A SIMPLE INDUCTIVE APPROACH TO THE PROBLEM OF CONVERGENCE OF CLUSTER EXPANSIONS OF POLYMER MODELS

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ABSTRACT. We explain a simple inductive method for the analysis of the convergence of cluster expansions (Taylor expansions, Mayer expansions) for the partition functions of "polymer models". We give a very simple proof of the "Dobrushin–Kotecký–Preiss criterion" and formulate its generalization usable for situations where a successive expansion of the partition function has to be used.

In this short note we explain a new and simple inductive method for the analysis of the convergence of cluster expansions of so called *polymer models*. The notion of a polymer model goes back to [GK] (see also [BCF,Br,M,S] and it was Dobrushin [D1] who first fully exploited the fact (already pointed out by Gruber and Kunz) that cluster expansions of these models are actually Taylor expansions of the logarithm of the partition function, taken w.r. to the fugacities of the considered polymers. Our new approach was already used in a recent paper [NOZ] but here we further simplify the argument and extend it in order to be applicable also to partially expanded polymer models and thus multi-scale expansions. This is important e.g. in the study of models with random impurities (see e.g. [BK,BoKu]) and in other situations where the "expandability" of a given "large" polymer (contour) Γ may be clarified only after expanding all the contours "smaller than Γ". In these situations, sequential expansions are indispensable and it is thus important to know that even in the case of an ordinary polymer models, the sequential approach gives an equally good control of the situation as the expansion "at once".

POLYMER MODELS. THE DOBRUSHIN–KOTECKÝ–PREISS CRITERION

Let $\mathcal{P}$ be a set whose elements $P_1, \dots, P_{|\mathcal{P}|}$ are called *polymers* (we should emphasize that the name *polymer* is used solely for historical reasons and may be misleading. For our present purposes, the P_i are just labels for the elements of the finite set and we might as well label them in the standard way by integers).

Key words and phrases. cluster expansion, Taylor expansion, Mayer expansion, polymer models.

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We suppose given a binary symmetric relation c of "compatibility" between the polymers. This means that in the product $\mathcal{P} \times \mathcal{P}$ we give a symmetric subset of "compatible" pairs (note that this compatibility relation encodes all what remains of the structural properties of the physical models). Two polymers which are not compatible are said to be incompatible and We write $P_1 \; c \; P_2$, $P_1 \; \iota \; P_2$ when $\{P_1, P_2\}$ is a compatible resp. an incompatible pair. We will assume that $P \, \iota \, P$ for all $P \in \mathcal{P}$.

Following Dobrushin [D1], we will associate to each polymer P a complex variable w_P and introduce the *polymer partition function*

$$(1) \qquad Z \equiv Z_{\mathcal{P}} \equiv Z_{\mathcal{P},w} = \sum_{\{P_1,\dots,P_n\}_c: \; P_i \in \mathcal{P}} \prod_{i=1}^{n} w_{P_i}$$

where the sum is over all families $\{P_1, \dots, P_n\}_c$ of pairwise compatible polymers from $\mathcal{P}$. The $n = 0$ term in (1) (no polymers at all) is set equal to 1. Note that $Z_{\mathcal{P}}$ is a function of the $|\mathcal{P}|$ *complex variables* $w_1, \dots, w_{|\mathcal{P}|}$.

Remark. In many applications, there is a spatial structure such that it is possible to associate a "support" $\operatorname{supp} P$, namely a finite subset of $\mathbb{Z}^d$, to a polymer P. In those cases the compatibility $P \; c \; \tilde{P}$ is usually just some geometrical property of the supports, typically polymers are compatible if their supports do not interact. Also the polymer activities arise as some simple functions depending on the "shape" of the polymer of the of the temperature and interaction potentials. To avoid confusion, let us stress, however, that in the abstract polymer models we study now, we do not consider these "physical" activities, but all polymers activities w_P are now treated as independent complex variables. The relation to the physical activities is made only later when thermodynamic functions of the abstract polymer model are evaluated at the physical values of the activities.

As usual we are interested in the computation of the logarithm of the partition function. We can write its Taylor series around zero[1]

$$(2) \qquad \log Z = \log Z_{\mathcal{P},\,w} = \sum_{I \in \mathcal{I}(\mathcal{P})} w_I \qquad \text{with} \qquad w_I = C_I \prod_{P \in \operatorname{supp} I} w_P^{I_P}$$

where the sum is over the collection $\mathcal{I}(\mathcal{P})$ of all "multi-indices" I (integer valued functions on $\mathcal{P}$). The Taylor coefficients C_I, $I = (I_P)_{P \in \mathcal{P}}$ are

$$(3) \qquad C_I = (I_{P_1}! \dots I_{P_N}!)^{-1} \frac{\partial^{I_{P_1} + \dots + I_{P_N}} \log Z_{\mathcal{P}}}{\partial^{I_{P_1}} w_{P_1} \dots \partial^{I_{P_N}} w_{P_N}} | \, w_{P_i} \equiv 0$$

where the derivative of $\log Z_{\mathcal{P}}$ is taken at $\{w_{P_i} \equiv 0, P_i \in \mathcal{P}\}$, $\mathcal{P} = \{P_1, \dots, P_N\}\}$.

A multi-index I on $\mathcal{P}$ can be regarded as a collection of polymers where multiple copies of a single polymer P *are* allowed. Then the non-negative integer I_P represents the multiplicity of P in I. Write $I \equiv (\mathcal{S}, I_{\mathcal{S}})$ where $\mathcal{S} = \operatorname{supp} I$ is the "support of I", $\mathcal{S} = \{P \in \mathcal{P} : I_P \geq 1\}$ of the multi-index I. Given any subset $\mathcal{S} = \{P_1, \dots P_n\} \subset \mathcal{P}$ and activities w_P, we denote by $w^{\mathcal{S}}$ modified activities such that $w_P^{\mathcal{S}} = w_P, P \in \mathcal{S}$ and $w_P^{\mathcal{P}} = 0, P \notin \mathcal{S}$.

[1] Its convergence is clear for sufficiently small activities w_P; see Theorem 1 for more details.

Notice that we have for any function F given by a power series in variables $\{w_P, P \in \mathcal{P}\}$ the relation $\partial_{\mathcal{S}} F\{w_P, P \in \mathcal{P}\} = \partial_{\mathcal{S}} F\{w_P, P \in \mathcal{S}\}$ where the symbol $\partial_{\mathcal{S}}$ denotes a derivative (of any order) taken w.r. to the variables $\{w_P, P \in \mathcal{S}\}$ at $\{w_P \equiv 0, P \in \mathcal{P}\}$. Hence the coefficients C_I are functions of the multi-index I (on the system of polymers $\{P \in \operatorname{supp} I\}$, with the compatibility relation $P \, c \, P'$) only.

In fact, nonzero values of C_I appear only for *indecomposable* multi-indices I; decomposability means that there exists a partition of $S = \operatorname{supp} I$ into two sets $S = S_1 \cup S_2$ such that $\{P_1, P_2\}$ is a compatible pair $\forall P_1 \in S_1, P_2 \in S_2$. We will also use a name *cluster* for such an indecomposable multi-index (collection of polymers) I. The collection of all clusters on $\mathcal{P}$ will be denoted by $\mathcal{C} = \mathcal{C}(\mathcal{P})$.

Indeed, analyze the coefficient C_I for a decomposable index $I = (S, I_S)$. If $S = S_1 \cup S_2$ with $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_1 \cap S_2 = \emptyset$ and every pair (P_1, P_2) with $P_1 \in S_1$, $P_2 \in S_2$ is compatible, then $\log Z_{S,w} = \log Z_{S_1,w} + \log Z_{S_2,w}$ and so $C_I = 0$, by (3).

Our main aim is to give a simple proof of a result below which is a slightly weaker version of the general result of [KP] (see also Chapter 5 of [U] for a good presentation). However, compared to existing proofs our arguments will be really elementary and straightforward. No complicated combinatorics, estimates by Cauchy formulas etc. will be used, only a simple induction argument. As our emphasis is on the simplicity of estimates, we do not try to replace the constant L (and E) below by $L = 1$ as in [KP]. Put

$$L = L(\delta) = \sup_{x \in (0,\delta)} \left\{ \frac{-\log(1-x)}{x} \right\} = \frac{-\log(1-\delta)}{\delta}. \tag{4}$$

In usual applications, δ will be small and so $L = 1 + O(\delta)$.

Theorem 1. *Assume that there is a function $\{a_P \geq 0\}$ on $\mathcal{P}$ such that*

$$|w_P| e^{a(P)} \leq \delta \tag{5}$$

holds for any polymer $P \in \mathcal{P}$. Moreover, assume that for any polymer $Q \in \mathcal{P}$

$$\sum_{P \iota Q} |w_P| e^{a(P)} \leq \frac{a(Q)}{L} \tag{6}$$

where L is from (4). Then, for any polymer $Q \in \mathcal{P}$, the following bound holds for the sum over all clusters (connected multi-indices) $I \in \mathcal{C}(\mathcal{P})$ containing Q:

$$\sum_{I \in \mathcal{C}(\mathcal{P}):\, I \ni Q} |w_I| \leq L |w_Q| e^{a(Q)} \quad \text{with} \quad w_I = C_I \prod_{P \in \operatorname{supp} I} w_P^{I_P}. \tag{7}$$

As a consequence one also has a bound, for the sum over all clusters I that are incompatible with a selected polymer $Q \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P}):\ I \iota Q} |w_I| \leq a(Q). \tag{8}$$

Remarks.

a) If polymers P are just points of the lattice $\mathbb{Z}^\nu$ and $\#\{j : j \,\iota\, i\} \leq k$ for each i then the condition $|w_i| \leq e^{-a} a/k$ i.e. $\sum_{j \iota i} |w_j| \leq a e^{-a}$ implies $\sum_{I \iota i} |w_I| \leq a$.

b) If polymers P are "bonds" $b = \{i, j\}$ and compatibility of bonds means just their non-intersection, then the condition (6) (valid e.g. if $\sum_{b \ni i} |w_b| \le e^{-a} a/2$) implies $\sum_{I \ni b} |w_I| \le a$.

c) For the low temperature Peierls contours [P,D2,Gr,MS] of the two dimensional Ising model, a natural choice of the function a is $a(P) = a|P|$ with $a = 2\beta J - C$ and a suitable C. Then the condition (6)

$$\sum_{P \iota 0} e^{-2\beta J|P|} e^{a|P|} \le 4 \sum_{n \ge 4} 3^n e^{-(2\beta J - a)n} \le a$$

is obviously verified for $2J\beta$ sufficiently larger than $\log 3$ (with suitable C).

d) An important application of the convergence result occurs when the abstract polymer model results from a contour representation in the context of the Pirogov-Sinai theory [PS,Z1,Z2]. In that case the polymers represent local deviations from some ground state configuration g. To each polymer P one then associates a subset of the lattice, $\underline{P} \subset \mathbb{Z}^d$. The incompatibility relation usually refers then to a sufficient distance between the corresponding "supports". The function $a(P)$ should then be chosen proportional to the volume of the support of P chosen as $a(P) = c|\underline{P}|$. In such a situation the convergence of the polymer expansion guarantees the existence of a Gibbs states μ_g corresponding to the ground state g in the following sense: Let $S_g(R)$ be the set of spin configurations such that there exists a point x at a distance less then R from the origin such that one can reach infinity from x on a path along which the configuration g is realized. Let $S_g = \cup_{R<\infty} S_g(R)$. Then $\mu_g(S_g) = 1$. To see this, note that $S_g(R)$ is not realized only if there is a contour P those interior contains the ball of radius R. We call $\mathcal{P}_R$ the collection of all such polymers. Then by standard arguments we have that

$$1 - \mu_g(S_g(R) \le \sum_{P \in \mathcal{P}(R)} w_P e^{\sum_{I^* \iota P} w_{I^*}} \le \sum_{P \in \mathcal{P}(R)} w_P e^{a(P)}$$

which will go to zero if R goes to infinity. In cases one has several such polymer models (namely in the Pirogov Sinai theory) this means that a rigorous control of the phase coexistence is established.

e) A stronger variant of (6) gives an exponential decay of the correlations in the given polymer model. Namely, the statement (8) gives also an information on the *decay* of the terms $w_I = C_I \prod_{P \in \text{supp}\, I} w_P^{I_P}$. Take $\tilde{w}_P = w_P e^{d(P)}$ with $d(P) = C \operatorname{diam} P$ and require (6) to hold *even for* $\tilde{w}_P$. Then (8) reads

$$a(Q) \ge \sum_{I \iota Q} |\tilde{w}_I| \ge \sum_{I \iota Q} |w_I| e^{\sum_{P \in \text{supp}\, I} d(P)} \ge \sum_{I \iota Q} |w_I| e^{C \operatorname{diam} \operatorname{supp} I}.$$

This tells us that the sum $\sum_I^N |w_I|$ taken over clusters of diameter at least N is of order at most e^{-dN}, and only those terms w_I appear in the formulas expressing the correlation between two cylindrical events having a distance $\ge N$. In the case of two dimensional low temperature Ising contours a convenient choice of a, d is such that $a + d$ is suitably smaller than $2\beta J - \log 3$. Then the correlation length is proven to be of the order $1/d$ (or less).

Proof of Theorem 1. Our proof uses, following [NOZ], an induction over the cardinality $|\mathcal{P}|$ of the system $\mathcal{P}$ of all available polymers. Suppose that we already have (by induction assumption) the bound (8), with $w_I = C_I \prod_{P \in \text{supp}\, I} w_P^{I_P}$, for the sum of Taylor coefficients of *any* polymer model employing a *smaller* number of polymers than $|\mathcal{P}|$. (It trivially holds for a model employing no polymers at all.) Then we want to prove the same bound for a model constructed over $\mathcal{P}$.

Select a polymer Q, denote by $\mathcal{P} \setminus\setminus Q = \{P \in \mathcal{P} : P\ c\ Q\}$ and consider the partition functions $Z_{\mathcal{P}\setminus Q}$ and $Z_{\mathcal{P}\setminus\setminus Q}$ of the model "without Q" resp. "employing only polymers compatible with Q"

$$Z_{\mathcal{P}\setminus Q} = \sum_{\{P_1,\dots,P_m\}_c\ P_i \in \mathcal{P},\ Q \notin \{P_i\}} \prod_i w_{P_i} \ ; \ Z_{\mathcal{P}\setminus\setminus Q} = \sum_{\{Q,P_1,\dots,P_m\}_c\ P_i \in \mathcal{P}} \prod_i w_{P_i}.$$

We can obviously decompose the partition function for the set $\mathcal{P}$ in the form

$$Z_{\mathcal{P}} = Z_{\mathcal{P}\setminus Q} + w_Q Z_{\mathcal{P}\setminus\setminus Q}$$

by writing first the sum over all terms that do not contain Q and then placing Q and summing over all remaining collections compatible with Q. Taking the logarithm we get

$$\log Z_{\mathcal{P}} = \log Z_{\mathcal{P}\setminus Q} + \log\left(1 + w_Q \frac{Z_{\mathcal{P}\setminus\setminus Q}}{Z_{\mathcal{P}\setminus Q}}\right). \tag{9}$$

Since the first summand here counts the sum over all clusters that do not make use of Q, the second term is necessarily equal to the sum of all clusters containing Q, i.e. the sum we want to control in (8).

On the other hand, the term $Z_{\mathcal{P}\setminus\setminus Q}/Z_{\mathcal{P}\setminus Q}$ appearing in the second logarithm is already "under control" because it uses partition functions of polymer models with less than $|\mathcal{P}|$ polymers. That is to say we have on the one hand that

$$\log\left(1 + w_Q \frac{Z_{\mathcal{P}\setminus\setminus Q}}{Z_{\mathcal{P}\setminus Q}}\right) = \sum_{I \in \mathcal{C}(\mathcal{P}):\ I \ni Q} w_I \tag{10}$$

and on the other hand

$$\log\left(1 + w_Q \frac{Z_{\mathcal{P}\setminus\setminus Q}}{Z_{\mathcal{P}\setminus Q}}\right) = \log\left(1 + w_Q \exp\left(-\sum_{I^*} w_{I^*}\right)\right) \tag{11}$$

where the sum $\sum_{I^*} w_{I^*}$ is precisely over all the clusters I^*, from the $\mathcal{P} \setminus Q$ model, which are *incompatible* with Q i.e. which contain some polymer $\tilde{Q}$ incompatible with Q. The sum $\sum_{I^*} w_{I^*}$ can be estimated, using the induction assumption (6), for any $\tilde{Q}$ (notice that the clusters I^* are taken from a "smaller", $\mathcal{P} \setminus Q$ model) as

$$\sum_{I^*} |w_{I^*}| \le L \sum_{\tilde{Q} \iota Q} |w_{\tilde{Q}}| e^{a(\tilde{Q})} \le a(Q). \tag{12}$$

Now we will use the following important fact: Consider the Taylor expansion in the variables w_Q, w_{I^*} of the function $\log\left(1 + w_Q \exp\left(-\sum_{I^*} w_{I^*}\right)\right)$ and replace all the coefficients in the resulting sum (of products of w_Q and w_{I^*}) by their absolute values. Use the following simple observation.

Lemma. *Denote by $f \prec g$ the relation, between functions of variables $x_1, x_2, \dots, x_n$, defined by the requirement that absolute values of all Taylor coefficients of f at $x_i \equiv 0$ are bounded from above by the corresponding positive Taylor coefficients of g. For any monomial $y_j = a_j \prod x_i^{N_i^j}$ denote by $\widetilde{y}_j = |a_j| \prod x_i^{N_i^j}$. Then, for any choice $\{y_j\}$ of monomials we have the relation, interpreting both sides as functions of $\{x_i\}$,*

$$\log\left(1 + x_1 \exp\left(\sum_j y_j\right)\right) \prec -\log\left(1 - x_1 \exp\left(\sum_j \widetilde{y}_j\right)\right).$$

Proof. Just notice that the Taylor coefficients of e^x and $-\log(1-x)$ are all positive.

Therefore[2],

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} w_I \prec -\log\left(1 - w_Q \exp\left(\sum_{I^*} \widetilde{w}_{I^*}\right)\right)$$

and finally, using (12), monotonicity of $-\log(1-x)$ and the notation (4)

$$\sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} |w_I| \le -\log\left(1 - |w_Q| \exp\left(\sum_{I^*} |w_{I^*}|\right)\right) \le L\ |w_Q| e^{a(Q)} \tag{13}$$

which proofs the inductive step for the desired bound (7). We recall that (8) then follows from (7) by summing the bound (7) over all $\tilde{Q} \in \mathcal{P}$ incompatible with Q, because any index I incompatible with Q contains at least one polymer $\tilde{Q}$ incompatible with Q.

Partially expanded polymer models. Generalized D–K–P criterion

Consider now a more general model where in addition to polymers $P \in \mathcal{P}$ (the "big ones", satisfying some compatibility relation) having weights w_P one also has a "cluster field" i.e. a collection of complex fugacities $\{w_G,\ G \in \mathcal{G}\}$ indexed by objects G which we will call "chains". Assume that a relation $G \,\iota\, P$ resp. $G \,c\, P$ of (in)compatibility between the chains and polymers is given. Note that on the contrary, that all chains are compatible with each other.

Given complex fugacities $\{w_P\}$ and $\{w_G\}$ we define the "mixed" (in the terminology of [Z2] and [HZ]) partition function

$$Z = Z_{\mathcal{P},\mathcal{G},w} = \sum_{\{P_1,\dots,P_n\}_c : P_i \in \mathcal{P}} \prod_{i=1}^n w_{P_i} \exp\left(\sum_{G:\, G \,c\, P_i\ \forall i} w_G\right)$$

where the first sum is again over all families $\{P_1, \dots, P_n\}_c$ of pairwise compatible polymers in $\mathcal{P}$ and the second sum $\sum_G w_G$ is over all chains G compatible with any polymer P_i.

[2] Recall that the expression $\log\left(1 + w_Q \exp\left(-\sum_{I^*} w_{I^*}\right)\right)$ which is a function of variables w_Q and $\{w_{I^*}\}$ is identified then also as a function of variables w_Q and $\{w_P;\ P \,\iota\, Q\}$.

Remark. Mixed partition functions arise from partially expanded polymer models. Assume that we have a polymer ensemble of the form $\mathcal{P} = \mathcal{P}_l \cup \mathcal{P}_s$ (in most applications, these are the "large" and "small" polymers separated according to some criterion). Then

$$Z_{\mathcal{P}} = \sum_{\substack{P_1,\dots,P_n : P_i \in \mathcal{P}_l, Q_1,\dots,Q_m : Q_i \in \mathcal{P}_s \\ \{P_1,\dots,P_n,Q_1,\dots,Q_m\}_c}} \prod_{i=1}^{n} w_{P_i} \prod_{j=1}^{m} w_{Q_i} \tag{14}$$

Here the entire collection $\{P_1, \dots, P_n, Q_1, \dots, Q_m\}$ must be compatible. For a given collection $P_1, \dots, P_n$, we can now take the logarithm of the sum

$$\log \left(\sum_{\substack{Q_1,\dots,Q_m : Q_i \in \mathcal{P}_s \\ \{P_1,\dots,P_n,Q_1,\dots,Q_m\}_c}} \prod_{i=1}^{n} w_{P_i} \prod_{j=1}^{m} w_{Q_i} \right) = \sum_{G : G c P_i, \forall_i} w_G$$

according to the procedure outlined in the previous section, the notion of compatibility of the cluster G with a polymer P_i meaning that each component of the cluster is compatible with O_i. The result of this procedure is precisely a mixed partition function as defined above. In most applications one now wants to further expand another subclass of the "large polymers" that remain. To do this, one must be able to compute the logarithm of the mixed polymer partition function.

We now investigate the Taylor series of the logarithm of partition function (14)

$$\log Z = \sum_{I \in \mathcal{I}(\mathcal{P} \cup \mathcal{G})} w_I \qquad \text{with} \qquad w_I = C_I \prod_{P \in \operatorname{supp} I} w_P^{I_P} \prod_{G \in \operatorname{supp} I} w_G^{I_G} \tag{15}$$

where the sum is over the collection $\mathcal{I} = \mathcal{I}(\mathcal{P} \cup \mathcal{G})$ of all multi-indices I (integer valued functions on $\mathcal{P} \cup \mathcal{G}$) and C_I are given like in (3).

It can be shown, similarly as in the previous section that nonzero C_I appear only for "connected", indecomposable multi-indices I called *clusters*. Here, the *decomposability* of a multi-index with a support $(\mathcal{P}_1 \cup \mathcal{G}_1) \cup (\mathcal{P}_2 \cup \mathcal{G}_2)$ means that any polymer resp. chain from the first system is compatible with any one from the second one. Since all the chains G are mutually compatible, incompatibility can only occur between two polymers or between a polymer and a chain.[3]

In the sequel, the *support* of a multi-index (in particular of a cluster) I will be defined as $\operatorname{supp} I = \mathcal{P} \cup \mathcal{G}$ where $\mathcal{P} = \{P : I_P > 0\}$ and $\mathcal{G} = \{G : I_G > 0\}$. Denote by $\mathcal{C}(\mathcal{P} \cup \mathcal{G})$ the collection of all clusters on $\mathcal{P} \cup \mathcal{G}$.

We define the functions

$$L \equiv L(\delta) = \frac{-\log(1-\delta)}{\delta}, \ E \equiv E(\delta) = \frac{e^{\delta} - 1}{\delta}, \quad \tilde{L} \equiv \tilde{L}(\delta) = L^2 E.$$

Note that with these definitions we have, for all $x \geq 0$,

$$1 + LE(e^{\frac{Lx}{\tilde{L}}} - 1) \leq e^{x}. \tag{16}$$

[3]A collection $\mathcal{P} \cup \mathcal{G}$ is a cluster if the graph whose bonds are pairs $P \,\iota\, P'$ and $P \,\iota\, G$ is *connected*.

Theorem 2.. *Assume that there is a function* $\{a(G) > 0,\ G \in \mathcal{P} \cup \mathcal{G}\}$ *such that,*

$$|w_P| e^{a(P)} \le \delta \quad \text{and} \quad (e^{|w_G|} - 1)\ (e^{b(G)} - 1) \le \delta \tag{17}$$

holds for any polymer $P \in \mathcal{P}$ *resp. chain* $G \in \mathcal{G}$. *Moreover assume, for a suitable function* $b(Q) \le a(Q)$ *on* $\mathcal{P} \cup \mathcal{G}$, *the validity of the following two bounds:*
1) Polymer fugacities satisfy a bound, for any chain or polymer $G \in \mathcal{G} \cup \mathcal{P}$

$$\sum_{P \in \mathcal{P}: P \,\iota\, G} |w_P| e^{a(P)} \le \frac{b(G)}{\tilde{L}}. \tag{18}$$

2) Chain fugacities w_G *satisfy a bound*

$$\sum_{G \in \mathcal{G}: G \,\iota\, Q} |w_G| e^{b(G)} \le a(Q) - b(Q) \tag{19}$$

for any polymer $Q \in \mathcal{P}$.[4]
Then the following bounds are valid:
a) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ *containing a polymer* $Q \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I \ni Q} |w_I| \le L |w_Q| e^{a(Q)} \quad \text{where} \quad w_I = C_I \prod_{P \in \operatorname{supp} I} w_P^{I_P}. \tag{20}$$

b) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ *containing a chain* G

$$\sum_{I \in \mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I \ni G} |w_I| \ \le\ |w_G| e^{b(Q)}. \tag{21}$$

c) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ *incompatible with a chain* $G \in \mathcal{C}$

$$\sum_{I \in \mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I \,\iota\, G} |w_I| \le \frac{L}{\tilde{L}} b(Q) \le b(Q). \tag{22}$$

d) For the sum of all clusters $I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G})$ *incompatible with a polymer* $Q \in \mathcal{P}$

$$\sum_{I \in \mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I \,\iota\, Q} |w_I| \le a(Q). \tag{23}$$

Proof. As before, we will use the induction over the total number $N = |\mathcal{P}| + |\mathcal{G}|$ of polymers and chains used in the model. The case $N = 0$ is trivial.

a) Proof of (20): This is an analogy of (7). Denote by Z the partition function of the "full model", by $Z_{\mathcal{P}\setminus Q, \mathcal{G}}$ the partition function of the model with polymer Q removed (i.e. with $w_Q = 0$). We have the relation (here and below, $Z \equiv Z_{\mathcal{P}, \mathcal{G}}$)

$$\sum_{I \in \mathcal{C}(\mathcal{P}\cup\mathcal{G}):\ I \ni Q} w_I = \log Z_{\mathcal{P},\mathcal{G}} - \log Z_{\mathcal{P}\setminus Q,\mathcal{G}} = \log \left(1 + w_Q \exp\left(- \sum_{I^*} w_{I^*} \right) \right) \tag{24}$$

[4]Notice that while in (19) the sum is over chains incompatible with a *polymer* Q, the sum in (18) is over polymers incompatible with a *polymer or chain* Q.

where the sum $\sum_{I^*} w_{I^*}$ is precisely over clusters I^* from the $\mathcal{P} \setminus Q$ model incompatible with Q. We proceed analogously as in the proof (9)–(11) of (7), but using the bound (18) instead of (6). By the induction assumption (23) for the $\mathcal{P} \setminus P$ model, we already have $\sum_{I^*} |w_{I^*}| \leq a(Q)$. Then, by (16) we get the required induction step

$$(25) \qquad \sum_{I \in \mathcal{C}(\mathcal{P}): I \ni Q} |w_I| \leq -\log\left(1 - |w_Q| \exp\left(\sum_{I^*} |w_{I^*}|\right)\right) \leq L|w_Q| \ \exp(a(Q)).$$

b) Denote by $Z_{\mathcal{P},\mathcal{G} \setminus G}$ resp. $Z_{\mathcal{P} \setminus\setminus G, \mathcal{G} \setminus G}$ the partition function of the model with G removed resp. partition function of the model where only polymers compatible with the chain G are allowed. For both models we may assume the validity of (20) – (23) by the induction assumption, and moreover we have $Z_{\mathcal{P} \setminus\setminus G} = e^{w_G} Z_{\mathcal{P} \setminus\setminus G, \mathcal{G} \setminus G}$. Notice that the term $\sum_{I \in \mathcal{P} \cup \mathcal{G}: I \ni G} w_I$ equals

$$\begin{aligned}
\sum_{I: I \ni G} w_I &= \log Z - \log Z_{\mathcal{P},\mathcal{G} \setminus G} \\
&= \log\left(Z_{\mathcal{P},\mathcal{G} \setminus G} + (e^{w_G} - 1) Z_{\mathcal{P} \setminus\setminus G, \mathcal{G} \setminus G}\right) - \log Z_{\mathcal{P},\mathcal{G} \setminus G} \\
&= \log\left(1 + (e^{w_G} - 1) \frac{Z_{\mathcal{P} \setminus\setminus G, \mathcal{G} \setminus G}}{Z_{\mathcal{P},\mathcal{G} \setminus G}}\right) \\
&= \log\left(1 + (e^{w_G} - 1) \exp\left(- \sum_{I^*: I^* \ni P, P \,\iota\, G} w_{I^*}\right)\right)
\end{aligned}$$

i.e.

$$(26) \qquad \sum_{I: I \ni G} w_I = w_G + \log\left(1 + (1 - e^{-w_G})\left(\exp\left(- \sum_{I^*: I^* \ni P, P \iota G} w_{I^*}\right) - 1\right)\right)$$

where the sum $\sum_{I^*} w_{I^*}$ is over clusters I^* containing a *polymer* P incompatible with G. Thus (and this is in fact the argument, used below in c), proving (22) from (20) and (18)) we have $\tilde{L} \sum_{I^*} |w_{I^*}| \leq Lb(G)$ and we can continue in the estimate (of the sum of absolute values in the expansion of the r.h.s of (26))

(27)

$$\begin{aligned}
\sum_{I \in \mathcal{C}(\mathcal{P} \cup \mathcal{G}): I \ni G} |w_I| &\leq |w_G| - \log\left(1 - (e^{|w_G|} - 1)\left(\exp\left(\sum_{I^*: I^* \ni P, P \iota G} |w_{I^*}|\right) - 1\right)\right) \\
&\leq |w_G| - \log\left(1 - (e^{|w_G|} - 1)(e^{\frac{L}{\tilde{L}} b(G)} - 1)\right) \\
&\leq |w_G| \ \left(1 + LE(e^{\frac{L}{\tilde{L}} b(G)} - 1)\right) \leq |w_G| e^{b(G)}
\end{aligned}$$

by (17). This proves (21).

c) If a cluster I touches a chain G then there is some *polymer* $Q \in \mathcal{P}$ incompatible with G such that $I \ni Q$. Summing the r.h.s of (20) in (18), over all such $Q \,\iota\, G$ we arrive to (22) analogously as from (6) to (8).

d) If a cluster I touches a polymer Q then i) either I contains a polymer $\tilde{Q}$ incompatible with Q ii) or I contains a chain G incompatible with Q. The sum

$\sum_I^{i)} |w_I|$ corresponding to the first case is bounded as $\leq L/\tilde{L}\, b(Q)$ just by inserting (20) into (18). Analogously, the sum $\sum_I^{ii)} |w_I|$ corresponding to the second case is bounded as $\leq a(Q) - b(Q)$, by inserting (21) into (19).

So we get the desired bound (23).

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