Catalytic and mutually catalytic super-Brownian motions

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Abstract. Catalytic branching processes describe the evolution of two types of material (populations) called catalyst and reactant. The catalyst evolves autonomously, but catalyzes the reactant. The individuals of both populations share the features of motion, growth and death. In mutually catalytic models however there is an additional feedback from the reactant to the catalyst destroying completely the basic independence assumption of branching theory. Recent results for continuum models of this type are surveyed.

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1. Introduction: Discrete models

We are dealing with large stochastic systems, we call *populations*, which are placed in some space, have features of motion, growth, and death. Traditionally in branching theory, the basic assumption is that disjoint parts develop independently. This independence assumption allows the use of a lot of mathematical tools, which has made possible the development of a huge mathematical theory. Probabilists have been busy with this for a long time, nevertheless research is still ongoing in this field, and there remain many interesting problems.

1.1. Critical binary branching simple random walk

Let us mention first one of the simplest models of this kind, the *critical binary branching simple random walk* in Z^2. Initially, start with a chaotic population of particles, placed in Z^2, we imagine as (possibly multiple) green points, say. We impose two kinds of dynamics on this population, namely motion and branching. So first of all, the particles (points) move chaotically. This means, each particle, *independently* of the others, may jump with a fixed rate to one of its neighboring sites, chosen at random (that is, with equal probability). But additionally, also with a certain constant rate, called the *branching rate*, say ϱ, each particle may split in a critical binary fashion. This means, with an exponential clock of rate ϱ, it doubles, or dies, both with the same chance. As mentioned, the particles evolve *independently*, in particular, this holds for newly born particles (created by the splitting). The only dependence assumption is, that the (two) offspring start from the same position where they had been created.

As time goes on, at some places the particles completely disappear, by the killing aspect of the branching, and in other regions of space, the population grows by the doubling of particles. This way *clusters* are formed (see Figure 1), which move, grow, but may also disappear.

Models of this type have been studied in great detail in one or another setting, in particular in terms of the related diffusion approximation, the continuous super-Brownian motion, and a lot is known about them.

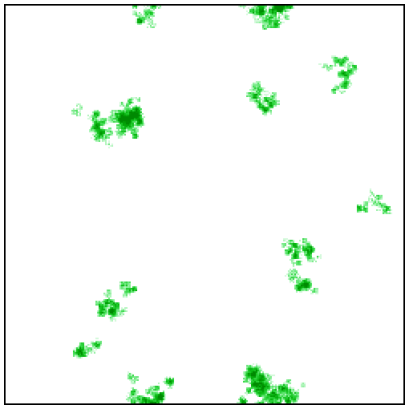

FIGURE 1. Simulation picture of the critical binary branching simple random walk (produced by Achim Klenke, Erlangen)

1.2. Imposing a random medium

The basic independence assumption, of course, is not so exciting, and one would like to get rid of it. How to do this? A first step in this direction is to additionally impose a *random medium*, thus dependencies occur by the randomness of the

medium. In the previous model, the evolution mechanisms had been the same everywhere in time and space. We now imagine that they might change in time and space, even in a random way. But for simplification, we keep the rule, that given the medium, the population evolves according to the basic independence assumptions of branching theory, more precisely, we retain the *conditional independence*.

For instance, in the previous model of critical binary branching simple random walk, we proceed as follows. We allow the *branching rate* ϱ *to fluctuate* in time and space. So we imagine, branching is controlled by some *catalyst*. In some part of space, there might be a catalyst, which allows branching of the population, we now call the *reactant*, and in other parts there might be no catalyst, and the reactant will only move chaotically there. More precisely, the branching rate of the reactant is assumed to be given by the amount $\varrho_t(b)$ of catalyst at time t at site $b \in \mathsf{Z}^2$, and, in particular, $\varrho_t(b)$ might also be zero, suppressing in this case the branching of the reactant.

So first we need a model for the catalyst ϱ. Why not take the previous model in the constant medium as the catalyst?

1.3. Critical binary branching simple random walk reactant with a critical binary branching simple random walk catalyst

In this scenario the imagined green points are now our catalyst, to which we add an initially independent chaotic population of reactant particles thought of as red points situated in Z^2. Then both, catalyst and reactant, are assumed to perform critical binary branching simple random walks on Z^2. More precisely, the catalyst is assumed to evolve autonomously, with branching rate which is constant in time and space, say a constant $\gamma > 0$, whereas the reactant is assumed to be controlled by the catalyst, that is, its branching rate is given by the varying number $\varrho_t(b)$ of catalytic particles at the present time t at the site $b \in \mathsf{Z}^2$. The resulting model is a simple example of a *catalytic branching random walk,* denoted by X^{ϱ}, say.

What long-term effect will the catalyst have on the reactant population? Recall that at some places the catalyst ϱ will be absent, then the reactant X^{ϱ} will only move chaotically. So outside the green clumps, we have only dispersion of the red reactant, similarly as in the heat flow. In other regions we have a large concentration of catalyst, giving a large branching rate for the reactant, hence speeding up the critical branching of reactant. Thus, inside the catalytic clumps, the huge branching rate mainly causes killing of the reactant, so holes of the reactant are created. But at the boundary layer of the green clumps, where the catalyst highly fluctuates, there also the reactant highly fluctuates, creating a kind of *hot spots* of reactant.

1.4. Mutually catalytic critical binary branching simple random walk

The real challenge is to handle branching models with still more interaction. For instance, concerning the previous model the natural desire would be, that both substances catalyze each other. Let us recall that the green particles had been

identified with the local branching rate of the red ones. But why should not conversely the red particles serve as the local branching rate of the green ones, instead of the constant in time and space branching rate γ so far? Then we would have two types of matter with a true interaction. In this case, we speak of a *mutually catalytic branching random walk.*

Consequently, initially we start with two chaotic substances, imagined as blue and red points, say. They move chaotically, but, in addition, catalyze each other. So the blues are the local branching rate for the reds, and vice versa.

What behavior do we expect in the long run? Again some clumps of materials should occur, which could lead to a local extinction of the other substance. So we could have some effect of separation of matter. But if a substance disappears locally, the interaction is gone locally, and the remaining population is only smeared out in such an area. On the other hand, at the *interface* of both materials a lot of activities are expected, presumably leading again to hot spots of both types, which are expected to play a more important role than in the former one-sided catalytic branching model.

1.5. Outline

In the remainder of this paper, models of the types described above will be made precise. With this aim, *continuum models* will be used as much as possible. This, in particular, requires replacement of the lattice space by Euclidean space, replacement of the simple symmetric random walk by a Brownian motion of particles, and passing to a high density/small particle mass limit, simultaneously speeding up the branching rates, etc. (*diffusion approximation*).

After modelling, some of the features already indicated will be expressed in terms of theorems, and some comments concerning the methods involved are in order.

For basic facts on superprocesses we refer to [Daw93]. Recent surveys on catalytic processes are given in [FM99], [Kle99a], and [DF99]. A few simulation pictures on the catalytic branching model of Subsection 1.3 are given in [FK99], and on both, the catalytic and mutually catalytic branching model, can be found via the home page of Achim Klenke:

`http://www.mi.uni-erlangen.de/~klenke`.

2. Catalytic branching

In this section we will start with the modelling of the super-Brownian catalyst, followed by the super-Brownian reactant with a super-Brownian catalyst. For the latter model, results on the long-term behavior and on the local structure are surveyed and compared with those concerning the ordinary super-Brownian motion.

2.1. Super-Brownian catalyst ϱ

Consider a *continuous super-Brownian motion (SBM)* $\{\varrho_t : t \geq 0\}$ in R^d starting with the Lebesgue measure ℓ, that is, with a uniformly smeared out mass. This

ϱ is a certain continuous measure-valued diffusion process. Let us first recall the intuitive description: Each infinitesimally small part of the population, we still call a "particle", moves chaotically, that is Brownian, but additionally, with an everywhere constant rate, say $\gamma > 0$, we have a critical binary splitting, also in a diffusion limit sense.

The latter effect alone can be made precise by the following diffusion ζ in R_+:

$$\mathrm{d}\zeta_t \;=\; \sqrt{\gamma\,\zeta_t}\;\mathrm{d}W_t\,, \quad \zeta_0 = 1, \tag{1}$$

with W a one-dimensional standard Wiener process, and $\gamma > 0$ a constant, the so-called *branching rate*. This is the famous *Feller's branching diffusion* without drift, arising as a high density small mass speeded up limit of critical binary Galton-Watson processes. Loosely speaking, this ζ can be seen as a "zero-dimensional super-Brownian motion".

Adding to (1) a chaotic motion component in space, leads to the formal stochastic equation

$$\mathrm{d}\varrho_t(b) \;=\; \frac{1}{2}\Delta\varrho_t(b)\,\mathrm{d}t \;+\; \sqrt{\gamma\,\varrho_t(b)}\;\mathrm{d}W_t(b)\,, \quad \varrho_0(b) \equiv 1, \qquad b \in \mathsf{R}^d, \tag{2}$$

with $\mathrm{d}W_t(b)$ a d-dimensional standard white noise. Roughly speaking, we keep Feller's branching diffusions, "independently" at each site $b \in \mathsf{R}^d$, but allow the particles additionally to move chaotically. Actually this equation makes sense rigorously only in dimension $d = 1$. Nevertheless, interpreting $\varrho_t(b)$ as generalized densities of measures $\varrho_t(\mathrm{d}b)$, and applying formally Itô's formula, from (2) one can derive the following representation of *log-Laplace transition functionals:*

$$-\log E\left\{\exp\langle\varrho_t\,,-\varphi\rangle \;\middle|\; \varrho_r\right\} \;=\; \big\langle\varrho_r, v(r,\,\cdot\,)\big\rangle\,, \qquad 0 \le r \le t. \tag{3}$$

Here given the populations state ϱ_r at time r, a suitable test function $\varphi : \mathsf{R}^d \to \mathsf{R}_+$ is integrated with the desired measure $\varrho_t(\mathrm{d}b)$ at time t. On the right hand side of (3), the so-called *log-Laplace function* $v(r,a)$, as a function of the time variable $r \ge 0$ and space variable $a \in \mathsf{R}^d$, is the unique solution to the following relatively simple *diffusion-reaction equation:*

$$\begin{gathered} -\frac{\partial}{\partial r}v(r,a) \;=\; \frac{1}{2}\,\Delta v(r,a) - \frac{1}{2}\,\gamma\,v^2(r,a), \qquad 0 < r < t, \quad a \in \mathsf{R}^d, \\ \text{with terminal condition} \quad v(t-,\,\cdot\,) = \varphi \ge 0. \end{gathered} \tag{4}$$

Here the Laplacian Δ acts on the space variable a, and reflects the chaotic motion of particles, whereas the square term is related to the critical binary splitting, independently at each space point. Recall, the constant $\gamma > 0$ is called the branching rate.

This procedure actually indicates one possibility of a rigorous approach to continuous SBM as the (unique in law) continuous measure-valued Markov process ϱ with log-Laplace transition functional (3) where the log-Laplace function v solves uniquely the diffusion-reaction equation (4). This continuous SBM ϱ is nowadays counted as one of the basic models in probability theory.

2.2. Super-Brownian reactant X^ϱ

Recall that this continuous SBM ϱ will serve as our rigorous description of the catalyst. What about a related reactant which we will denote by X^ϱ? We want it to also be a super-Brownian motion, but with the constant rate γ replaced by $\varrho_r(\mathrm{d}b)$. So first of all, in the diffusion-reaction equation (4) we replace formally the constant γ by the randomly fluctuating rate $\varrho_r(\mathrm{d}a)$:

$$-\frac{\partial}{\partial r}v(r,a) \;=\; \frac{1}{2}\,\Delta v(r,a) - \frac{1}{2}\,\varrho_r(\mathrm{d}a)\,v^2(r,a), \qquad 0<r<t, \quad a\in \mathsf{R}^d, \tag{5}$$
$$\text{with terminal condition} \quad v(t-,\,\cdot\,)=\varphi\geq 0.$$

Consequently, we have a diffusion-reaction equation with a random coefficient in the reaction term. The point is, that these reaction term coefficients are now even measures $\varrho_r(da)$. Actually, we should write more carefully $\frac{\varrho_r(\mathrm{d}a)}{\mathrm{d}a}(a)$ instead of $\varrho_r(da)$. Of course, such derivatives would be relatively harmless, if these measures were absolutely continuous. But the super-Brownian motion ϱ has in fact singular states, except in dimension $d=1$. Indeed, by the critical binary splitting component within ϱ, a lot of extinction is going on, which creates measures which are "thin" in a sense (if $d\geq 2$). On the other hand, these measures $\varrho_r(da)$ are well-known to have carrying Hausdorff dimension $2\wedge d$ in any dimension d of space. Note that these singularities of SBM fit very well to mimic situations where catalysts are distributed heterogeneously in space as, for instance, enzyme catalysts in glycolysis, see [PT88].

In any case, by this irregularity of the catalyst ϱ, the diffusion-reaction equation (5) with random coefficients is not a standard equation. Nevertheless, it can be handled, namely as an integral equation. In fact, if one formally integrates this equation, in terms of the semigroup $\{S_t : t\geq 0\}$ of the heat flow, and writes the heat flow S probabilistically as the expectation of functionals of a Brownian path W (independent of ϱ), then the following formal quantity will be involved:

$$\mathrm{d}r \int_{\mathsf{R}^d} \varrho_r(\mathrm{d}b)\,\delta_b(W_r) \;=:\; L_{[W,\varrho]}(\mathrm{d}r). \tag{6}$$

Intuitively, one measures the time points r where a typical Brownian reactant particle's path W is at the position b of a catalytic particle. Such a quantity (if meaningful) is called the *collision local time* $L_{[W,\varrho]}$ of W and ϱ. Since the $\varrho_r(db)$ are rather singular in higher dimensions, it is not immediately clear that this quantity makes sense rigorously. At the first sight, one might think, $L_{[W,\varrho]}$ exists non-trivially only in dimension one, since two independent Brownian particles, one from the catalyst and one from the reactant, will meet only in dimension 1. But we have to take into account, that ϱ describes a whole cloud of particles, which results after taking a diffusion limit. So ϱ is a more involved object than a countable number of moving and branching Brownian particles. Recall that each ϱ_t has carrying Hausdorff dimension $d\wedge 2$.

Actually, the random measures $L_{[W,\varrho]}(\mathrm{d}r)$ make sense (via a regularization procedure) and are non-trivial if and only if $d\leq 3$ ([BEP91], [EP94]). In higher

dimensions, they degenerate to 0; then a typical Brownian reactant particle W does not meet the thin catalyst ϱ, that is, it goes through it without hitting. More precisely, the occupation density of W at ϱ is 0, if $d > 3$.

So from now on we want to restrict to dimensions $d \leq 3$. And in these dimensions, a super-Brownian reactant X^ϱ with catalyst ϱ really exists as a continuous measure-valued Markov process whose log-Laplace transition functional can be expressed in terms of a log-Laplace function which is the unique solution (in a mild sense) of the diffusion-reaction equation (5) with random coefficients. For simplification, let's start with a uniform initial state: $X_0^\varrho = \ell$, the Lebesgue measure on R^d. Detailed statements follow in the next theorem taken from ([DF97a]). Let $\big(W, \Pi_{r,a}\,,\, r \geq 0,\, a \in \mathsf{R}^d\big)$ denote a standard Brownian motion in R^d (independent of ϱ), and p the related heat kernel:

$$\mathrm{p}_s(a) \;:=\; (2\pi s)^{-d/2} \exp\Big[-\frac{|a|^2}{2s} \Big], \qquad s > 0, \quad a \in \mathsf{R}^d, \tag{7}$$

corresponding to the heat flow semigroup S.

Theorem 2.1 (non-degenerate existence of X^ϱ)**.** *Assume $d \leq 3$. Given the super-Brownian catalyst ϱ with $\varrho_0 = \ell$, the super-Brownian reactant X^ϱ with $X_0^\varrho = \ell$ exists non-trivially as a (time-inhomogeneous) continuous measure-valued Markov process $\{X_t^\varrho : t \geq 0\}$, with log-Laplace transition functional*

$$-\log E\left\{\exp\langle X_t^\varrho, -\varphi\rangle \;\middle|\; X_r^\varrho\right\} \;=\; \big\langle X_r^\varrho, v(r,\cdot)\big\rangle\,, \qquad 0 \leq r \leq t, \tag{8}$$

where, for each suitable test function $\varphi \geq 0$, the log-Laplace function v is the unique solution of the integral equation

$$v(r,a) \;=\; \Pi_{r,a}\bigg[\varphi(W_t) - \frac{1}{2}\int_r^t L_{[W,\varrho]}(\mathrm{d}s)\, v^2(s, W_s)\bigg], \qquad 0 \leq r \leq t, \quad a \in \mathsf{R}^d. \tag{9}$$

X^ϱ has the heat flow as its expectation:

$$E\left\{X_t^\varrho \;\middle|\; X_r^\varrho\right\} \;=\; S_{t-r} X_r^\varrho, \qquad 0 \leq r \leq t, \tag{10}$$

and, for test functions φ_1 and φ_2, as well as $0 \leq r \leq t_1, t_2$, the covariance is given by

$$\begin{aligned} &Cov\left\{\big\langle X_{t_1}^\varrho, \varphi_1\big\rangle, \big\langle X_{t_2}^\varrho, \varphi_2\big\rangle \;\middle|\; X_r^\varrho\right\} \\ &= \int_r^{t_1 \wedge t_2} \mathrm{d}s \int_{\mathsf{R}^d} \varrho_s(\mathrm{d}a) \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_{s-r}(b-a)\, S_{t_1-s}\varphi_1\,(b)\; S_{t_2-s}\varphi_2\,(b)\,. \end{aligned} \tag{11}$$

On the other hand, if $d \geq 4$, the solution v to (9) degenerates to the heat solution (since $L_{[W,\varrho]} = 0$), hence X^ϱ from (8) degenerates to the heat flow. Recall that by a formal differentiation (9) reduces to (5). Let us skip further details on the construction of X^ϱ.

This reactant process X^ϱ has interesting features, and it is now relatively well understood.

2.3. Long-term behavior of X^ϱ

After construction of X^ϱ, first the long-term behavior was studied. The following theorem is a combination of results taken from [DF97a, DF97b, EF98, FK99, FK00]. Recall that we assumed for simplicity uniform initial states $\varrho_0 = \ell = X_0^\varrho$.

Theorem 2.2 (persistent convergence)**.** *In all dimensions* $(d \leq 3)$*, we have persistent convergence in law:*

$$X_T^\varrho \xrightarrow[T\uparrow\infty]{\mathcal{L}} \text{ some } X_\infty^\varrho \text{ with } EX_\infty^\varrho = \ell. \tag{12}$$

Additionally,

$$X_\infty^\varrho = \begin{cases} \ell, & \text{if } d = 1, \\ \zeta\,\ell \text{ with } V\mathrm{ar}\,\zeta > 0, & \text{if } d = 2, \\ \text{has countably infinite local biodiversity,} & \text{if } d = 3. \end{cases} \tag{13}$$

Here *countably infinite local biodiversity* means, that given ϱ, the random measure X_∞^ϱ does not have a deterministic component, and, in the cluster representation of the infinitely divisible random measure X_∞^ϱ, infinitely many clusters contribute to each finite region ([FK00]).

Consequently, in all dimensions, a limit population X_∞^ϱ exists, and has full expectation ℓ. Moreover, in dimension one the limit population is uniform and deterministic, since the catalytic clumps disappear locally; in $d = 2$ the limit X_∞^ϱ is still uniform but with a random intensity ζ, since the large but rare catalytic clumps come back from time to time, creating some randomness even in the limit; whereas in $d = 3$ (where, more precisely, the catalyst is assumed to be in equilibrium from the beginning), X_∞^ϱ exhibits countably infinite local biodiversity (genetic abundance). We mention that in the *annealed* model (if one mixes the laws of the reactant with respect to the catalyst's law), the variance $V\mathrm{ar}\,\zeta$ of the random intensity ζ is even infinite.

Let us stress that in all three dimensions the reactant X^ϱ behaves quite differently from the ordinary SBM ϱ (catalyst). In fact, as well-known,

$$\varrho_T \xrightarrow[T\uparrow\infty]{\mathcal{L}} \text{ some } \varrho_\infty \text{ for all } d \geq 1, \tag{14}$$

however

$$\varrho_\infty = \begin{cases} 0, & \text{if } d \leq 2, \\ \text{steady state with } \textit{finite} \text{ local biodiversity,} & \text{if } d \geq 3. \end{cases} \tag{15}$$

Let us mention that it was a matter of dispute for a while whether in dimension two the mean will be preserved for the reactant X^ϱ at $T = \infty$ (persistence), since here huge catalytic clumps come back again and again to a finite window of observation causing an extreme killing of reactant there.

2.4. Local structure of the reactant X^ϱ off the catalyst

In contrast to the usual SBM ϱ, the reactant X^ϱ always has absolutely continuous states, and moreover in dimension two and three the density functions are almost everywhere extremely smooth and actually satisfy the heat equation ([FK99]):

Theorem 2.3 (smooth density field). *For $d = 2, 3$, given ϱ,*

$$X_t^\varrho(\mathrm{d}b) \;=\; \xi_t^\varrho(b)\,\mathrm{d}b, \qquad t > 0, \tag{16}$$

where, with probability one, ξ^ϱ is Lebesgue almost everywhere a $\mathcal{C}^\infty$–function (in both variables t and b) and satisfies the heat equation (both off the catalyst ϱ, which lives on a Lebesgue zero set in $\mathsf{R}_+ \times \mathsf{R}^d$).

Of course, these heat solutions ξ^ϱ might be rather irregular at the boundaries where the catalyst ϱ acts. Note that this theorem does not tell anything about the boundary layer behavior. In particular, it does not explain the occurrence of hot spots at the interface of reactant and catalyst seen in the mentioned simulations.

2.5. On the hot spots

One possible way to gain information about the hot spots is to study the collision local time $L_{[\varrho, X^\varrho]}$ between catalyst and reactant, which measures the times r where ϱ_r and X_r^ϱ are at the same sites $a \in \mathsf{R}^d$ (in the sense of occupation densities). $L_{[\varrho, X^\varrho]}$ exists as a limiting measure on $\mathsf{R}_+ \times \mathsf{R}^d$ in a non-trivial way in all dimensions. The approximating measures are formed by means of the heat kernels p from (7). This is part of the following theorem, taken from [DF98].

Theorem 2.4 (collision local time $L_{[\varrho, X^\varrho]}$).

(a) (non-trivial existence): *In $d \le 3$, for almost all (ϱ, X^ϱ), the collision local time*

$$L_{[\varrho, X^\varrho]}\big(\mathrm{d}[r, a]\big) \;=\; \lim_{\varepsilon \downarrow 0} \mathrm{d}r\, \varrho_r(\mathrm{d}a) \int_{\mathsf{R}} X_r^\varrho(\mathrm{d}b)\, \mathrm{p}_\varepsilon(b - a) \tag{17}$$

between ϱ and X^ϱ exists non-trivially.

(b) (spatial marginal collision densities): *In $d = 2$ and for $T > 0$,*

$$L_{[\varrho, X^\varrho]}\big([0, T], \mathrm{d}b\big) \quad \textit{is absolutely continuous a.s.} \tag{18}$$

(c) (random ergodic limit): *In $d = 2$,*

$$\frac{1}{T}\, L_{[\varrho, X^\varrho]}\big([0, T] \times (\cdot)\big) \underset{T \uparrow \infty}{\overset{\mathcal{L}}{\Longrightarrow}} \eta\, \ell(\cdot), \tag{19}$$

where with respect to the annealed law, η has expectation one and a variance in $(0, \infty)$.

Consequently, besides existence in all dimensions, for $d = 2$ the spatial marginal measures are even absolutely continuous. Finally, in the long run one has a kind of ergodic limit, but with some additional noise: The averaging (in $d = 2$) does not lead to a deterministic quantity, the expectation ℓ. Roughly speaking, in the time average, the hot spots are everywhere present, and they grow in all scales. Often this is called a *diffusive behavior*, see, for instance, [CG86].

Remark 2.5 (open problem)**.** The L^2–methods used in [DF98] to derive the $d = 2$ results of (b) and (c), do *not* apply in three dimensions, and the behavior for $d = 3$ remains *open.* ◇

We mention that the collision local time $L_{[\varrho, X^\varrho]}$ is also important for the formulation of the martingale problem for the reactant X^ϱ (see [DF98]).

2.6. Long-term behavior for finite reactant populations

So far we assumed that X^ϱ starts from the Lebesgue measure. But also in the case of finite reactant populations the reactant process X^ϱ exists for almost all ϱ, and has interesting features different from the ordinary super-Brownian motion ϱ. Recall that for finite initial measures ϱ_0, the total mass process $t \mapsto \|\varrho_t\| := \varrho_t(\mathsf{R}^d)$ is Feller's branching diffusion (1) without drift. Hence, with probability one,

$$\varrho_T \;=\; 0 \quad \text{for some } T < \infty. \tag{20}$$

In contrast to this finite time extinction of the ordinary super-Brownian motion ϱ, from [DF97a] and [FK00] we have the following results.

Theorem 2.6 (finite reactant populations)**.** *Assume that* $0 < \|X_0^\varrho\| < \infty$. *Then for almost all* ϱ *(with* $\varrho_0 = \ell$*), the following statements are true.*

(a) (persistent convergence): *If* $d = 1$, *then*

$$\|X_T^\varrho\| \;\xrightarrow[T\uparrow\infty]{\mathcal{L}}\; \textit{some } \|X_\infty^\varrho\|, \tag{21}$$

where $\|X_\infty^\varrho\|$ *has full expectation* $E\|X_0^\varrho\|$ *and a variance in* $(0, \infty)$.

(b) (long-term extinction): *If* $d = 2, 3$, *then*

$$\|X_T^\varrho\| \;\xrightarrow[T\uparrow\infty]{}\; 0 \;\; a.s. \tag{22}$$

(c) (finite time survival): *If* $d = 3$ *and* $X_0^\varrho \neq 0$, *then*

$$X_T^\varrho \neq 0 \quad \textit{for all } T < \infty, \;\; a.s. \tag{23}$$

In fact, in dimension 1, an intrinsic reactant particle has only a *finite time of interference* with the catalyst ([DF97a]), which explains at least heuristically the persistent convergence in (a). In $d = 2, 3$, however, the total collision local time $L_{[W,\varrho]}(\mathsf{R}_+)$ of an intrinsic reactant particle with path W and the catalyst ϱ is *infinite* almost surely ([FK00]), which intuitively makes plausible the long-term extinction as in (b). This can actually be turned into a rigorous proof by exploiting the fact that, via an individual time change, Feller's branching diffusion is "*embedded*" in the reactant's total mass process ([DFM00]). Finally, in $d = 3$, the finite time survival of (c) is caused by some *instantaneous propagation of reactant matter* property ([FK00]), which is based on the fact that lines are polar for the support of three-dimensional ordinary super-Brownian motion.

Remark 2.7 (open problem)**.** The latter fact raises the interesting question whether or not the complement of two-dimensional ordinary super-Brownian motion is *connected*: A positive answer would extend the finite time survival of (c) also to $d = 2$. ◇

3. Mutually catalytic branching

Before we will turn to a mutually catalytic model, let us have an alternative look at the model we just dealt with.

3.1. Heuristic writing of (ϱ, X^ϱ)

Symbolically, the previous model can be described by the following system of stochastic evolution equations:

$$\begin{aligned} \text{catalyst:} \qquad \mathrm{d}\varrho_t &= \frac{1}{2}\,\Delta\varrho_t\,\mathrm{d}t + \sqrt{\gamma\,\varrho_t}\,\mathrm{d}W_t^1, \\ \text{reactant:} \qquad \mathrm{d}X_t^\varrho &= \frac{1}{2}\,\Delta X_t^\varrho\,\mathrm{d}t + \sqrt{\varrho_t\,X_t^\varrho}\,\mathrm{d}W_t^2, \end{aligned} \tag{24}$$

where the $\mathrm{d}W^i$ are independent space-time white noises. In fact, the first of these equations is a repetition of (2). Recall, the Laplacian Δ stands for the chaotic motion, so the catalytic measures are smeared out, whereas the root term reflects the critical binary branching of catalyst, in fact in the diffusion limit sense. The second equation in (24) is the symbolic counterpart concerning the reactant: the constant branching rate γ is simply replaced by the measure-valued path ϱ.

Clearly, these symbolic equations (24) can be turned into martingale problems, which indeed can successfully be handled. In fact, for the reactant case, the collision local time $L_{[\varrho, X^\varrho]}$ of Theorem 2.4 occurs ([DF98]) instead of the suspicious product $\varrho_t\,X_t^\varrho$ of generalized densities in (24).

Altogether, one has 2 substances, the first one, the catalyst, is autonomous, whereas the second one, the reactant, is catalyzed by the first one. Sometimes such situation is also called a "one-way interaction". However we now want the two substances to catalyze each other. A formal symmetrization of the previous model (ϱ, X^ϱ) would be a pair $\mathbf{X} = \left(X^1, X^2\right)$ of processes satisfying

$$\mathrm{d}X_t^i = \frac{1}{2}\,\Delta X_t^i\,\mathrm{d}t + \sqrt{X_t^1\,X_t^2}\,\mathrm{d}W_t^i\,, \qquad i = 1, 2, \tag{25}$$

(which was proposed by Carl Mueller years ago, as remarked in [DP98]).

At the first sight, there is a serious mathematical problem: How to make sense of such a model (25)? The main obstacle is, that the basic assumption of branching theory is lost, namely the branching property: In the earlier model (ϱ, X^ϱ) we had an independent evolution of disjoint reactant parts, given the catalyst. And this basic conditional independence assumption was the source for the log-Laplace calculus, say. All this is gone in such type of a model $\mathbf{X} = \left(X^1, X^2\right)$. In particular,

one loses the very useful connection to diffusion-reaction equations. In other words, a completely new mathematical approach is needed.

3.2. Mutually catalytic simple super-random walk $\mathbf{X}$ on Z^d

After some years of struggle, [DP98] succeeded in constructing such a mutually catalytic model, in fact, first of all in a discrete space setting. To make this precise, we call a function f defined on Z^d *tempered*, if

$$\|f\|_\lambda \;:=\; \sum_{b\in\mathsf{Z}^d} \big|f(b)\big|\, \mathrm{e}^{-\lambda\,|b|} \;<\; \infty \quad \text{for all } \lambda>0. \tag{26}$$

Topologize the set $\mathcal{M}_{\mathrm{tem}} = \mathcal{M}_{\mathrm{tem}}(\mathsf{Z}^d)$ of all non-negative tempered functions (considered also as measures) by

$$\mathrm{d}^{\mathcal{M}}_{\mathrm{tem}}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n}\left(\|f-g\|_{1/n}\wedge 1\right), \qquad f,g\in\mathcal{M}_{\mathrm{tem}}\,. \tag{27}$$

Theorem 3.1 (existence of $\mathbf{X}$ on Z^d)**.** *On the lattice space Z^d of each dimension $(d\geq 1)$, the mutually catalytic simple super-random walk $\mathbf{X}=\big(X^1,X^2\big)$ exists: To each $\mathbf{x}=\big(x^1,x^2\big)\in\mathcal{M}^2_{\mathrm{tem}}$, there is a weakly unique continuous $\mathcal{M}^2_{\mathrm{tem}}$–valued solution $\mathbf{X}=\big(X^1,X^2\big)$ to the following system of stochastic equations*

$$\mathrm{d}X^i_t(b) \;=\; \frac{1}{2}\,\Delta X^i_t(b)\,\mathrm{d}t \;+\; \sqrt{\gamma\, X^1_t(b)\, X^2_t(b)}\;\mathrm{d}W^i_t(b), \quad b\in\mathsf{Z}^d,\;\; i=1,2, \tag{28}$$

satisfying $\mathbf{X}_0=\mathbf{x}$, where $\big\{W^i(b):\; b\in\mathsf{Z}^d, \quad i=1,2\big\}$ are independent Wiener processes, and $\gamma>0$ is a constant, the so-called collision rate.

In other words, in this model the motion of "particles" is the simple random walk on Z^d, in (28) expressed by the discrete Laplacian Δ. On the other hand, the noise is modelled by independent Wiener processes. Thus, first of all, independently at each space site b one has Feller's branching diffusions without drift [as in (1)], but "additionally" with a rate controlled by the local concentration of mass of the opposite substance.

A solution to (28) can be constructed by standard methods as in [SS80], for instance. For uniqueness of solutions however, as stressed in [DP98], a self-duality property was essentially used, recently established in [Myt98].

So first of all, such mutually catalytic process $\mathbf{X}$ in Z^d makes sense non-trivially. Moreover, it has interesting features, which can be found in [DP98]. Some of them are related in a sense to the previous (ϱ, X^ϱ) model, others are significantly different. But we want to skip all these things at this stage, since our main interest here concerns related continuum models that we now want to discuss.

3.3. Existence of a one-dimensional continuum version

The existence of a one-dimensional continuum version was established in [DP98], with uniqueness also based on [Myt98]. In order to make this precise, let $\mathcal{C}_{\mathrm{tem}} =$

$\mathcal{C}_{\rm tem}(\mathsf{R})$ denote the set of all *tempered* continuous functions f defined on R, that is continuous functions such that

$$|f|_\lambda \;:=\; \sup_{b\in\mathsf{R}} \big|f(b)\big|\,\mathrm{e}^{-\lambda|b|} \;<\; \infty \quad \text{for all } \lambda>0. \tag{29}$$

Topologize $\mathcal{C}_{\rm tem}$ by

$$\mathrm{d}^{\mathcal{C}}_{\rm tem}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n}\left(|f-g|_{1/n}\wedge 1\right), \qquad f,g\in\mathcal{C}_{\rm tem}\,. \tag{30}$$

Theorem 3.2 (existence of $\mathbf{X}$ on R)**.** *The mutually catalytic super-Brownian motion $\mathbf{X}=\big(X^1,X^2\big)$ in R exists: To each $\mathbf{x}=\big(x^1,x^2\big)\in\big(\mathcal{C}^+_{\rm tem}\big)^2$, there is a weakly unique continuous $\big(\mathcal{C}^+_{\rm tem}\big)^2$–valued solution $\mathbf{X}=\big(X^1,X^2\big)$ to the following system of stochastic partial differential equations:*

$$\mathrm{d}X^i_t(b) \;=\; \frac{1}{2}\Delta X^i_t(b)\,\mathrm{d}t \;+\; \sqrt{\gamma\, X^1_t(b)\, X^2_t(b)}\;\mathrm{d}W^i_t(b), \quad b\in\mathsf{R}, \quad i=1,2, \tag{31}$$

satisfying $\mathbf{X}_0=\mathbf{x}$. Here $\mathrm{d}W^1$, $\mathrm{d}W^2$ are independent white noises, and the constant $\gamma>0$ is the collision rate.

The construction of a solution $\mathbf{X}$ to (31) can be provided starting from approximations $\{{}^n\mathbf{X}:\,n\geq 1\}$, where ${}^n\mathbf{X}=({}^nX^1,{}^nX^2)$ has the property that on small time periods $\big[\frac{j}{n},\frac{j+1}{n}\big)$, $j\geq 0$, given ${}^n\mathbf{X}_{j/n}$ the two subpopulations ${}^nX^1,{}^nX^2$ behave as *independent* one-dimensional catalytic super-Brownian motions with frozen, smoothed and truncated branching rate functions (catalyst) $\gamma\left(S_{1/n}{}^nX^2_{i/n}\wedge n\right)$ and $\gamma\left(S_{1/n}{}^nX^1_{i/n}\wedge n\right)$, respectively. (This partly relies on ideas of [Shi94].)

Uniqueness however is based on Mytnik's self-duality property we next want to deal with. We first mention that analogously to Theorem 3.2 the mutually catalytic super-Brownian motion in R can also be defined as a $\big(\mathcal{C}^+_{\rm rap}\big)^2$–valued process ([DP98]). Here $\mathcal{C}_{\rm rap}=\mathcal{C}_{\rm rap}(\mathsf{R})$ is the set of all *rapidly decreasing* continuous functions f defined on R, that is continuous functions such that $|f|_{-\lambda}<\infty$ for all $\lambda>0$ [recall (29)], topologized by

$$\mathrm{d}^{\mathcal{C}}_{\rm rap}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n}\left(|f-g|_{-1/n}\wedge 1\right), \qquad f,g\in\mathcal{C}_{\rm rap}\,. \tag{32}$$

Introduce the pairing

$$\langle f,\tilde f\rangle \;:=\; \int_{\mathsf{R}} \mathrm{d}b\; f(b)\,\tilde f(b), \qquad f\in\mathcal{C}^+_{\rm tem}\,, \quad \tilde f\in\mathcal{C}^+_{\rm rap}\,. \tag{33}$$

Moreover, write A for the orthonormal matrix $2^{-1/2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, and introduce the duality function

$$\Psi(\mathbf{x},\widetilde{\mathbf{x}}) \;:=\; \exp\Big[-\Big\langle(\mathbf{x}A)^1,(\widetilde{\mathbf{x}}A)^1\Big\rangle + i\Big\langle(\mathbf{x}A)^2,(\widetilde{\mathbf{x}}A)^2\Big\rangle\Big], \tag{34}$$

$\mathbf{x} \in \left(\mathcal{C}^+_{\rm tem}\right)^2$, $\widetilde{\mathbf{x}} \in \left(\mathcal{C}^+_{\rm rap}\right)^2$. From [Myt98] we now quote the following result, implying uniqueness in the martingale problem related to (31) by a standard procedure.

Theorem 3.3 (self-duality)**.** *Let* $\mathbf{X}$ *and* $\widetilde{\mathbf{X}}$ *be independent mutually catalytic super-Brownian motions in* R *with* $\mathbf{X}_0 \in \left(\mathcal{C}^+_{\rm tem}\right)^2$ *and* $\widetilde{\mathbf{X}}_0 \in \left(\mathcal{C}^+_{\rm rap}\right)^2$. *Then the following self-duality formula holds*

$$E\Psi(\mathbf{X}_t\,, \widetilde{\mathbf{X}}_0) \;=\; E\Psi(\mathbf{X}_0\,, \widetilde{\mathbf{X}}_t), \qquad t \geq 0. \tag{35}$$

This identity is actually simply to verify by using Itô's formula.

3.4. Long-term behavior of the mutually catalytic SBM in R

Recall that in $d = 1$, starting from finite initial measures,

$$\left(\|\varrho_T\|, \|X^\varrho_T\|\right) \;\xrightarrow[T\uparrow\infty]{\mathcal{L}}\; \left(0, \|X^\varrho_\infty\|\right), \tag{36}$$

where $\|X^\varrho_\infty\|$ has full expectation and is non-trivial if $X^\varrho_0 \neq 0$ [Theorem 2.6 (a) and formula (20)], whereas in the case of Lebesgue initial measures,

$$(\varrho_T\,, X^\varrho_T) \;\xrightarrow[T\uparrow\infty]{\mathcal{L}}\; (0, \ell) \tag{37}$$

[Theorem 2.2 and formulas (14)/(15)]. For the mutually catalytic model we have instead the following persistent convergence statements ([DP98]).

Theorem 3.4 (global segregation of types)**.** *Recall that* $d = 1$.

(a) (finite $\mathbf{X}_0$): *If* $\mathbf{X}_0 \in (\mathcal{C}^+_{\rm rap})^2$, *then*

$$\|\mathbf{X}_T\| \;:=\; \left(\|X^1_T\|, \|X^2_T\|\right) \;\xrightarrow[T\uparrow\infty]{\mathcal{L}}\; \|\mathbf{X}_\infty\| \;:=\; B_\tau \tag{38}$$

where $t \mapsto B_t$ *is a Brownian motion in* R^2_+ *starting from* $\|\mathbf{X}_0\|$, *and* τ *is its first hitting time of the boundary of* R^2_+. *In particular,*

$$E\|\mathbf{X}_\infty\| \;=\; E\|\mathbf{X}_0\| \quad (\textit{persistence}), \tag{39}$$

and

$$\|X^1_\infty\| \cdot \|X^2_\infty\| = 0 \;\; \textit{a.s.} \quad (\textit{global segregation}). \tag{40}$$

(b) (uniform initial measures): *If* $\mathbf{X}_0 = \mathbf{c}\ell$, $\mathbf{c} \in \mathsf{R}^2_+$, *instead, then*

$$\mathbf{X}_T \;\xrightarrow[T\uparrow\infty]{\mathcal{L}}\; \mathbf{X}_\infty \;:=\; B_\tau\,\ell \tag{41}$$

where B *now starts from* $\mathbf{c}$.

Note that this global segregation (non-coexistence) of types can be seen as a "symmetrization" of (36).

The statement (b) follows from (a) via the self-duality Theorem 3.3. The verification of (a) however requires some efforts. First of all, since $T \mapsto \|X^1_T\| +$

$\|X_T^2\|$ is a uniformly integrable non-negative martingale, it converges as $T \uparrow \infty$. This implies that its square function

$$T \mapsto A_T := \gamma \int_0^T \mathrm{d}t \, \langle X_t^1, X_t^2 \rangle \tag{42}$$

converges to a finite value as $T \uparrow \infty$. Then the strategy of proof is that the assumption that $\|X_T^1\| \cdot \|X_T^2\|$ is bounded away from zero as $T \uparrow \infty$ implies that A_T converges to infinity, giving a contradiction. For the involved details, see [DP98], or [FX99], where one can find a generalization of the model to more then two types.

3.5. Finite time survival

After describing the behavior in the long run, let us turn to the more delicate question about finite time survival of both types. The answer is actually dependent on the initial state. A Z^d–version of the following theorem is given in [MP99], whereas the present R–version follows as a special case of [FX99].

Theorem 3.5 (finite time behavior)**.** *Let* $d = 1$ *and* $0 < \|\mathbf{X}_0\| < \infty$.

(a) (finite time survival): *Under certain additional conditions on* $\mathbf{X}_0$,

$$\|X_T^1\| \cdot \|X_T^2\| > 0 \quad \textit{for all } T > 0, \ \textit{a.s.} \tag{43}$$

(b) (finite time extinction of a type with high probability): *Similarly, under a certain additional condition on* X_0^1, *for each* $\varepsilon \in (0, 1]$ *and* $T > 0$, *one can choose* X_0^2 $(\neq 0)$ *so small, that*

$$P\left(X_t^2 = 0 \textit{ for all } t \geq T\right) \geq 1 - \varepsilon.$$

A sufficient condition for the situation is (a) is given if X_0^1 and X_0^2 are separated in different half lines of R and have there sufficiently large tails at infinity. The proof of (a) is essentially based on a variance estimate and Chebychev's inequality.

The verification of (b) is more involved and relies on the following philosophy. First of all, because 0 is an absorbing state for the subprocess $t \mapsto X_0^2$, it suffices to consider $\mathbf{X}$ on a possibly smaller time interval $[0, T]$. Since initially X^2 is not too large by assumption and serves as the catalyst for X^1, then X^1 should not be very small on $[0, T]$. But since X^1 serves as the catalyst for X^2, then X^2 should have the chance to die by time T. Actually, under the present conditions and on a suitable time scale, some norm of X_T^2 can be bounded from above by a *supercritical* Feller's branching diffusion which certainly dies by a given time $T > 0$ with positive probability. Finally, by making the initial state of this diffusion sufficiently small, the latter extinction probability can be forced to be at least $1-\varepsilon$.

We mention that for the case of uniform initial states, where by Theorem 3.4 (b) global segregation occurs in the long run, one can nevertheless expect that the predominant type near the origin *oscillates* as time goes on, just as proved in [CK99] for the Z^d–model, $d \leq 2$.

So far we discussed only a one-dimensional continuum mutually catalytic branching model. Why this restriction to dimension one? Could one perhaps hope for the non-trivial existence of some mutually catalytic measure-valued processes in higher dimensional continua?

3.6. First arguments against a non-trivial existence in R^2

Actually, there are a couple of arguments against non-trivial existence in the higher-dimensional case. First of all, variances blow up in some variance calculations working with approximations. But perhaps more important, non-trivial existence seems to imply some *paradoxes* that we will now explain.

Recall that in the constant medium case, the SBM ϱ in dimensions $d \geq 2$ has singular states. The same is true in the case of a regular medium, that is, if we have a catalyst, described by absolutely continuous measures. Consequently, a regular catalyst leads to a singular reactant (in $d \geq 2$). On the other hand, if the catalyst is singular enough, the reactant may become absolutely continuous states ([DF95, Kle99b]). Summarizing, in catalytic models of dimension $d \geq 2$, roughly speaking, catalyst and reactant exhibit an "*alternation property*", say. In particular, locally only one of the substances can be absolutely continuous.

What should follow from this for the symmetric, mutually catalytic model in R^2 we want to have? Might such alternation property be reasonable in the symmetric case?

In dimension two, starting with uniform initial states, the continuum model should be *self-similar*, as one can easily check by formal calculations. But it should also have the same long-term behavior as the Z^2–model ([DP98]). That is, Theorem 3.4 (b) should be true also in $d = 2$. But by the self-similarity, this global statement about $\mathbf{X}_\infty$ should be turned into a local property of $\mathbf{X}_1$, say:

(i) The limiting uniform states should be turned into *absolutely continuous* measure states $\mathbf{X}_1$, say, which then do *not* have the alternation property discussed above.

(ii) More important, the global segregation should lead to a *local segregation* of types: Locally only one type should be present. In other words, if two types were locally present, then one of the types should immediately kill the other one. On the other hand, the sought-after non-trivial stochastic model, if it existed, would live from the interaction!

How to resolve such paradoxes, other than to conclude, that the higher-dimensional continuum model would degenerate to the heat flow? At least, if the model existed, it should be rather delicate!

These kind of questions attracted several persons, and now there is consensus about the non-trivial existence of the 2-dimensional continuum model, provided that the collision rate γ is sufficiently small.

3.7. Mutually catalytic super-Brownian motion in R^2

The following theorem will be proved in [DEF$^+$00].

Theorem 3.6 (non-degenerate existence)**.** *Suppose that* $d = 2$*, that* $\gamma > 0$ *is sufficiently small, and that* $\mathbf{X}_0 = \mathbf{c}\ell$*,* $\mathbf{c} \in \mathsf{R}_+^2$*. Then* $\mathbf{X} = \big(X^1, X^2\big)$ *exists as a pair of non-trivial continuous measure-valued processes, described by a* martingale problem*, involving the collision local time* $L_{\mathbf{X}}$*. That is, for smooth test functions* $\varphi^i \geq 0$*,*

$$t \;\mapsto\; M_t^i(\varphi^i) \;:=\; \big\langle X_t^i, \varphi^i \big\rangle - c^i \big\langle \ell, \varphi^i \big\rangle - \int_0^t \mathrm{d}s \, \Big\langle X_s^i \,, \, \frac{1}{2}\Delta\varphi^i \Big\rangle, \tag{44}$$

$i = 1, 2$*, are orthogonal, square integrable martingales, with square functions*

$$t \;\mapsto\; \big\langle\!\big\langle M^i(\varphi^i) \big\rangle\!\big\rangle_t \;=\; \gamma \int_{[0,t]\times\mathsf{R}^2} L_{\mathbf{X}}\big(\mathrm{d}[s,b]\big) \; (\varphi^i)^2(b), \tag{45}$$

where $L_{\mathbf{X}}$ *is the collision local time of* X^1 *and* X^2*, which is the limit in law as* $\varepsilon \downarrow 0$ *of the measures*

$$\mathrm{d}r \, X_r^1(\mathrm{d}a) \int_{\mathsf{R}} X_r^2(\mathrm{d}b) \, \mathrm{p}_\varepsilon(b-a) \tag{46}$$

on $\mathsf{R}_+ \times \mathsf{R}^2$*.*

Note that the existence of the collision local time $L_{\mathbf{X}}$ is an essential part of the martingale problem.

The key tool for the construction of $\mathbf{X}$ are expensive 4^{th} moment estimates. In fact, there is a closed system of equations for the 4^{th} moments of approximating processes, which can successfully be handled.

The restriction to small γ comes from the fact that moments of order higher than 2 exhibit a phase transition: They are infinite for sufficiently large $\gamma > 0$. (Imagine that too much reaction leads to very large fluctuations in the sense that moments of order three and more explode.)

Since the model exists non-trivially according to this theorem, how can the "paradoxes" mentioned earlier be resolved? In other words, what features the model does have, and how can these be understood?

3.8. Resolving the paradoxes

The following theorem is also part of the program in [DEF$^+$00].

Theorem 3.7 (local properties)**.** *Consider the process* $\mathbf{X}$ *from Theorem* 3.6*, and fix a time point* $t > 0$*.*

(a) (density functions): *Both states* X_t^i *are absolutely continuous:*

$$X_t^i(\mathrm{d}b) \;=\; X_t^i(b)\,\mathrm{d}b, \qquad i = 1, 2. \tag{47}$$

(b) (law of $\mathbf{X}_t(x)$**):** *For almost all* $b \in \mathsf{R}^2$*, the law of* $\mathbf{X}_t(b)$ *coincides with the state* B_τ *of a Brownian motion in* R_+^2*, starting from* $\mathbf{c}$*, at its hitting time* τ *of the boundary of* R_+^2*. In particular:*

$$\mathit{V}\mathrm{ar}\, X_t^i(b) \equiv \infty. \tag{48}$$

(c) (local segregation of types): *For almost all* b,

$$X_t^1(b)\, X_t^2(b) \;=\; 0 \quad a.s. \tag{49}$$

Consequently, the states of $\mathbf{X}$ are absolutely continuous, as we heuristically concluded in the previous subsection. Moreover, the law of the random density in a point can explicitly be described. The infinite variance of the hitting state can also be obtained by calculations starting from the Z^2–model. Finally, the intuitively predicted local segregation property is confirmed.

Note that the law of the random densities $\mathbf{X}_t(b)$ is independent of t and b. This follows from the spatial shift invariance (in law) and the *self-similarity*

$$X_{K^2 t}(K\,\cdot) \;\stackrel{\mathcal{L}}{=}\; X_t(\,\cdot\,), \qquad t, K > 0. \tag{50}$$

But if the types are separated as described in (49), how can it be explained that there is still a non-trivial interaction going on? For this purpose, it might be helpful to first consider the following simplified example (see also Figure 2). For measures μ and ν on R, define the *collision measure* $C_{[\mu,\nu]}$ on R by

$$\int_{\mathsf{R}} C_{[\mu,\nu]}(\mathrm{d}a)\, \varphi(a) \;=\; \lim_{\varepsilon \downarrow 0} \int_{\mathsf{R}} \mathrm{d}a\; \varphi(a)\; \mathrm{p}_\varepsilon{*}\mu(a)\; \mathrm{p}_\varepsilon{*}\nu(a) \tag{51}$$

provided that the limit exists in R_+ for all continuous functions $\varphi \geq 0$ with compact support.

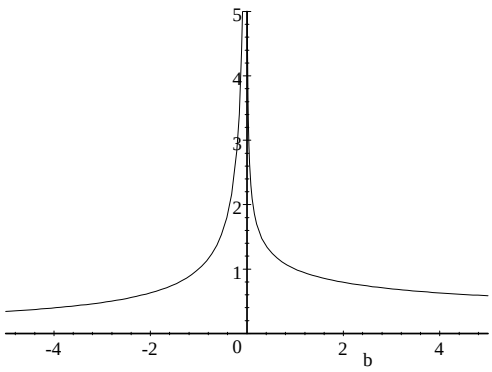

FIGURE 2. Segregated measure densities with $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$

Example 3.8 (segregated measures with a non-trivial collision)**.** For $\alpha, \beta \in (0,1)$, consider the segregated locally finite measures μ and ν having the density functions

$$f(b) \;:=\; \mathbf{1}_{(-\infty,0)}\, |b|^{-\alpha} \quad \text{and} \quad g(b) \;:=\; \mathbf{1}_{(0,\infty)}\, b^{-\beta}, \qquad b \in \mathsf{R}, \tag{52}$$

respectively. Then, for $\varepsilon > 0$, as well as φ continuous and with compact support,

$$\begin{aligned} &\int_{\mathsf{R}} \mathrm{d}b\, \varphi(b)\ \mathrm{p}_\varepsilon{*}\mu(b)\ \mathrm{p}_\varepsilon{*}\nu(b) \\ &= \int_{\mathsf{R}} \mathrm{d}a\ f(a) \int_{\mathsf{R}} \mathrm{d}a'\ g(a') \int_{\mathsf{R}} \mathrm{d}b\, \varphi(b)\ \mathrm{p}_\varepsilon(a-b)\, \mathrm{p}_\varepsilon(a'-b), \end{aligned} \tag{53}$$

and by Brownian scaling we may continue with

$$= \varepsilon^{\frac{1}{2}(1-\alpha-\beta)} \int_{\mathsf{R}} \mathrm{d}a\ f(a) \int_{\mathsf{R}} \mathrm{d}a'\ g(a')\, \varphi\big(\sqrt{\varepsilon}\, b\big)\ \mathrm{p}_1(a-b)\, \mathrm{p}_1(a'-b). \tag{54}$$

But as $\varepsilon \downarrow 0$, the latter integral expression converges to $\varphi(0)\, c$ with

$$0 \ < \ c \ := \ \int_{\mathsf{R}} \mathrm{d}a\ f(a) \int_{\mathsf{R}} \mathrm{d}a'\ g(a')\, \mathrm{p}_2(a-a') \ < \ \infty. \tag{55}$$

Consequently, for $\alpha+\beta > 1$ the approximated collision measures explode, whereas for $\alpha + \beta < 1$ they will finally disappear, as suggested by the formal expression $f(b)g(b)\,\mathrm{d}b \ \equiv \ 0$ for the collision measure. But in the *critical* case $\alpha + \beta = 1$ (as in Figure 2), the collision measure $C_{[\mu,\nu]} = c\,\delta_0$ exists *non-trivially*, with the constant c [from (55)] depending on the approximation procedure.

Note that lifting up μ, ν to measure-valued paths $t \mapsto \mu_t$ and $t \mapsto \nu_t$ which are constant in time, analogous statements hold for the collision local time $L_{[\mu,\nu]}$. In particular, for critical exponents α, β, the collision local time $L_{[\mu,\nu]}$ exists and is *non-trivial*, namely

$$L_{[\mu,\nu]}\big(\mathrm{d}[s,b]\big) \ = \ \mathrm{d}s\, c\, \delta_0(\mathrm{d}b) \quad \text{with } 0 < c < \infty \text{ from (55)}, \tag{56}$$

despite the paths μ, ν are separated. ◇

The last example illuminates that the local segregation property (49) is not in contradiction with the existence of a non-trivial collision local time $L_{\mathbf{X}}$ as in Theorem 3.6, thus resolving an apparent paradox. Consequently, due to the large fluctuations at the interface of the two types (remember the hot spots mentioned in Subsection 1.4), the locally segregated types actually interact non-trivially, keeping the mutually catalytic process $\mathbf{X}$ in R^2 stochastically alive.

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