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On inhomogeneous velocity boundary conditions in the Förste model of a radiating, viscous, heat conducting fluid

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Abstract

For the mathematical model of a three-dimensional flow of a radiating, viscous and heat conducting fluid due to J. Förste, we consider existence and uniqueness of weak solutions in case of inhomogeneous Dirichlet boundary conditions for the velocity, and in dependence on the physical parameters.

1 Introduction

In [2], a model has been proposed for the stationary flow of a radiating, viscous and heat conducting fluid. Apparently, this is the only paper in which, simultaneously, such important characteristics of real industrial processes have been taken into account, as: three-dimensionality, influence of temperature and radiation on fluid flow.

The paper of Förste shows a way to prove existence and uniqueness of a weak solution under homogeneous velocity boundary conditions, and also contains the assertion that the approach ensures uniqueness for heat conduction and viscosity coefficients sufficiently large and for absorption coefficients and solution domain sufficiently small; moreover, it announces that it should be possible to handle inhomogeneous Dirichlet boundary conditions for the velocity.

When going through the arguments of J. Förste in our paper [3], we found it necessary to inspect all constants in the estimates in order to prove the uniqueness. The result was that uniqueness can be shown under the single condition of a sufficiently small solution domain; uniqueness for appropriate coefficients remained unclear, and the question of inhomogeneous velocity boundary conditions was not tackled.

In the present paper we generalize the existence theorem of [2] to the case of inhomogeneous Dirichlet data for the velocity. Moreover, we prove a result on uniqueness concretizing the original assertion of Förste.

2 The Förste model and its weak solution

Let $\Omega \subset R^3$ be a bounded domain with Lipschitz boundary Γ. For $x = (x_1, x_2, x_3) \in \Omega$ we consider the following system of equations [2], which represent the physical conservation laws of impulse, mass, inner and radiated energy:

$$
\begin{aligned}
\rho(\vec{v}\,\mathrm{grad})\vec{v} + \mathrm{grad}\, p &= \mu\Delta\vec{v} + \vec{f}_0(T - T_0), && (1)\\
\mathrm{div}\,\vec{v} &= 0, && (2)\\
(\vec{v}\,\mathrm{grad})T &= \lambda\Delta T - 4\alpha_P(\sigma T^4 - \pi I_m), && (3)\\
0 &= \Delta I_m + \frac{3\alpha_R\alpha_P}{\pi}(\sigma T^4 - \pi I_m). && (4)
\end{aligned}
$$

Along with these differential equations, the following boundary conditions are considered:

$$\vec{v} = q_\Gamma,\ T = \vartheta,\ I_m = I_{m,0}, \quad x \in \Gamma.$$

Above, we have used the following notations for the unknowns to be determined:

- $\vec{v} = (v_1, v_2, v_3)^T$ is the velocity vector,
- T is temperature,
- I_m denotes the radiation intensity.

Moreover, the following constants are occuring :

- ρ is the density of the fluid, μ its viscosity, $\vec{f}_0$ the vector of earth acceleration multiplied by the extension coefficient (as resulting from the Boussinesq approximation),
- λ is the coefficient of heat conductivity, α_P and α_R are the Planck and the Rosseland absorption coefficients,
- σ is the Stefan-Boltzmann constant.

With the usual notations for Sobolev spaces [1], we assume that $I_{m,0}, \vartheta \in H^{1/2}(\Gamma)$ and hence can be continued into all of Ω to define functions of $H^1(\Omega)$; we further suppose $T_0 \in L_2(\Omega)$.

In order to be able to take into account inflow and outflow across Γ, as a generalisation of the boundary condition $\vec{v}_{|\Gamma} = 0$, in this paper we consider the inhomogeneous boundary condition $\vec{v}_{|\Gamma} = \vec{q}_{|\Gamma}$. In [4] a similar investigation has been performed for the Navier-Stokes equations.

The velocity space is then

$$\vec{V} := \{\vec{v} \in (H^1(\Omega))^3,\ (\mathrm{div}\,\vec{v}, p)_0 = 0 \text{ for all } p \in L_2(\Omega)\}$$

instead of

$$\vec{V}_0 := \{\vec{u} \in (H_0^1(\Omega))^3;\ (\mathrm{div}\,\vec{u}, p)_0 = 0 \text{ for all } p \in L_2(\Omega)\},$$

which serves here as the space of the velocity test functions $\vec{w}$.

Concerning the velocity boundary conditions, we assume that $\vec{q}_{|\Gamma} \in (H^{1/2}(\Gamma))^3$ and satisfies the solvability condition $\int_\Gamma \vec{n} \cdot \vec{q}_{|\Gamma} \mathrm{ds} = 0$. Then

$\vec{q}_{|\Gamma}$ can be continued into Ω defining there a function $\vec{q} \in \vec{V}$ with trace $\vec{q}_{|\Gamma}$ and with the property

$$\|\vec{q}\|_{H^1} \leq c_{1/2} \|\vec{q}_{|\Gamma}\|_{1/2,\Gamma}. \tag{5}$$

We now look for weak solutions $\vec{v} = \vec{q} + \vec{z} \in \vec{V},\ T, I_m \in H^1(\Omega)$. Then $\vec{z} := \vec{v} - \vec{q} \in \vec{V}_0,\ \ \tau := T - \vartheta, i := I_m - I_{m,0} \in H_0^1(\Omega)$.

We shall denote both the $L_2(\Omega)$ and the $(L_2(\Omega))^3$ scalar products by $(\cdot,\cdot)_0$, and both the $H_0^1(\Omega)$ and $(H_0^1(\Omega))^3$ scalar products by $(\cdot,\cdot)_1$, e.g.

$$\begin{aligned}
(\tau, t)_1 &:= \int \sum_{k=1}^{3} \operatorname{grad} \tau \operatorname{grad} t \mathrm{d}\Omega, \quad \tau, t \in H_0^1, && (6)\\
(\vec{v}, \vec{w})_1 &:= \int \sum_{k=1}^{3} \operatorname{grad} v_k \operatorname{grad} w_k \mathrm{d}\Omega, \quad \vec{v}, \vec{w} \in \vec{V}_0, && (7)
\end{aligned}$$

whereas for the corresponding norms, we use the notation $\|\cdot\|_{L_2}$ and $|\cdot|_1$. Further, when $|\cdot|$ will be applied to a constant vector resp. to Ω, then it denotes the euclidean norm resp. the volume.

Finally, we introduce for $\vec{u}, \vec{v}, \vec{w} \in \vec{V}$ the trilinear form

$$a_1(\vec{u}, \vec{v}, \vec{w}) := \int \sum_{k=1}^{3} (\vec{u} \cdot \operatorname{grad} v_k) w_k \mathrm{d}\Omega. \tag{8}$$

Then, the weak solution $(\vec{v}, \tau, i) \in \vec{V} \times H_0^1 \times H_0^1$ is defined by the following variational problem in which $\vec{v} = \vec{q} + \vec{z}$ with $\vec{z} \in H_0^1$, and $\vec{w}, t, j$ are test functions from $\vec{V}_0 \times H_0^1 \times H_0^1$:

$$\begin{aligned}
\mu(\vec{v}, \vec{w})_1 &= -\rho a_1(\vec{v}, \vec{v}, \vec{w}) + (\vec{f}_0(\tau + \vartheta - T_0), \vec{w})_0, && (9)\\
\lambda(\tau, t)_1 &= ((\tau + \vartheta)\vec{v}, \operatorname{grad} t)_0 - \lambda(\vartheta, t)_1 - \\
&\quad -\alpha(\sigma|\tau + \vartheta|^3(\tau + \vartheta) - \pi(i + I_{m,0}), t)_0, && (10)\\
(i, j)_1 &= \alpha\beta(\sigma|\tau + \vartheta|^3(\tau + \vartheta) - \pi(i + I_{m,0}), j)_0. && (11)
\end{aligned}$$

In the variational problem (9)-(11), we have introduced the constants $\alpha := 4\alpha_P$ and $\beta := \frac{3}{4\pi}\alpha_R$; the equation of mass conservation has been absorbed into the definition of $\vec{V}_0$. We remark that in [3] instead of $\alpha\beta$ the notation β was used in the i-equation, here labeled (11).

We list also the misprints of that paper:

On p. 370, between formulae (13) and (14), after the sentence "Here we take $t = j = \beta\lambda\tau + \alpha i$ to get", in the next formula there is $+$ sign instead of $-$.

Before formula (17), on the same p. 370, there is a reference to formula (6) instead of (16).

In formula (25), on p. 371, the power of $\|T\|_{L_5}$ on the left-hand side must be 5 instead of 2.

In formula (28), on p. 372, instead of the min shown there must be a max.

All these misprints have no influence on the conclusions or on the constants γ_i used in the existence argument.

On p. 373, fourth line from below, the estimate for c_2 must be $c_2^2 \leq d^2/6$ where d is the diameter of Ω. A correction of this misprint has the result that the estimates (38) and (39) of Lemma 2 for the constants c_q of the continuous imbedding $H_0^1 \hookrightarrow L_q$, i.e.

$$\|u\|_{L_q} \leq c_q |u|_1, \tag{12}$$

can be united into

$$c_q \leq O(d^{(6-q)/(2q)}), \;\; 1 \leq q \leq 6. \tag{13}$$

Moreover, due to this misprint, the exponents of d in the proof of the uniqueness theorem on p. 374 are wrong but remain positive, and hence the conclusion remains true.

Remarks. 1. For the constants $\overline{c}_q$ of the continuous imbedding $H^1 \hookrightarrow L_q$:

$$\|u\|_{L_q} \leq \overline{c}_q \|u\|_{H^1} = \overline{c}_q \{\|u\|_{L_2}^2 + |u|_1^2\}^{1/2}, \tag{14}$$

we have $\overline{c}_q \geq |\Omega|^{\frac{2-q}{2q}}$ for $q \geq 2$ as follows from (14) by inserting $u \equiv 1$. This fact must be taken into account when we are going to prove a uniqueness theorem like in [3] for a sufficiently small diameter of Ω: we must avoid using (14) and split functions from H^1 into a boundary part and a H_0^1-part (like $T = \vartheta + \tau$) and use imbedding only for this latter part, see (12) and (13).

2. For the imbedding constant $c_{p,q}$ of $L_q \hookrightarrow L_p$ (where $q > p$) i.e.:

$$\|u\|_{L_p(\Omega)} \leq c_{p,q} \|u\|_{L_q(\Omega)} \text{ for all } u \in L_q(\Omega) \tag{15}$$

we have

$$c_{p,q} = |\Omega|^{\frac{q-p}{qp}}. \tag{16}$$

In fact, the upper estimate $c_{p,q} \leq |\Omega|^{\frac{q-p}{qp}}$ follows from an application of a Hölder inequality to $\|u\|_{L_p}$, whereas the corresponding lower estimate is obtained by inserting $u \equiv 1$ into (15). •

3 Boundedness and existence in case of inhomogeneous boundary values of the velocity

Our investigation parallels that of [3] for homogeneous boundary values of the velocity.

To derive an estimate for possible solutions of (9)–(11), we remember that, in (9), $\vec{v} = \vec{z} + \vec{q}$ with $\vec{z} \in \vec{V}_0$, and inserting $\vec{w} = \vec{z}$ we obtain

$$\begin{aligned} \mu(\vec{z}, \vec{z})_1 \;&=\; -\rho\{a_1(\vec{z}, \vec{z}, \vec{z}) + a_1(\vec{q}, \vec{z}, \vec{z}) + a_1(\vec{z}, \vec{z}, \vec{q}) + a_1(\vec{q}, \vec{z}, \vec{q})\} + \\ &\quad + (\vec{f}_0(T - T_0), \vec{z})_0 - \mu(\vec{q}, \vec{z})_1. \end{aligned} \tag{17}$$

Since $a_1(\vec{u}, \vec{z}, \vec{z}) = 0$ is well known for $\vec{u} \in \vec{V}$ and $\vec{z} \in \vec{V}_0$, it follows that

$$\begin{aligned} (\mu - \rho c_4 \|\vec{q}\|_{(L_4)^3}) |\vec{z}|_1^2 \;&\leq\; \rho \|\vec{q}\|_{(L_4)^3}^2 |\vec{z}|_1 + \\ &\quad + \; |\vec{f}_0| \|T - T_0\|_{L_2} c_2 |\vec{z}|_1 + \mu \|\vec{q}\|_{\vec{V}} |\vec{z}|_1. \end{aligned}$$

As earlier, here c_q denotes the imbedding constant of $H^1_0 \hookrightarrow L_q$, We assume now

$$\mu - \rho c_4 \|\vec{q}\|_{(L_4)^3} \geq \mu - \rho c_4 \overline{c}_4 c_{1/2} \|\vec{q}_{|\Gamma}\|_{1/2,\Gamma} > 0 \tag{18}$$

where $\overline{c}_q$ is the imbedding constant of $H^1 \hookrightarrow L_q$, and find then

$$\begin{aligned} |\vec{z}|_1 \quad &\leq \quad \overline{\gamma_1}\|T\|_{L_2} + \overline{\gamma_2}, \;\; \overline{\gamma_1} := \frac{|\vec{f}_0|c_2}{\mu - \rho c_4 \|\vec{q}\|_{(L_4)^3}}, \\ &\overline{\gamma_2} := \overline{\gamma_1}\|T_0\|_{L_2} + \frac{\rho\|\vec{q}\|^2_{(L_4)^3} + \mu\|\vec{q}\|_{\vec{V}}}{\mu - \rho c_4\|\vec{q}\|_{(L_4)^3}}. \end{aligned} \tag{19}$$

Concerning (18) we remark that the expression $\mu - \rho c_4 \overline{c}_4 c_{1/2} \|\vec{q}_{|\Gamma}\|_{1/2,\Gamma}$ comes from estimating $\rho a_1(\vec{z}, \vec{z}, \vec{q})$ on the right-hand side of (17) by $\rho c_4 \|\vec{q}\|_{L_4} |\vec{z}|_1^2 \leq \rho c_4 \overline{c}_4 \|\vec{q}\|_{H^1} |\vec{z}|^2_1$ and then using (5).

Instead, since it is well known that for $\vec{z} \in \vec{V}_0$ there holds $a_1(\vec{z}, \vec{z}, \vec{q}) = -a_1(\vec{z}, \vec{q}, \vec{z})$, we may also use Lemma 1.8 in [6] or the corresponding result in [5] stating that the function $\vec{q} \in \vec{V}$ which continues the boundary values of $\vec{q}_{|\Gamma}$ into Ω can be chosen in such a way as to satisfy $|a_1(\vec{z}, \vec{q}, \vec{z})| \leq \epsilon |\vec{z}|^2_1$ for any positive ϵ. Hence (18) can be weakened, but it has the advantage to stress that there is a condition on the possible boundary values.

If $\lambda_1 = \lambda_1(-\Delta)$ is the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions, then from the Friedrichs inequality we have

$$\left(\frac{\lambda_1}{1+\lambda_1}\right)^{1/2} \|\vec{z}\|_{\vec{V}} \leq |\vec{z}|_1,$$

and since $\|\vec{v}\|_{\vec{V}} \leq \|\vec{q}\|_{\vec{V}} + \|\vec{z}\|_{\vec{V}}$, we get from (19)

$$\begin{aligned} \|\vec{v}\|_{\vec{V}} \quad &\leq \quad \gamma_1 \|T\|_{L_2} + \gamma_2, \\ \gamma_1 := c_{v,1}\overline{\gamma_1} \quad &, \quad \gamma_2 := c_{v,1}\overline{\gamma_2} + \|\vec{q}\|_{\vec{V}}, \;\; c_{v,1} := \left(\frac{1+\lambda_1}{\lambda_1}\right)^{1/2}. \end{aligned} \tag{20}$$

Adding next (10) multiplied by β to (11) and substituting $t = j = \beta\lambda\tau + i$, we get the inequalities

$$\begin{aligned} |\beta\lambda\tau + i|_1 \quad &\leq \quad \beta\|\vec{v}\|_{(L_4)^3}\|T\|_{L_4} + \gamma_3, \\ &\leq \quad \beta(\|\vec{q}\|_{(L_4)^3} + c_4(\overline{\gamma_1}\|T\|_{L_2} + \overline{\gamma_2}))\|T\|_{L_4} + \gamma_3, \\ &\leq \quad \gamma_4\|T\|_{L_2}\|T\|_{L_4} + \gamma_5\|T\|_{L_4} + \gamma_3, \\ \gamma_3 \quad &:= \quad \beta\lambda|\vartheta|_1, \;\; \gamma_4 := \beta c_4\overline{\gamma_1}, \;\; \gamma_5 := \beta(\|\vec{q}\|_{(L_4)^3} + c_4\overline{\gamma_2}). \end{aligned}$$

Then, using the triangle inequality, there follows

$$|i|_1 \leq \gamma_4\|T\|_{L_2}\|T\|_{L_4} + \gamma_5\|T\|_{L_4} + \gamma_6|\tau|_1 + \gamma_3, \tag{21}$$

where $\gamma_6 := \beta\lambda$.

Concerning γ_3 we remark that ϑ in general is not zero on Γ (we obtained ϑ just by continuation into Ω from the boundary values for the temperature T). Hence, $|\vartheta|_1$ is only a semi-norm and zero for constant ϑ.

From (21) we get like in [3] the estimate

$$\begin{aligned} |i|_1^2 &\leq \gamma_7 \left(\lambda|\tau|_1^2 + \alpha\sigma\|T\|_{L_5}^5 + \gamma_8\right). \qquad (22) \\ \gamma_7 &:= \frac{6c_{4,5}^2}{5\alpha\sigma}\left(\gamma_4^2 c_{2,5}^2 + \gamma_5^2\right) + \frac{\gamma_6^2}{\lambda} + 1, \quad \gamma_8 := \gamma_3^2 + \frac{2}{3}\alpha\sigma. \end{aligned}$$

We now find an estimate of $|\tau|_1$ by putting $t = \tau$ in (10):

$$\begin{aligned} \lambda|\tau|_1^2 &= \int \{T\vec{v}\,\mathrm{grad}\,\tau - \lambda\,\mathrm{grad}\,\vartheta\,\mathrm{grad}\,\tau \\ &\quad -\alpha\left(\sigma|T|^3T - \pi(i + I_{m,0})\right)\tau\}\mathrm{d}\Omega. \end{aligned}$$

Here, the first term on the right-hand side contains $\tau\,\mathrm{grad}\,\tau\vec{v} = \mathrm{grad}(\frac{1}{2}\tau^2)\vec{v}$ the integral of which is zero, since even in the presence of inhomogeneous boundary conditions of $\vec{v}$ we have

$$\begin{aligned} \int \tau\vec{v}\,\mathrm{grad}\,\tau\mathrm{d}\Omega &= \int \vec{v}\,\mathrm{grad}(\frac{1}{2}\tau^2)\mathrm{d}\Omega \\ &= \frac{1}{2}\int_\Gamma \tau^2\vec{v}\cdot\vec{n}\mathrm{ds} - \frac{1}{2}\int \tau^2\,\mathrm{div}\,\vec{v}\mathrm{d}\Omega = 0, \qquad (23) \end{aligned}$$

because of $\tau \in H_0^1$ and $\vec{v} \in \vec{V}$. Hence

$$\begin{aligned} \lambda|\tau|_1^2 &\leq \|\vartheta\|_{L_4}\|\vec{v}\|_{(L_4)^3}|\tau|_1 + \lambda|\vartheta|_1|\tau|_1 + \\ &\quad +\alpha\pi\left|\int(i + I_{m,0})\tau\mathrm{d}\Omega\right| - \alpha\sigma\int|T|^3T\tau\mathrm{d}\Omega \qquad (24) \\ &\leq \|\vartheta\|_{L_4}\|\vec{v}\|_{(L_4)^3}|\tau|_1 + \lambda|\vartheta|_1|\tau|_1 + \\ &\quad +\alpha\pi\|i + I_{m,0}\|_{L_{5/4}}\|T\|_{L_5} + \alpha\pi\|i + I_{m,0}\|_{L_{5/4}}\|\vartheta\|_{L_5} - \\ &\quad -\alpha\sigma\|T\|_{L_5}^5 + \alpha\sigma\|T\|_{L_5}^4\|\vartheta\|_{L_5}, \qquad (25) \end{aligned}$$

where the two last terms in (24) have been estimated using Hölder inequalities and $\tau = T - \vartheta$. Next, using (19) and imbedding theorems, we find for the first term in (25)

$$\|\vartheta\|_{L_4}\|\vec{v}\|_{(L_4)^3}|\tau|_1 \leq \|\vartheta\|_{L_4}\left(\|\vec{q}\|_{(L_4)^3} + c_4(\overline{\gamma_1}c_{2,5}\|T\|_{L_5} + \overline{\gamma_2})\right)|\tau|_1.$$

Together with (25), this gives

$$\begin{aligned} \lambda|\tau|_1^2 + \alpha\sigma\|T\|_{L_5}^5 &\leq \gamma_9\|T\|_{L_5}|\tau|_1 + \gamma_{10}|\tau|_1 + \alpha\sigma\|\vartheta\|_{L_5}\|T\|_{L_5}^4 \\ &\quad +\alpha\pi\left(\|i\|_{L_{5/4}} + \|I_{m,0}\|_{L_{5/4}}\right)\|T\|_{L_5} \\ &\quad +\alpha\pi\|\vartheta\|_{L_5}\|i\|_{L_{5/4}} + \gamma_{11}, \end{aligned}$$

where

$$\begin{aligned}\gamma_9 &:= \|\vartheta\|_{L_4} c_4 \overline{\gamma_1} c_{2,5}, \quad \gamma_{10} := \|\vartheta\|_{L_4}(\|\vec{q}\|_{(L_4)^3} + c_4\overline{\gamma_2}) + \lambda|\vartheta|_1, \\ \gamma_{11} &:= \alpha\pi\|I_{m,0}\|_{L_{5/4}}\|\vartheta\|_{L_5}.\end{aligned}$$

Applying ϵ-inequalities like in [3] we arrive at

$$\lambda|\tau|_1^2 + \alpha\sigma\|T\|_{L_5}^5 \le \gamma_{14}|i|_1^{5/4} + \gamma_{13}, \tag{26}$$

where

$$\begin{aligned}\gamma_{14} := 2\gamma_{12}c_{5/4} \quad &, \quad \gamma_{12} = \gamma_{12}(\epsilon) := \frac{4\alpha\pi}{5\epsilon^{5/4}} + \frac{4\epsilon^{5/4}}{5}\alpha\pi, \\ \gamma_{13} = \gamma_{13}(\epsilon) \quad &:= \quad 2\left(\frac{3}{5}\frac{\gamma_9^{10/3}}{(\lambda\epsilon)^{5/3}} + \frac{\gamma_{10}^2}{\lambda} + \alpha\sigma\frac{\|\vartheta\|_{L_5}^5}{5\epsilon^5} + \right. \\ &\left. \quad + \frac{4\alpha\pi}{5\epsilon^{5/4}}\|I_{m,0}\|_{L_{5/4}}^{5/4} + \alpha\pi\frac{\|\vartheta\|_{L_5}^5}{5\epsilon^5} + \gamma_{11}\right),\end{aligned}$$

and ϵ is the unique positive solution of

$$k(\epsilon) := \frac{2}{5}\epsilon^{5/2} + \frac{4}{5}\alpha\sigma\epsilon^{5/4} + \frac{2}{5}\epsilon^5\alpha\pi = \frac{\alpha\sigma}{2}.$$

From (26) and (22) we get

$$\begin{aligned}|i|_1^2 \quad &\le \quad \gamma_{15}|i|_1^{5/4} + \gamma_{16}, \\ \gamma_{15} := \gamma_7\gamma_{14} \quad &, \quad \gamma_{16} := \gamma_7(\gamma_{13} + \gamma_8).\end{aligned} \tag{27}$$

and further, like in [3], there follows a bound on $|i|_1$:

$$|i|_1 \le \max\left(\gamma_{16}^{1/2}\left(\frac{8}{3}\right)^{4/5}, \left(\gamma_{15}^{5/3} + \frac{5}{3}\gamma_{16}^{5/8}\right)^{4/5}\right) =: K_i. \tag{28}$$

Now we get a bound K_τ for $|\tau|_1$ from (26), whereafter, for $1 < q \le 6$,

$$\begin{aligned}\|T\|_{L_q} &\le \|\vartheta\|_{L_q} + \|\tau\|_{L_q} \le \|\vartheta\|_{L_q} + c_q|\tau|_1 \\ &\le \|\vartheta\|_{L_q} + c_q K_\tau =: K_{T,q}.\end{aligned} \tag{29}$$

Finally, from (19) and (20), we have the estimates

$$\begin{aligned}|\vec{z}|_1 &\le \overline{\gamma_1}K_{T,2} + \overline{\gamma_2} =: K_z, \\ \|\vec{v}\|_{\vec{V}} &\le \gamma_1 K_{T,2} + \gamma_2 =: K_V, \\ \|\vec{v}\|_{(L_q)^3} &\le \|\vec{q}\|_{(L_q)^3} + \|\vec{z}\|_{(L_q)^3} \\ &\le \|\vec{q}\|_{(L_q)^3} + c_q K_z =: K_{v,q}.\end{aligned} \tag{30}$$

Since the presence of inhomogeneous boundary conditions for the velocity does not influence the complete continuity of the operators in the operator equations which can be defined on the basis of (9)–(11), from the above result on boundedness of possible weak solutions there follows their existence invoking the Leray–Schauder fixed point theorem, see [2], [3].

4 Uniqueness

To solve the question of uniqueness of a weak solution under more general conditions than in [3], we modify and generalize the approach taken there.

Consider two solutions $(\vec{v}, \tau, i)$ and $(\vec{v}', \tau', i')$ of (9)-(11) in $\vec{V} \times H_0^1 \times H_0^1$, subtract the corresponding variational equations, define

$$\vec{U} := \vec{v} - \vec{v}', \quad \Theta := \tau - \tau', \quad J := i - i',$$

and put $\vec{w} = \vec{U},\ t = \Theta,\ j = J$ in (9)-(11). Then, there results

$$\mu|\vec{U}|_1^2 = \int \left\{ \rho \sum_{k=1}^{3} (U_k \vec{v} + v_k' \vec{U}) \operatorname{grad} U_k + \vec{U} \vec{f}_0 \Theta \right\} \mathrm{d}\Omega, \tag{31}$$

$$\lambda|\Theta|_1^2 = \int \Big\{ (T\vec{U} + \Theta\vec{v}') \operatorname{grad} \Theta - \tag{32}$$

$$- \alpha[\sigma(|T|^3 T - |T'|^3 T') - \pi J]\Theta \Big\} \mathrm{d}\Omega,$$

$$|J|_1^2 = \alpha\beta \int [\sigma(|T|^3 T - |T'|^3 T') - \pi J] J \mathrm{d}\Omega. \tag{33}$$

Here we have used

$$\Theta = T - T' \quad , \quad J = I_m - I_m',$$
$$v_k \vec{v} - v_k' \vec{v}' = U_k \vec{v} + v_k' \vec{U} \quad , \quad T\vec{v} - T'\vec{v}' = T\vec{U} + \Theta \vec{v}'.$$

In (31) resp. in (32), the integrals over $\sum_{k=1}^{3} U_k \vec{v} \operatorname{grad} U_k$ resp. over $\Theta \vec{v}' \operatorname{grad} \Theta$ are zero since $\vec{v} \in \vec{V}$ and $U_k, \Theta \in H_0^1$, compare with (23). Taking this into account, we first estimate the right-hand side of (31):

$$\mu|\vec{U}|_1^2 \le \rho \|\vec{U}\|_{(L_4)^3} \|\vec{v}'\|_{(L_4)^3} |\vec{U}|_1 + |\vec{f}_0| \|\vec{U}\|_{(L_2)^3} \|\Theta\|_{L_2}. \tag{34}$$

Remember that $|\vec{f}_0|$ denotes the euclidean norm of $\vec{f}_0$. To derive an estimate from (32), we remark that

$$||T|^3 T - |T'|^3 T'| \le |T - T'| P_3(|T|, |T'|) = |\Theta| P_3(|T|, |T'|), \tag{35}$$

where $P_3(x, y) := x^3 + x^2 y + x y^2 + y^3$. Then we obtain

$$\lambda|\Theta|_1^2 \le \|T\|_{L_4} \|\vec{U}\|_{(L_4)^3} |\Theta|_1 +$$
$$+ \alpha\sigma \|P_3(|T|, |T'|)\|_{L_2} \|\Theta\|_{L_4}^2 + \alpha\pi \|J\|_{L_2} \|\Theta\|_{L_2}. \tag{36}$$

Using Lemma 1 from [3], we have

$$\|P_3(|T|, |T'|)\|_{L_2} \le P_3(\|T\|_{L_6}, \|T'\|_{L_6}). \tag{37}$$

Applying to (36) also the theorem on continuous imbedding (12), it follows that

$$\lambda|\Theta|_1 \le c_4 \|T\|_{L_4} |\vec{U}|_1 + \alpha\sigma P_3(\|T\|_{L_6}, \|T'\|_{L_6}) c_4^2 |\Theta|_1$$
$$+ \alpha\pi c_2^2 |J|_1. \tag{38}$$

Next, applying (12) also to (34), we find

$$\mu|\vec{U}|_1 \leq \rho c_4 \|\vec{v}'\|_{(L_4)^3} |\vec{U}|_1 + c_2^2 |\vec{f}_0| |\Theta|_1. \tag{39}$$

Finally, we see from (33), (35) and from (37) that

$$\begin{aligned} |J|_1^2 &\leq \alpha\beta\sigma \|\Theta\|_{L_4} \|P_3(|T|, |T'|)\|_{L_2} \|J\|_{L_4} \\ &\leq \alpha\beta\sigma c_4^2 |\Theta|_1 P_3(\|T\|_{L_6}, \|T'\|_{L_6}) |J|_1 \end{aligned} \tag{40}$$

We now introduce the vector

$$y := (|\vec{U}|_1, |\Theta|_1, |J|_1)^T$$

with the aim to show that under suitable conditions there holds $y = 0$. For this, we first take into account in (38)–(40) the boundedness estimates (29)–(30) which we here summarize as

$$\begin{aligned} \|T\|_{L_4}, \|T'\|_{L_4}, \|T\|_{L_6}, \|T'\|_{L_6}, &\leq K_T, \\ \|\vec{v}\|_{(L_4)^3}, \|\vec{v}'\|_{(L_4)^3} &\leq K_v. \end{aligned}$$

E.g., (37) can now be continued by $P_3(\|T\|_{L_6}, \|T'\|_{L_6}) \leq 4K_T^3$ due to the definition of P_3.

Then we rewrite the estimates (38)-(40) in vector form as follows :

$$0 \leq y \leq \mathcal{A}y, \text{ where } \mathcal{A} := \begin{pmatrix} \frac{\rho}{\mu} c_4 K_v & \frac{1}{\mu} c_2^2 |\vec{f}_0| & 0 \\ \frac{1}{\lambda} c_4 K_T & 4\frac{\alpha\sigma}{\lambda} c_4^2 K_T^3 & \frac{\alpha}{\lambda}\pi c_2^2 \\ 0 & 4\alpha\beta\sigma c_4^2 K_T^3 & 0 \end{pmatrix}. \tag{41}$$

These inequalities are to be understood componentwise. It turns now out that the conditions listed by Förste as sufficient uniqueness conditions can be separated.

Theorem. The Förste model (9)-(11) has at most one solution $\vec{v} \in \vec{V}$, $T = \tau + \vartheta \in H^1$, $I_m = i + I_{m,0} \in H^1$ when either of the following two conditions holds:

1) the diameter d of Ω is sufficiently small;

2) α and β are sufficiently small, and λ and μ are sufficiently large, moreover, for some positive constants κ_1, κ_2 there holds

$$\kappa_1 \leq \alpha\lambda, \text{ and } \beta^2\lambda \leq \kappa_2 \lambda^\kappa, \tag{42}$$

where $\kappa := 3/10$.

Proof. 1) As (41) shows, every nonzero element of $\mathcal{A}$ contains an imbedding constant which goes to zero when d goes to zero, see (13). We must therefore clarify the possible growth of K_T, K_v for decreasing d.

Hence, taking into account (13) and (16) and considering values like $\|\vec{q}\|_{(L_4)^3}, |\vartheta|_1$ as $O(1)$, we check all constants γ_i for their dependence on d and find that $\overline{\gamma_1}, \gamma_1, \gamma_4, \gamma_9, \gamma_{14}, \gamma_{15}$ with d go to zero whereas the remainder and K_i, K_τ, K_T, K_v are $O(1)$.

Thus, the spectral radius of $\mathcal{A}$ becomes less 1 for sufficiently small diameter of Ω, and then there follows $y = 0$ from (41) – as in [3].

Similarly, for fixed d, since every nonzero element of $\mathcal{A}$ contains either the absorption coefficients α or β, or $1/\lambda$ or $1/\mu$, we also have uniqueness in the second case, provided the bounds K_T and K_v don't grow with $\lambda, \mu, 1/\alpha, 1/\beta$. For this, we trace the constants γ_i of the estimates in Section 3 under condition 2 and find the following relations when assuming $\kappa \geq 0$ in (42):

$$\begin{aligned}
\gamma_1, \gamma_9 &\leq O(\mu^{-1}), \;\; \gamma_2 \leq O(1 + \mu^{-1}) = O(1), \;\; \gamma_3, \gamma_6 \leq O(\beta\lambda), \\
\gamma_4 &\leq O(\beta\mu^{-1}), \;\; \gamma_5 \leq O(\beta + \beta\mu^{-1}) \leq O(\beta), \\
\gamma_7 &\leq O(1 + \beta^2\lambda + \beta^2\alpha^{-1} + \beta^2\mu^{-2}\alpha^{-1}) \leq O(1 + \beta^2\lambda), \\
\gamma_8 &\leq O(\beta^2\lambda^2 + \alpha) \leq O(\lambda^{1+\kappa}), \;\; \gamma_{10} \leq O(1 + \mu^{-1} + \lambda) \leq O(\lambda), \\
\gamma_{11} &\leq O(\alpha), \;\; \gamma_{12}, \gamma_{14} \leq O(\alpha^{1/2}), \;\; \epsilon = O(\alpha^{2/5}), \\
\gamma_{13} &\leq O(\lambda + \alpha^{-1}) \leq O(\lambda), \\
\gamma_{15} &\leq O(\alpha^{1/2}(1 + \beta^2\lambda)) \leq O(\lambda^{\kappa - 1/2}), \\
\gamma_{16} &\leq O((1 + \beta^2\lambda)(\beta^2\lambda^2 + \lambda + \alpha^{-1})) \leq O(\lambda^{1+2\kappa}).
\end{aligned}$$

Then, from (28)–(30) there result the estimates

$$\begin{aligned}
|i|_1 &\leq O(\lambda^{\kappa+1/2}) =: K_i, \\
\lambda|\tau|_1^2 &\leq O(\alpha^{1/2}\lambda^{(\kappa+1/2)\frac{5}{4}} + \lambda) = O(\lambda), \\
|\tau|_1 &\leq O(1), \;\; \|T\|_{L_q} \leq O(1) =: K_T, \\
\|\vec{v}\|_{(L_q)^3} &\leq O(1) =: K_v.
\end{aligned} \tag{43}$$

The specific value of κ arises from (43) when requiring $(\kappa + 1/2)\frac{5}{4} = 1$. This also avoids the appearance of an upper bound for $\alpha\lambda$ in (42).

Observe that (due to the linearity of the Förste model in I_m), K_i does not appear in $\mathcal{A}$. •

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