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Surface Waves at an Interface Separating a Saturated Porous Medium and a Liquid

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Abstract

Surface waves at an interface between a saturated porous medium and a liquid are investigated. Existence and peculiarities of surface wave propagation are revealed. Three types of surface waves are proven to be possible: true Stonely surface wave propagating almost without dispersion, leaky slow pseudo-Stonely wave, and leaky generalized Rayleigh wave. True Stonely and generalized Rayleigh waves correspond to those waves, which exist at a free interface of a saturated porous medium with vacuum.

1 Introduction

In this paper we proceed to study the effect of fluid-filled porous media on the velocity and attenuation of surface waves, which propagate along the interface separating porous and liquid half-spaces.

Let us recall that at a single component solid-liquid interface there exist two surface waves: a generalized Rayleigh wave and a true Stonely wave (sometimes called the Scholte wave) [1-3]. The phase velocity of generalized Rayleigh wave, which is a system of three waves (one in the liquid and two in the solid), is higher than the wave velocity in the fluid alone. This surface wave radiates energy continuously into the liquid, forming therein an inhomogeneous wave departing from the boundary. Since the energy flows across the interface (leaky wave), the wave attenuates along the propagation direction.

Stonely surface wave consists of an inhomogeneous wave in the liquid and two inhomogeneous waves in the solid and it propagates parallel to the boundary without attenuation. Its velocity is lower than all the bulk velocities in the solid, and in the liquid.

Due to the presence of a second compressional wave in a fluid-saturated porous medium, the properties of surface waves at interfaces of fluid-filled porous solid in contrast to solid-liquid interface are different. Three types of surface waves are proven to be possible. Before we proceed to investigate these waves, it is instructive to compare the approach presented in [1], where wave number k is defined as $k = k(\omega)$, and approach, presented in this paper and [4], where frequency ω is defined as $\omega = \omega(k)$.

Traditionally, for the research of inhomogeneous boundary-value problems (see [1]) frequency ω is choosen to be real and wave number $k = k(\omega)$ is sought as a solution of corresponding dispersion equation and can be complex. It is easy to show that complex roots $k \in C$, $|k| \leq \mathrm{Const}|\omega|^{1+\delta}$, $\delta > 0$, can be represented in terms of complex roots $\tilde{\omega}$, which have been derived in [4] and in Part 4 of this paper, in the following form:

$$k = \frac{\omega}{\tilde{\omega}} + o(1/|\omega|), \quad |\omega| \gg 1 \tag{1.1}$$

Both approaches [1,4] lead to the same structure of solution for isolated waves, namely

$$U^{\pm} = A(y)\exp(i(kx - \omega t)), \tag{1.2}$$

where indices $\pm$ denote half-spaces $y > 0$ and $y < 0$ respectively and an amplitude $A(y) \to 0$ as $|y| \to \infty$. However, a linear function ikx, which can be written as

$$ikx = i\frac{\omega x}{|\tilde{\omega}|^2}\mathrm{Re}\tilde{\omega} + \frac{\omega x}{|\tilde{\omega}|^2}\mathrm{Im}\tilde{\omega} \tag{1.3}$$

increases for $x > 0$ and decreases for $x < 0$ (or vice versa, depending on sign $\mathrm{Im}\tilde{\omega}$). Therefore, surface waves with complex wave number k cannot exist as isolated waves and following [5] in order to get bounded solutions one should consider Fourier integrals with respect to k. The latter means that these surface waves exist only as a result of interaction with the bulk waves.

In contrast to [1], our approach, where k is real and ω has to be defined as a function of k, (see [4] and present paper) allows one in the case of the complex roots of the dispersion equation to consider isolated surface waves without interaction with the bulk waves. Most likely, this approach also allows one to investigate the stability of isolated surface waves and, consequently, to prove the existence of surface waves for nonlinear problems.

2 Mathematical Model and Boundary Conditions

Consider two semi-infinite spaces Ω^I and Ω^{II} having a common interface Γ. Let the region Ω^I is occupied by a saturated porous medium and the region Ω^{II} is occupied by the liquid. In dimensionless variables the set of field equations describing the porous medium has the form ($x \in R^3, t \in [0,T]$) [4,6]:

Mass conservation equations

$$\frac{\partial}{\partial t}\varrho_f + \mathrm{div}(\varrho_f \mathbf{v}_f) = 0,$$

$$\frac{\partial}{\partial t}\varrho_s + \mathrm{div}(\varrho_s \mathbf{v}_s) = 0. \tag{2.1}$$

Here ϱ is the mass density, $\mathbf{v}$ is the velocity vector and indices f and s indicate a fluid or solid phases, respectively.

Momentum conservation equations

$$\varrho_f \left[\frac{\partial}{\partial t} + (v_{fj}, \frac{\partial}{\partial x_j})\right] v_{fi} - \frac{\partial}{\partial x_j} T^f_{ij} + \pi(v_{fi} - v_{si}) = 0,$$

$$\varrho_s \left[\frac{\partial}{\partial t} + (v_{sj}, \frac{\partial}{\partial x_j})\right] v_{si} - \frac{\partial}{\partial x_j} T^s_{ij} - \pi(v_{fi} - v_{si}) = 0. \tag{2.2}$$

Here T^f_{ij} and T^s_{ij} are the stress tensors, π is a positive constant. The stress tensor in the fluid is assumed to be given by the following linear law:

$$T^f_{ij} = -p_f \delta_{ij} - \beta \Delta_m \delta_{ij}, \quad p_f = p_{f0} + \kappa(\varrho_f - \varrho_{f0}), \tag{2.3}$$

where p_f is the partial fluid pressure. p_{f0} and ϱ_{f0} are the initial values of this pressure and fluid mass density, respectively. κ is the constant compressibility coefficient of the fluid depending only on equilibrium value of the porosity m_E. $\Delta_m = m - m_E$ is the change of the porosity. β denotes the coupling coefficient of the components.

The stress tensor in skeleton has the following form:

$$T^s_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} + \beta \Delta_m \delta_{ij}, \tag{2.4}$$

where λ and μ are the Lame constants of the skeleton, which depend only on m_E, and e_{ij} is the strain tensor of small deformations.

Equation for the change of porosity

$$\frac{\partial}{\partial t}\Delta_m + (v_{si}, \frac{\partial}{\partial x_i})\Delta_m + m_E \mathrm{div}(\mathbf{v}_f - \mathbf{v}_s) = -\frac{\Delta_m}{\tau}, \tag{2.5}$$

where τ is the relaxation time of porosity.

For the strain tensor one has:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_{si}}{\partial x_j} + \frac{\partial u_{sj}}{\partial x_i}\right), \tag{2.6}$$

where $\mathbf{u}_s$ is the displacement vector for the solid phase with $\mathbf{v}_s = \partial \mathbf{u}_s / \partial t$.

Basic conservation equations describing the liquid in the region Ω^{II} have the form:

Mass conservation equation

$$\frac{\partial}{\partial t}\varrho^-_f + \mathrm{div}\left(\varrho^-_f \mathbf{v}^-_f\right) = 0. \tag{2.7}$$

Here upper index "$-$" indicates the region Ω^{II}.

Momentum conservation equation

$$\varrho^-_f \left[\frac{\partial}{\partial t} + \left(v^-_{fj}, \frac{\partial}{\partial x_j}\right)\right] v^-_{fi} - \frac{\partial}{\partial x_j} T^-_{ij} = 0. \tag{2.8}$$

Here T^-_{ij} is the stress tensor:

$$T^-_{ij} = -p^-_f \delta_{ij}, \qquad p^-_f = p^-_{f0} + \kappa^- \left(\varrho^-_f - \varrho^-_{f0}\right). \tag{2.9}$$

Liquid is assumed to be compressible with constant compressibility coefficients κ^-. p^-_{f0} and ϱ^-_{f0} are the initial values of pressure and liquid density.

After linearization about equilibrium state with constant values $\varrho^-_f = \varrho^-_{f0}$ and $\mathbf{v}^-_f = 0$, equations (2.7), (2.8) take the following form:

$$\frac{\partial}{\partial t}\varrho^-_f + \varrho^-_{f0}\frac{\partial}{\partial x_i}\left(\frac{\partial u^-_{f_i}}{\partial t}\right) = 0, \tag{2.10}$$

$$\varrho^-_{f0}\frac{\partial^2 u^-_{f_i}}{\partial t^2} + \frac{\partial}{\partial x_j} p^-_f \delta_{ij} = 0, \tag{2.11}$$

where $\mathbf{u}^-_f$ is the displacement vector for the liquid with $\mathbf{v}^-_f = \partial \mathbf{u}^-_f / \partial t$

In what follows we consider 2D-problem of propagation of surface waves along the interface $y = 0$, which separates the porous medium (semi-infinite space $y > 0$) and the liquid (semi-infinite space $y < 0$).

On the interface $y = 0$ the following linearized boundary conditions, which are the consequence of the general conditions [4], have to be satisfied:

a) the continuity of total stresses:

$$\left(T^s_{ij} + T^f_{ij}\right) n_j \Big|^I = T^-_{ij} n_j \Big|^{II}, \tag{2.12}$$

i.e.

$$\left(\frac{\partial u_{s1}}{\partial y} + \frac{\partial u_{s2}}{\partial x}\right)\Big|_{y=0} = 0 \tag{2.13}$$

and

$$\lambda \mathrm{div}\ \mathbf{u}_s + 2\mu \frac{\partial u_{s2}}{\partial y} - \kappa\left(\varrho_f - \varrho_{f0}\right)\Big|_{y=0} = -\kappa^- \left(\varrho^-_f - \varrho^-_{f0}\right)\Big|_{y=0} \tag{2.14}$$

b) the continuity of mass flux across the interface

$$\varrho_{f0}\frac{\partial}{\partial t}\left(u_{f_2} - u_{s2}\right)\Big|_{y=0} = \varrho^-_{f0}\frac{\partial}{\partial t}\left(u^-_{f_2} - u_{s2}\right)\Big|_{y=0} \tag{2.15}$$

c) proportionality between discontinuity in pressure and relative velocity of the fluid with respect to solid phase

$$-\varrho_{f0}\left(v_{f_2} - v_{s2}\right)\Big|_{y=0} = \alpha\left(p_f - m_E p^-_f\right)\Big|_{y=0} \tag{2.16}$$

In order to prove that boundary value problem (2.11)-(2.15) [4],(2.10)-(2.16) has solutions as surface waves, we will investigate the propagation of harmonic wave whose frequency is ω, wave number is k, and amplitude depends on y. It should be noted here that as in [4] we consider the solutions of (2.1)-(2.9) in the absence of external forces which are defined uniquely by Cauchy data. In this case it is natural to derive ω as a function with respect to real wave number $k \in R^1$. Thus, $\mathrm{Re}\omega/k$ defines the phase velocity of the waves, while $\mathrm{Im}\omega$ defines attenuation.

3 Construction of Solution

As in the case of the free interface of the porous medium, solution is sought in the following form:

$$\begin{aligned}
&\mathbf{u}_f = \nabla\varphi_f + \Big((\psi_f)_y, -(\psi_f)_x\Big), \qquad \mathbf{u}_s = \nabla\varphi_s + \Big((\psi_s)_y, -(\psi_s)_x\Big)\\
&\varphi_f = A_f(y)\exp\Big(i(kx-\omega t)\Big), \qquad \varphi_s = A_s(y)\exp\Big(i(kx-\omega t)\Big)\\
&\psi_f = B_f(y)\exp\Big(i(kx-\omega t)\Big), \qquad \psi_s = B_s(y)\exp\Big(i(kx-\omega t)\Big)\\
&\varrho_f - \varrho_{f0} = A_{\varrho,f}(y)\exp\Big(i(kx-\omega t)\Big)\\
&\varrho_s - \varrho_{s0} = A_{\varrho,s}(y)\exp\Big(i(kx-\omega t)\Big)\\
&\Delta_m = A_{\Delta m}(y)\exp\Big(i(kx-\omega t)\Big)
\end{aligned} \tag{3.1}$$

Consequently, solution (3.32) [4] remains to be valid.

For the liquid, occupying region $y < 0$, the solution has the form:

$$\mathbf{u}_f^- = \nabla\varphi_f^-$$

$$\begin{aligned}
\varphi_f^- &= A_f^-(y)\exp\Big(i(kx-\omega t)\Big)\\
\varrho_f^- - \varrho_{f0}^- &= A_{\varrho,f}^-(y)\exp\Big(i(kx-\omega t)\Big)
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (2.10), (2.11) one gets:

$$-\varrho_{f0}^-\omega^2 A_f^- + \kappa^- A_{\varrho,f}^- = 0, \tag{3.3}$$

whence

$$A^-_{\varrho,f} = \left(k^-_f\right)^2 \varrho^-_{f0} A^-_f, \qquad \left(k^-_f\right)^2 = \frac{\omega^2}{\kappa^-} \tag{3.4}$$

and

$$A^-_{\varrho,f} + \varrho^-_{f0}\left(\frac{d^2}{dy^2} - k^2\right) A^-_f = 0. \tag{3.5}$$

Taking into account (3.4), (3.5) has the following bounded solution

$$A^-_f = C^-_1(0)\exp(\gamma^-_1 y), \quad \gamma^-_1 = \sqrt{k^2 - (k^-_f)^2} = |k|\sqrt{1 - \frac{\tilde{\omega}^2}{\kappa^-}}, \tag{3.6}$$

where $\tilde{\omega} = \omega/k$.

In order to derive dispersion relation and define the frequencies of the surface waves, one should substitute solutions (3.32) [4] and (3.6) into boundary conditions (2.13)-(2.16). We proceed to do so.

4 Dispersion Relation

Substituting the solution into boundary conditions for the case $\beta = 0$ and $|k| \gg 1$ one gets the following system of equations with respect to unknown constants $C_1(0),\ C_2(0),\ C_s(0)$ [see 4] and $C^-_1(0)$:

$$(\lambda + 2\mu)(\tilde{\gamma}_2^2 - 1)C_2 + 2\mu C_2 + 2\mu i\tilde{\mu}_s C_s - \tilde{\omega}^2 \varrho_{f0} C_1 = -\tilde{\omega}^2 \varrho^-_{f0} C^-_1, \tag{4.1}$$

$$\tilde{\gamma}_2 C_2 + \frac{i}{2}\left(\tilde{\mu}_s^2 + 1\right) C_s = 0, \tag{4.2}$$

$$-\tilde{\gamma}_1 C_1 + \tilde{\gamma}_2 C_2 + iC_s = \frac{\varrho^-_{f0}}{\varrho_{f0}} \tilde{\gamma}^-_1 C^-_1, \tag{4.3}$$

$$i\left(\tilde{\gamma}^-_1 C^-_1 + \tilde{\gamma}_2 C_2 + iC_s\right) = \alpha\tilde{\omega}\frac{\varrho_{f0}}{\varrho^-_{f0}}\left(C_1 - C^-_1\right), \tag{4.4}$$

where

$$\tilde{\gamma}_1 = \sqrt{1 - \frac{\tilde{\omega}^2}{\kappa}}, \qquad \tilde{\gamma}_2 = \sqrt{1 - \frac{\tilde{\omega}^2}{a_{s1}^2}},$$

$$\tilde{\mu}_s = \sqrt{1 - \frac{\tilde{\omega}^2}{a_{s2}^2}}, \qquad \tilde{\gamma}_1^- = \sqrt{1 - \frac{\tilde{\omega}^2}{\kappa^-}}. \tag{4.5}$$

The condition that the determinant of the system (4.1)-(4.4) must vanish yields the dispersion equation for the definition of frequencies of the surface waves:

$$\left(- \frac{\mu}{\tilde{\gamma}_1 \tilde{\gamma}_2} \mathcal{P}_v + \varrho_{f0}^- \frac{\tilde{\omega}^4}{\tilde{\gamma}_1 a_{s2}^2} \right) \left(\tilde{\gamma}_1^- - i\alpha\tilde{\omega} \left(\frac{\tilde{\gamma}_1^-}{\tilde{\gamma}_1} + m_E \right) \right)$$
$$- \varrho_{f0}^- \left(1 + \frac{\tilde{\gamma}_1^-}{\tilde{\gamma}_1} \right) \frac{\tilde{\omega}^4}{a_{s2}^2} \left(1 - \frac{i}{\tilde{\gamma}_1} \alpha\tilde{\omega}(1 - m_E) \right) = 0. \tag{4.6}$$

Here $\mathcal{P}_v$ is the dispression relation, corresponding to the case of surface waves at a free interface of a porous medium [4]:

$$\mathcal{P}_v = \tilde{\gamma}_1 \mathcal{P}_R + \tilde{\gamma}_2 \frac{\varrho_{f0}}{\varrho_{s0}} \frac{\tilde{\omega}^4}{a_{s2}^4}, \tag{4.7}$$

whereas $\mathcal{P}_R$ is a classical Rayleigh equation [4]:

$$\mathcal{P}_R = \left(2 - \frac{\tilde{\omega}^2}{a_{s2}^2} \right)^2 - 4\tilde{\gamma}_2 \tilde{\mu}_s. \tag{4.8}$$

Obviously, (4.6) includes radicals $\tilde{\gamma}_1, \tilde{\gamma}_1^-, \tilde{\gamma}_2, \tilde{\mu}_s$, which are multi-valued functions. In order to make these function single-valued, consider Riemann surface of $\tilde{\omega}$ with the cuts outgoing from the points $\pm\kappa, \pm\kappa^-, \pm a_{s2}, \pm a_{s1}$. In the following we will consider this Riemann surface, where the signs at radicals on the real axis satisfy radiation condition [1]. The latter means that solutions (3.1) and (3.2) are bounded.

Next consider for simplicity the case when the liquid, saturating a porous medium, and the liquid, occupying half-space $y < 0$, are the same and, consequently, $\kappa^- = \kappa$ as well as $\tilde{\gamma}_1^- = \tilde{\gamma}_1$.

Let either

Condition 1

$$1 > \max Re \left(\frac{\tilde{\omega}^2}{a_{f1}^2}, \frac{\tilde{\omega}^2}{a_{s2}^2}, \frac{\tilde{\omega}^2}{a_{s1}^2} \right) \tag{4.9}$$

and, consequently, $\tilde{\gamma}_1, \tilde{\gamma}_2$ and $\tilde{\mu}_s$ are defined as in (4.5),

or

Condition 2

$$Re\frac{\tilde{\omega}^2}{a_{f1}^2} > 1 > \max Re\left(\frac{\tilde{\omega}^2}{a_{s2}^2}, \frac{\tilde{\omega}^2}{a_{s1}^2}\right), \tag{4.10}$$

hold. Then $\tilde{\gamma}_2$ and $\tilde{\mu}_2$ are defined as above. However

$$\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1} \tag{4.11}$$

for the first strip of the Riemann surface (see Appendix 1 for detailed explanation).

Next we will show that dispersion equation (4.6) has three roots satisfying either (4.9) or (4.10). In what follows we investigate the dependence of the roots of (4.6) on parameters α and ϱ_{f0}^- and consider the correspondence of these roots with those odtained in [4] for the case porous medium-vacuum.

5 Asymptotics of the Roots

First let us prove that there exists a root $\tilde{\omega}_1$ of (4.6) satisfying (4.9), i.e. $\mathrm{Re}\tilde{\omega}_1 \in [0, \sqrt{\kappa})$. The solution is sought in the following form:

$$\tilde{\omega} = \sqrt{\kappa}(1 - c_1\kappa^2 + \cdots). \tag{5.1}$$

Substituting (5.1) into (4.6) one gets:

$$\begin{aligned} -\mu\sqrt{2c_1}\kappa\Bigg[-2\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)\kappa + 4\left(\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)c_1\right)\kappa^3\Bigg] \\ \times\Big(\sqrt{2c_1}\kappa - i\alpha\sqrt{\kappa}(1 - c_1\kappa^2)(1 + m_E)\Big) \\ \frac{\kappa^2(1 - c_1\kappa^2)^4}{a_{s2}^2}\Bigg[(\varrho_{f0}^- - \varrho_{f0})\Big(\sqrt{2c_1}\kappa - i\alpha\sqrt{\kappa}(1 - c_1\kappa^2)(1 + m_E)\Big) \\ -2\varrho_{f0}^-\Big(\sqrt{2c_1}\kappa - i\alpha\sqrt{\kappa}(1 - c_1\kappa^2)(1 - m_E)\Big)\Bigg] + O(\kappa^{13}) = 0. \end{aligned} \tag{5.2}$$

Here we consider outer expansion $\alpha \sim 1$ of the roots with respect to κ. Thus, the coefficient of lowest power of κ

$$-2\mu\sqrt{2c_1}\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s_1}^2}\right)(1 + m_E) + \frac{(1 + m_E)\varrho_{f0} + (1 - 3m_E)\varrho_{f0}^-}{a_{s2}^2}\varrho_{f0}^- = 0 \tag{5.3}$$

and, consequently,

$$\sqrt{2c_1} = \frac{(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^{-}}{2a_{s2}^2\mu\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s_1}^2}\right)(1+m_E)}. \tag{5.4}$$

By virtue of physical reasons,

$$(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^{-} > 0, \tag{5.5}$$

and, consequently, $c_1 > 0$. Therefore

$$\mathrm{Re}\tilde{\omega}_1 = \sqrt{\kappa}\Big(1 - c_1\kappa^2 + O(\kappa^3)\Big) \in [0, \sqrt{\kappa}). \tag{5.6}$$

This phase velocity corresponds to very slow surface wave (true Stonely wave), propagating almost without dispersion. Its speed is less than the velocities of all bulk waves in the porous medium and in the liquid and has order $O(\sqrt{\kappa})$.

Next we will show that dispersion equation (4.6) has also two complex roots, satisfying Condition (4.10). These roots correspond to the localized with respect to y surface waves whose phase velocities are close to $\sqrt{\kappa}$ and a_{s2} respectively.

First of them is sought in the following form:

$$\tilde{\omega} = \sqrt{\kappa}(1 + c_2\kappa + c_3\kappa^{3/2} + \cdots). \tag{5.7}$$

Substitution of (5.7) into (4.6) yields:

$$\begin{aligned}
&-\mu(\sqrt{2c_2}\sqrt{\kappa} + \frac{c_3}{\sqrt{2c_2}}\kappa)\left[-2\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)\kappa + 4c_2\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)\kappa^2\right] \\
&\times\left(-\sqrt{2c_2}\sqrt{\kappa} - \frac{c_3}{\sqrt{2c_2}}\kappa + \alpha\sqrt{\kappa}(1 + c_2\kappa + c_3\kappa^{3/2})(1+m_E)\right) - i\frac{\kappa^2(1 + c_2\kappa + c_3\kappa^{3/2})^4}{a_{s2}^2} \\
&\times\Bigg\{(\varrho_{f0}^{-} - \varrho_{f0})\tilde{\gamma}_2\Big(-\sqrt{2c_2}\sqrt{\kappa} - \frac{c_3}{\sqrt{2c_2}}\kappa + \alpha\sqrt{\kappa}(1 + c_2\kappa + c_3\kappa^{3/2})(1+m_E)\Big) \\
&-2\varrho_{f0}^{-}\tilde{\gamma}_2\Big(-\sqrt{2c_2}\sqrt{\kappa} - \frac{c_3}{\sqrt{2c_2}}\kappa + \alpha\sqrt{\kappa}(1 + c_2\kappa + c_3\kappa^{3/2})(1-m_E)\Big)\Bigg\} + O(\kappa^{11}) = 0.
\end{aligned} \tag{5.8}$$

It should be noted here, that for $\tilde{\gamma}_1$ the following branch was taken: $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1}$.

For the lowest $O(\kappa^2)$ approximation one gets:

$$-\sqrt{2c_2} + \alpha(1+m_E) = 0 \tag{5.9}$$

and, consequently,

$$c_2 = \frac{\alpha^2}{2}(1+m_E)^2 > 0. \tag{5.10}$$

For the next $O(\kappa^{5/2})$ approximation one has:

$$-2\mu\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)c_3 - \frac{i}{a_{s2}^2}\left[-2\bar{\varrho_{f0}}\left(-\sqrt{2c_2} + \alpha(1-m_E)\right)\right] = 0, \tag{5.11}$$

whence

$$c_3 = -i\frac{2}{a_{s2}^2}\frac{\bar{\varrho_{f0}}\alpha m_E}{\mu\left(\frac{1}{a_{s2}^2} - \frac{1}{a_{s1}^2}\right)}. \tag{5.12}$$

Finally, one gets the following expansion for the second root of dispersion relation (4.6):

$$\tilde{\omega}_2 = \sqrt{\kappa}\left(1 + c_2\kappa + c_3\kappa^{3/2} + O(\kappa^2)\right), \tag{5.13}$$

where coefficients c_2 and c_3 are defined above. This root defines slightly dispersive surface wave (pseudo-Stonely wave), whose phase velocity is close but somewhat more than $\sqrt{\kappa}$. This is a leaky wave, thus reradiation of energy occurs across the interface.

As it was mentioned already, dispersion equation (4.6) has one more complex root, satisfying also (4.10). It corresponds to generalized Rayleigh wave with phase velocity $c_{R'} \to c_R$ as $\bar{\varrho_{f0}} \to 0$, where c_R is a velocity of the classical Rayleigh wave in elastic half-space. Taking into account that here we have to choose the following branch $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1}$ and, due to the fact that $c_{R'}$ is close to a_{s2}, $\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1} \approx \frac{\tilde{\omega}}{\sqrt{\kappa}}$, equation (4.6) can be rewritten as:

$$\begin{aligned}\left(\mathcal{P}_R - i\sqrt{\kappa}\tilde{\gamma}_2\frac{\varrho_{f0}}{\varrho_{s0}} + i\sqrt{\kappa}\tilde{\gamma}_2\frac{\bar{\varrho_{f0}}}{\varrho_{s0}}\frac{\tilde{\omega}^3}{a_{s2}^4}\right)\left(1 - \sqrt{\kappa}\alpha(1+m_E)\right)\\ -i2\tilde{\gamma}_2\sqrt{\kappa}\frac{\bar{\varrho_{f0}}}{\varrho_{s0}}\frac{\tilde{\omega}^3}{a_{s2}^4}\left(1 - \sqrt{\kappa}\alpha(1-m_E)\right) = 0.\end{aligned} \tag{5.14}$$

The solution is sought in the following form:

$$\tilde{\omega} = \Omega_0 + \sqrt{\kappa}\Omega_1 + \cdots \tag{5.15}$$

It is easy to see that leading part Ω_0 of expansion (5.15) satisfies the Rayleigh equation: $\mathcal{P}_R(\Omega_0) = 0$, i.e. $\Omega_0 = c_R$. For the next term Ω_1 one gets the following equation:

$$i\sqrt{1-\frac{\Omega_0^2}{a_{s1}^2}}\frac{\varrho_{f0}^- + \varrho_{f0}}{\varrho_{s0}}\frac{\Omega_0^3}{a_{s2}^4}$$

$$-\left[\frac{4}{a_{s2}^4}\Omega_0^3 - \frac{8}{a_{s2}^2}\Omega_0 - 4\frac{d}{d\tilde{\omega}}\left(\sqrt{1-\frac{\tilde{\omega}^2}{a_{s1}^2}}\sqrt{1-\frac{\tilde{\omega}^2}{a_{s2}^2}}\right)\Big|_{\tilde{\omega}=\Omega_0}\right]\Omega_1 = 0. \tag{5.16}$$

Finally, one has:

$$\tilde{\omega}_{R'} = c_R + \sqrt{\kappa}\Omega_1 + O(\kappa) \tag{5.17}$$

where Ω_1 is imaginary and is determined by (5.16). It is easy to estimate that $\mathrm{Im}\Omega_1 > 0$. Thus, solutions (3.1),(3.2) decrease as $y \to \pm\infty$, i.e. indeed we obtained the solution in the form of surface waves. It should be noted that complex roots $\tilde{\omega}_2$ and $\tilde{\omega}_{R'}$ lie at the first strip of the Riemann surface, while $\tilde{\omega}_1$ lies at the upper (second) strip of the Riemann surface. Moreover, if $\alpha \to 0$ and $\varrho_{f0}^- \to 0$ (limit passage to the vacuum), the roots $\tilde{\omega}_1$ and $\tilde{\omega}_{R'}$ continuously pass to the corresponding roots deriveded in [4] for the case of a free interface of a saturated porous medium. Simultaneously, the root $\tilde{\omega}_2$ tends to $\sqrt{\kappa}$, i.e. this surface wave is transformed into the bulk compressional wave of the second kind (see Appendix 2).

Because of two-parametrical dependence of dispersion equation (4.6) on α and ϱ_{f0}^-, bifurcation of its roots may happen. That is why we have to consider asymptotics of the roots with respect to "main" parameter α (see Appendix 2).

6 Conclusions

The results presented in the paper concern surface waves which propagate on an interface separating a saturated porous media and a liquid. The present research reveals new features of surface waves in porous media in comparison with the case of liquid/elastic medium interface. In contrast to this classical case, where two surface wave exist, namely true Stonely wave and generalized Rayleigh wave, in porous materials three types of surface waves are proven to be possible. They are due to the combination of four waves: three waves in porous medium and one wave in liquid.

The first mode is a true Stonely wave, which, as shown in (5.6), propagates almost without dispersion. Asymptotic analysis showed that its velocity is less than the velocities of all bulk waves in unbounded porous medium and in a liquid and is influenced primarily by the compressibility coefficient of the liquid phase. If $\alpha \to 0$ and $\varrho_{f0}^- \to 0$ (limit passage to the vacuum), this wave passes to the analogous one at a free interface of a porous medium.

The second type of surface waves are leaky pseudo-Stonely waves. Its velocity is close but somewhat more than velocity of the lowest logitudinal wave of the second kind and is defined, as for the true Stonely wave, by the compressibility coefficient of the liquid phase. If $\alpha \to 0$, than this wave is degenerated to the bulk logitudinal wave of the second kind.

The third mode is a leaky generalized Rayleigh wave since its phase velocity is close to the velocity of the classical Rayleigh wave. In typical case of an interface between liquid and elastic half-spaces a generalized Rayleigh wave is carried mostly by the elastic half-space and radiates some of its energy into the liquid. In porous materials most likely the main part of its energy is absorbed by a slow compressional wave. However, this statement has solely a physical nature and could not yet be proven. If $\varrho_{f0}^{-} \to 0$, this wave is transformed to the generalized Rayleigh wave at a free interface of a porous medium.

Leaky modes are the intermediate waves between surfaces waves and bulk waves. It is obvious that due to energy radiation into the bulk of the medium, they can exist only in the limited domain (localized waves).

Appendix 1

Here we expalin in detail how to choose the sign at the radical $\tilde{\gamma}_1$ for the upper (second) and first strips of the Riemann surface.

Locally, in the small neighborhood of $\tilde{\omega} = \sqrt{\kappa}$ Riemann surface consists of two strips. Any point at this surface $\tilde{\omega} = \sqrt{\kappa}(1+z)$ can be described by the pair $(\varrho, \varphi(mod 4\pi))$, $\varrho \in (0, \delta], \varphi \in R^1$, where $z = \varrho \exp(i\varphi)(1+o(\kappa))$, $0 < \varrho < \sqrt{\kappa}$, $-\frac{3}{2}\pi \le \varphi < \frac{1}{2}\pi$ for the first stript and $\frac{1}{2}\pi \le \varphi < \frac{5}{2}\pi$ for the second strip. Then

$$\sqrt{1 - \frac{\omega^2}{\kappa}} = \sqrt{-2\varrho \exp(i\varphi)(1+o(\kappa))} = i \exp(\frac{1}{2} i\varphi) \sqrt{2\varrho(1+o(\kappa))}$$

for the first strip $-\frac{3}{2}\pi \le \varphi < \frac{1}{2}\pi$ and

$$\sqrt{1 - \frac{\omega^2}{\kappa}} = \sqrt{-2\varrho \exp(i\varphi)(1+o(\kappa))} = i \exp(\frac{1}{2} i(2\pi + \varphi)) \sqrt{2\varrho(1+o(\kappa))}$$
$$= -i \exp(\frac{1}{2} i\varphi) \sqrt{2\varrho(1+o(\kappa))}$$

for the second strip $-\frac{3}{2}\pi \le \varphi < \frac{1}{2}\pi$ (this strip is called upper in [1]).

Appendix 2

True Stonely and generalized Rayleigh waves

Let us prove that for the case $\alpha \to 0$, $\varrho_{f0}^{-} \to 0$ (limit passage to the vacuum) the roots $\tilde{\omega}_1$ and $\tilde{\omega}_{R'}$ of dispersion equation (4.6) continuously pass to the roots $\tilde{\omega}_1$ and $\tilde{\omega}_{R'}$ of dispersion relation derived in [4] for the free interface of a porous medium.

Regarding generalized Rayleigh wave this statement simply follows from (5.16) if $\varrho_{f0}^{-} \to 0$. Than (5.17) coincides with (4.26) [4].

In order to prove this statement for true Stonely wave, we have to consider inner expansion of the roots with respect to κ, i.e. let $\alpha = \alpha_0\sqrt{\kappa}, \quad \alpha_0 \geq 0$. Then for ω_1 (see (5.1)) dispersion relation (4.6) takes the form:

$$-\frac{1}{a_{s2}^2}\Big\{2\mu(1-\nu)(\sqrt{2c_1})^2 - \sqrt{2c_1}(\varrho_{f0} + \varrho_{f0}^{-} + 2i\alpha_0\mu(1+m_E)(1-\nu))$$
$$+i\alpha_0\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^{-}\Big)\Big\}\kappa^3 + O(\kappa^{7/2}), \qquad (A2.1)$$

where $\nu = a_{s2}^2/a_{s1}^2$. From the leading part of (A2.1) it follows that $\sqrt{2c_1} = Z$ satisfies the equation

$$2\mu(1-\nu)Z^2 - \Big(\varrho_{f0} + \varrho_{f0}^{-} + 2i\alpha_0\mu(1+m_E)(1-\nu)\Big)Z$$
$$+i\alpha_0\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^{-}\Big) = 0. \qquad (A2.2)$$

Since $\mathrm{Re}c_1$ should be positive, we look for the roots such that $\mathrm{Re}Z > 0$ and $\mathrm{Re}Z^2 > 0$.

For $\alpha_0 = \varrho_{f0}^{-} = 0$ one has:

$$Z\Big(Z - \frac{\varrho_{f0}}{2\mu(1-\nu)}\Big) = 0 \qquad (A2.3)$$

and, consequently, $Z = 0$ or

$$Z = \frac{\varrho_{f0}}{2\mu(1-\nu)} \quad \text{and} \quad c_1 = \frac{\varrho_{f0}^2}{8\mu^2(1-\nu)^2}. \qquad (A2.4)$$

Thus, (A2.4) defines the first approximation of

$$\tilde{\omega}_1^{vac} = \sqrt{\kappa}(1 - c_1\kappa^2 + ...) \qquad (A2.5)$$

for the case porous medium-vacuum (see (4.22) [4]). Obviously, (A2.3) has the unique root such that $\mathrm{Re}Z > 0$ and $\mathrm{Re}Z^2 > 0$.

Now we will prove that for $\alpha_0 \sim 0$ equation (A2.2) has two roots $\hat{Z}^+$ and $\hat{Z}^-$, satisfying conditions $\mathrm{Re}\hat{Z}^{\pm} > 0$ and $\mathrm{Re}(\hat{Z}^+)^2 > 0$ and $\mathrm{Re}(\hat{Z}^-)^2 < 0$. Let

$$\hat{Z}^{\pm} = a_0^{\pm} + \alpha_0 a_1^{\pm} + \alpha_0^2 a_2^{\pm} + ... \qquad (A2.6)$$

Substituting (A2.6) into (A2.2) one gets:

$$a_0^+ = \frac{\varrho_{f0} + \varrho_{f0}^-}{2\mu(1-\nu)}, \quad a_1^+ = i\frac{4m_E\varrho_{f0}^-}{\varrho_{f0} + \varrho_{f0}^-}, \tag{A2.7}$$

i.e. $\mathrm{Re}\hat{Z}^+ > 0$ and $\mathrm{Re}(\hat{Z}^+)^2 > 0$, and

$$a_0^- = 0, \quad a_1^- = i\frac{(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-}{\varrho_{f0} + \varrho_{f0}^-},$$

$$a_2^- = \frac{8\mu(1-\nu)\,m_E}{(\varrho_{f0} + \varrho_{f0}^-)^3}\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big). \tag{A2.8}$$

Using (5.5) one gets

$$(\hat{Z}^-)^2 \sim -\frac{\Big[(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big]^2}{(\varrho_{f0} + \varrho_{f0}^-)^2}\alpha_0^2 + O(\alpha_0^3) \tag{A2.9}$$

and $\mathrm{Re}\hat{Z}^- > 0$ and $\mathrm{Re}(\hat{Z}^-)^2 < 0$.

Lemma 1 *The roots $Z(\alpha_0, \varrho_0^-)$ of equation (A2.2) for any parameters $\alpha_0 > 0$, $\varrho_0^- \geq 0$ do not belong to real and imaginary axes of complex plane.*

Proof. Let us show that for any α_0, ϱ_0^- roots Z do not intersect imaginary axis. Set

$$Z = iQ, \;\; Q = Q(\alpha_0, \varrho_0^-) \in \; R^1. \tag{A2.10}$$

Substitution of (A2.10) into (A2.2) yields

$$-2\mu(1-\nu)Q^2 + 2\alpha_0\mu(1+m_E)(1-\nu)Q$$
$$= i\Big[(\varrho_{f0} + \varrho_{f0}^-)Q + \alpha_0((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-)\Big] \tag{A2.11}$$

Equation (A2.11) has real solution, if Q satisfies the following system:

$$\begin{cases} (\varrho_{f0} + \varrho_{f0}^-)Q + \alpha_0((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-) & = \;\; 0 \\ \alpha_0(1+m_E) & = \;\; Q, \end{cases} \tag{A2.12}$$

i.e

$$(1+m_E)\varrho_{f0} + (1-m_E)\varrho_{f0}^- = 0. \tag{A2.13}$$

Since (A2.13) never holds, equation (A2.11) has no real roots.

Analogously, let us show also that for any α_0, ϱ_0^- roots Z do not intersect real axis. Setting $Z = Q_1, \;\; Q_1 \in R^1$, one gets:

$$2\mu(1-\nu)Q_1^2 - (\varrho_{f0} + \varrho_{f0}^-)Q_1$$

$$= i\alpha_0\Big[(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^- + 2\mu(1-\nu)(1+m_E)Q_1\Big] \quad (A2.14)$$

and (A2.14) has a real root under the same condition (A2.13). As we have mentioned already, (A2.13) never holds. Thus (A2.14) has no real roots.

Finally we have that equation (A2.2) has two roots $Z^{\pm}(\alpha_0, \varrho_{f0}^-)$ such that for any $\alpha_0 > 0, \varrho_0^- > 0$ they lie in the first quater of complex plane.

Let us denote as Z^+ the root such that $\mathrm{Re}(Z^+)^2 \geq 0$ (and Z^- with $\mathrm{Re}(Z^-)^2 < 0$). From (A2.2) one gets that

$$\mathrm{Re}(Z^+)^2 = \frac{1}{2\mu(1-\nu)}\Big[(\varrho_{f0} + \varrho_{f0}^-)\mathrm{Re}Z^+ - 2\alpha_0\mu(1+m_E)(1-\nu)\mathrm{Im}Z^+\Big] > 0 \quad (A2.15)$$

if

$$\mathrm{Re}Z^+ > \frac{2\alpha_0\mu(1+m_E)(1-\nu)}{\varrho_{f0} + \varrho_{f0}^-}\mathrm{Im}Z^+. \qquad (A2.16)$$

Obviously, (A2.16) is always true for $\alpha_0 \sim 0$, that is confirmed by the root $\hat{Z}^+$ (see (A2.7)).

Next we have to consider the behaviour of roots of (A2.2) for the case $\alpha_0 \to \infty$. Let $Z = \alpha_0 W$. Then

$$2\mu(1-\nu)W^2 - \Big[\frac{\varrho_{f0} + \varrho_{f0}^-}{\alpha_0} + 2i\mu(1+m_E)(1-\nu)\Big]W$$

$$= -\frac{i}{\alpha_0}\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big). \qquad (A2.17)$$

Substituting

$$W = b_0 + \frac{b_1}{\alpha_0} + \frac{b_2}{\alpha_0^2} + ... \qquad (A2.18)$$

into (A2.17) one gets two roots $\tilde{Z}^+$ and $\tilde{Z}^-$ with

$$b_0^+ = 0, \;\; b_1^+ = \frac{((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-)}{2\mu(1+m_E)(1-\nu)},$$

$$b_2^+ = \frac{2i\varrho_{f0}^- b_1^+}{\mu(1+m_E)^2(1-\nu)} \qquad (A2.19)$$

and

$$b_0^- = i(1+m_E),\ b_1^- = \frac{\varrho_{f0}^-}{\mu(1+m_E)(1-\nu)}. \tag{A2.20}$$

Due to (5.5) $\tilde{Z}^\pm > 0$, and, obviously, $\mathrm{Re}(\tilde{Z}^+)^2 > 0$ and $\mathrm{Re}(\tilde{Z}^-)^2 < 0$. Thus, it was shown that, as for the case $\alpha_0 \sim 0$, if $\alpha_0 \to \infty$, $\varrho_{f0}^- > 0$ there exist two roots $\tilde{Z}^\pm(\alpha_0, \varrho_{f0}^-)$ of equation (A2.2) with $\mathrm{Re}\tilde{Z}^+ > 0$, $\mathrm{Re}(\tilde{Z}^+)^2 > 0$, and $\mathrm{Re}(\tilde{Z}^-)^2 < 0$. Moreover, as it is clear from (A2.19) and (5.4), asymptotics for the case $\alpha = \alpha_0\sqrt{\kappa}$, $\alpha_0 \to \infty$ coincides with asymptotics for the case $\alpha \sim 1$ if $\alpha_0 = \alpha/\sqrt{\kappa}$.

It is interesting to note, that $\mathrm{Re}(\hat{Z}^+)^2 > \mathrm{Re}(\tilde{Z}^+)^2$. This means that if α increases, than surface wave propagates faster.

Next we will investigate the behaviour of the roots $Z^\pm$ of equation (A2.2) for any $\alpha_0 > 0$, $\varrho_{f0}^- > 0$ and will show that roots $\hat{Z}^\pm(\alpha_0, \varrho_{f0}^-)$ and $\tilde{Z}^\pm(\alpha_0, \varrho_{f0}^-)$ are continuously connected.

Lemma 2 *For any $\alpha_0 > 0$, $\varrho_{f0}^- > 0$ from domain $\Omega_\gamma = R^{++}\backslash\gamma$, where*

$$R^{++} = \{\alpha_0 > 0,\ \varrho_{f0}^- > 0\},$$

$$\gamma = \{\alpha_0,\ \varrho_{f0}^-;\ \varrho_{f0}^- \geq (\varrho_{f0}^-)^*,\ \alpha_0 = \alpha_0^*(\varrho_{f0}^-)\}, \tag{A2.21}$$

the roots $Z^\pm(\alpha_0,\ \varrho_{f0}^-)$ depend continuously and uniquely on parameters α_0, ϱ_{f0}^- so that $Re(Z^+)^2 > 0$ and $Re(Z^-)^2 < 0$. Moreover, at the upper and lower sides of the cut γ the root Z^+ equals to $Q^\pm(1+i)$, respectively, where

$$Q^\pm = \frac{1}{4\mu(1-\nu)}\Big(\varrho_{f0} + \varrho_{f0}^- \pm \sqrt{\frac{\varrho_{f0} + \varrho_{f0}^-}{1+m_E}}\sqrt{-(1+m_E)\varrho_{f0} - (1-7m_E)\varrho_{f0}^-}\Big) \tag{A2.22}$$

Proof. Let us assume that

$$Z = Q(1+i),\ Q \in R^1,\ Q > 0, \tag{A2.23}$$

i.e. the roots of (A2.2) lie at the main diagonal of complex plane. Substiuting (A2.23) into (A2.2) one gets the following equation system with respect to Q:

$$\left\{\begin{array}{lcl} -(\varrho_{f0} + \varrho_{f0}^-)Q + 2\alpha_0\mu(1+m_E)(1-\nu)Q & = & 0 \\ 4\mu(1-\nu)Q^2 - \Big(\varrho_{f0} + \varrho_{f0}^- + 2\alpha_0\mu(1+m_E)(1-\nu)\Big)Q & & \\ \qquad +\alpha_0\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big) & = & Q, \end{array}\right. \tag{A2.24}$$

It is obvious that if $\alpha_0 \neq \alpha_0^*(\varrho_{f0}^-)$, where

$$\alpha_0^*(\varrho_{f0}^-) = \frac{\varrho_{f0} + \varrho_{f0}^-}{4\mu(1-\nu)(1+m_E)}, \tag{A2.25}$$

than system (A2.24) has no roots. Otherwise, $Q^\pm$ is defined as in (A2.22) and $Q^\pm$ will be real, if $\varrho_{f0}^- \geq (\varrho_{f0}^-)^*$, where

$$(\varrho_{f0}^-)^* = \frac{(1+m_E)\varrho_{f0}}{7m_E - 1}. \tag{A2.26}$$

It is easy to estimate that for the real materials with porosity of order 0.2-0.5, $(\varrho_{f0}^-)^* > 0$.

Next we will show that roots $Z^\pm$ intersect the main diagonal of complex plain only at the points $Q^\pm(1+i)$ respectively. Moreover, with the change of parameter α_0 the root $Z^+(\alpha_0,\ \varrho_{f0}^-)$, $\varrho_{f0}^- > (\varrho_{f0}^-)^*$, passes at the point $Q^+(1+i) = Z^+(\alpha_0^*(\varrho_{f0}^-),\ \varrho_{f0}^-)$ to the root $Z^-(\alpha_0,\ \varrho_{f0}^-)$, $\alpha_0 > \alpha_0^*$, lying above the main diagonal of the complex plain. Simultaneously, the root $Z^-(\alpha_0,\ \varrho_{f0}^-)$ at the point $Q^-(1+i) = Z^-(\alpha_0^*(\varrho_{f0}^-),\ \varrho_{f0}^-)$ passes to the root $Z^+(\alpha_0,\ \varrho_{f0}^-)$, $\alpha_0 > \alpha_0^*$, lying below the main diagonal.

As it is clear from (A2.24) for any $\varrho_{f0}^- > 0$, $\alpha_0 \neq \alpha_0^*$, the roots $Z^\pm(\alpha_0,\ \varrho_{f0}^-)$ of (A2.2) do not intersect the main diagonal of the complex plain (Fig.1) and $\mathrm{Re}(Z^+)^2 > 0$, $\mathrm{Re}(Z^-)^2 < 0$.

If $\alpha_0 = \alpha_0^*$ and $\varrho_{f0}^- = (\varrho_{f0}^-)^*$, than equation (A2.2) has a multiple root Z, such that

$$Z = Z^+(\alpha_0^*,\ (\varrho_{f0}^-)^*) = Z^-(\alpha_0^*,\ (\varrho_{f0}^-)^*) = Q(1+i), \tag{A2.27}$$

$$Q = \frac{\varrho_{f0} + (\varrho_{f0}^-)^*}{4\mu(1-\nu)}$$

and $\mathrm{Re}Z^2 = 0$ (i.e. Z belongs to main diagonal of the complex plain, Fig.2).

If $\alpha_0 = \alpha_0^*(\varrho_{f0}^-)$ and $\varrho_{f0}^- > (\varrho_{f0}^-)^*$, then equation (A2.2) has two roots $Z^\pm = Q^\pm(1+i)$ which lie at the main diagonal. We have to show that each root $Z^\pm(\alpha_0,\ \varrho_{f0}^-)$ with growing α_0 intersects the main diagonal only at one point $Q^\pm(1+i)$ respectively. It is not difficult to check, that

$$\mathrm{Re}\frac{dZ}{d\alpha_0}\Big|_{Z=Q^-(1+i)} = -\frac{(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-}{2(4\mu(1-\nu)Q^- - \varrho_{f0} - \varrho_{f0}^-)} > 0, \tag{A2.28}$$

$$\mathrm{Re}\frac{dZ}{d\alpha_0}\Big|_{Z=Q^+(1+i)} = -\frac{(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-}{2(4\mu(1-\nu)Q^+ - \varrho_{f0} - \varrho_{f0}^-)} < 0 \tag{A2.29}$$

and

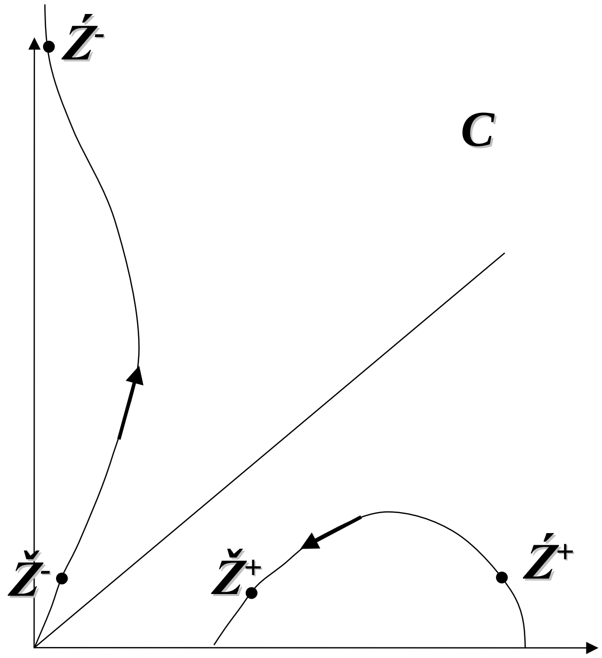

Fig.1

$$\left.\frac{dZ}{d\alpha_0}\right|_{Z=Q^{\pm}(1+i)} = -\frac{(1+m_E)\varrho_{f0}+(1-3m_E)\varrho_{f0}^{-}}{2(4\mu(1-\nu)Q^{\pm}-\varrho_{f0}-\varrho_{f0}^{-})}(1+i)$$

$$+2i\frac{\mu(1+m_E)(1-\nu)Q^{\pm}}{4\mu(1-\nu)Q^{\pm}-\varrho_{f0}-\varrho_{f0}^{-}} \qquad (A2.30)$$

Thus, at the point $Z = Q^{+}(1+i)$ the vector $\left.\frac{dZ}{d\alpha_0}\right|_{Z=Q^{+}(1+i)}$ is directed into domain above the main diagonal, while at the point $Z = Q^{-}(1+i)$ the vector $\left.\frac{dZ}{d\alpha_0}\right|_{Z=Q^{-}(1+i)}$ is directed into domain below the main diagonal. As far as for some $\varrho_{f0}^{-} > (\varrho_{f0}^{-})^{*}$ $\frac{dZ^{\pm}}{d\alpha_0}$ is a velocity along the branches $Z^{\pm}(\alpha_0,\ \varrho_{f0}^{-})$, than $Z^{-}(\alpha_0,\ \varrho_{f0}^{-})$ with growing α_0 intersects the main diagonal at the point $Q^{-}(1+i),\ \alpha_0 = \alpha_0^{*}$ and passes to the root $Z^{+}(\alpha_0,\ \varrho_{f0}^{-}),\ \alpha_0 > \alpha_0^{*}$, lying below the main diagonal. Analogously, $Z^{+}(\alpha_0,\ \varrho_{f0}^{-})$ with growing α_0 intersects the main diagonal at the point $Q^{+}(1+i),\ \alpha_0 = \alpha_0^{*}$ and passes to the root $Z^{-}(\alpha_0,\ \varrho_{f0}^{-}),\ \alpha_0 > \alpha_0^{*}$, lying above the main diagonal (Fig.3).

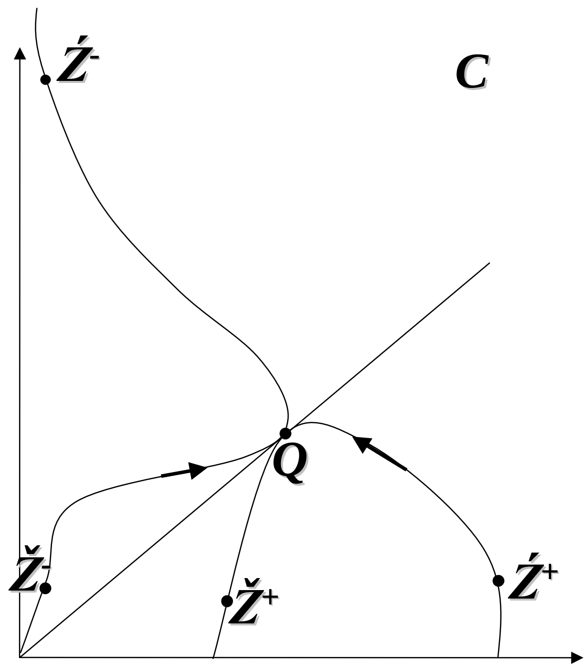

Fig.2

Thus, for $\varrho_{f0}^- > (\varrho_{f0}^-)^*$ and $\alpha_0 = \alpha_0^*(\varrho_{f0}^-)$ we have different limit values of Z^+ from the left and right of $\alpha_0 = \alpha_0^*$, namely

$$\lim_{\alpha_0 \nearrow \alpha_0^*} Z^+(\alpha_0,\ \varrho_{f0}^-) = Q^+(1+i) \qquad (A2.31)$$

and

$$\lim_{\alpha_0 \searrow \alpha_0^*} Z^+(\alpha_0,\ \varrho_{f0}^-) = Q^-(1+i). \qquad (A2.32)$$

Analogously, for the root $Z^-(\alpha_0,\ \varrho_{f0}^-)$ we have:

$$\lim_{\alpha_0 \nearrow \alpha_0^*} Z^-(\alpha_0,\ \varrho_{f0}^-) = Q^-(1+i) \qquad (A2.33)$$

and

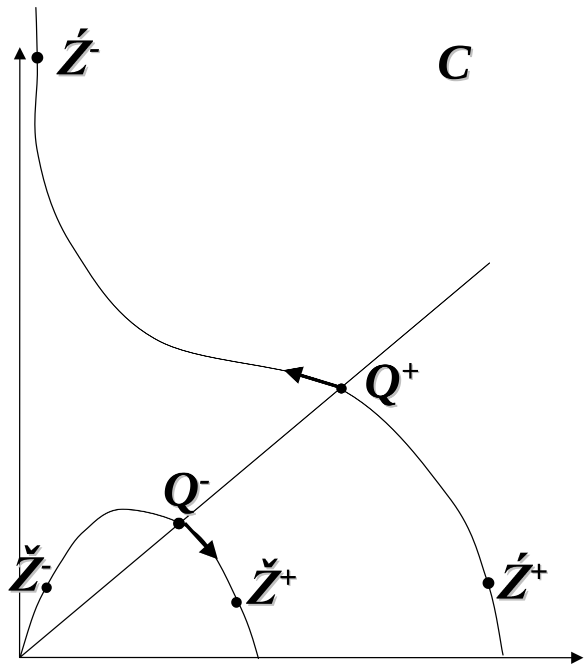

Fig.3

$$\lim_{\alpha_0 \searrow \alpha_0^*} Z^-(\alpha_0,\ \varrho_{f0}^-) = Q^+(1+i). \tag{A2.34}$$

In order to get single-valued roots $Z^\pm(\alpha_0,\ \varrho_{f0}^-)$ it is necessary to make a cut γ in the domain R^{++} of parameters $(\alpha_0,\ \varrho_{f0}^-)$ such that

$$\gamma = \{\alpha_0,\ \varrho_{f0}^-;\ \varrho_{f0}^- \geq (\varrho_{f0}^-)^*,\ \alpha_0 = \alpha_0^*(\varrho_{f0}^-)\} \tag{A2.35}$$

(see Fig.4). Then in the domain $\Omega_\gamma = R^{++} \backslash \gamma$ the roots $Z^\pm$ depend uniquely and continuously on parameters $(\alpha_0,\ \varrho_{f0}^-)$ and $\mathrm{Re}(Z^+)^2 > 0, \quad \mathrm{Re}(Z^-)^2 < 0$ in Ω_γ.

Let us note also that roots $Z^\pm$ smoothly connect the limit values $\hat{Z}^\pm$ and $\tilde{Z}^\pm$, i.e.

$$\hat{Z}^\pm(0,\ \varrho_{f0}^-) = \lim_{\alpha_0 \searrow 0} Z^\pm(\alpha_0,\ \varrho_{f0}^-) \tag{A2.36}$$

and

$$\tilde{Z}^{\pm}(\alpha_0,\ \varrho_{f0}^{-}) = \lim_{\alpha_0 \nearrow \frac{\alpha}{\sqrt{\kappa}}} Z^{\pm}(\alpha_0,\ \varrho_{f0}^{-}). \qquad (A2.37)$$

Remark. The necessity to make a cut γ in parameter plane $(\alpha_0,\ \varrho_{f0}^{-})$ results from the appearance of bifurcation of the roots of dispersion equation (4.6) at critical values $\alpha = \alpha_0^*(\varrho_{f0}^{-})\sqrt{\kappa}$ and $\varrho_{f0}^{-} = (\varrho_{f0}^{-})^*$.

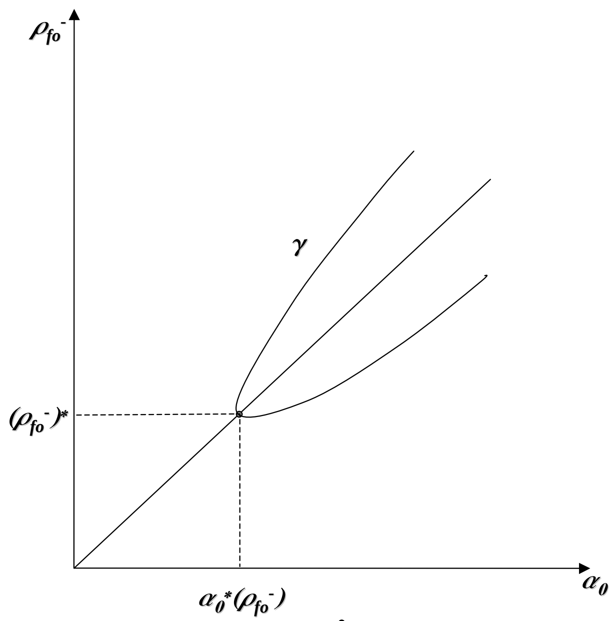

Fig.4

Pseudo-Stonely wave

Finally, we consider the root $\tilde{\omega}_2$ and show that pseudo-Stonely surface wave is degenerated into bulk slow compressional wave as $\alpha \to 0$.

Let $\alpha = \alpha_0\sqrt{\kappa}, \quad \alpha_0 \geq 0$ and

$$\tilde{\omega}_2 = \sqrt{\kappa}(1 + c_2\kappa^2 + \cdots). \qquad (A2.38)$$

Taking into account that $\tilde{\gamma}_1 = i\sqrt{\frac{\tilde{\omega}^2}{\kappa} - 1}$ disperion relation (4.6) takes the form:

$$-\frac{1}{a_{s2}^2}\Big\{2\mu(1-\nu)(\sqrt{2c_2})^2-\sqrt{2c_1}(-i(\varrho_{f0}+\varrho_{f0}^-)+2\alpha_0\mu(1+m_E)(1-\nu))$$

$$-i\alpha_0\Big((1+m_E)\varrho_{f0}+(1-3m_E)\varrho_{f0}^-\Big)\Big\}\kappa^3+O(\kappa^{7/2}). \qquad (A2.39)$$

From the leading part of (A2.39) it follows that $\sqrt{2c_2}=Z$ satisfies the equation

$$2\mu(1-\nu)Z^2-\Big(-i(\varrho_{f0}+\varrho_{f0}^-)+2\alpha_0\mu(1+m_E)(1-\nu)\Big)Z$$

$$i\alpha_0\Big((1+m_E)\varrho_{f0}+(1-3m_E)\varrho_{f0}^-\Big)=0. \qquad (A2.40)$$

Since $\mathrm{Re}c_2$ should be positive, we look for the roots such that $\mathrm{Re}Z>0$ and $\mathrm{Re}Z^2>0$. Next we will prove that for $\alpha_0\sim 0$ equation (A2.40) has two roots $\hat{Z}^+$ and $\hat{Z}^-$, satisfying conditions $\mathrm{Re}\hat{Z}^\pm>0$ and $\mathrm{Re}(\hat{Z}^+)^2>0$ and $\mathrm{Re}(\hat{Z}^-)^2<0$. Let

$$\hat{Z}^\pm=a_0^\pm+\alpha_0a_1^\pm+\alpha_0^2a_2^\pm+... \qquad (A2.41)$$

Substituting (A2.41) into (A2.40) one gets:

$$a_0^+=0,\ a_1^+=\frac{(1+m_E)\varrho_{f0}+(1-3m_E)\varrho_{f0}^-}{\varrho_{f0}+\varrho_{f0}^-},\ a_2^+=-1\frac{8\mu(1-\nu)m_E\varrho_{f0}^-a_1^+}{(\varrho_{f0}+\varrho_{f0}^-)^2} \qquad (A2.42)$$

and

$$a_0^-=-i\frac{\varrho_{f0}+\varrho_{f0}^-}{2\mu(1-\nu)},\quad a_1^-=\frac{4m_E\varrho_{f0}^-}{\varrho_{f0}+\varrho_{f0}^-},$$

$$a_2^-=-i\frac{2\mu(1-\nu)a_1^-}{(\varrho_{f0}+\varrho_{f0}^-)^2}\Big((1+m_E)\varrho_{f0}+(1-3m_E)\varrho_{f0}^-\Big). \qquad (A2.43)$$

Finally we have:

$$\hat{Z}^+=\frac{(1+m_E)\varrho_{f0}+(1-3m_E)\varrho_{f0}^-}{\varrho_{f0}+\varrho_{f0}^-}\alpha_0-i\frac{8\mu(1-\nu)m_E\varrho_{f0}^-a_1^+}{(\varrho_{f0}+\varrho_{f0}^-)^2}\alpha_0^2+O(\alpha_0^3), \quad (A2.44)$$

$$\hat{Z}^-=-i\frac{\varrho_{f0}+\varrho_{f0}^-}{2\mu(1-\nu)}+\frac{4m_E\varrho_{f0}^-}{\varrho_{f0}+\varrho_{f0}^-}\alpha_0+O(\alpha_0^2) \qquad (A2.45)$$

Thus, $\mathrm{Re}\hat{Z}^\pm>0$ and $\mathrm{Re}(\hat{Z}^+)^2>0$, $\mathrm{Re}(\hat{Z}^-)^2<0$ and both roots lie in the fourth quarter of the complex plane.

Obviously, $\hat{Z}^+ \to 0$ as $\alpha_0 \to 0$. The latter means that phase velocity of pseudo-Stonely wave tends to $\sqrt{\kappa}$ and this mode is degenerated into bulk slow compressional wave.

Lemma 3 *The roots $Z(\alpha_0, \varrho_0^-)$ of equation (A2.40) for any parameters $\alpha_0 > 0$, $\varrho_0^- \geq 0$ do not belong to real and imaginary axes of complex plane.*

(The proof is analogous to Lemma 1.)

Next consider the behaviour of the roots $Z(\alpha_0, \varrho_0^-)$ of equation (A2.40) for $\alpha_0 \to \infty$. Let $Z = \alpha_0 W$. Then

$$2\mu(1-\nu)W^2 - \Big[-i\frac{\varrho_{f0} + \varrho_{f0}^-}{\alpha_0} + 2\mu(1+m_E)(1-\nu)\Big] W$$

$$= \frac{i}{\alpha_0}\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big). \tag{A2.46}$$

Substituting

$$W = b_0 + \frac{b_1}{\alpha_0} + \frac{b_2}{\alpha_0^2} + ... \tag{A2.47}$$

into (A2.46) one gets two roots $\tilde{Z}^+$ and $\tilde{Z}^-$ with

$$b_0^+ = 1 + m_E,\ b_1^+ = -i\frac{2m_E\varrho_{f0}^-}{\mu(1+m_E)(1-\nu)},$$

$$b_2^+ = \frac{m_E\varrho_{f0}^-\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big)}{\mu^2(1-\nu)^2(1+m_E)^3} \tag{A2.48}$$

and

$$b_0^- = 0,\ b_1^- = -i\frac{(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-}{2\mu(1+m_E)(1-\nu)},$$

$$b_2^- = \frac{m_E\varrho_{f0}^-\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^-\Big)}{\mu^2(1-\nu)^2(1+m_E)^3}. \tag{A2.49}$$

Finally,

$$\tilde{Z}^+ = \alpha_0(1+m_E) - i\frac{2m_E\varrho_{f0}^-}{\mu(1+m_E)(1-\nu)} + O\Big(\frac{1}{\alpha_0}\Big), \tag{A2.50}$$

$$\tilde{Z}^{-} = -i\frac{(1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^{-}}{2\mu(1+m_E)(1-\nu)}$$

$$+\frac{1}{\alpha_0}\frac{m_E\varrho_{f0}^{-}\Big((1+m_E)\varrho_{f0} + (1-3m_E)\varrho_{f0}^{-}\Big)}{\mu^2(1-\nu)^2(1+m_E)^3}O\Big(\frac{1}{\alpha_0^2}\Big). \qquad (A2.51)$$

Taking into account (5.5), we have $\tilde{Z}^{\pm} > 0$, and, obviously, $\mathrm{Re}(\tilde{Z}^{+})^2 > 0$ and $\mathrm{Re}(\tilde{Z}^{-})^2 < 0$. Both roots lie in the fourth quarter of the complex plane.

Moreover, as it is clear from (A2.50) and (5.13), asymptotics for the case $\alpha = \alpha_0\sqrt{\kappa}$, $\alpha_0 \to \infty$ coincides with asymptotics for the case $\alpha \sim 1$ if $\alpha_0 = \alpha/\sqrt{\kappa}$.

Lemma 4 *The roots $Z(\alpha_0, \varrho_0^{-})$ of equation (A2.40) for any parameters $\alpha_0 \geq 0$, $\varrho_0^{-} \geq 0$ do not intersect the secondary diagonal of complex plane.*

(The proof is analogous to Lemma 2.)

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