, $n, k \neq 0$.

Corollary 4.1. *For every $k, k' \in N$, $n \in \mathbb{Z}\backslash\{0\}$*

$$(L_\mu^{(n)}\phi_k, \phi_{k'}) = \lambda_{n,k}\delta_{k,k'} \tag{4.20}$$

where $\delta_{k,k} = 1$ and $\delta_{k,k'} = 0$ if $k \neq k'$.

Indeed the formula (4.20) follows from the expression (4.16) and from the fact (4.19).

Lemma 4.5. *The unstable index $N(\mu)$ possesses the following estimate*

$$N(\mu) \geq M(\mu) \equiv \#\{(n,k),\ n \in \mathbb{Z}\backslash\{0\},\ k \in \mathbb{N} : \lambda_{n,k} > 0\} \tag{4.21}$$

Indeed, it follows from Lemma 4.3 and from Corollary 4.1 that the functions

$$\sin(4\pi k x_1)e^{ip_n x_2}, \quad (n,k) \in M(\mu) \tag{4.22}$$

satisfy all assumptions of Lemma 4.2.

To complete the proof of the theorem we need the following combinatorical result.

Lemma 4.6. *The number $M(\mu)$ possesses the following estimate:*

$$M(\mu) \geq C_1\mu - C_2 \tag{4.23}$$

for the appropriate positive constants C_i.

The assertion of the lemma is more or less evident and so we omit it's proof.

The assertion of Theorem 4.2 follows now from Lemmata 4.5 and 4.6. Theorem 4.2 is proved. □

Combining the estimate (3.23) with Theorems 4.1 and 4.2 we obtain the following result.

Theorem 4.3. *Let the assumptions of Theorem 4.2 hold. Then the dimension of the attractor $\mathcal{A} = \mathcal{A}_\mu$ of the problem (0.1) possesses the following estimates:*

$$C_1\mu - C_2 \leq \dim_H \mathcal{A}_\mu \leq C_3\mu^2 \tag{4.24}$$

5. The numerical method

The numerical examples are carried out by the pdelib/sysconlaw code [FKL98, Fuh00] for the solution of nonlinear systems of viscous conservation laws.

5.1. **The finite volume discretization ansatz.** We will describe the finite volume scheme for the following class of problems: We look for a vector valued function $u(x,t) : \Omega \times [0,T] \to \mathbb{R}^\nu$ such that

$$\begin{aligned} \partial_t b(u) + \nabla \cdot \mathbf{q}(u) &= 0 \\ \mathbf{q}(u) &= -k(u)\nabla u + \mathbf{v}(u). \end{aligned} \tag{5.1}$$

Here, $b(\cdot), k(\cdot) : \mathbb{R}^\nu \to \mathbb{R}^\nu$ are vector valued functions depending on vectors, and $\mathbf{q}(\cdot), \mathbf{v}(\cdot) : \mathbb{R}^\nu \to \mathbb{R}^{\nu d}$ are $\nu \times \nu$-tensor valued functions. Using a Voronoi box based finite volume ansatz, we can base our discretization on a simplicial tesselation of the domain Ω admitting a conforming Delaunay property [BE92]. By joining the circumcenters of the simplices, we define the so-called Voronoi boxes as the balance volumina which are now aligned to the nodes of the tesselation. For the two-dimensional case, the data involved are well illustrated in 5.1 and explained in table 1. Please see [Mac53] for the appearantly oldest reference to the method, and

$\gamma_{ij} = \overline{\omega_i} \cap \overline{\omega_j}, \quad i \neq j$	interior Voronoi box faces
$\mathbf{n}_i$	outward unit normal to the Voronoi box boundary
$\mathbf{n}_{ij}, \quad i \neq j$	outward unit normal to the interior Voronoi box face
$h_{ij} = \lvert\mathbf{x}_i - \mathbf{x}_j\rvert, \quad i \neq j$	edge lengths

TABLE 1. Geometrical data used .

[FL98] for a more thorough explanation of the scalar case of the approach used here. Please just note that the whole description can be made dimension independent. Introducing the vector valued flux functions

$$g(\mathbf{h}, u, v) = (g_1(\mathbf{h}; u_1 \dots u_\nu; v_1 \dots v_\nu) \dots g_\nu(\mathbf{h}; u_1 \dots u_\nu; v_1 \dots v_\nu))$$

which should be approximations to the cell-to-cell fluxes generated by the projection of the main part of our system onto the mesh edges, we can approximate the balance of the l-th equation of (5.1) over the space-time cell $\omega_i \times (t_n, t_{n+1})$ by:

$$\begin{aligned}
0 &= \int_{t^n}^{t^{n+1}} \int_{\omega_i} \left(\frac{\partial b_l(u)}{\partial t} + \nabla \cdot \mathbf{q}_l - f_l \right) d\omega d\tau \\
&= \int_{\omega_i} \left(b_l(u^{n+1}) - b_l(u^n) - f_l \right) d\omega + \int_{\partial\omega_i} \mathbf{q}_l \cdot \mathbf{n}\, ds \qquad (5.2) \\
&\approx |\omega^i| \left(b_l(u_i^{n+1}) - b_l(u_i^n) - f_l \right) + \tau^n \sum_{j \in \mathrm{nb}_{\mathcal{N}}(i)} \frac{|\gamma_{ij}|}{h_{ij}} g^l(\mathbf{h}_{ij}, u_i^{n+1}, u_j^{n+1})
\end{aligned}$$

The problem now is characterized by the mass terms b and flux function g corresponding to each material. The approximation chosen corresponds to an implicit Euler scheme. An existence and stability analysis of this scheme for the scalar case has been carried out in [FL98]. To approximate system 0.1, we first show that it fits into the problem class described here. Let $u = (P, T)$. Then we have

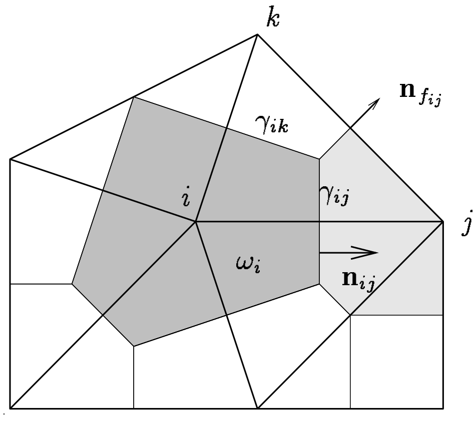

FIGURE 5.1. Geometrical coefficients of the finite volume scheme

$$\begin{aligned}
b_1(P, T) &= 0 \\
b_2(P, T) &= T
\end{aligned} \qquad (5.3)$$

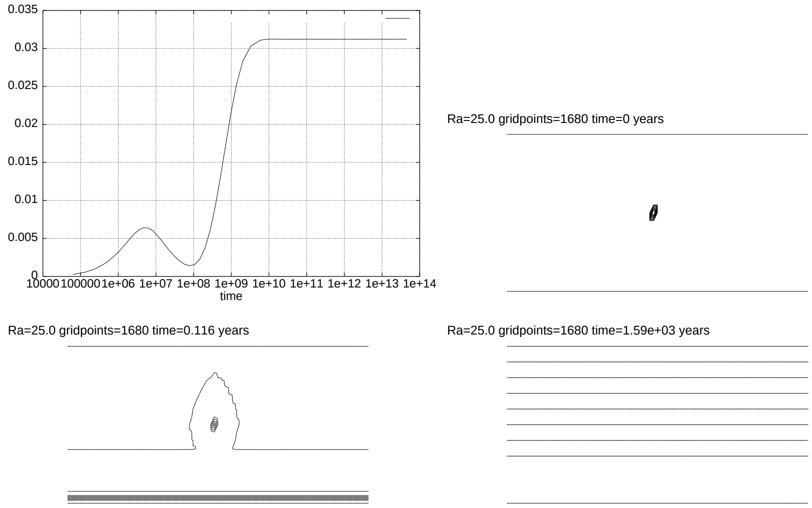

FIGURE 5.2. Ra=25: Stable equilibrium solution. Temperature at $(150, 75)$ and isothermes at three stages of the solution process.

and

$$
\begin{aligned}
q_1(P,T) &= \nabla P - \mu\gamma T \\
q_2(P,T) &= \nabla T - T q_1(P,T)
\end{aligned} \tag{5.4}
$$

Correspondingly, we chose the following flux functions

$$
g_1 = g_1(\mathbf{h},(p_1,T_1),(p_2,T_2)) = = p_1 - p_2 - \begin{cases} \mu T_1 \gamma \cdot \mathbf{h}, & \gamma \cdot \mathbf{h} > 0 \\ \mu T_2 \gamma \cdot \mathbf{h}, & \gamma \cdot \mathbf{h} < 0 \end{cases}
$$

$$
g_2(\mathbf{h},(p_1,T_1),(p_2,T_2)) = T_1 - T_2 + \begin{cases} T_1 g_1, & g_1 > 0 \\ T_2 g_1, & g_1 < 0 \end{cases}
$$

and the mass functions from (5.3). This first oder upwinding ansatz characterized by the sign dependent terms in the definition of the flux functions is aimed at a temperature maximum principle which we believe is an essential physical property.

5.2. **The solution method.** The solution method is described only shortly. In each time step, we have to solve a system of nonlinear equations. This is done using Newton's method [KA59] using an affine invariant monotonicity test [DH79]. In the presented numerical examples, we use the direct sparse matrix solver PARDISO [SGF00], and we solved the nonlinear equations up to machine precision. In general we use fixed space grids and an adaptive time step control scheme which is aimed at holding the L^∞ norm of the change of the solution constant over all timesteps. There are two exceptions from this scheme. First, this scheme appears to be to insensitive to catch the periodic behaviour described below, so we had to introduce a maximal timestep. Second, timesteps are lowered if Newton's method fails for a given timestep.

5.3. **The numerical examples.** To be consistent to the theoretical investigations in the present paper, we modify the classical Horton-Rogers-Lapwood problem [HR45, Lap48] in such a way that we use only Dirichlet boundary conditions. The computational domain is $\Omega = (0,300) \times (0,150)$. On $(0,300) \times 0 \subset \partial\Omega$ we pose the Dirichlet boundary conditions $p = 0, T = T_c$ for a given temperature T_c. On $(0,300) \times 150 \subset \partial\Omega$ we set $p = T = 0$. The boundary parts $0 \times (0,150)$ and

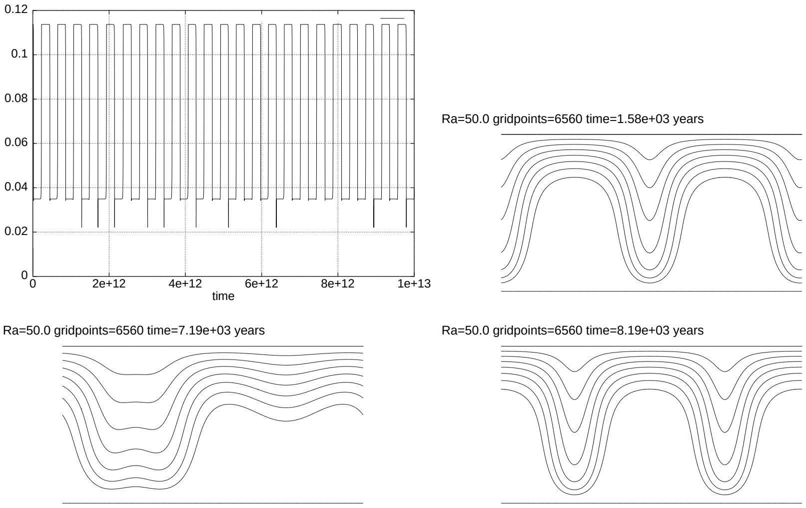

FIGURE 5.3. Ra=50: Time periodic solution. Temperature at $(150, 75)$ and isothermes for two metastable patterns, and the transition period.

$300 \times (0, 150)$ are glued together to yield periodic boundary conditions instead of the no flow boundary condition in the original problem. As an initial condition, we take $T = p = 0$, perturbed by $T(150, 75) = T_c$. The domain is discretized by a rectangular grid. To get a simplicial tesselation as described above, each rectangle is subdivided into two triangles by the lower-left-upper-right diagonal. With all other parameters fixed, we define T_c directly from Ra by $T_c = Ra/400.38$. Here, some more physical data are involved whose introduction would take to much space in this paper. During our numerical experiments, we detected the following types of regimes:

Stable equilibrium: For low Rayleigh numbers, solutions converge to the equilibrium solution which is also reached if we perturb the initial value. See fig. 5.2.

Time periodic: A main characteristic of the time periodic regime appears to be the existence of relatively long, stably looking states between which the solution switches in rapid transition periods. See fig. 5.3.

Chaotic: Here, we see a time and space dependent behaviour, which with increasing Rayleigh number looks more and more chaotic. See fig. 5.4.

Stable non-equilibrium: This is a non-constant in space solution which is reached after some time. See 5.6

The behaviour of the method in dependence on the space and time grid suggests the following hypothesis:

Hypothesis 5.1. *In reality, there are only two regimes for this problem. We have a only the stable equilibrium and the time periodic behaviour. Both other regimes are numerical artefacts. The stable nonequilibrium solution may be due to coarse time discretization or due to accumulation of roundoff errors on fine time grids, respectively. The chaotic solution may be due to a too coarse space discretization.*

However, there still is another possibility:

Hypothesis 5.2. *In reality, there are only two regimes for this problem. We have only the stable equilibrium and the stable nonequilibrium behaviour. Both other regimes are numerical artefacts. The time periodic solution is an artefact due to accumulation of roundoff errors caused by too coarse meshes. The flow* *switches between two energetically close stationary patterns which cannot. The chaotic solution may be due to a too coarse space discretization.*

This second possibility appears to be less probable, as in the case of a box with Neumann boundary conditions, time periodic solutions of this problem have been found [RW91]. This poses the following general questions for further research. The answer to these questions is essential for the understanding of applications mentioned in the introduction.

- Is analysis able to tell which regimes really take place ?
- Hoe the meta-stable behaviour can be characterized ?
- How this behaviour will change for more realistic equations (non-Boussinesq, temperature dependent viscosity), and for more realistic geometries and boundary conditions ?
- Is an analysis of the discrete dynamics possible, can it help to detect well- or misbehaviour of the method ?
- How the numerical method can be enabled to to reflect the right dynamics in a guaranteed way or at least to detect that the dynamical behaviour of a problem cannot be resolved with the given discretization parameters. Are classical adaptive methods with their inaccurate mass conservation between time levels able to catch these phenomena ?
- Can attractors be calculated numerically?

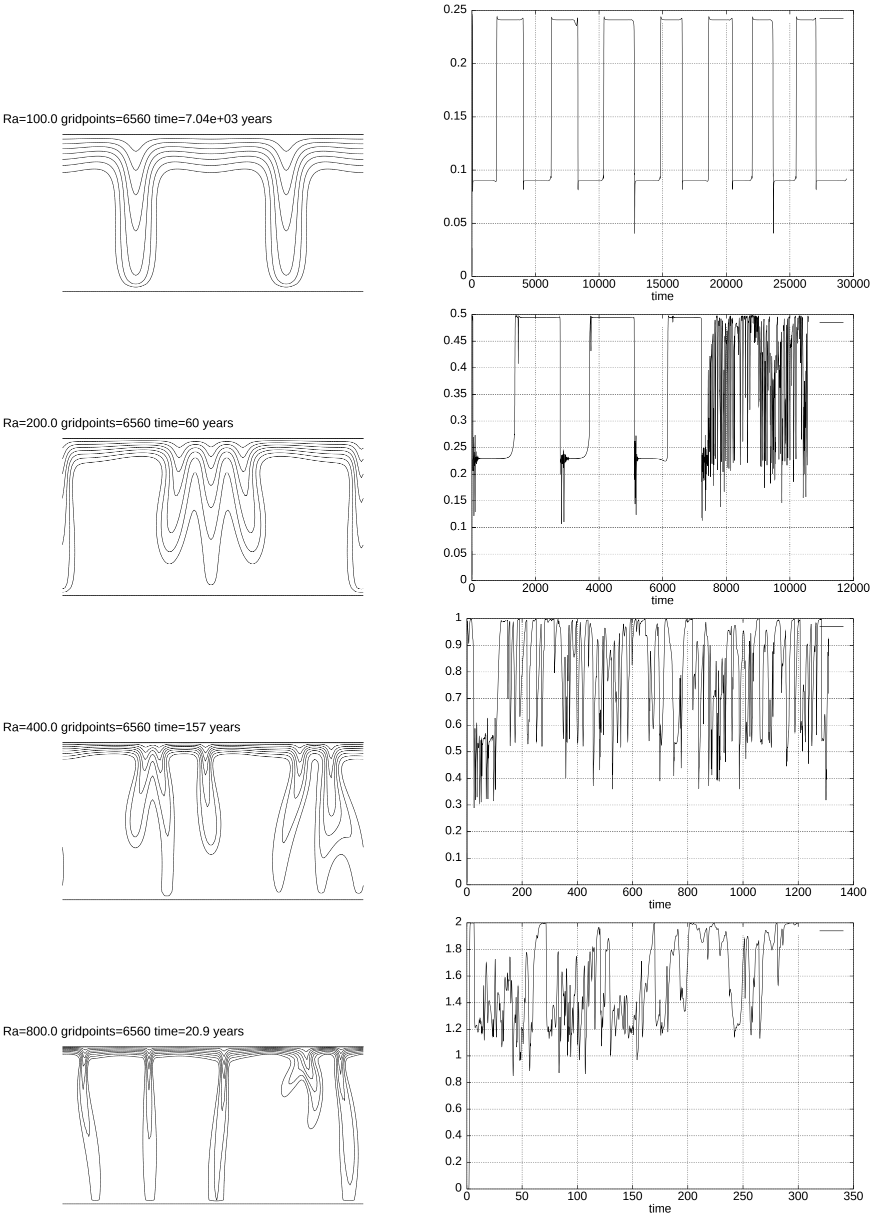

FIGURE 5.4. From periodicity to chaos with increasing Rayleigh number. Typical flow patterns and temperature logs.

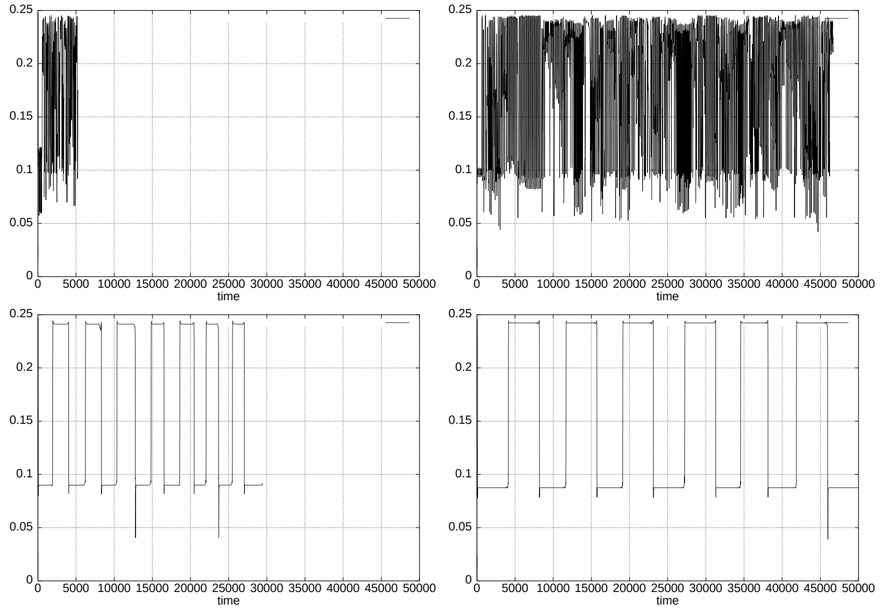

FIGURE 5.5. From chaos to periodicity with finer space grids ? Temperature at (150,75) on different grids at Ra=100.

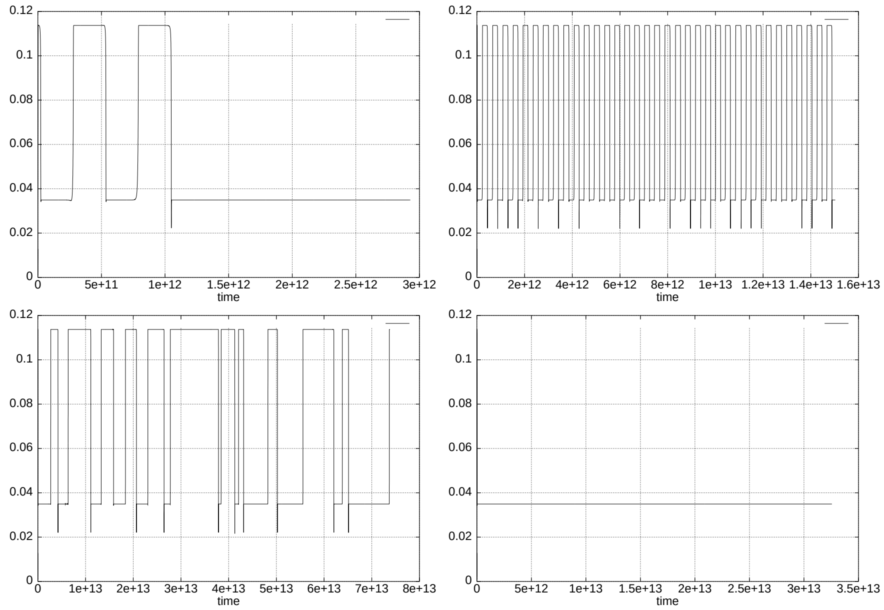

FIGURE 5.6. From stationary nonequilibrium solution to periodicity to stationary nonequilibrium ? Temperature at (150,75) for Ra=50 on a fixed space grid with time step sizes 10,100,1000,10000 years.

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