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A POSTERIORI ERROR ESTIMATORS FOR ELLIPTIC EQUATIONS WITH DISCONTINUOUS DIFFUSION COEFFICIENTS

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ABSTRACT. We regard linear elliptic equations with discontinuous diffusion coefficients in two and three space dimension with varying boundary conditions. The problem is discretized with linear Finite Elements. We propose the treatment of the arising singularities within an adaptive procedure based on a posteriori error estimators. Within this concept no a priori data like the degree of the singularity is needed.

We introduce the class of quasi-monotone distributed diffusion coefficients. Within this class an interpolation operator as well as a posteriori error estimators with bounds which are independend of the variation of the diffusion coefficients are derived. In numerical examples we confirm robustness of the error estimators and show that on adaptively refined meshes the reduction of the error is optimal with respect to the number of unknows.

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1. INTRODUCTION

It is known that the solution of an elliptic boundary value problem is smooth (i.e. from a Sobolev space H^2) if the boundary is smooth or the domain is convex and the data are smooth. If the domain is neither convex nor the boundary is smooth or if the diffusion coefficient has jump discontinuities , then the solution is not smooth

(i.e. $\notin H^2$) anymore. Nonsmooth solutions can occur even for the Laplace equation in convex domains if the boundary conditions change at a point on the boundary. It turns out that such solutions can be decomposed into a smooth part and and some singular functions of regularity $H^s, 1 \leq s < 2$. Approximating the solution with Finite Elements yields then usually a wrong convergence, due to the bad regularity.
Several approaches take these singularities into account to get better convergence. One could locally refine the mesh around a singularity [14] or add special functions to the Galerkin space. In all of these approaches one needs some a priori data to get the best performance. Mostly this is the degree of the singularity that means the biggest number s with $\varphi \in H^\lambda$, $\lambda < s$, where φ is the singular function. Except in some special cases the degree of singularity is not known explicitly.
We present an approach relying on a posteriori error estimates which allows for treatment of singularities in two and three space dimensions, which works without the knowledge of parameters aligned with the singularities. Applying mesh refinement based on a posteriori error estimates, numerical experiments show convergence rates of the same order as for regular problems. Moreover, this approach allows for estimates of the error between the solution and its approximation. These estimates are reliable and efficient and for a large class problems also robust. Robustness means that variations of the diffusion, i.e. the amount of the jump discontinuity does not enter in the bounds. In some cases we have to introduce locally some weighting factors which depend on the local behaviour of the diffusion. Then robustness is not proven.
Our results are similar to the very recent paper [27]. There independendly an equivivalent a posteriori error estimator was derived for a class of problems which is somewhat smaller then ours (see Remark 2 on page 10). Deriving the a posteriori error estimators we use the framework developed in [25].
The outline of this paper is as follows. The problem setting and the notation used are introduced in section 2. We describe the nature of the singularities in more detail and discuss approximation properties of finite elements (section 3). To prepare the proof for the upper bound of the error by the error estimator we need interpolation results in weighted Sobolev spaces (section 4). Interpolation results are necessary in other domains of Numerical Analysis and are therefore of interest not only in connection with a posteriori error estimators. For this reason section 4 is written to be selfcontained. Here we present known results and extend them slightly.
The main results of this work are presented in section 5. The reader interested in a posteriori error estimators may skip the preceeding section 4 since we recall the results in an instant manner. We introduce so called residual based error estimators and estimators which are based on solving local problem and show that they are reliable and efficient. We also discuss a known approach which relies on hierarchical bases in section 6 and a Zienkiewicz-Zhu like estimator in section 7. The section 8 is devoted to various numerical experiments.

2. PROBLEM SETTING

2.1. **Continuous and discrete Problem.** We are interested in linear elliptic problems with varying diffusion coefficients and boundary conditions of Dirichlet and Neumann type in open polygonal (polyhedral) domains $\Omega \subset R^d, d = 2, 3$. Let the boundary be decomposed in $\partial\Omega = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$ and for definiteness let $meas(\Gamma_D) > 0$. Let $g \in H^{1/2}(\Gamma_D)$ given. There is an extension of g onto a function defined in $H^1(\Omega)$ and having g as trace on Γ_D. Let us denote this extension also by g. Let $h \in L^2(\Gamma_N)$ and $f \in L^2(\Omega)$ be given.

These problems can be written in variational form: with the space $V = \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}$ look for u, fulfilling $u - g \in V$ and satisfying:

$$\int_\Omega k\nabla u\nabla v \ = \int_\Omega fv + \int_{\Gamma_N} hv \ , \ \forall v \in V \quad . \tag{1}$$

Let the diffusion coefficient k

$$\varepsilon \leq k \leq \varepsilon^{-1} \tag{2}$$

be bounded. In what follows our aim is that the parameter ε does not enter into our bounds. If ε enters at some place we will point this out and provide argumentation why one can not neglect ε. We suppose that Ω can be partitioned in disjoint open polygonal (polyhedral) subdomains $\Omega_l, l = 1,..,n_l$ on which the diffusion coefficient is constant. Problems with discontinuous diffusion coefficients are known as interface or transmission problems.
We introduce a discrete problem through a finite element space $V_h \subset V$ with continuous and piecewise linear functions. The underlying grid is referred to T_h. We assume that the grid is aligned with the partition of Ω, that means that the boundary $\partial\Omega_l$ is made up of faces from simplices in T_h.
Let g_h an finite element approximation of g on Γ_D. Then solution of the discrete problem u_h satisfies $u_h - g_h \in V_h$ and

$$\int_\Omega k\nabla u_h\nabla v_h \ = \int_\Omega fv_h + \int_{\partial\Omega} hv_h \quad , \ \forall v_h \in V_h \quad . \tag{3}$$

We disregard problems arising from the approximation of non homogeneous boundary conditions, that means we set $g = g_h$ on Γ_D.

2.2. **Assumptions and Notation.** The space dimension will be denoted by d. We use the terminus "2D case" or "3D" to indicate that $d = 2$ or $d = 3$. Unless states otherwise all results cover the 2D and the 3D case simultaneously.
For a subset $\Omega' \subset \Omega$ we will use the usual Sobolev (semi-)norms

$$||u||^2_{L^2(\Omega')} \ = \int_{\Omega'} u^2 \quad , \quad |u|_{H^1(\Omega')} \ = \ ||\nabla u||_{L^2(\Omega')} \quad .$$

We define the weighted (semi-)norm

$$|v|^2_{kH^1(\Omega')} \ := \int_{\Omega'} k\,(\nabla v)^2 \quad .$$

We assume that the measure of the Dirichlet part of the boundary Γ_D is not vanishing. This implies that $|v|_{kH^1(\Omega)}$, $v \in V$ is a norm equivalent up to ε with $|v|_{H^1(\Omega)}$ and hence solvability of (1) and (3). The space defined by the semi-norm $|\cdot|_{kH^1(\Omega)}$ will be refered by us as weighted Sobolev space.

Although Ω is open we use the notation $\overset{\circ}{\Omega}$ to make clear that we really exclude $\partial\Omega$. We call $\bigcup_{l=1,..,n_l} \partial\Omega_l/\partial\Omega$ the interface, which we denote by Γ. If a point $x \in \Omega$ belongs to more then two subdomains $\bar{\Omega}_l$ then x is called a crosspoint.
We make the following assumptions for the discrete problem (3): The underlying triangulation T_h is shape regular but not necessary uniform. We denote by h the local cell diameter.
By $meas_d(\cdot)$ we denote the d-dimensional Lebesgue measure.
We use the notation $a \approx b, a \preceq b$ and $a \succeq b$ to indicate the relations $c_1 a \leq b \leq c_2 a$, $c_3 a \leq b$ and $a \leq c_4 b$ where c_i are further not specified constants which depend only on the shape regularity of the finite element mesh T_h. We underline that such constants also don't depend on the difsfusion coefficient nor on the solutions u and u_h. Such behaviour will be called *robust*.

The set of nodes of triangulation will be denoted by N_h. We include also nodes lying on the Dirichlet boundary. For a subset $S \subset \bar{\Omega}$ we denote with $N_h(S)$ the nodes on S. Similary for the set of all simplices T, faces F and edges E will be denoted by T_h, F_h and E_h. All simplices $T \in T_h$, faces $F \in F_h$ and edges $E \in E_h$ are closed as usually. In 2D $F_h = E_h$.
For a node x resp. an edge E or face F we define S_x resp S_E or S_F as the union of all simplices T which have x resp. E or F in common.
We define a simplexwise constant approximation f_h of f by $f_h|_T := \|f\|_{L^1(T)}/\ \|f\|^2_{L^2(T)}$. We denote the value of k on a simplex T by k_T.
For a simplex $T \in T_h$ let T^+ consists of T and a finite number of neighbors of T. Similarly for a face F let F^+ contain some neighboring simplices of F. The number of neighbors is bounded by the shape regularity. See section 4 for the precise definition.
We denote by $F \subset \partial T$ a face of simplex $T \in T_h$. If F is a boundary face then S_F will contain only one simplex.
For a face F denote by T_F a simplex from S_F such that $k_{T_F} \geq k_{T'}$, where $T' \subset S_F, T' \neq T_F$. If $F \subset \partial\Omega$ then T_F will be the only simplex with face F.

Definition 1. *Let $F \subset \mathring{\Omega} \cup \Gamma_D$ not lying in Γ_N. Denote with n_T and $n_{T'}$ the outward normal of $F \subset \partial T$ resp. $F \subset \partial T'$. The jump of the normal fluxes across this face F is defined as*

$$\left[k\frac{\partial u_h}{\partial n}\right]_F := k_T\frac{\partial u_h}{\partial n_T} + k_{T'}\frac{\partial u_h}{\partial n_{T'}} \quad .$$

Let $meas_{d-1}(F \cap \Gamma_N) > 0$ a face on the Neumann boundary. Denote with n_F the outward normal of $F \subset \partial\Omega$. The jump of the normal fluxes across this face F is defined as

$$\left[k\frac{\partial u_h}{\partial n}\right]_F := h - k_T\frac{\partial u_h}{\partial n_F} \quad .$$

Sometimes we use a shorter notation and write j_F instead of $\left[k\frac{\partial u_h}{\partial n}\right]_F$.
The finite element shape functions aligned with the node x_i are denoted by λ_{x_i} and also by λ_i. Note that here x_i may belong to $N_h(\Gamma_D)$ thus $\lambda_i \notin V_h$. We define so called bubble functions. These are nonnegative shape functions not contained in V_h with a small support. An element bubble can be defined as $\phi_T := d^{-d}\prod_{i=1..d+1}\lambda_i$. A face bubble function ϕ_F for the face F will have S_F as support. One can take the product of all barycentric coordinates λ_i excluding λ_j vanishing on F, scaled with $(d-1)^{1-d}$.

3. Interface problems and its approximation with Finite Elements

In this section we discuss the influence of discontinuous diffusion coefficients on the regularity of the solution. We restrict ourself to the regularity in the interior of Ω and thus do not discus the influence of the boundary or boundary conditions on the regularity. In the case of the Laplace equation it is known [8] that the solution is in the inner of the domain contained in C^∞, if the right hand side is. But in the case of discontinuous diffusion coefficients we can not expect such regularity.
The first restriction on the regularity is due to the discontinuity of the normal derivatives across the interface. Clearly one has

$$k_i\frac{\partial u_h}{\partial n} = k_j\frac{\partial u_h}{\partial n} \quad . \tag{4}$$

on the interface, where k_i and k_j are the diffusion coefficients on both sides of the interface and n is a vector normal to the interface.

This physically condition prevents the solution from belonging to $H^{3/2}(\Omega)$. In [22] one finds that this restriction is sharp, namely that u belongs to $H^{1+\lambda}(\Omega), \lambda < 1/2$ (with homogeneous Dirichlet conditions, $f \in L^2(\Omega)$ and two subdomains, in 2D,3D). But the lacking continuity of the normal derivatives not necessary influences the approximation properties of a finite element space. Imagine a smooth solution vanishing at the interface. Multiplication with piecewise constant diffusions coefficients yields a function which is not smooth anymore, but in a suitable problem setting the finite elements will converge with the same order as before.
More important is the piecewise regularity. In the two dimensional case we refer to [14] for an overview and [17],[10],[8]. In the two and three dimensional case see [21] and [20]. Their results state that the solution admits a decomposition into piecewise H^2 regular functions and some so called singular functions. Each of the singular functions will be aligned with a so called singular point.
The results about the decomposition of the solution u are of the following type. With sufficiently smooth boundary data and a load function $f \in L^2(\Omega)$ u can be decomposed as

$$u = w + \sum_j c_j \psi_j \quad , \tag{5}$$

where $w \in H^2(\Omega_i)$ and $\psi_j \in H^{1+\lambda}(\Omega_i)$ are singular functions and $c_j \in R$. Further one has piecewise $\Delta w \subset L^2(\Omega_i)$ and $\Delta \psi_j = 0$. Both w and ψ_j satisfy the interface condition (4) and are continuous along the interface. We want also to mention that the decomposition (5) is not unique. Multiplying with a smooth cut off function one can assure that ψ_j vanishes outside a neighborhood of the singular point.
A general result from [13], [12] states

Theorem 1. *The solution u from (1) is contained in $H^{1+\alpha}(\Omega)$ for a certain $\alpha > 0$ depending on Ω, Γ_D and k.*

In some special cases the behaviour of the singular part is known. We want to illustrate the behaviour of the singular solution in two cases. Denote by $r(x, y), \varphi(x, y)$ the polar coordinates.
In the first example the interface will be an angle [8]. Let $\Omega = [-1, 1] \times [-1, 1]$ be decomposed by intersection of a cone with angle $\theta, \theta \leq \pi$. $\Omega_2 := \{(x, y) : 0 < \varphi(x, y) < \theta, (x, y) \in \Omega\}$ and $\Omega_1 = \Omega/\Omega_2$. The diffusion coefficient is piecewise constant:

$$k(x, y) := \begin{cases} 1, & \text{for} \quad (x, y) \in \Omega_1 \\ k_2, & \text{for} \quad (x, y) \in \Omega_2 \end{cases}$$

The point where the interface is not smooth, here $(0, 0)$ will be the singular point. Then the singular function behaves as

$$\psi = r^\lambda \begin{cases} cos(\lambda(\varphi - \theta/2)) & \text{for}(x, y) \in \Omega_2 \\ \beta cos(\lambda(\pi - |\varphi - \theta/2|)) & \text{otherwise} \end{cases} \tag{6}$$

where λ, β depend on k.
Simple calculations show that $\psi \in H^{1+\varepsilon}(\Omega_i), \varepsilon < \lambda$. For $k_2 < 1$ there is no singularity in the sense that ψ now belongs to $H^2(\Omega_i)$.
In the case of a right interface angle, $\theta = \pi/2$, the coefficients λ and β are defined explicitly by:

$$\lambda = \frac{4}{\pi} \arctan\left(\sqrt{\frac{3 + k_2}{1 + 3k_2}} \right) \quad , \quad \beta = -k_2 \frac{sin(\lambda \frac{\pi}{4})}{sin(\lambda \frac{3\pi}{4})} \quad .$$

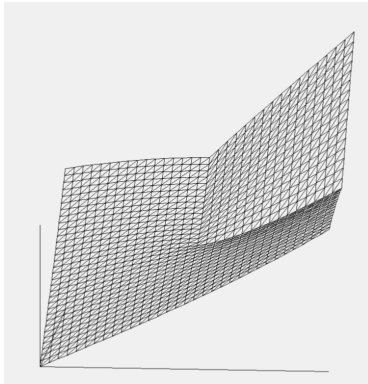

FIGURE 1. singular solution ψ for $k_2 = 0.01$, $\psi \in H^2(\Omega_i)$

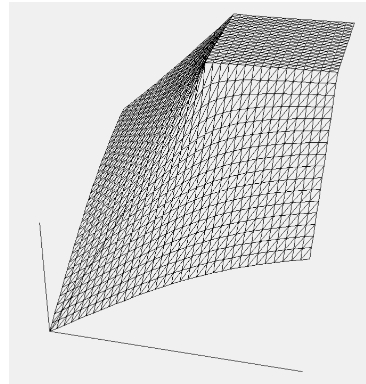

FIGURE 2. singular solution ψ for $k_2 = 100$, ψ not contained in $H^2(\Omega_i)$

In the case $\theta \neq \pi/2$ they are implicitly given. See [21], [20] for numerically calculated λ.

Example 1. *Set $\theta = \pi/2$. Let $\Omega = [-1,1] \times [-1,1]$ and ψ be defined as in (6). As above set $k_1 = 1$ and take k_2 as parameter.*

In the figures 2, 1 we show ψ for $k_2 = 100, 0.01$.
Accordingly to the above formula for λ on sees that the higher k_2 the lower the regularity. In the limit $k_2 \to \infty$ the problem reduces to a problem on Ω_1 with homogeneous Dirichlet conditions on the interface only. The limiting value is then $\lambda = 3/4$. Thus in this case the regularity of an interface problem is not worse then that of an the associated boundary value problem defined on Ω_1, with homogenuous Dirichlet conditions on the interface.
In the last example the solution of the interface problem belonged to $H^{3/2}(\Omega)$ independent of the jump discontinuity of the diffusion coefficient. We now show an example where the best regularity result one could get independent of k is $u \in H^1(\Omega)$. Such situations can occur if in 2D there are more then three domains sharing a interior point. In [15] the situation where the interface looks locally like two intersecting lines is analyzed. Set $\Omega = [-1,1] \times [-1,1]$ and define $\Omega_1 := \{(x,y) : 0 < \varphi < \theta \;\; or \;\; \pi < \varphi < \theta + \pi, (x,y) \in \Omega\}$ and $\Omega_2 = \Omega/\Omega_1$. Set $k = k_1$ on Ω_1 and $k = k_2$ on Ω_2.
Here the singular function behaves like $\psi = r^\lambda s(\varphi)$ where

$$s(\varphi) := \begin{cases} cos(\lambda(\pi - \theta - c))\, cos(\lambda(\varphi - \theta + b)) & \text{for } 0 \leq \varphi \leq \theta \\ cos(\lambda b)\, cos(\lambda(\varphi - \pi + c)) & \text{for } \theta \leq \varphi \leq \pi \\ cos(\lambda c)\, cos(\lambda(\varphi - \pi - b)) & \text{for } \pi \leq \varphi \leq \pi + \theta \\ cos(\lambda(\theta - b))\, cos(\lambda(\varphi - \theta - \pi - c)) & \text{for } \pi + \theta \leq \varphi \leq 2\pi \end{cases} \tag{7}$$

The parameter $\theta \in (0, \pi/2]$ is the intersection arc between the two lines of the interface. One can vary λ between $(0,1]$ to get different exponents of the singular function. The remaining parameters are $b = 0.5\theta, c = \pi/2(1 + \frac{1}{\lambda}) - b$.The corresponding values for the diffusion are $k_1 = -tan(\lambda c)$ and $k_2 = tan(\lambda b)$. In this case one could have the regularity parameter $\lambda > 0$ arbitrary close to 0.

Example 2. *Take $\Omega = [-1,1] \times [-1,1]$ and $\psi = r^\lambda s(\varphi)$, where s is defined in (7) . Set $\theta = \pi/2$ and vary λ as a parameter.*

With $\theta = \pi/2$ we have $k_1 \to \left(\lambda\frac{\pi}{4}\right)^{-1}$ and $k_2 \to \lambda\frac{\pi}{4}$ with $\lambda \to 0$ In figure 3 we show

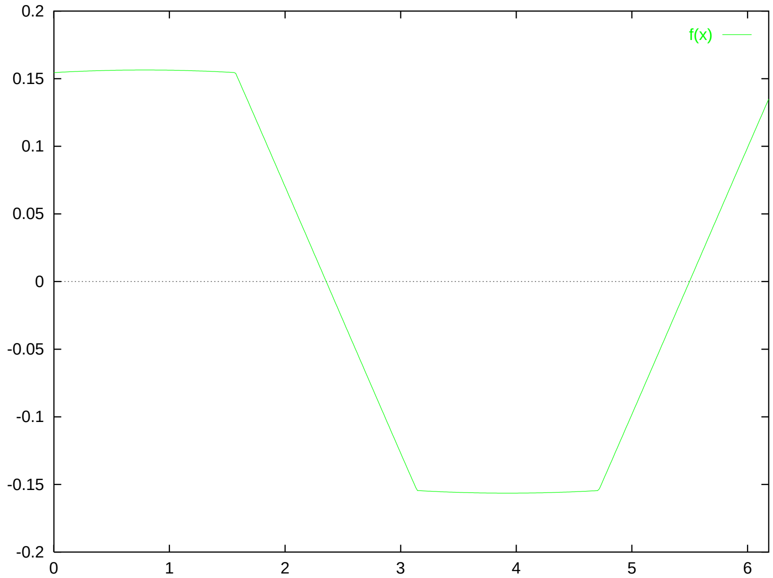

FIGURE 3. the angular part $s(\varphi)$ of the singular function (7) for $\lambda = 0.2, \theta = \pi/2,\ k_1 = 6.31375, k_2 = 0.158384$

a plot of $s(\varphi)$ for $\lambda = 0.2$. It is clear that for smaller λ the angular part $s(\varphi)$ of the singular function would become more and more constant on Ω_1 and linear on Ω_2.

3.1. **Convergence rates for finite elements on uniform meshes.** One expects that the regularity of u decides about the convergence rate of u_h, when solving (3) on uniform meshes. We show that at least in the 2D case it is the piecewise regularity and not the global regularity which bounds the convergence rate.
Regularity results from lemma 1 show $u \in H^{1+\varepsilon}(\Omega)$. By Sobolev embeddings we know that u is continuous in Ω. Thus the interpolation operator $I_N : V \to V_h$ which is given by taking the values in the nodal points is well defined. Exploiting arguments given in Example 3 in [24] one has the following interpolation results in fractional Sobolev spaces

Lemma 2. *For all simplices $T \in T_h$ it holds:*

(8) $$||u - I_h(u)||_{H^1(T)} \le h^\lambda |u|_{H^{1+\lambda}(T)} \quad .$$

Combining this bound with Galerkin orthogonality shows the desired estimate in terms of piecewise contributions if $u \in H^{1+\lambda}(\Omega_i)$:

(9) $$|u - u_h|^2_{kH^1(\Omega)} \le |u - I_h(u)|^2_{kH^1(\Omega)} = \sum_i k_i\, |u - I_h(u)|^2_{H^1(\Omega_i)} \preceq h^{2\lambda} \sum_i k_i |u|^2_{H_{1+\lambda}(\Omega_i)} \quad .$$

This bound shows that finite elements make in a certain sense use of piecewise regularity. Since global regularity is restricted to $H^{3/2}(\Omega)$ it follows that for $\lambda > 1/2$ the local regularity and not the global bounds the convergence.
Let $d = 2$ and suppose that there is only one singular point. Let $\psi|_{\Omega_i} = b_i r^\lambda cos(\lambda\varphi + c_i), 0 < \lambda$ where r, φ are polar coordinates with respect to the singular point. Then it yields for small h

(10) $$h^\lambda c(\psi) \preceq |\psi - v_h|_{H^1(T)} \quad , \ \forall v_h \in V_h \quad .$$

See [16],p.265, [28] for linear finite elements. This is even true for higher order finite elements [10] Theorem 3.1..

Remark 1. *In view of higher order convergence in regions where the solution is in H^2, the singularity will asymptotically dominate on uniform meshes. Suppose that the solution u admits a decomposition $u|_{\Omega_i} = w|_{\Omega_i} + \psi|_{\Omega_i}, w|_{\Omega_i} \in H^2(\Omega_i)$, where ψ is as above. Let u_h be a Galerkin approximation of u on uniform meshes with mesh size h and finite elements which are continuous and piecewise polynomials of order $m \geq 1$. Then for sufficiently small h*

$$c(u)h^\lambda \preceq |u - u_h|_{kH^1(\Omega)} \preceq h^{\lambda+\varepsilon}c_1(u) \ ,$$

where $c(u), c_1(u)$ depend on u, $0 < \varepsilon$.

4. Interpolation operators for weighted Sobolev spaces

In this section we derive an interpolation operator for weighted Sobolev spaces. As the results of this section are of interest not only in the context of error estimators we recall necassary notation in order to make this section selfcontaining.
Let $\Omega \subset R^d$ be a Lipschitz domain with polygonal (polyhedral) boundary which is partitioned in $\Gamma := \partial\Omega = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$. The parts of the boundary Γ_D and Γ_N may be refered by Dirichlet and Neumann-boundary (resp.). Define $V = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$. Let T_h be a shape regular triangulation and $V_h \subset V$ the space of piecewise linear finite elements.
From the literature [5] interpolation operators from $I_C : V \to V_h$ are known. The approximation properties of these operators were not only of optimal order but also allow for local approximations. The results in [5] state essentially that

$$\begin{aligned} ||u - I_C(u)||_{L^2(T)} &\leq ||u||_{L^2(T_C^+)} \\ ||u - I_C(u)||_{L^2(T)} &\leq h\ |u|_{H^1(T_C^+)} \\ ||u - I_C(u)||_{L^2(F)} &\leq h^{1/2}\ |u|_{H^1(F_C^+)} \ , \end{aligned} \tag{11}$$

where T_C^+ (resp.F_C^+) consists of simplices sharing a node with T (resp. F). The operator $I_C : V \in V_h$ is defined by the values in the nodal points. For a node $x \in N_h(\overset{\circ}{\Omega} \cup \Gamma_N)$ define

$$I_C(u)(x) = \frac{1}{|S_x|}\int_{S_x} u \tag{12}$$

Recall that S_x contains all simplices with node x.
Suppose we have given a finite partition $\Omega = \bigcup \Omega_j$ in polygonal (polyhedral) subdomains with weights k_j constant on each subedomain. Further the boundaries $\partial\Omega_j$ consists of faces of the underlying triangulation. It may be convinient to look on the weights as on a global weight function k. Denote by k_T the value of the weight on the simplex T and with k_F the sum of the weights of neighboring simplices of the face F.
Straightforeward use of the relation

$$\varepsilon \leq k \leq \varepsilon^{-1}$$

yields interpolation estimates for weighted Sobolev spaces equipped with the weighted (semi)norm $|v|^2_{kH^1(\Omega)} = ||k(\nabla v)^2||_\Omega$, where k enters into the bounds:

Lemma 3. *Let Let $u|_{\Gamma_D} = 0$. Let the weights be bounded locally*

$$\exists\, d \in R \ , \ \exists\, \varepsilon > 0 \ , \ \forall\, T' \subset T_C^+, F_C^+\ d\,\varepsilon \leq k_{T'} \leq d\,\varepsilon^{-1}$$

Then for each simplex $T \in T_h$ and for each face $F \in F_h$ hold the following bounds

$$\begin{aligned} \|I_C(u)\|_{L^2(T)} &\preceq \|u\|_{L^2(T_C^+)} \\ \|u - I_C(u)\|_{L^2(T)} &\preceq h\, |u|_{H^1(T_C^+)} \ \preceq \varepsilon^{-1}\, h\, k_T^{-1/2}\, |u|_{kH^1(T_C^+)} \\ |u - I_C(u)|_{H^1(T)} &\preceq |u|_{H^1(T_C^+)} \\ k_F^{1/2}\, \|u - I_C(u)\|_{L^2(F)} &\preceq \varepsilon^{-1} h^{1/2}\, |u|_{kH^1(F_C^+)} \quad , \end{aligned}$$

where $T_C^+ = \bigcup_{x \in T} S_x$ and $F_C^+ = \bigcup_{x \in F} S_x$.

Of course one would like to have estimates where k does not enter into the bounds. Such operators are derived in [29], [11] under some conditions on the subdomains Ω_j and on k_j.
All subdomains in R^2, R^3 which we regard in future consist of some simplices. Their shape depends on the local geometry of the triangulation which is essentially confined through the shape regularity of the triangulation. Thus all constants arising in trace inequalities or Poincare inequalities depend on the shape regularity of the triangulation and the local mesh-size only.
Note that one can not expect robust interpolation results for arbitrary weights. This was shown in [30]. For example for a checkerboard like distribution of weights ε and 1 there are no robust interpolation results.
We need a scaled version of a standard trace inequality. It states that

Lemma 4. *Let $\Omega_0 \subset R^d, d = 2, 3$ be a domain with diameter h and Lipschitz boundary. Let $F \subset \partial\Omega_0$. Then for $u \in H^1(\Omega_0)$*

$$\|u\|^2_{L^2(F)} \ \preceq h^{-1}\, \|u\|^2_{L^2(\Omega_0)} \ + h\, |u|^2_{H^1(\Omega_0)} \quad .$$

PROOF
Use homogeneous scaling $F : \Omega_0 \to \widehat{\Omega}_0$ with the factor $1/h$ to transform Ω_0 to $\widehat{\Omega}_0$ with diameter 1 and define $\widehat{u} = u \circ F^{-1}$. Then a standard trace inequality [9] states

$$\|\widehat{u}\|^2_{L^2(\widehat{F})} \ \leq \ \|\widehat{u}\|^2_{L^2(\partial\widehat{\Omega}_0)} \ \preceq \ \|\widehat{u}\|^2_{L^2(\widehat{\Omega}_0)} \ + \ |\widehat{u}|^2_{H^1(\widehat{\Omega}_0)} \quad .$$

Observing that

$$\begin{aligned} h^{d-1}\, \|\widehat{u}\|^2_{L^2(\widehat{F})} &\approx \|u\|^2_{L^2(F)} \\ h^d\, \|\widehat{u}\|^2_{L^2(\partial\widehat{\Omega}_0)} &\approx \|u\|^2_{L^2(\partial\Omega_0)} \\ h^{d-2}\, |\widehat{u}|^2_{H^1(\widehat{\Omega}_0)} &\approx |u|^2_{H^1(\Omega_0)} \end{aligned}$$

one gets the assertion. ∎

4.1. **An Interpolation operator for varying boundary conditions with stability in $L^2(\Omega)$.**

In the case that in the elliptic equation there is present a mass term we need an interpolation operator which is continuous in $L^2(\Omega)$. Such stability does not hold in the case of the interpolation operator defined in [29].
We allow also for more general boundary conditions, that means for the case $meas(\Gamma_D) \neq 0$ and $meas(\Gamma_N) \neq 0$. Therefore we choose another way to define an interpolation operator then in [29]. Note that our interplolation operator differs from the one presented in [29] only for nodes on the boundary $\partial\Omega$.
In order to prepare the next definition denote by S_x as before all simplices which share the node $x \in N_h$. The set of simplices containing an edge E will be denoted by S_E.

Definition 2. *For a node $x \in N_h$ denote by C_x all simplices $T \subset S_x$ where the weights k_T achieves the maximum in S_x.*

FIGURE 4. The distribution of weights $k_T, T \subset S_x$ is quasi-monotone with respect to the node x on the left picture but not on the right figure. The simplex T is colored dark and the set $T_{x,qm}$ is colored with different levels of grey in the left picture

Definition 3. *Chose a node $x \in N_h(\bar{\Omega})$. The distribution of weights $k_{T'}, T' \subset S_x$ will be called* quasi-monotone with respect to a node x of a triangulation T_h which is given by the Finite Element space V_h *if the following conditions are fullfilled.*
For each simplex $T \subset S_x$ there exists a Lipschitz set $T_{x,qm}$ with $T \cup C_x \subset T_{x,qm} \subset S_x$, such that

$$k_T \leq k_{T'} \quad , \quad \forall T' \subset T_{x,qm} \quad .$$

If $x \in N_h(\Gamma_D)$ we demand additionally that $meas_{d-1}(\partial T_{x,qm} \cap \Gamma_D) > 0$.

If there is no danger of confusion we will simply say that the distribution of weights is quasi-monotone with respect to a node x if the above definition is fullfilled for x. We say that the distribution of weights $k_T, T \in T_h$ is quasi-monotone if it is quasi-monotone with respect to all nodes of N_h.
Here we use the definition of Lipschitz domains as given in [4]. It follows that the sum of two simplices sharing a node is a Lipschitz domain iff they share a face.
In case of an interior node an equivivalent condition would be that the weight function as a function restricted to a sphere contained in S_x with center x has exactly one local maximum.
We illustrate the quasi-monotonicity condition in figure 4 and 5. Of course the Lipschitz set $T_{x,qm}$ from the above definition could contain more simplices from S_x then just T and this with maximal weights. See figure 4 for an example, where x is an interior node of Ω.
In figure 5 the situation is illustrated where the node x_0 lies on Γ_D. Comparing the introduced quasi-monotonicity condition with that of [29] one notices that both definitions are the same for interior nodes and nodes on Γ_D.
It follows from the definition that C_x a Lipschitz domain if the distribution of weights is quasi-monotone with respect to the node $x \in N_h$. To see this take $T \subset C_x$. In this case $T_{x,qm} = C_x$. Hence a checker board like distribution of the weights is not quasi-monotone.

Remark 2. *Suppose $\Gamma_N = \emptyset$ and that there is no nodal point which belongs to the closure of more then three subdomains. Then there are no robust interpolation results in weighted Sobolev spaces without additional conditions on the weights* [30]. *We describe counter examples where there are no robust interpolation results.*
Regard in 2D a point on the boundary which belongs to $\bar{\Omega}_i, i = 1, 2, 3$. Let the weights be 1 in domains $\partial\Omega_i \cap \Gamma_D \neq \emptyset, i = 1, 3$ and greater then 1 in the domain Ω_2.
In 3D let Ω_1, Ω_2 be simplices which share only an interior vertex. Let $\Omega_3 = \Omega/(\Omega_1 \cup \Omega_2)$. Let the weight be 1 in Ω_3 and greater 1 in Ω_1, Ω_2.

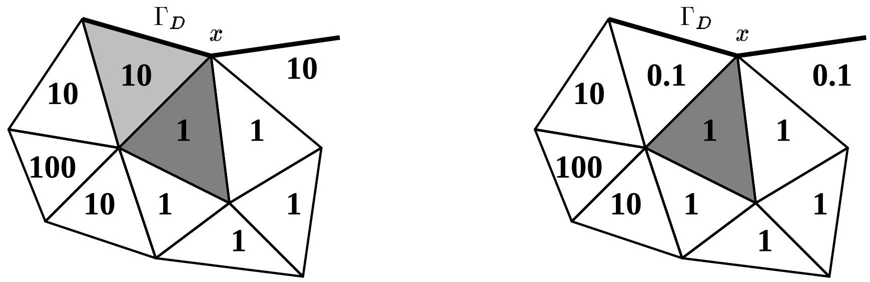

FIGURE 5. the distribution of weights $k_T, T \subset S_x$ is quasi-monotone with respect to the node $x \in N_h(\Gamma_D)$ on the left figure but not on the right. The simplex T is colored dark and the set $T_{x,qm}$ with different levels of grey in the left figure

Not even in the 3D case that there is a boundary point which belongs to the closure of two subdomains there are robust interpolation results. Take Ω_2 as a simplex which touches the boundary in a vertex and let the weight be 1 outside the simplex and greater 1 in the simplex. Note that the quasi-monotonicity condition is weaker then demanding that for any simplices $T_1, T_n \subset S_x$ there is a Lipschitz set $\{T_i, i = 1, .., n, T_i \subset S_x\}$ fullfilling $k_{T_i} \leq k_{T_{i+1}}$, $i = 0, .., n-1$. This is seen from the left picture in figure 4.

We call a triangulation $T_{h'}$ a refined triangulation of T_h if for the according Finite Element spaces holds $V_h \subset V_{h'}$. In 2D quasi-monotonicity is preserved during refinement of a triangulation.
In 3D this is not true. For example regard the case of a domain $[-1,1] \times [-1,1] \times [0,3]$ subdived into three horizontal layers. In the bottom and the upper layer the weight is 100. The middle layer is subdivided into 4 cubes sharing a common edge. There the weights 1 and 10 are distributed alternatingly.
To assure quasi-monotonicity also for triangulations obtained by refinement we introduce the following

Definition 4. *Chose a node $x \in T_h$. The distribution of the weights $k_{T'}, T' \subset S_x$ will be called* stable quasi-monotone with respect to the node x of a triangulation T_h which is given by the Finite Element space V_h*, if it is quasi-monotone with respect to the node x and the following conditions are fullfilled:*
Denote with C_E the union of simplices $T \subset S_E$ where k_T achieves the maximum in S_E.
If $x \in N_h(\mathring{\Omega} \cup \Gamma_N)$ and for each edge E with endpoint x it holds for each simplex $T \subset S_E$ there exist a Lipschitz set $T_{x,qm,E}$ with $T \cup C_E \subset T_{x,qm,E} \subset S_E$, such that

$$k_T \leq k_{T'} \quad , \quad \forall T' \subset T_{x,qm,E} \quad .$$

If $x \in N_h(\Gamma_D)$ and for each edge $E \subset \Gamma_D$ with endpoint x it holds for each simplex $T \subset S_E$ there exist a Lipschitz set $T_{x,qm,E,\Gamma}$ with $T \cup C_E \subset T_{x,qm,E} \subset S_E$ such that $meas_{d-1}(\partial T_{x,qm,E} \cap \Gamma_D) > 0$ and

$$k_T \leq k_{T'} \quad , \quad \forall T' \subset T_{x,qm,E} \quad .$$

To illustrate this condition denote by G_E a 1-dimensional sphere perpendicular to E, with center on E and contained in S_E. This definition states that for interior edges E the weight function has only one local maximum on G_E. Obviously stable quasi-monotonicity is stronger then quasi-monotonicity and one checks that this property is preserved during refinement.
Choose a simplex T. Let us introduce the set T^+ which contains some neighboring simplices of T. For nodes $x \in N_h(T)$ where the distribution of weights is

quasimonotone with respect to x we add $T_{x,qm}$ to T^+. For other nodes we add S_x. Number the nodes of T in such a way that the distribution of weights is quasi-monotone with respect to nodes $x_i \in N_h(T), i = 0,..,m$ and not quasi-monotone for $x_i \in N_h(T), i = m+1,..,d$ where $0 \leq m \leq d$. For a simplex T and a face F define Lipschitz sets containing some neighboring simplices of T resp. F

$$\begin{aligned} T^+ &:= \bigcup_{x_i \in N_h(T), i \leq m} T_{x,qm} \cup \bigcup_{x_i \in N_h(T), i > m} S_x \quad , \\ F^+ &:= T_F^+ \quad . \end{aligned}$$

If the distribution of weights is quasi-monotone with respect to all nodes of T then the weight function restricted to T^+ has a local minimum on T that is

$$k_T \leq k_{T'} \quad , \quad \forall T' \subset T^+ \quad .$$

The interpolation operator is defined by

$$I_L u := \sum_{x \in N_h(\mathring{\Omega} \cup \Gamma_N)} \lambda_x p_x \quad , \quad \text{where } \; p_x := \frac{1}{|C_x|} \int_{C_x} u \quad , \quad x \in N_h(\mathring{\Omega} \cup \Gamma_N) \quad .$$

and λ_x are the finite element shape functions of V_h. Thus we don't take shape functions aligned with points on the Dirichlet boundary. Hence $I_L : V \to V_h$.
But for convenience set $p_x = 0$ for nodal points $x \in \Gamma_D$ so that in fact $I_L u := \sum_x \lambda_x p_x$ where the sum is taken also over the finite element shape functions which are aligned with points on Dirichlet part on the boundary.

Lemma 5. *Let $d = 2, 3$. Let $u \in V$. Chose a simplex $T \in T_h$ and a face $F \in F_h$. Let the distribution of weights $k_{T'}, T' \subset S_x$ be quasi-monotone with respect to x for nodes of T and T_F. Then the following bounds hold*

$$||I_L(u)||_{L^2(T)} \preceq ||u||_{L^2(T^+)} \tag{13}$$

$$||u - I_L(u)||_{L^2(T)} \preceq h \; |u|_{H^1(T^+)} \preceq k_T^{-1/2} h \; |u|_{kH^1(T^+)} \tag{14}$$

$$|u - I_L(u)|_{H^1(T)} \preceq |u|_{H^1(T^+)} \tag{15}$$

$$k_F^{1/2} \; ||u - I_L(u)||_{L^2(F)} \preceq h^{1/2} \; |u|_{kH^1(F^+)} \quad . \tag{16}$$

PROOF The proof is similar to that in [29]. Since in the definition of I_L only integrals on simplices and no integrals on faces enter it is possible to bound the L^2-norm of I_L in terms of the L^2-norm.
Choose a simplex T and number its nodes with $x_i, i = 0,..,d$. Let $x \in N_h(\mathring{\Omega} \cup \Gamma_N)$. Note that p_x can be written as $P_x(u)$ where P_x is the L^2 orthogonal projections on constant functions in $L^2(C_x)$. Exploiting the property of this projection it yields for nodes $x_i \in N_h(T/\Gamma_D)$ and any $c \in R$

(17)
$$||p_{x_i} - c||^2_{L^2(T)} \; \leq \; ||p_{x_i} - c||^2_{L^2(C_{x_i})} \; = \; ||P_{x_i}(u-c)||^2_{L^2(C_{x_i})} \; \preceq \; ||u - c||^2_{L^2(C_{x_i})}$$

We have $I_L(u) = \sum_{i=0}^d \lambda_i p_{x_i}$, where λ_i are the nodal basis functions. We conclude from (17) with $c = 0$

$$||I_L(u)||^2_{L^2(T)} \; \approx \sum_i \; ||p_{x_i}||^2_{L^2(T)} \; \leq \sum_i \; ||u||^2_{L^2(C_{x_i})} \; \preceq \; ||u||^2_{L^2(T^+)} \quad .$$

This shows (13)

Now let's prove (14). We use $u = \sum_{i=0}^{d} \lambda_i u$ and $I_L(u) = \sum_{i=0}^{d} \lambda_i p_{x_i}$. This shows

$$\|u - I_L(u)\|^2_{L^2(T)} = \sum_{i=0}^{d} \|\lambda_i(u - p_{x_i})\|^2_{L^2(T)} \preceq \sum_{i=0}^{d} \|u - p_{x_i}\|^2_{L^2(T)} \quad . \tag{18}$$

Inequality (17) applied to nodes $x_i \in N_h(T/\Gamma_D)$ yields for any $c \in R$

$$\begin{aligned} \|u - p_{x_i}\|^2_{L^2(T)} &\leq \|u - c\|^2_{L^2(T)} + \|p_{x_i} - c\|^2_{L^2(T)} \\ &\preceq \|u - c\|^2_{L^2(T)} + \|u - c\|^2_{L^2(C_{x_i})} \end{aligned}$$

Recall that from definition $C_x \subset T_{x,qm} \subset T^+$. Summing up contributions from nodes $x_i \in N_h(T/\Gamma_D)$ and applying Poincare inequality to the Lipschitz set T^+ yields since c is arbitrary

$$\begin{aligned} \sum_{x_i \in N_h(T/\Gamma_D)} \|u - p_{x_i}\|^2_{L^2(T)} &\preceq \|u - c\|^2_{L^2(T)} + \sum_{x_i \in N_h(T/\Gamma_D)} \|u - c\|^2_{L^2(C_{x_i})} \\ &\preceq \|u - c\|^2_{L^2(T+)} \preceq h^2 \, |u|^2_{H^1(T+)} \quad . \end{aligned} \tag{19}$$

For nodes x_i from T lying on the Dirichlet boundary we use $p_{x_i} = 0$ and the fact that u vanishes on $\partial T_{x,qm} \cap \Gamma_D$. Another Poincare inequality yields

$$\|u - p_{x_i}\|^2_{L^2(T)} \leq \|u\|^2_{L^2(T_{x_i,qm})} \preceq h^2 \, |u|^2_{H^1(T_{x_i,qm})} \quad . \tag{20}$$

Collecting inequalities (19) and (20) shows together with (18)

$$\begin{aligned} \|u - I_L(u)\|^2_{L^2(T)} &\preceq h^2 \left(|u|^2_{H^1(T+)} + \sum_{x_i \in N_h(T \cap \Gamma_D)} |u|^2_{H^1(T_{x_i,qm})} \right) \\ &\preceq h^2 \, |u|^2_{H^1(T+)} \quad . \end{aligned} \tag{21}$$

It remains to use the quasi-monotonicity conditions which states

$$k_T \leq k_{T'} \ , \ \forall T' \subset T^+ \quad . \tag{22}$$

With this bound we prove

(23)

$$|u|^2_{H^1(T+)} = \sum_{T' \subset T^+} k_{T'}^{-1} \, |u|^2_{kH^1(T')} \leq \sum_{T' \subset T^+} k_T^{-1} \, |u|^2_{kH^1(T')} = k_T^{-1} \, |u|^2_{kH^1(T+)}$$

which yields due to (21) assertion (14).

For showing (15) we use as before the decomposition of the unity $\sum_{i=0}^{d} \lambda_i = 1$.

$$|u - I_L(u)|^2_{H^1(T)} \leq \sum_{i=0}^{d} |\lambda_i(u - p_{x_i})|^2_{H^1(T)} \quad .$$

The properties of λ_i imply then

$$\begin{aligned} |\lambda_i(u - p_{x_i})|^2_{H^1(T)} &\preceq \|(\nabla \lambda_i)(u - p_{x_i})\|^2_{L^2(T)} + \|\lambda_i \nabla (u - p_{x_i})\|^2_{L^2(T)} \\ &\preceq h^{-2} \, \|u - p_{x_i}\|^2_{L^2(T)} + |u|^2_{H^1(T)} \quad . \end{aligned}$$

We combine again (19) and (20) to bound

$$h^{-2} \sum_{i=0}^{d} \|u - p_{x_i}\|^2_{L^2(T)} \preceq |u|^2_{H^1(T+)} \quad .$$

From the last three inequalities follows now (15).

The trace inequality (4) and inequalities (14) , (15) show

$$\|u - I_L(u)\|^2_{L^2(F)} \preceq h^{-1} \, \|u - I_L(u)\|^2_{L^2(T_F)} + h \, |u - I_L(u)|^2_{H^1(T_F)} \preceq h \, |u|^2_{H^1(T_F^+)} \quad .$$

Multiplication with $k_{T_F} \approx k_F$ proofs due to (22) with T^+ substituted by T_F^+ the last assertion (16). ∎

Without demanding quasi-monotonicity we can show only the *nonrobust* bounds:

Lemma 6. *Let $u \in V$. Let a face $F \in F_h$ and a simplex $T \in T_h$ be given. Let the weights be bounded locally*

$$\exists\, d \in R \;\;,\;\; \exists\, \varepsilon > 0 \;\;,\;\; \forall T' \subset T^+, F^+ \;\;,\;\; d\,\varepsilon \le k_{T'} \le d\,\varepsilon^{-1} \quad .$$

Then the following nonrobust bound hold

$$||I_L(u)||_{L^2(T)} \preceq ||u||_{L^2(T^+)} \tag{24}$$

$$||u - I_L(u)||_{L^2(T)} \preceq c(T)\, h\, |u|_{H^1(T^+)} \preceq c(T)\, h k_T^{-1/2}\, |u|_{kH^1(T^+)} \tag{25}$$

$$|u - I_L(u)|_{H^1(T)} \preceq c(T)\, |u|_{H^1(T^+)} \tag{26}$$

$$k_F^{1/2}\, ||u - I_L(u)||_{L^2(F)} \preceq c(T_F)\, h^{1/2}\, |u|_{kH^1(F^+)} \quad . \tag{27}$$

The constants $c(T), c(T_F)$ are defined to be equal 1 if the distribution of weights $k_T, \subset S_x$ is quasi-monotone with respect to x for nodes x of T and T_F.
Otherwise $c(T) := \varepsilon^{-1}$.

PROOF Proceed as in the proof of lemma 5. In the case that the distribution of weights is not quasi-monotone with respect to x substitute $T_{x,qm}$ with S_x. Use instead of (23) the nonrobust bound

$$\begin{aligned} |u|^2_{H^1(T^+)} &= \sum_{T' \subset T^+} k_{T'}^{-1}\, |u|^2_{kH^1(T')} \\ &\le \varepsilon^{-2} \sum_{T' \subset T^+} k_T^{-1}\, |u|^2_{kH^1(T')} = \varepsilon^{-2} k_T^{-1}\, |u|^2_{kH^1(T^+)} \quad . \end{aligned}$$

∎

The question arises whether there are functions $b(\varepsilon) < \varepsilon^{-1}$ such that lemma 6 holds with ε^{-1} replaced by $b(\varepsilon)$. Calculations for example 2 shows that $b(\varepsilon) \succeq \varepsilon^{-1/2}$.

5. Error estimators for equations with discontinuous diffusion coefficients

5.1. **Theoretical basis for a posteriori error estimators.** A insight in papers on a posteriori error estimators [25], [26], [23], [19], [2] reveals the necessity of interpolation results. In the case of linear equations interpolation results play the crucial role for the derivation of an upper bound for the error by a posteriori error estimators. If the diffusion is approximately constant i.e. in the cases of the Laplace equation the norm involved is the usual Sobolev (semi) norm $H^1(\Omega)$. Since we deal with non constant diffusion coefficients and since we do not want the bounds $\varepsilon \le k \le \varepsilon^{-1}$ to enter in our estimates we introduced weighted Sobolev norms. The interpolation operator has to work in such weighted spaces.
The interpolation operator is a linear operator $I_C : V \rightarrow V_h$ with interpolation properties. Such operators are defined in section 4 . It turns out that the distribution of the diffusion coefficients rules and restricts interpolation properties. We recall results from lemma 6 from section 4. Under the assumptions of the above lemma for each $v \in V$ there is a $v_h \in V_h$ with

$$\begin{aligned} k_T^{1/2}\, ||v - v_h||_{L^2(T)} &\preceq c(T)\, h\, |v|_{kH^1(T^+)} \\ k_F^{1/2}\, ||v - v_h||_{L^2(F)} &\preceq c(T_F)\, h^{1/2}\, |v|_{kH^1(F^+)} \quad , \end{aligned}$$

where T^+ resp. F^+ contains some neighbors of $T \in T_h$ resp. $F \in F_h$. See section 4 for a precise definition. The parameter $c(T)$ depends on the diffusions coefficients the neighborhood of x. If this distribution is quasi-monotone (see section 4) with respect to all nodes of T then $c(T) = 1$. This is for instance the case if there are only interior crosspoints and there is no crosspoint contained in $\mathring{\Omega}$ which belongs to more then three in 2D or two in 3D subdomains Ω_l. For crosspoints on the boundary the maximal number of subdomains to which x may belong if there are no restrictions on the diffusion coefficients is two in 2D or even one in 3D. See remark 2 in subsection 4.1.
If the distribution of diffusion coefficients is not quasi-monotone with respect to some nodes then there may be no robust interpolation operators in weighted Sobolev spaces. See [30] and example 2 in section 3. Therefore we had to admit a constant $c(T)$ depending on ε from relation (2).
We define the residual $r(u) \in V^\star$

(28) $$r(u)(v) := \int_\Omega k\nabla\left(u-u_h\right)\nabla v \;=\int_\Omega fv + \int_{\Gamma_N} hv - \int_\Omega k\nabla u_h \nabla v \text{ for } v\in V \quad .$$

The following decomposition of the residual will be used in the derivation of an upper bound for the error

Lemma 7. *For any $v \in V$ and any $v_h \in V_h$ we have the following representation of the residual:*

(29)
$$\int_\Omega k\nabla\left(u-u_h\right)\nabla v \;=\sum_{T\in T_h}\left\{\int_T f\left(v-v_h\right) \;+\sum_{F\subset \partial T/\Gamma_D} b_F\int_F \left[k\frac{\partial u_h}{\partial n}\right]_F \left(v-v_h\right)\right\} \quad ,$$

where $b_F = 1$ if $F \subset \bar{\Gamma}_N$ and $b_F = 1/2$ otherwise.

PROOF Recall that we suppose $g = g_h$, that means that problem (1) and (3) fulfill the same Dirichlet conditions. Integration by parts allows for splitting the residual into local contributions:

$$\begin{aligned}
&\int_\Omega k\nabla\left(u-u_h\right)\nabla v\\
&=\int_\Omega fv + \int_{\Gamma_N} hv \;-\int_\Omega k\nabla u_h\nabla v\\
&=\int_\Omega fv + \int_{\Gamma_N} hv \;-\sum_{T\in T_h}\int_T k\nabla u_h\nabla v\\
&=\sum_{T\in T_h}\int_T fv \;+\sum_{F\in\Gamma_N}\int_F hv \;+\sum_{T\in T_h}\int_T k_T\Delta u_h v \;-\int_{\partial T/\Gamma_D} k\frac{\partial u_h}{\partial n}v\\
&=\sum_{T\in T_h}\left\{\int_T fv \;+\sum_{F\subset\partial T/\Gamma_D} b_F\int_F\left[k\frac{\partial u_h}{\partial n}\right]_F v\right\} \quad .
\end{aligned}$$

In the last step we exploited the fact that Δu_h vanishes for linear functions and the definition of $\left[k\frac{\partial u_h}{\partial n}\right]_F$ where we included the function h. Galerkin orthogonality

$$\int_\Omega k\nabla\left(u-u_h\right)\nabla v_h = 0$$

allows for substitution of v by $v - v_h$ with $v_h \in V_h$. ■

5.2. **A residual based error estimator.** Extending an estimator from [25] we define an residual based estimator η_R. The global estimator η_R consists of the sum of local estimators $\eta_{R,T}$.

Definition 5.

$$\eta_R^2 := \sum_{T \in T_h} \eta_{R,T}^2 \quad .$$

$$\eta_{R,T}^2 := \frac{h^2}{k_T} \, ||f_h||_{L^2(T)}^2 \; + \sum_{F \subset \partial T / \Gamma_D} \frac{h}{k_F} \, || \left[k \frac{\partial u_h}{\partial n} \right]_F ||_{L^2(F)}^2 \quad .$$

The next theorem is the main result in this section. We show the estimator to be reliable and efficient. Efficiency is robust, that means that the constants in this bound do not depend on parameters of the problem like meshsize and especially the diffusion coefficients. Reliability is proofen to be robust if the distribution of diffusion coefficients is quasi-monontone. If it is not, additional constants enter in the upper bound for the estimator. See remark 3 for a discussion about these constants.

Theorem 8. *Let $d = 2, 3$.*
Then for the solution of (1) and (3) it holds under the assumption $g = g_h$ that the estimator η_R is globally reliable, that is

$$|u - u_h|_{kH^1(\Omega)}^2 \; \preceq \sum_{T \in T_h} d(T)^2 \, \eta_{R,T}^2 \; + \sum_{T \in T_h} c(T)^2 \frac{h^2}{k_T} \, ||f - f_h||_{L^2(T)}^2 \quad , \tag{30}$$

where $d(T)$ is defined as the maximum of $c(T)$ (defined in lemma 6) for all simplices sharing with T a face (including T).
Without any assumptions about the distribution of the diffusion coefficient the estimator η_R is locally efficient, that is for a simplex $T \in T_h$

$$\eta_{R,T}^2 \; \preceq \; |u - u_h|_{kH^1(T_N)}^2 \; + \sum_{T' \subset T_N} \frac{h^2}{k_{T'}} \, ||f - f_h||_{L^2(T')}^2 \quad . \tag{31}$$

where T_N contains all simplices sharing with T a face.

Corellary 1. *Let $d = 2, 3$ and let a initial triangulation T_{h_0} be given and let the distribution of diffusion coefficients $k_{T'}, T' \subset S_x$ be stable quasimonotone with respect to all nodes in $x \in N_{h_0}$. Then for each finite element space V_h obtained by refining T_{h_0} that is for V_h with $V_{h_0} \subset V_h$ (where $V_{h_0} \subset V$ is the finite element space of aligned with T_{h_0}) the error estimator is robust reliable, that means*

$$|u - u_h|_{kH^1(\Omega)}^2 \; \preceq \sum_{T \in T_h} \eta_{R,T}^2 \; + \sum_{T \in T_h} \frac{h^2}{k_T} \, ||f - f_h||_{L^2(T)}^2 \quad ,$$

Remark 3. *The factor $c(T)$ is equal to 1 if the distribution of diffusion coefficients $k_{T'}, T' \subset S_x$ is quasi-monotone with respect to all nodes of T. Otherwise $c(T)$ will depend on ε from equation (2), i.e. from the variation of the diffusion coefficient and the resulting estimates are not robust anymore.*

PROOF The technics used in the proof are essentially those of [25].
Define $v = u - u_h$, where u and u_h are solutions of (1) and (3) and observe that v vanishes on Γ_D. Let $v_h := I_L(v)$ be the result of the interpolation operator defined in section 4. Reliability will be shown using the representation of the residual 7 and

lemma 6 from section 4 for bounding the terms $v - v_h$. By Lemma 7 we split the residual

(32)
$$\int_\Omega k\nabla(u-u_h)\nabla v = \sum_{T\in T_h}\left\{\int_T f(v-v_h) + \sum_{F\subset\partial T} b_F \int_F \left[k\frac{\partial u_h}{\partial n}\right]_F (v-v_h)\right\} ,$$

where $b_F \in \{1/2, 1\}$.
We use lemma 6 from section 4 to bound

(33)
$$\int_T f(v-v_h) \le \|f\|_{L^2(T)} \; \|v-v_h\|_{L^2(T)} = c(T)\frac{h}{k_T^{1/2}}\|f\|_{L^2(T)} \; |v|_{kH^1(T^+)} .$$

In a second step we estimate the terms in (32) associated with the normal derivatives. Using the approximation inequality (27) from lemma 6 from section 4 we have the local estimate

(34)
$$\begin{aligned}\int_F \left[k\frac{\partial u_h}{\partial n}\right]_F (v-v_h) &\le \left\|\left[k\frac{\partial u_h}{\partial n}\right]_F\right\|_{L^2(F)} \; \|v-v_h\|_{L^2(F)} \\ &\preceq c(T_F)\left(\frac{h}{k_T+k_{T'}}\right)^{1/2} \left\|\left[k\frac{\partial u_h}{\partial n}\right]_F\right\|_{L^2(F)} \; |v|_{kH^1(F^+)} .\end{aligned}$$

At the end we take separately for the equations (33) and (34) the sum over all simplices. After application of the Cauchy Schwarz inequality we make use of the fact that each simplex T is covered at most by finite number of T^+ or F^+ which depends on the shape regularity of the mesh T_h to obtain

$$\int_\Omega k\nabla(u-u_h)\nabla v = |v|^2_{kH^1(\Omega)} \preceq |v|_{kH^1(\Omega)} \cdot$$
$$\left\{\sum_{T\in T_h}\left(c(T)^2\frac{h^2}{k_T}\|f\|^2_{L^2(T)} + \sum_{F\subset\partial T/\Gamma_D} c(T_F)^2\frac{h}{k_T+k_{T'}}\left\|\left[k\frac{\partial u_h}{\partial n}\right]_F\right\|^2_{L^2(F)}\right)\right\}^{1/2} .$$

Cancelation and the triangle inequality $\|f\|_{L^2(T)} \le \|f_h\|_{L^2(T)} + \|f-f_h\|_{L^2(T)}$ finishes the proof of (30).
The proof of the lower bound goes in two steps. We have the representation

(35)
$$\int_\Omega k\nabla(u-u_h)\nabla v = \sum_{T\in T_h}\left(\int_T f_h v + \int_T (f-f_h)\,v\right) + \sum_{F\in F_h/\Gamma_D} b_F\int_F\left[k\frac{\partial u_h}{\partial n}\right]_F v .$$

First we estimate the element residual. We denote by φ_T an element bubble function vanishing outside T as defined in section 2.2.
Note that since f_h is a constant on each simplex the local equivalences

$$|f_h\varphi_T|_{H^1(T)} \approx \|f_h\|_{L^2(T)}\, h^{-1} , \quad \|f_h\varphi_T\|_{L^2(T)} \approx \|f_h\|_{L^2(T)} .$$

follow directly from the relation $|\varphi_T|_{H^1(T)} \approx h^{-1}\,\|1\|_{L^2(T)}$ and $\|\varphi_T\|_{L^2(T)} \approx \|1\|_{L^2(T)}$.

Using $v = f_h \varphi_T$ as a test function in (35) the local equivalences imply

$$
\begin{aligned}
\|f_h\|^2_{L^2(T)} &\approx \int_T f_h\,(f_h \varphi_T) \\
&= \int_T k\,\nabla(u-u_h)\ \nabla(f_h\varphi_T) \ - \int_T (f-f_h)\,(f_h\varphi_T) \\
&\le k_T\ |u-u_h|_{H^1(T)}\ |f_h\varphi_T|_{H^1(T)} \ + \ \|f-f_h\|_{L^2(T)}\ \|f_h\varphi_T\|_{L^2(T)} \\
&= \left(\frac{h}{k_T^{1/2}}\right)^{-1} |u-u_h|_{kH^1(T)}\ \|f_h\|_{L^2(T)} \ + \ \|f-f_h\|_{L^2(T)}\ \|f_h\|_{L^2(T)} \quad .
\end{aligned}
$$

Simplifying the last expression we get an upper bound for the local consistency error:

$$
\frac{h}{k_T^{1/2}}\,\|f_h\|_{L^2(T)} \ \preceq\ |u-u_h|_{kH^1(T)} \ + \frac{h}{k_T^{1/2}}\,\|f-f_h\|_{L^2(T)} \quad . \tag{36}
$$

To make the estimate from below complete we need to estimate the jump term. For a face F not contained in Γ_D we denote by φ_F a bubble function vanishing at the boundary of S_F as defined in section 2.2 and with T, T' simplices which share this face. For shorter notation let j_F denote the constant $\left[k\frac{\partial u_h}{\partial n}\right]_F$.
We will exploit local equivalences of the type

$$
\|j_F\varphi_F\|_{L^2(T)} \approx h^{1/2}\,\|j_F\|_{L^2(F)} \quad , \quad |j_F\varphi_F|_{H^1(T)} \approx h^{-1/2}\,\|j_F\|_{L^2(F)} \quad .
$$

This equivalences follow from $\|\varphi_F\|_{L^2(T)} \approx h^{1/2}\,\|1\|_{L^2(F)}$ and $|\varphi_F|_{H^1(T)} \approx h^{-1/2}\,\|1\|_{L^2(F)}$.
We insert $j_F\varphi_F$ as a test function in (35) and obtain together with the above equivalences

$$
\begin{aligned}
\|j_F\|^2_{L^2(F)} &\approx \int_F j_F\,(j_F\varphi_F) \\
&= \int_{S_F} f_h\,(j_F\varphi_F) \ - \int_{S_F} (f-f_h)\,(j_F\varphi_F) \int_{S_F} k\,\nabla(u-u_h)\,\nabla(j_F\varphi_F) \\
&\le \sum_{T\subset S_F} \Big\{ \big(\|f_h\|_{L^2(T)} \ + \ \|f-f_h\|_{L^2(T)} \big)\ \|j_F\varphi_F\|_{L^2(T)} \\
&\qquad + k_T\ |u-u_h|_{H^1(T)}\ |j_F\varphi_F|_{H^1(T)} \Big\} \\
&\preceq \sum_{T\subset S_F} \Big\{ \big(\|f_h\|_{L^2(T)} \ + \ \|f-f_h\|_{L^2(T)} \big)\ \|j_F\|_{L^2(F)}\ h^{1/2} \\
&\qquad + k_T\ |u-u_h|_{H^1(T)}\ \|j_F\|_{L^2(F)}\ h^{-1/2} \Big\} \quad .
\end{aligned}
$$

Cancelation and insertion of (36) into the last right hand side gives

$$
\begin{aligned}
h^{1/2}\,\|j_F\|_{L^2(F)} &\preceq \sum_{T\subset S_F} \Big\{ h\,\|f-f_h\|_{L^2(T)} \ + k_T^{1/2}\,|u-u_h|_{kH^1(T)} \Big\} \\
&\le \Big\{ h\,\|f-f_h\|_{L^2(S_F)} \ + k_F^{1/2}\,|u-u_h|_{kH^1(S_F)} \Big\} \quad .
\end{aligned}
$$

Simplifying we get finally the upper bound for the jump term

$$
\begin{aligned}
\frac{h}{k_F}\,\|j_F\|^2_{L^2(F)} &\preceq |u-u_h|^2_{kH^1(S_F)} \ + \frac{h^2}{k_F}\,\|f-f_h\|^2_{L^2(S_F)} \\
&\le |u-u_h|^2_{kH^1(S_F)} \ + \sum_{T\subset S_F} \frac{h^2}{k_T}\,\|f-f_h\|^2_{L^2(T)} \quad .
\end{aligned}
\tag{37}
$$

Combing (36) and (37) we obtain the upper bound for $\eta_{R,T}$

$$\eta_{R,T}^2 \preceq |u - u_h|^2_{H^1(T_N)} + \sum_{T' \subset T_N} \frac{h^2}{k_{T'}} \|f - f_h\|^2_{L^2(T')} \quad ,$$

where T_N contains all simplices sharing with T a face. ■

5.3. **Error estimators based on local problems.**

5.3.1. *Error estimators based on local Dirichlet problems.* In the literature one finds estimators which are based on solving local problems of finite dimensions. There the domain Ω is covered by patches of simplices. On each patch one defines a space of bubble functions and solves within this space a local analogon of the continuous problem. The energy norm of the resulting solution is taken as an error estimator. Naturally one is interested in keeping the computation as cheap as possible which means that the patch should consists only of some simplices with few bubble functions. The local problems can be classified into Dirichlet and Neumann problems. See [25] for examples.
We present here an approach with Dirichlet boundary conditions on a patch consisting of two neighboring simplices. For a face $F \subset \Omega \cup \Gamma_N$ let T and T' be simplices sharing the face F. If $F \subset \Gamma_N$ regard only one simplex T with face F. The Galerkin space consists of two element bubble functions $\phi_T, \phi_{T'}$ vanishing outside T resp. T' and one bubble function aligned with the face ϕ_F vanishing outside S_F. Let V_D be the space spanned by three bubble functions $\phi_T, \phi_{T'}, \phi_F$, Such bubble functions are defined in subsection 2.2.
We look for $v_D \in v_D$ such that:

$$(38) \qquad \int_{S_F} k\nabla v_D \nabla\phi = \int_{S_F} f_h\phi + \int_{F\cap\Gamma_N} h\phi - \int_{S_F} k\nabla u_h \nabla\phi \quad , \quad \forall\phi \in V_D \quad .$$

Let $F \subset \overset{\circ}{\Omega}$. The solution v_D can be viewed as a solution of a Dirichlet problem with $v_D|_{\partial S_F} = u_h|_{\partial S_F}$ and with the Galerkin space V_D.
We define the estimator η_D as:

Definition 6. *For a face $F \in F_h(\Omega \cup \Gamma_N)$ we define*

$$\eta_{D,F} := |v_D|_{kH^1(S_F)} \quad .$$

Similar estimators were also proposed in [3].
We show that the residual based estimator η_R and the estimator η_D are equivalent.

Theorem 9. *Let $d = 2, 3$. For each face F not contained in Γ_D and neighboring simplices T, T' we have*

$$\eta_{D,F} \preceq \eta_{R,T} + \eta_{R,T'} \text{ and } \eta_{R,T} \preceq \sum_{F\subset\partial T/\Gamma_D} \eta_{D,F} \quad .$$

PROOF The proof uses technics from [25] developed for the case $k = 1$. We fix a face $F \in F_h/\Gamma_D$ and denote with T, T' neighboring simplices.
Recall the definition of $\left[k\frac{\partial u_h}{\partial n}\right]_F$ and the convention that if $F \subset \Gamma_N$ then S_F contains only one simplex T. Integration by parts yields therefore

$$(39) \qquad \int_F \left[k\frac{\partial u_h}{\partial n}\right]_F \phi = \int_{F\cap\Gamma_N} h\phi - \int_{S_F} k\nabla u_h \nabla\phi \quad , \quad \phi \in V_D \quad .$$

Insertion of v_D in (38) as trial function yields together with (39)

(40)
$$\eta_{D,F}^2 = \int_{S_F} k\nabla v_D \nabla v_D = \int_{S_F} f_h v_D - \int_F \left[k\frac{\partial u_h}{\partial n}\right]_F v_D$$
$$\leq \sum_{T\subset S_F} \|f_h\|_{L^2(T)} \ \|v_D\|_{L^2(T)} + \| \left[k\frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)} \ \|v_D\|_{L^2(F)} =: S_f + S_j \quad .$$

The upper bound for $\eta_{D,F}$ follows in two steps from (40). For the first sum on the last right hand side S_f we have using local equivalences $\|v_D\|_{L^2(T)} \approx h\ |v_D|_{H^1(T)}$

(41)
$$S_f = \sum_{T\subset S_F} \|f_h\|_{L^2(T)} \ \|v_D\|_{L^2(T)} \approx \sum_{T\subset S_F} \|f_h\|_{L^2(T)} \ |v_D|_{H^1(T)}\ h$$
$$= \sum_{T\subset S_F} \frac{h}{k_T^{1/2}} \|f_h\|_{L^2(T)} \ |v_D|_{kH^1(T)} \leq |v_D|_{kH^1(S_F)} \sum_{T\subset S_F} \frac{h}{k_T^{1/2}} \|f_h\|_{L^2(T)}$$
$$\leq \eta_{D,F} \sum_{T\subset S_F} \eta_{R,T} \quad .$$

While for the second sum S_j for $T_F \subset S_F$ again by local equivalences $\|v_D\|_{L^2(F)} \approx h^{1/2}\ |v_D|_{H^1(T)}$

$$S_j = \| \left[k\frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)} \ \|v_D\|_{L^2(F)} \approx \| \left[k\frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)}\ h^{1/2}\ |v_D|_{H^1(T_F)}$$
$$= \| \left[k\frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)} \ \frac{h^{1/2}}{k_{T_F}^{1/2}}\ |v_D|_{kH^1(S_F)} \quad .$$

Since $k_{T_F} \approx k_F$ the last inequality implies

$$S_j \preceq \eta_{R,T} \ \ \eta_{D,F} \tag{42}$$

Collecting inequalities (40),(41) and (42) we obtain the upper bound for $\eta_{D,F}$

$$\eta_{D,F} \leq \eta_{R,T} + \eta_{R,T'} \quad .$$

To obtain a lower bound we make use of certain trial functions:

$$w_f = f_h\phi_T \text{ and } w_j = \left[k\frac{\partial u_h}{\partial n}\right]_F \phi_F \quad .$$

We use w_f as a trial function in (38) and make use of the fact, that w_f vanishes at the boundary of T. Local equivalences $|w_f|_{H^1(T)} \approx h^{-1}\ \|f_h\|_{L^2(T)}$ show:

$$\|f_h\|_{L^2(T)}^2 \approx \int_T f_h w_f = \int_T k_T \nabla v_D \nabla w_f$$
$$\leq k_T\ |v_D|_{H^1(T)} \ |w_f|_{H^1(T)} \approx k_T\ |v_D|_{H^1(T)} \ \|f_h\|_{L^2(T)}\ h^{-1}$$

Thus it follows a lower bound for $\eta_{D,F}$

$$\frac{h}{k_T^{1/2}} \|f_h\|_{L^2(T)} \preceq |v_D|_{kH^1(T)} \quad . \tag{43}$$

In the next step we insert w_j as test function in (38). Denote $j_F := \left[k\frac{\partial u_h}{\partial n}\right]_F$. Observe that in (39) ϕ_F can be replaced by w_j. Using this and the equivalences

$\|w_j\|_{L^2(T)} \approx h^{1/2} \, \| \left[k \frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)}$ and $|w_j|_{H^1(T)} \approx h^{-1/2} \, \| \left[k \frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)}$ gives

$$\begin{aligned}
&\|j_F\|^2_{L^2(F)} \approx \int_F j_F w_j = \int_{S_F} f_h w_j - \int_{S_F} k \nabla v_D \nabla w_j \\
\leq &\sum_{T \subset S_F} \left(\|f_h\|_{L^2(T)} \, \|w_j\|_{L^2(T)} + k_T \, |v_D|_{H^1(T)} \, |w_j|_{H^1(T)} \right) \\
\approx &\sum_{T \subset S_F} \left(\|f_h\|_{L^2(T)} \, \|j_F\|_{L^2(F)} \, h^{1/2} + k_T \, |v_D|_{H^1(T)} \, \|j_F\|_{L^2(F)} \, h^{-1/2} \right) \\
= &\, h^{-1/2} \, \|j_F\|_{L^2(F)} \sum_{T \subset S_F} \left(h \, \|f_h\|_{L^2(T)} + k_T \, |v_D|_{H^1(T)} \right)
\end{aligned}$$

Simplifying the last inequality and using (43) one gets:

$$\begin{aligned}
h^{1/2} \, \| \left[k \frac{\partial u_h}{\partial n}\right]_F \|_{L^2(F)} &\preceq \sum_{T \subset S_F} \left(h \, \|f_h\|_{L^2(T)} + k_T \, |v_D|_{H^1(T)} \right) \\
&\preceq \sum_{T \subset S_F} k_T^{1/2} \, |v_D|_{kH^1(T)} \\
&\leq (k_T + k_{T'})^{1/2} \, |v_D|_{kH^1(S_F)} \quad .
\end{aligned}$$

Finally we conclude

$$\frac{h}{k_T + k_{T'}} \, \| \left[k \frac{\partial u_h}{\partial n}\right]_F \|^2_{L^2(F)} \leq \eta_{D,F} \quad . \tag{44}$$

Combining (43) and (44) we obtain the lower bound for $\eta_{D,F}$, namely

$$\eta_{R,T} \leq \sum_{F \subset \partial T} \eta_{D,F} \quad .$$

■

5.3.2. *Error estimators based on a local Neumann problems.* It may be desirable to construct an estimator based on a local problems with just one simplex as support for the bubble functions. Then one has to impose Neumann boundary conditions. In the case of the Laplace equation one sets the Neumann conditions to $\left[\frac{\partial u_h}{\partial n}\right]$. For varying coefficients it is not straightforward how to impose these boundary conditions in order to keep the resulting estimator robust. One has to scale the boundary terms appropriately.

We propose an estimator with one bubble function w_F per face F_i and one w_T for the simplex as done in the estimator η_N in [25]. The Galerkin space spanned by this functions is denoted by V_N. Let's by k_i denote the value of k on the neighboring simplex which shares with T the face F_i.

We look for $v_N \in v_N$ such that:

(45)

$$\int_T k_T \nabla v_N \nabla \phi = \int_T f_h \phi - \sum_{F_i \subset \partial T / \Gamma_D} \left(\frac{k_T}{k_F}\right)^{1/2} \int_F \left[k \frac{\partial u_h}{\partial n}\right]_F \phi \quad , \quad \forall \phi \in V_N \quad .$$

The estimator η_N will then be defined as:

Definition 7.

$$\eta_{N,T} := |v_N|_{kH^1(T)} \quad .$$

In distinction to local problems with Dirichlet boundary conditions this problem seems not be very natural and there is no intuitive interpretation for this local problem.

Theorem 10. *The estimators $\eta_{R,T}$ and $\eta_{N,T}$ are locally equivalent:*

$$\eta_{N,T}^2 \approx \eta_{R,T}^2 \quad .$$

PROOF The proof is done as the proof of theorem 9. ■

Remark 4. *The additional scaling factor $\left(\frac{k_T}{k_F}\right)^{1/2}$ is necessary for proving robustness. But is it also necessary in numerical experiments ?*
We carried out numerical experiments with an interface problem. The version of η_N without additional scaling revealed a good behaviour as long as the solution of the interface problem was locally not *in $H^2(\Omega_l)$. But for solutions in $H^2(\Omega_l)$ an over refinement occured along the interface. This over refinement is in agreement with the observation that without additional scaling the resulting estimator will be bigger. Thus this scaling is indeed necessary.*

6. ESTIMATORS BASED ON HIERARCHICAL BASES

6.1. **A general approach for estimators based on hierarchical bases.** In distinction from residual based error estimators, where one needs interpolation results for the derivation of upper bounds for the error, there is an alternative approach based on so called hierarchical bases, see [3], [18]. Here the upper bound is shown by the so called saturation assumption.
In [3] one finds an analysis for error estimators based on hierarchical bases. The problem considered there was the similar to (1). But the authors made not clear that in some proofs they really did not need to use global bounds on k, that is the ε from (2). Thus without changing one line of their proofs important theorems in [3] return robust results for diffusion coefficients with large jumps along the element boundaries. For the diffusion coefficient we assume:

$$\forall T \in T_h \ \ \exists k_T : \ \ k_T \ |v|_{H^1(T)}^2 \ \ \approx \ \ |v|_{kH^1(T)}^2 \quad .$$

For completeness we repeat important for us theorems of [3]. (For readers familiar with [3]: we restrict ourself to the case where the algebraic error is zero, that means u_h is known and $\tilde{u} = u_h$. This is done only for brevity.) We restrict ourself to the case of hierarchical bases given by piecewise quadratic finite elements.
Let V_h be as before the space of continuous, piecewise linear functions. For brevity we restrict ourself to the case of homogeneous Dirichlet boundary conditions. Let $Q \subset V$ a subspace of piecewise quadratic finite elements. The hierarchical extension $\mathcal{V}$ is defined by the splitting

$$Q = V_h \oplus \mathcal{V} \quad .$$

We define u_Q as the solution of the variational problem with Galerkin space Q: $u - u_Q \in Q$ with

$$\int_\Omega k \nabla u_Q \nabla v_Q \ = \int_\Omega f v_Q + \int_{\partial\Omega} h v_Q \quad , \quad \forall v_Q \in Q \quad . \tag{46}$$

Exploiting Galerkin orthogonality

$$\int_\Omega k \nabla \left(u_Q - u_h\right) \nabla \left(u_Q - u_h\right) \ = \int_\Omega k \nabla \left(u_Q - u_h\right) \nabla \left(u - u_h\right)$$

one has a lower bound for the error

$$|u_Q - u_h|_{kH^1(\Omega)} \ \le \ |u - u_h|_{kH^1(\Omega)} \quad .$$

On the other hand supposing the

Definition 8. *Saturation assumption*

$$|u - u_Q|_{kH^1(\Omega)} \leq \beta \, |u - u_h|_{kH^1(\Omega)} \quad \textit{for } \beta < 1$$

one has an upper bound as the following theorem states.

Theorem 11. *The saturation assumption is equivalent to*

$$|u - u_h|_{kH^1(\Omega)} \leq \left(1 - \beta^2\right)^{-1/2} \, |u_Q - u_h|_{kH^1(\Omega)} \quad .$$

It is clear that the computation of u_Q would require more effort then the computation of u_h. But it can be shown that is it not necessary to compute u_Q. It suffices to dispose of local projections (orthogonal with respect to the energy scalar product) of the error $u_Q - u_h$, which can be calculated cheaply.
We denote the basis of $\mathcal{V}$ with $\{\psi\}$. Let $v_\psi \in lin\,\{\psi\}$ be the solution of

$$\int_\Omega k\nabla v_\psi \nabla\psi = \int_\Omega f\psi + \int_{\partial\Omega} h\psi - \int_\Omega k\nabla u_h \nabla\psi \quad .$$

One sees that v_ψ is the orthogonal projection (in the energy scalar product) of $u - u_h$ and $u_Q - u_h$ on $lin(\psi)$.
We define now the error estimor.

Definition 9. *Given an hierarchical extension $\mathcal{V}$ we define the error estimator*

$$\eta_{\mathcal{V}}^2 := \sum_{\psi\in\mathcal{V}} \eta_{H,\mathcal{V}}^2 \ \textit{where} \ \eta_{H,\mathcal{V}} := |v_\psi|_{kH^1(\Omega)} \quad .$$

Note that

$$\eta_{H,\mathcal{V}}^2 = \frac{\left(\int_\Omega f\psi + \int_{\partial\Omega} h\psi - \int_\Omega k\nabla u_h \nabla\psi\right)^2}{|\psi|_{kH^1(\Omega)}^2} \quad ,$$

and can therefore be cheaply calculated.
One can now prove that the arguments exploited in [3] carry over also to the case diffusion coefficients with large jumps. Thus one has

Theorem 12. *For the solution u_h of (3) and u_Q of (46) it holds*

$$|u_Q - u_h|_{kH^1(\Omega)}^2 \approx \sum_{\psi\in S} \eta_{H,\mathcal{V}}^2 \quad .$$

Combing theorems 11 and 12 leads to

Lemma 13. *The saturation assumption is equivalent to the upper bound*

$$|u - u_h|_{kH^1(\Omega)}^2 \preceq \sum_{\psi\in S} \eta_{H,\mathcal{V}}^2 \quad .$$

What can be said about the validity of the saturation assumption? Obviously since Q_h is the greater space the inequality $|u - u_Q|_{kH^1(\Omega)} \leq |u - u_h|_{kH^1(\Omega)}$ holds due to Galerkin orthogonality. If $u \in H^{2+s}\Omega$, $s > 0$ one has the approximation inequality

$$|u - u_Q|_{kH^1(\Omega)} \preceq h^{1+s} |u|_{kH^{2+s}(\Omega)}$$

with order $o(h^{1+s})$. Compared with the approximation order for piecewise linear functions of order $o(h^1)$ the saturation assumption would be fulfilled at least asymptotically.
But since we deal with piecewise constant diffusion coefficients such regularity result usually does not hold. In the case $u \in H^{1+\lambda}(\Omega)$ with maximal $\lambda < 2$ the continuous solution will be approximated on uniform meshes by higher order functions

with the *same order* $o(h^\lambda)$ as for linear finite elements, see remark 1 or [10]. Thus we can not expect to control the constant β in the saturation assumption through higher order approximations in such a way.
If one is interested in the saturation assumption only, one can use reliability of the adjoint error estimator to proof the saturation assumption [3]. This can be done if the space $\mathcal{V}$ contains for each simplex $T \in T_h$ a polynomial with support contained T.

Remark 5. *In* [3] *one can find numerical experiments for various hierarchical extension spaces in 3-D. As it turns out the efficiency index, that is the ratio of the estimated error and the error, is closer 1 for bigger hierarchical extension spaces. But a comparison of the error reduction over the number of the nodes shows no differences between different extension spaces. That means on adaptive meshes with similar numbers of nodes the error was similar independend of the type of error estimator used in the adaptive procedure.*

6.2. **Hierarchical bases without the saturation assumption.** In this section we define two estimators which can be interpreted as hierarchical bases estimators. Following directly the lines of the proof from [25] one can extend the Proposition 1.14 an 1.15 from [25] to the case of varying diffusion coefficients.
For a better understanding we rewrite the definition of the estimators from [25] in a more intuitive form as norms of solutions of local problems. For a face $F \in F_h/\Gamma_D$ find $v_F \in lin\{\varphi_F\}$

$$\int_{S_F} k\nabla v_F \nabla\varphi_F = \int_{S_F} f\varphi_F + \int_{F\cap\Gamma_N} h\varphi_F - \int_{S_F} k\nabla u_h \nabla\varphi_F \tag{47}$$

Clearly this is eqvivalent to $v_F = \lambda_F \varphi_F$ where

$$\lambda_F = \left(\int_{S_F} f\varphi_F + \int_{F\cap\Gamma_N} h\varphi_F - \int_{S_F} k\nabla u_h \nabla\varphi_F\right) / \ |\varphi_F|^2_{kH^1(S_F)} \quad .$$

As an estimator we define

$$\eta_{J,F} := |v_F|_{kH^1(S_F)} \quad . \tag{48}$$

Here v_F is the result of an orthogonal projection of $u - u_h$ on $lin(\varphi_F)$.
This estimator seems similar to this based on a local Dirichlet problem in subsection 5.3.1. But here there is no element bubble function involved. For this reason the element residuals $||f||^2_{L^2(S_F)}$ will occur in the upper bound.

Lemma 14. *Let the assumptions of theorem 9 be fulfilled. With* $F \in F_h/\Gamma_D$ *and* $\eta_{J,F}$ *defined in (48)* $\eta_{J,F}$ *is efficient*

$$\eta_{J,F} \preceq |u - u_h|_{kH^1(S_F)}$$

and reliable

$$|u - u_h|^2_{kH^1(S_F)} \preceq \sum_{F\in T_h} c(T_F)^2 \eta^2_{J,F} + \sum_{T\subset T_h} d(T)^2 \frac{h^2}{k_T} ||f||^2_{L^2(T)} \quad ,$$

where $d(T)$ *is the maximum of* $c(T')$ *among all simplices* T' *sharing with* T *a face.*

PROOF The proof is essentially based on an analogical proof in [25] and on the proof of theorem 9. ∎

One can get rid of the element residuals $||f||^2_{L^2(S_F)}$ in Lemma 14 if one includes additional bubble functions. Each of these functions is aligned with a simplex and vanishes outside of this simplex. This was done already in subsection 5.3.1.
Again we rewrite the definition of the estimator. Find $v_T \in lin\{\varphi_T\}$

$$\int_T k\nabla v_T \nabla\varphi_T = \int_T f\varphi_T - \int_T k\nabla u_h \nabla\varphi_T = \int_T f\varphi_T \quad . \tag{49}$$

This is equivalent to $v_T = \lambda_T \varphi_T$ where

$$\lambda_T = \left(\int_T f v_T \right) / \ |\varphi_F|^2_{kH^1(S_F)} \quad .$$

As an estimator we define

$$\eta^2_{H,F} := |v_T|^2_{kH^1(T)} + |v_{T'}|^2_{kH^1(T')} + |v_F|^2_{kH^1(F)} \quad . \tag{50}$$

In this case v_T is the result of an orthogonal projection of $u - u_h$ on $lin(\varphi_T)$. Then we can improve Lemma 14 to

Lemma 15. *Let the assumptions of theorem 9 be fulfilled. Let η_H be defined in (50). Then η_H is efficient*

$$\eta_{H,F} \preceq |u - u_h|_{kH^1(S_F)}$$

and reliable

$$|u - u_h|^2_{kH^1(\Omega)} \preceq \sum_{F \in F_h / \Gamma_N} d(T_F)^2 \eta^2_H + \sum_{F \subset T_h} d(T) \frac{h^2}{k_T} \, ||f - f_h||^2_{L^2(S_F)} \quad ,$$

where $d(T)$ is the maximum of $c(T')$ among all simplices T' sharing with the simplex T a face.

PROOF The proof is done as in theorem 9 and relies on lemma 14. ■

7. Other estimators

7.1. **A Zienkiewicz-Zhu like estimator.** Especially among engineers estimators originating from Zienkiewicz and Zhu are popular. We will discuss an modification of a Zienkiewicz-Zhu estimator for the case of diffusion coefficients with discontinuities.
We describe shortly the Zienkiwicz-Zhu estimator for the Laplace equation and refer to [25] for details. The Zienkiwicz-Zhu estimator projects the gradient ∇u_h in a lumped L^2 scalar product on $G_{u_h} \in (V_h)^d$. The estimator is defined as

$$|G_{u_h} - \nabla u_h|_{H^1(T)} \quad . \tag{51}$$

Using this estimator in an adaptive procedure for elliptic equation with discontinuous diffusion coefficient will fail immediately. This is due to the fact that the above estimator will take large values for simplices at the interface, because there is a jump in the derivatives normal to the interface. This jump is physically all right but application of the Zienkiwicz-Zhu estimator will lead to an overrefinement along the interface.
One heuristic modification of (51) consists in replacing ∇u_h by $k \nabla u_h$. Define $|T| := meas(T)$ and let $|\omega_x|$ be the measure of simplices having x as node. Let $G_{u_h} \in (V_h)^d$ defined by

$$G_{u_h}(x) := \sum_{T \ni x} \frac{|T|}{\omega_x} k_T \nabla u_{h,T} \quad , \quad x \in N_h \quad . \tag{52}$$

An modified indicator could then be defined as

$$\eta^2_{ZZ,T} := \frac{1}{k_T} \, ||G_{u_h} - k_T \nabla u||^2_{L^2(T)} \quad . \tag{53}$$

But this indicator too can't serve as an estimator. Note that all of the so far presented estimators make use of the weighted gradient $k_T \nabla u_h$ but in distinction from η_{ZZ} they use only the normal part of this gradient. This is the reason why also the modified estimator will fail.

We show why this is the case. Let $d = 2$ and the diffusion coefficient be 1 and k_2. Suppose the discrete solution u_h is on a part of the interface not constant, that means the derivative parallel to the interface takes the value $t \neq 0$. For simplicity let the interface be parallel to the x-direction. In this case η_{ZZ} will be at least of order

$$|| \left(\frac{1+k_2}{2} - 1 \right) t ||_{L^2(T)} \approx ||k_2 t||_{L^2(T)} \quad ,$$

for simplices T on that side of the interface, where k takes the value 1. Choosing appropriate boundary data one can even take $u = u_h$, so that this indicator will be big, even if the error is 0.

From this we draw the conclusions that robust extensions of the Zienkiewicz-Zhu estimator should either contain only the derivatives normal to the interface or should ensure that the derivatives parallel to the interface are not weighted by k. In both cases one needs to have a local coordinate system aligned with the interface. Such a system would increase the computational cost for simplices which share only a vertex with the interface.

8. Numerical experiments

8.1. **Error reduction with uniform refinement.** We choose the problem setting in such a way that ψ is the solution if the interface problem from example 1 from section 3 . That is we choose a zero load function $f = 0$ and Dirichlet conditions $g = \psi$. Let $g_h \in V_h$ be an approximation of g on $\partial\Omega$. Then u_h with $u_h - g_h \in V_h$ will be the solution of the respective finite element problem

$$\int_\Omega k(x)\ \nabla u_h \nabla v_h\ dx = 0 \ , \quad \forall v_h \in V_h \quad . \tag{54}$$

We disregard influences from the approximation the boundary conditions.

We solve (54) on uniform meshes which are obtained by refining the former mesh globally once. Although u is known we had to apply quadrature rules to approximate $|u - u_h|_{kH^1(\Omega)}$. We interpolate u on a mesh obtained by three times globally refining the adaptive grid by the piecewise quadratic function u_Q. The resulting approximation $|u_Q - u_h|_{kH^1(\Omega)}$ of $|u - u_h|_{kH^1(\Omega)}$ is denoted with $\widehat{\eta}$.

In the figure 6 we plot the error $\widehat{\eta}$ for different values of $k_2 > 1$ over the number of nodes. On sees that it holds a relation $|u - u_h|_{kH^1(\Omega)} \approx \widehat{\eta} \approx N^{-\mu/2}$ for some μ depending on k. We also carried out calculations for $k_2 < 1$, that is for the case of piecewise H^2 regular solutions. Here the reduction of the error tooks place as expected with order $N^{-1/2}$.

As we can explicitly calculate $\lambda > 0$ such that $u \in H^{1+\nu}, \nu < \lambda$ we can compare the numerical convergence rate with the theoreatically expected convergence rate of $|u - u_h|_{kH^1(\Omega)} \approx h^\lambda \approx N^{-\lambda/2}$. A comparison of μ with λ shows a good accordance off both values. Here we take as μ the reduction of the error in the last refinement step.

k_2	theory λ	numerics μ
0.01	1.0	1.01
0.5	1.0	0.98
2	0.89	0.85
10	0.73	0.75
100	0.67	0.69

The numerical and theoretically predicted convergence rates are similar in both cases, when the singular solution is piecewise in H^2 and when it's not. Thus we confirmed theoretically and by numerical experiments, that the convergence rate is restricted by the piecewise regularity.

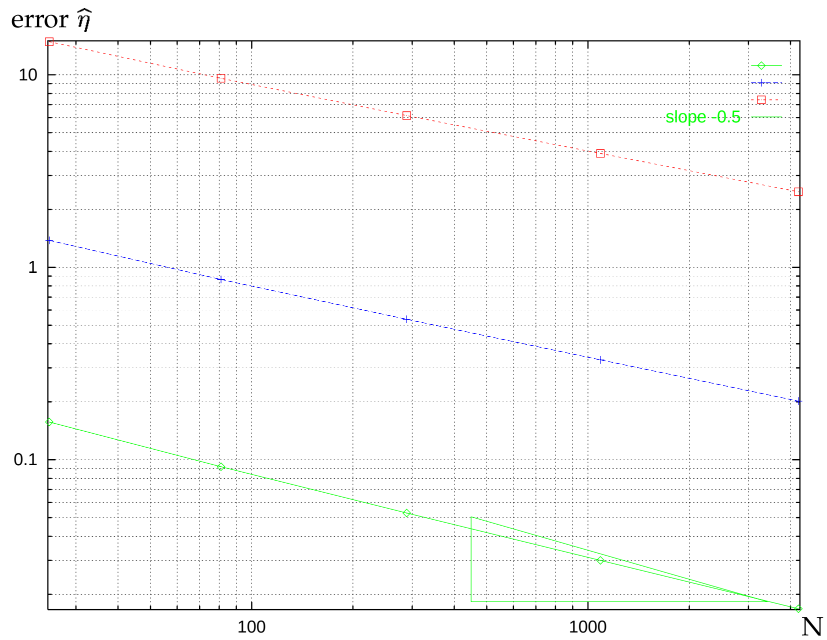

FIGURE 6. reduction of the error $\widehat{\eta}_T$ with uniform refinement, $k_2 \in \{2\diamond, 10+, 100\square\}$, with different rates, triangle has slope -1/2

8.2. **Error estimators and adaptive refinement.** One reason of using error estimators is that one wishes to control the error. For this one has to know that the error estimators is reliable, that is the error is bounded from above by the estimator with some constant:

$$|u - u_h|_{kH^1(\Omega)} \leq C\eta \quad . \tag{55}$$

Naturally one wishes to have the constant C close to 1. On the other hand efficiency of the estimator is desired, that is

$$c\eta \leq \ |u - u_h|_{kH^1(\Omega)} \quad , \tag{56}$$

for some constant c also near 1.
Thus an criterion for judging an estimator would be the efficiency index, that is the ratio

$$|u - u_h|_{kH^1(\Omega)} \,/\eta \quad .$$

Having an efficiency index close to 1 would be highly desirable. But there are examples of error indicators (with no theoretically basis) with efficiency index close to one just by chance. See experiments with a Zienkiewicz-Zhu like estimator. Accordingly the efficiency index can not be the only quality measure of an estimator. Actually one is interested in having relations like (55) and (56) locally. Unfortunately due to the global character of differential operators only local efficiency could be achieved. That means that is *not* possible to bound the error on a simplex from above by local terms. Since all of the so far presented estimators are based on local terms, we can not expect them to reflect the behavior of the error locally. Nevertheless results from [6] for the Laplace equation show that beginning from a certain point (when the mesh is sufficiently fine) and with a certain refinement procedure one can expect a geometrical decrease of the error compared with the refinement levels l, that is a relation error$\approx \alpha^l$.

One good criterion would be the reduction of the error in course of an refinement based on the estimator. For calculating the error one has to know the exact solution or one can try to get an approximation of it calculating a discrete solution on a mesh which is much finer then the adaptive.
Another possibility is to compare this convergence rate with an refinement based on the error itself. That means one calculates the error per simplex and uses this value in course of refinement. One can expect that such an refinement would be in "optimal" in a certain sense.
There are two necessary but not sufficient possibilities. The first would be the "picture norm". If refinement takes place in "relevant" parts this would be a good sign. A second possibility is to look of the reduction of the estimated error. A decrease other then then $O(N^{-1/2})$ indicates that the efficiency index $O(1)$ or a non optimal refinement strategy.
It is clear that one is interested in robust estimators, that is on estimators which behave well for a large class of equations. Therefore one has to carry out experiments with several parameters.
What functions should be chosen to in the numerical examples? As the error reduction for smooth functions (that means piecewise in H^2) is of optimal order already for uniform refinement we see that results obtained by applying mesh refinement based on error estimators results may not be of great importance. Therefore it is more interesting to test error estimators which functions, which are non smooth (not in piecewise H^2). Such functions occur if there a singularities due to changing boundary conditions, parts of the boundaries where the domain is locally non convex or discontinuous diffusion coefficients.
We want to point out, that it is much easier to produce refined meshes which look good then to assure an efficiency index which differs only with a small ε from 1.
In the case of regular solutions, that means if the solution is piecewise in H^2, a reduction of the error with order $O(N^{-1/2})$ can be achieved by uniform refinement. This order is optimal and therefore one can not hope to get better asymptotics. Nevertheless adaptive refinement can provide to lower errors then uniform refinement. See the numerical examples below. Such behaviour can be explained as follows. At the beginning of the refinement procedure one starts with a relatively coarse mesh, where the error is not equally distributed. Provided the estimator can recover local errors, refinement will take place in regions with higher errors. In such a way the overall error will be reduced super optimal, as long as there are simplices which are not yet refined. Then the asymptotic behaviour begins and the reduction of the error takes place with order $O(N^{-1/2})$. But at this point one can have better constants c then for the uniform refinement.

8.3. **Implementation issues.** The whole code is written within the package `pdelib`, which is developed in WIAS (see `http://www.wias-berlin.de/pdelib`) [7]. This package supports the idea of programming in a dimensionless manner. In such a way the error estimators are implemented. Thus they work in 2D as well as on 3D grids.
The adaptive procedure is organized as follows (see [6]). First we calculate the estimators. Then we mark simplices where the estimator takes the largest values. We mark until the sum of the squares of the estimators on marked simplices reaches a certain threshold (here 20%) of the square of estimated overall error.
Marked simplices will be refined by the adaptive kernel of the program `kaskade` from Konrad-Zuse-Zentrum für Informationstechnik Berlin (`http://www.zib.de/`)[1] which is called through to the grid interface of `pdelib`. The refinement is of red type, that means in 2D triangles are subdivided into four similar triangles. After red refinment a green closure is applied, that

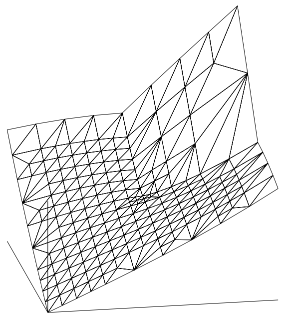

FIGURE 7. $k_2 = 0.01$, refined mesh for a regular solution $u \in H^2(\Omega_i)$, refinement in large areas

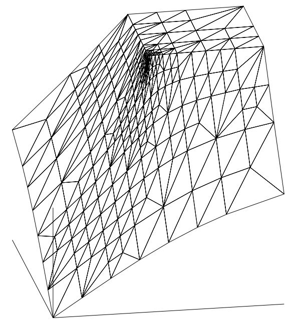

FIGURE 8. $k_2 = 100$, refined mesh for a non regular solution $u \notin H^2(\Omega_i)$, refinement takes place at the singularity

means neighboring triangles are subdivided into two triangles in order to make up a triangulation on which Finite Elements are properly defined. Furthermore in subsequent refinement steps green triangles are unrefined and refined red. This assures that the parameter of the shape regularity for subsequent adapted meshes is approximatly the same as for the initial triangulation.

8.4. **Error reduction rates and efficiency for an example with $H^{3/2}$ regularity.** The problem setting is as in example 1 from section 3 . We use the error estimators η_R, η_D and η_H which are defined in section 5. Again the error $|u - u_h|_{kH^1(\Omega)}$ is approximated by $\widehat{\eta}$, where $\widehat{\eta}$ is calculated on a finer mesh as before.
In the figures 7 and 8 we plotted a refined mesh for the case $k_2 \in \{0.01, 100\}$. The initial mesh is based on a 4×4 tensor grid with 16 squares, each subdivided in the same way into two triangles. In case $k_2 = 100$ we have $u \notin H^2(\Omega_i), i = 1, 2$ and the refinement takes place around the singularity. In the case $k_2 = 0.01$ it yields $u \in H^2(\Omega_i), i = 1, 2$ and the refinement proceeds in the whole domain (although the mesh is finer in the neighborhood of the origin).
In the figure 9 we plot the reduction of the error over the number of nodes for three different estimators for $k_2 = 100$. Our results show that all estimators reduce the error equally well and at least with the optimal convergence rate $O(N^{-1/2})$. The depicted triangle has a slope of $-1/2$,. Additionally we used the refinement based on $\widehat{\eta}$. We see that on relatively coarse meshes the reduction of the error is a little worse for "optimal" refinement. The effect vanishes with finer meshes. For comparison we plotted also the reduction of the error in course of uniform refinement. Here one sees clearly the advantage of the adaptive procedure. The same error is achieved on an adaptively refined mesh with 180 unknowns and an uniformly refined mesh with 1000 unknowns.
Let's take a look at the efficiency index in figure 10. We see that it moderately decreasing on about 50%.
Qualitatively the same results are obtained in the case $k_2 = 0.01$ (see figure 11). Remember that in this case u belongs to $H^2(\Omega_i)$. Again the depicted triangle has a slope of exactly $-1/2$. Thus we have a convergence rate note worse then $O(N^{-1/2})$. The uniform refinement reduces the error with an order not higher as the estimators do. There is only little advantage of the refinement based on the estimators.

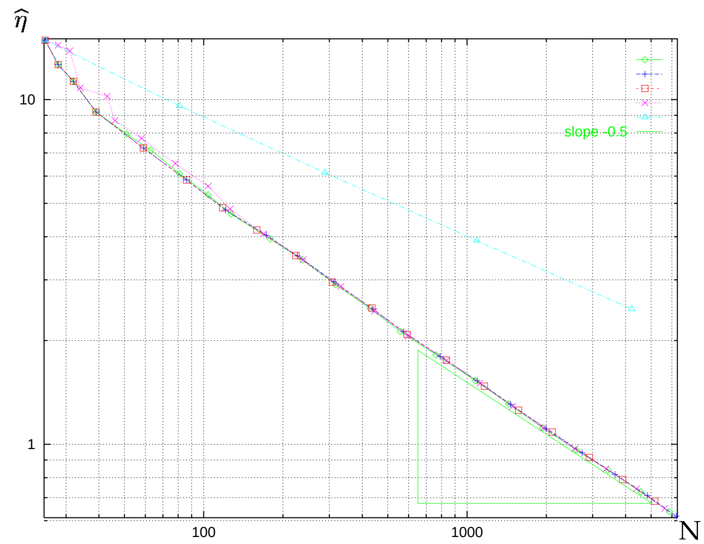

FIGURE 9. $k_2 = 100$, solution $u \in H^2(\Omega_i)$, optimal reduction of the error for refinement with $\eta_R(\diamond), \eta_H(+), \eta_D(\square)$ and $\widehat{\eta}(\times)$ refinement, uniform($\triangle$) refinement for comparision, triangle has slope -0.5

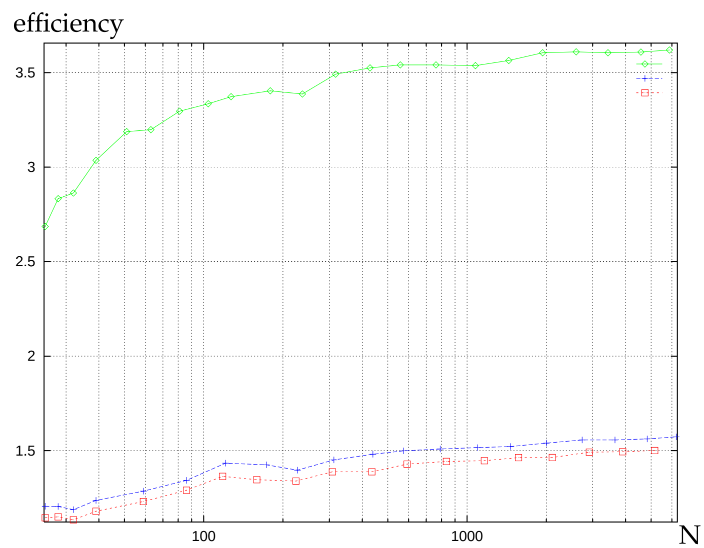

FIGURE 10. efficiency index for $\eta_R(\diamond), \eta_H(+), \eta_D(\square)$, $k_2 = 100$

The same error is achieved on an adaptively refined mesh with 850 and an uniformly refined mesh with approx. 1050 unknowns. It seems that adaptive refinement will be ahead also asymptotically.
The efficiency index is plotted in figure 12. It has smaller variations then in the irregular case (figure 10).

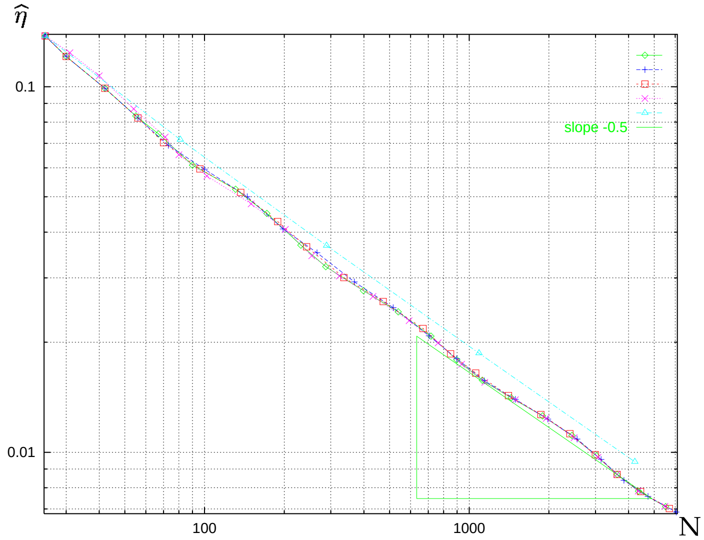

FIGURE 11. optimal reduction of the error for refinement with $\eta_R(\diamond), \eta_H(+), \eta_D(\square)$, $k_2 = 0.01$, solution $u \in H^2(\Omega_i)$, uniform($\triangle$) and optimal ($\times$) refinement, triangle has slope -0.5

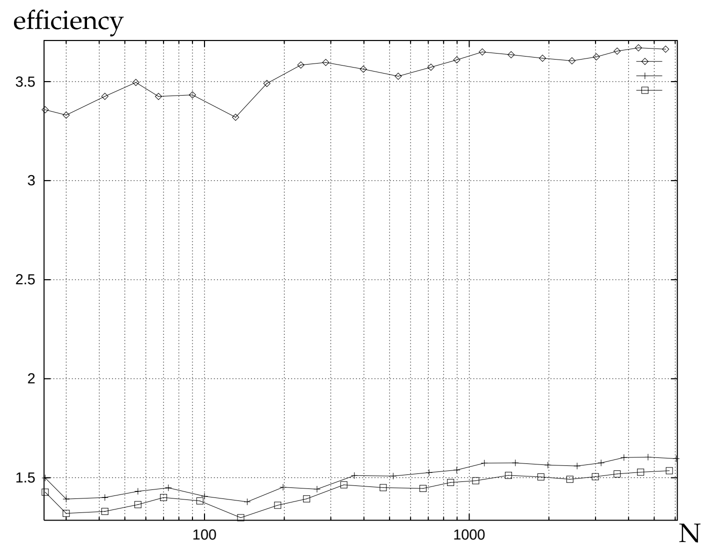

FIGURE 12. nearly constant efficiency index for $\eta_R(\diamond), \eta_H(+), \eta_D(\square)$, and $k_2 = 0.01$

8.5. **Robustness for an example with $H^{3/2}$ regularity.** The problem setting is as in example 1 in section 3. To confirm theoretically proofed robustness we carried out numerical experiments with $k_2 \in \{10^{-5}, 10^{-3}, 10^{-1}, 10^1, 10^3, 10^5\}$. In figure 13 one sees that while refining with η_R the error is reduced uniformly with order $O(N^{-1/2})$ for all values of k_2. For $k_2 \leq 1$ the error takes nearly the same values.

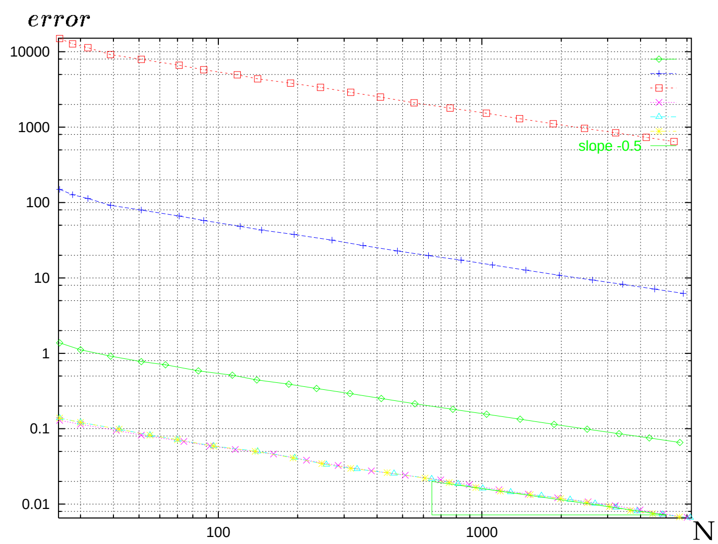

FIGURE 13. optimal reduction of the error for various $k_2 \in \{10^1(\diamond), 10^3(+), 10^5(\square),\ 10^{-1}(\times), 10^{-3}, 10^{-5}\}$, refinement with η_R, for $k_2 < 1$ the reduction has the same behaviour, triangle has slope -1/2

As depicted in figure 14 the efficiency index is not constant but there is only moderate dependency on k_2 and the number of nodes. For the estimators η_D and η_H we have similar results. The reduction of the error takes place also with optimal order $O(N^{-1/2})$ and the efficiency index behaves moderately constant as plotted in figure 15.

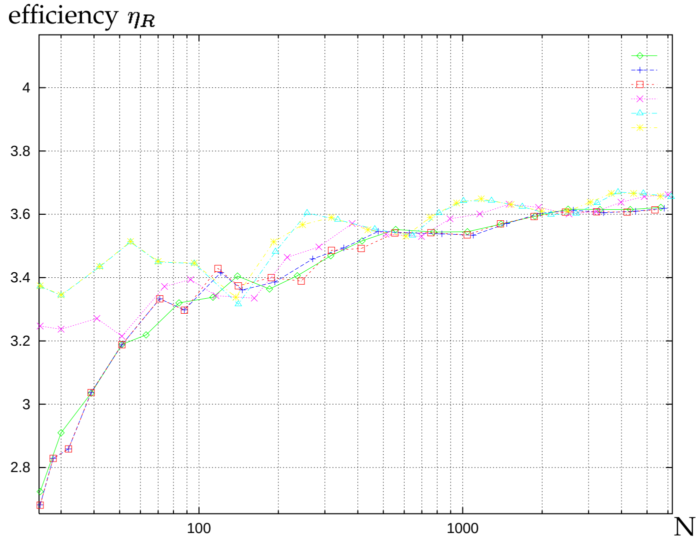

FIGURE 14. efficiency index for η_R and $k_2 \in \{10^1(\diamond), 10^3(+), 10^5(\square),\ 10^{-1}(\times), 10^{-3}(\triangle), 10^{-5}\}$

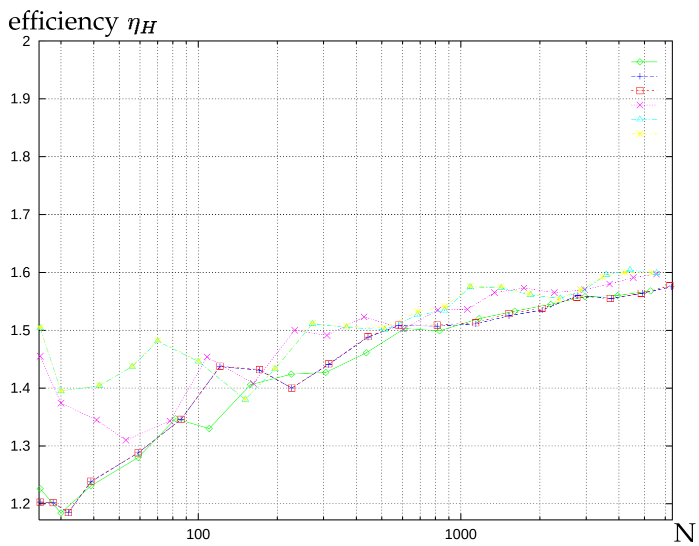

FIGURE 15. efficiency index for η_H and $k_2 \in \{10^1(\diamond), 10^3(+), 10^5(\square),\ 10^{-1}(\times), 10^{-3}(\triangle), 10^{-5}\}$

8.6. **An example with a sharp angled interface.** In this subsection we demonstrate that the derived error estimators are also robust with respect to the geometry of the interface. So far the interface was right angled. We show an example where the interface is an arc with angle about 20 degrees. The solution of the continuous problem is again given by (6) with $\lambda = 0.6$. Its maximal regularity is $H^{1.6-\varepsilon}$. An adapted grid is shown in figure 16. The refinement takes place around the singularity at the origin.

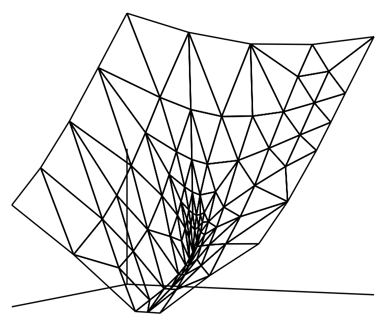

FIGURE 16. adaptive grid, $\lambda = 0.6$, refinement at the singularity at the origin

In figure 17 we plot the reduction of $\widehat{\eta}$. One sees the the error is reduced with optimal order $N^{-1/2}$ for various estimators.
The efficiency shows again a moderat behaviour as depicted in figure 18. This example demonstrates that the derived error estimators are well suited also for non right angled interfaces corners.

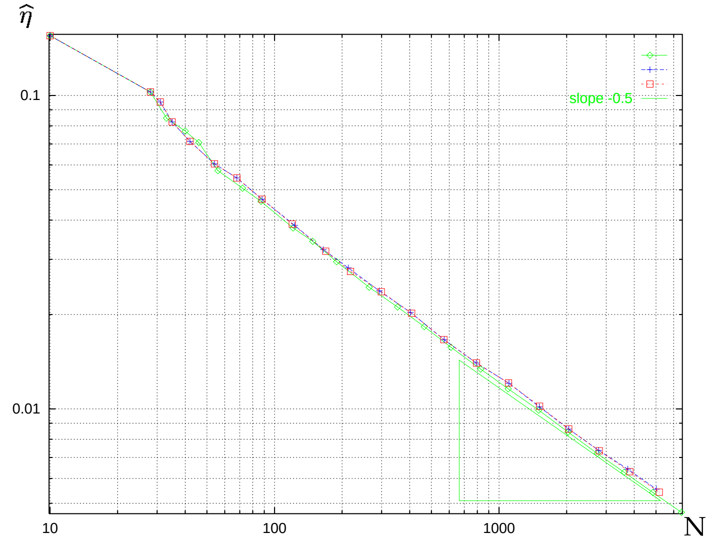

FIGURE 17. $\widehat{\eta}$ over N for refinement with estimators η_R, η_H and η_D, triangle has slope -1/2

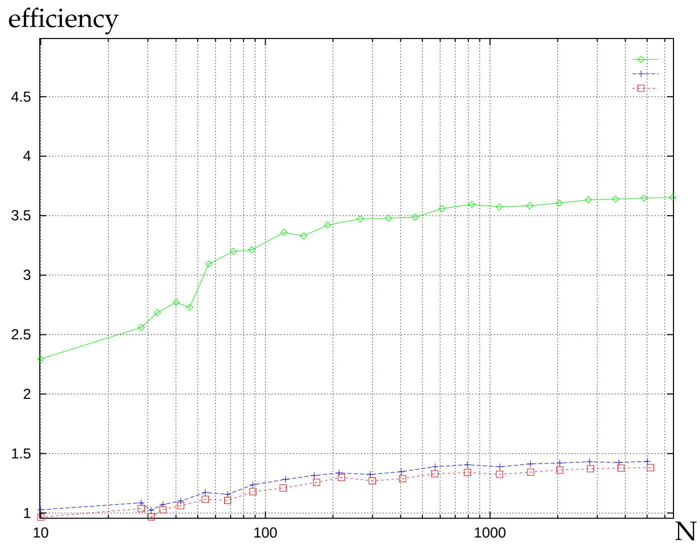

FIGURE 18. efficiency for $\eta_R(\diamond), \eta_H(+)$ and $\eta_D(\square)$

8.7. **Examples with a Zienkiewicz-Zhu type estimator.** As pointed out in subsection 7.1 the proposed error indicator η_{ZZ} will fail to recover the error in the neighborhood of the interface. When applied to the example of the preceding subsection it will lead to an unreasonable refinement along the interface. This is shown

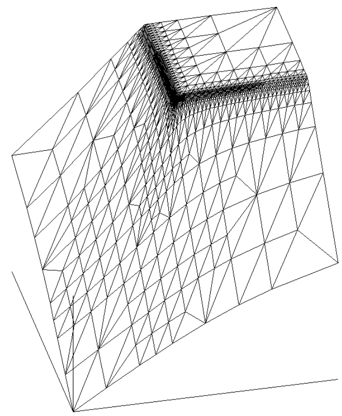

FIGURE 19. adaptive refinement done with η_{ZZ}, over-refinement along the interface

in figure 19. Comparison of the error reduction in an adaptive procedure where either η_{ZZ} or η_D are applied shows (figure 20) that the error is reduced by η_{ZZ} much worse the with η_D. Even though the indicator η_{ZZ} is proven to be bad we want to

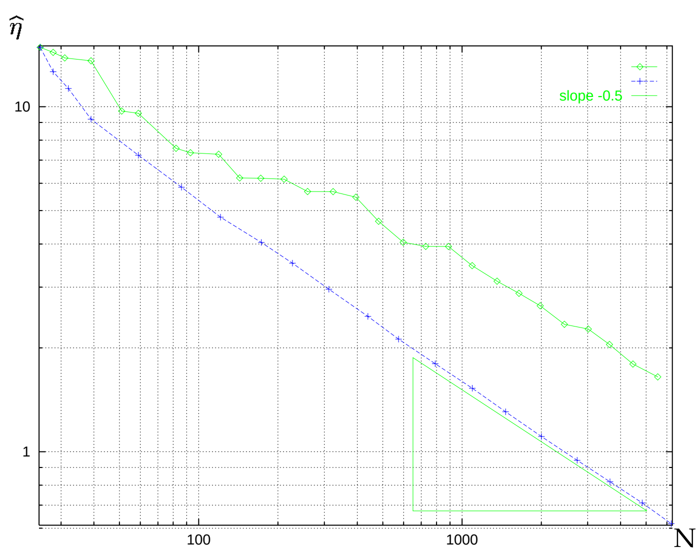

FIGURE 20. error reduction for refinement with $\eta_{ZZ}(\diamond)$ and with $\eta_D(+)$, triangle has slope -1/2

note that its "efficiency" is fairly good (21). This underlines that it is not sufficient to look only for a good efficiency index.

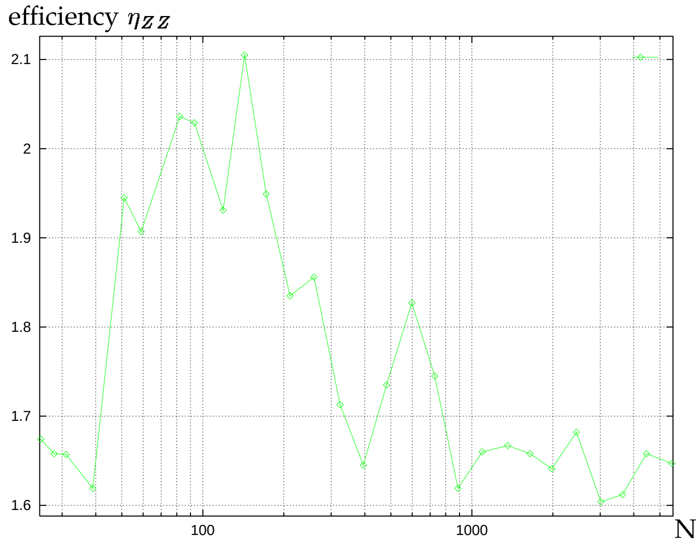

FIGURE 21. efficiency for indicator η_{ZZ}

8.8. **Examples with deteriorate regularity.** The continuous problem is chosen in such a way that the singular function from example 2 from section 3 is the solution. The discrete problem is solved with Dirichlet boundary conditions, such that the discrete solution takes the same values as the continuous solution in the nodes on the boundary (figure 22).

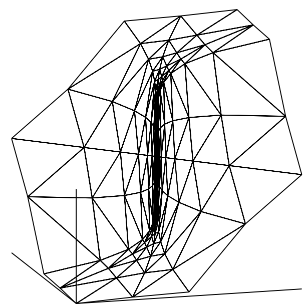

FIGURE 22. $u \in H^{1+\lambda'}(\Omega_i)$, $\lambda' < 0.1$, strong refinement around the singularity

Although the error $u - u_h$ is known it may be hard task to approximate it by quadrature formulas which are exact for piecewise quadratic polynomials only. This is because of the strong singularity for smaller values of λ. Our approach of approximating the energy norm of the error relies on subdividing recursively the adaptive mesh to a reference mesh and on piecewise quadratic interpolation of the solution u on the reference mesh. The interpolate will be used to calculate an approximation of the error $\widehat{T}$. Increasing the number simplices in the reference mesh yields smaller values for $\widehat{\eta}$ but no convergence of $\widehat{\eta}$ was observed. This is in distinction to the case where we have regularity better then $H^{3/2}$. That means that we may have with $\widehat{\eta}$ only a rough approximation of the energy norm of the error. But we have reason to believe that this approximation will be a upper bound since increasing the number of simplices leads to a mononton reduction of $\widehat{\eta}$. In such a way the efficiency would be bounded from below.
The reference grid was constructed as follows: simplices in the neighborhood of the singular point were refined into 4^{10} similar simplices. Other simplices where divided into at least 4^6 similar simplices. This yields meshes with a number of simplices between 10^6 and $6 \cdot 10^7$.
Recall that the solution $u \in H^{1+\varepsilon}(\Omega), 0 \leq \varepsilon < \lambda$ and $u \notin H^{1+\lambda}(\Omega)$. Thus the regularity drops down with decreasing λ. With decreasing regularity also the convergence rate will be smaller, see remark 1. This is confirmed through calculations done one uniform and subsequently globally refined meshes for different values of $\lambda \in \{0.8, 0.5, 0.2, 0.1\}$ as depicted in figure 23 where the error $\widehat{\eta}$ is plotted over the number of unknowns.
In the next table we compare the convergence rate for $\widehat{\eta}$ obtained between the last two levels with approx. 1000 and 4000 unknowns with the predicted asymptotical convergence rate.

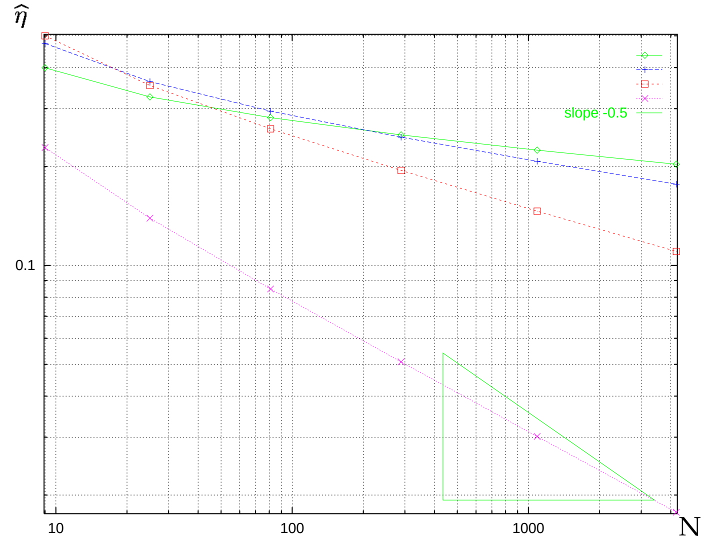

FIGURE 23. error reduction on uniform meshes for $\lambda \in \{\, 0.8(\times),\ 0.4(\square),\ 0.2(+),\ 0.1(\diamond),\ \}$ with decreases with decreasing regularity, triangle has slope -0.5

theory λ	numerics μ
0.8	0.78
0.4	0.42
0.2	0.23
0.1	0.14

The results are in good agreement.

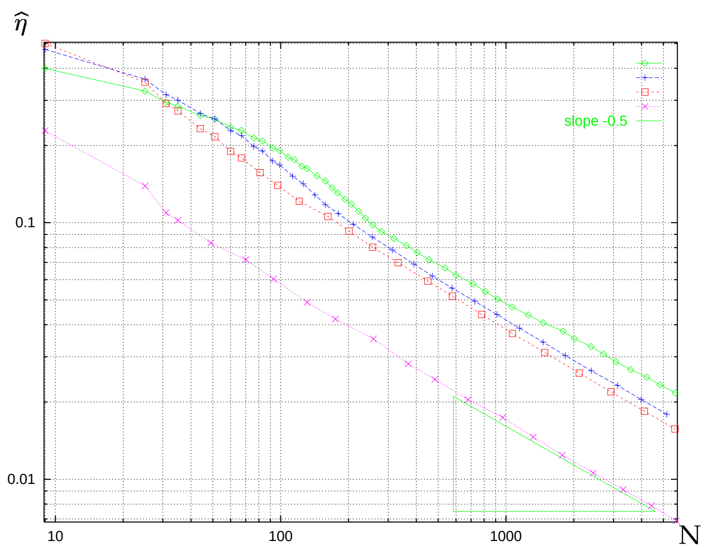

FIGURE 24. error reduction for $\lambda \in \{\, 0.8(\diamond),\ 0.4(+),\ 0.2(\square),\ 0.1(\times)\,\}$ with order $O(N^{-1/2})$, triangle has slope -1/2

In figure 24 we plot the reduction of $\widehat{\eta}$ over the number of unknowns. The approximation of the error $\widehat{\eta}$ is reduced with order $O(N^{-1/2})$ independent of the regularity.

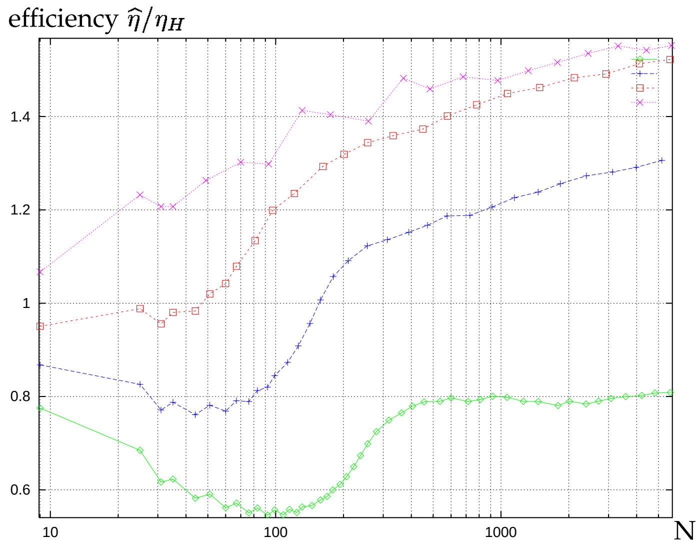

FIGURE 25. efficiency index for uniform and adaptive refinement, $\lambda \in \{\, 0.8(\times),\ 0.4(\square),\ 0.2(+),\ 0.1(\diamond),\ \}$ decreases with decreasing λ

In figure 25 the approximated efficiency index is plotted. We compare the efficiency for various parameters $\lambda \in \{0.8(\diamond), 0.4(+), 0.2(\square), 0.1(\times), 0.05(\triangle)\}$. One sees that the approximated efficiency gets smaller for smaller λ. But we observed that better approximations $\widehat{\eta}$ yield a higher approximated efficiency, so that we have reason to believe that the depicted approximated efficiency is only a lower bound for the efficiency.

In figure 26 we plot the minimal mesh size during the refinement. One sees that the smaller the λ the more the mesh is refined. For $\lambda = 0.1$ and 3000 unknowns the minimal mesh-size is 10^{-10}. For smaller values of λ the refiniment would be much higher. This shows that one reaches quickly boundaries where round off errors and may enter.

Moreover if the solution u is obtained in the context of a physical model the validity of the model could be exceeded by calculating on such fine meshes.

9. CONCLUSIONS, EXTENSIONS AND OUTLOOK

We presented and analyzed a posteriori error estimators for a linear elliptic model problem with possibly large variations of the diffusion coefficient. The error estimators could be shown to be robust reliable and efficient for quasi-monotone distributed diffusion coefficients. In the non quasi-monotone case robustness could not be proofen and it is not clear if holds at all.

We carried out numerical experiments for a large variety of parameters of the problems (1),(3). The obtained adapted meshes took existing singularities into acount and led to an optimal reduction of the error with order $O(N^{-1/2})$. The efficiency index turned out to be moderatly constant and independend on the investigated problems. There were no differences concerning the error reduction between the estimators η_R, η_D and η_H.

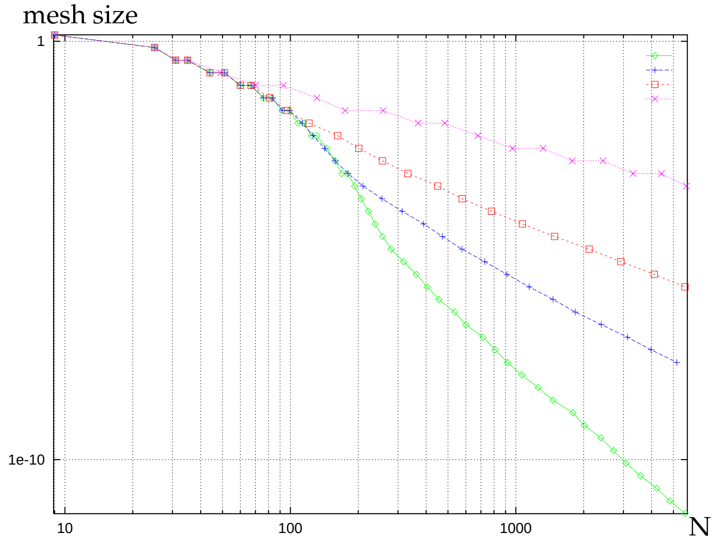

FIGURE 26. minimal mesh size for $\lambda \in \{\, 0.8(\times),\ 0.4(\square),\ 0.2(+),\ 0.1(\diamond),\ \}$, minimal mesh size decreases with decreasing λ

The results obtained in the quasi-monotone case confirmed therefore the theoretically stated properties of the error estimators η_R, η_D and η_H and showed in this manner robust reliability and efficiency as well as optimal error reduction rates together with a moderatly constant efficiency index for a large class of diffusion coefficients.

Similary to the model problem (1) one can analyze error estimators for the case that a mass term is present in the equation

$$\int_\Omega k\nabla u \nabla v + m\ u\ v\ = \int_\Omega fv + \int_{\Gamma_N} hv\ \ ,\ \ \forall v \in V \quad .$$

The residual based error estimator is defined as

$$\eta_{mR,T}^2 := \min\left\{\frac{h^2}{k_T}, m^{-1}\right\}\ ||f_T - mu_h||_{L^2(T)}^2$$
$$+ \sum_{F\subset \partial T/\Gamma_D} \frac{1}{k_F^{1/2}} \min\left\{\frac{h}{k_F^{1/2}}, m^{-1/2}\right\}\ ||\left[k\frac{\partial u_h}{\partial n}\right]_F||_{L^2(F)}^2 \quad .$$

Results analougous to this from theorem 8 hold. The analysis follows that of the present paper and uses the framework of [26].

The case of non quasi-monotone distributed of diffusion coefficients has to be investigated further.

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