Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Regularity of solutions for some problems of mathematical physics

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submitted: 29th June 1993

[1] This paper was written while the author was visiting the Institute of Applied Analysis and Stochastics.

Preprint No. 53
Berlin 1993

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1. Preface

This paper is devoted to some boundary value problems for systems of partial differential equations. In particular we consider the Stoke's system and the quasilinear elliptic degenerated systems of the divergent type with bounded nonlinearities. It was shown in [2] (see also [3] and [8]) that the question of regularity of weak solutions for quasilinear elliptic and parabolic systems is closely attached with the dispersion of the spectrum of the matrix, which defines the ellipticity (parabolicity) of the system. The upper bound for this dispersion is determined by some coercieve constants for elementary elliptic or parabolic operators. The explicit form of these constants leads to some conditions, which is easy to check in order to obtain the regularity of weak solutions. This approach can be applied for example to such important systems like Stoke's system. We divide the whole material in three paragraphs.

The first one is devoted to some constants concerning the operators Δ and $\varepsilon\partial_t - \Delta$, where ε is an arbitrary positive constant.

Let B be a unit ball in $R^m (m > 2)$ with the centrum at the origin and let $\alpha = 2 - m - 2\gamma$ $(0 < \gamma < 1)$. If $u(x)$ is equal zero on ∂B then the inequality

$$\int_B |D^2u|^2|x|^\alpha dx \leq \left[1 + \tfrac{m-2}{m+1} + 0(\gamma)\right] \int_B |\Delta u|^2 |x|^\alpha dx + \\ + C\left(\int_B |D^2 u|^2 |x|^\alpha dx\right)^{\frac{m}{m+2\gamma}} \left(\int_B |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}}$$

takes place. Here $|D^2u|^2$ and $|Du|^2$ are correspondendly the sum of all squared derivatives of u of the second and the first order.

This estimate could be obtained with the help of the E. Stein [4] result concerning the boundness of the singular integral operators in the weighted spaces $L_{2,\alpha}(\mathbb{R}^m)(|\alpha| < m)$. But this method doesn't give the explicit constant before the right-hand side integral containing Δu. The nonstationary case is also considered in this paragraph. Let $Q = (0,T) \times B, u = 0$ for $t = 0$ and ζ is a cut–off function. Then the inequality

$$\int_B |D^2u|^2|x|^\alpha \zeta dxdt \leq \frac{m}{2}\left[1 + \frac{m-2}{m+1} + 0(\gamma)\right] \int_B |\varepsilon\dot{u} - \Delta u|^2 |x|^\alpha \zeta dxdt +$$

$$C\left\{ \left(\int_B |D^2u|^2|x|^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left[\int_B (|Du|^2 + |u|^2) dxdt\right]^{\frac{2\gamma}{m+2\gamma}} + \int_B (|Du|^2 + |u|^2) dxdt \right\}$$

holds for $m \geq 3$ and the constant C doesn't depend on $\varepsilon > 0$.

In paragraph 2 we consider the Stoke's system both for stationary and nonstationary cases. Consider for example here only the stationary system

$$\Delta u + \nabla p = f,$$

$$\text{div } u = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^m$ with a smooth boundary and with $u = 0$ on $\partial\Omega$. Let x_0 be an arbitrary point of Ω, with dist $(x_0, \partial\Omega) > R_0 =$ const and $R < R_0$.

Then the estimates for the weak solution u, p

$$\int\limits_{B_R(x_0)} |\nabla p|^2 |x - x_0|^\alpha dx \leq \left[1 + \frac{(m-2)^2}{m-1} + 0(\gamma)\right] \int\limits_{B_R(x_0)} |f|^2 |x - x_0|^\alpha dx + C \int\limits_{B_R(x_0)} |p|^2 dx$$

and

$$\int\limits_{B_R(x_0)} |D^2 u|^2 |x - x_0|^\alpha \zeta dx \leq \left\{1 + \left[1 + \frac{(m-2)^2}{m-1}\right]^{1/2}\right\}^2 \left[1 + \frac{m-2}{m+1} + 0(\gamma)\right] \times$$

$$\int\limits_{B_R(x_0)} |f|^2 |x - x_0|^\alpha \zeta dx + C \left[\left(\int\limits_\Omega |D^2 u|^2 |x - x_0|^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_\Omega |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \right.$$

$$\left. + \int\limits_\Omega \left(|Du|^2 + |u|^2 + |p|^2\right) dx \right]$$

take place and C doesn't depend on x_0. The results of this paragraph were obtained in cooperation with A. Wagner (Cologne).
The third paragraph contains some results about the elliptic system

$$\sum_{i=1}^m D_i a_i(x; u, Du) - a_0(x; u, Du) = 0.$$

Under natural analytic conditions about coefficients $a_i(x, p)$ we assume that the eigenvalues λ_j of the symmetric matrix

$$A = \left\{\frac{\partial a_i}{\partial p_j}\right\} \quad (i, j = 0, \ldots, m)$$

satisfy the following inequalities

$$\frac{\lambda}{1 + |p|^s} \leq \lambda_j \leq \frac{\Lambda}{1 + |p|^s}$$

with $\Lambda, \lambda =$ const. > 0 and $0 \leq s < 1$.

It is proved for example, that if the inequality

$$\frac{\left(1 + \frac{m-2}{m+1}\right)[1 + (m-2)(m-1)]}{\left(1 + \frac{m-2}{m+1}\right)[1 + (m-2)(m-1)] - 1} \frac{\lambda}{\Lambda} > 1$$

holds then the "small" weak solution of the system satisfies the Hölder condition in $\overline{\Omega}$.

2. Some coercieve inequalities with explicit constants

Consider in R^m $(m \geq 2)$ a ball $B_R(x_0)$ with the center x_0 and radius R. The ball $B_1(0)$ we shall denote by B. In this ball an equation

$$\Delta u = f(x) \tag{2.1}$$

with a boundary condition

$$u|_{\partial B} = 0 \tag{2.2}$$

is given.

Suppose that $f \in L_{2,\alpha}(B)$, where $L_{2,\alpha}$ is the space of squared integrable functions with a weight $|x|^\alpha$. During all this paper we assume that $\alpha = 2-m-2\gamma$ $(0 < \gamma < 1)$ and $|x|$ is the distance from the origin. The norm in $L_{2,\alpha}(B)$ as usually is determined by

$$\left(\int_B |u|^2 |x|^\alpha dx\right)^{\frac{1}{2}}.$$

Denote

$$|Du|^2 = \sum_{i=1}^m u_i^2 \text{ and } \left|D^2 u\right|^2 = \sum_{i,k=1}^m u_{ik}^2, \tag{2.3}$$

where u_i are the derivatives with respect to x_i.

By $W_{2,\alpha}^{(2)}(B)$ we shall denote the Soboleff's space $W_2^{(2)}(B)$; the second derivatives of the elements of this space are squared summable with the weight $|x|^\alpha$. For the norm in this space could be taken for example the expression

$$\left(\int_B \left|D^2 u\right|^2 |x|^\alpha dx + \int |u|^2 dx\right)^{\frac{1}{2}}.$$

One of he aim of this paragraph is to prove for the solution of the problem (2.1), (2.2) the inequality

$$\int_B \left|D^2 u\right|^2 r^\alpha dx \leq C_\alpha^2 \int_B |f|^2 r^\alpha dx, \quad (r = |x|)$$

where C_α has an explicit form. For $\alpha' = m - 2 + 2\gamma$ such an inequality was proved by the author in [1].

At first we shall prove some lemmas.

Lemma 2.1. *If $u \in W_{2,\alpha}^{(2)}(B)$, then the inequalities*

$$|u(0)|^2 < \eta \int_B |Du|^2 r^\alpha dx + C_0(\eta) \int_B |u|^2 dx \tag{2.4}$$

and

$$\sum_{i=1}^m |u_i(0)|^2 < \eta \int_B \left|D^2 u\right|^2 r^\alpha dx + C_0(\eta) \int_B |u|^2 dx \tag{2.5}$$

hold.

Here η is as usual an arbitrary positive constant and

$$C_0(\eta) = 2m|S|^{-(m/2\gamma+1)}\gamma^{-\frac{m}{2\gamma}}\eta^{-\frac{m}{2\gamma}}, \tag{2.6}$$

where $|S|$ is the surface of the unit sphere in R^m and λ is the smallest absolute value of the eigenvalues for operator Δ in B with the condition (2.2).

Proof. Evidently

$$u(0) = u(x) - \int_0^r \frac{\partial u}{\partial \varrho} d\varrho.$$

Square both sides of this equality and integrate over the ball $B_\delta(0) = B_\delta$ with $\delta < 1$. We get

$$|u(0)|^2 |S| m^{-1} \delta^m \leq 2 \int_{B_\delta} \left| \int_0^r \frac{\partial u}{\partial \varrho} d\varrho \right|^2 dx + 2 \int_B u^2 dx.$$

The first term on the right hand side we can write in the equivalent form and get

$$|u(0)|^2 |S| m^{-1} \delta^m \leq 2 \int_{\partial B_\delta} dS \int_0^\delta \left| \int_0^r \frac{\partial u}{\partial \varrho} \varrho^{\frac{\alpha+m-1}{2}} \varrho^{-\frac{\alpha+m-1}{2}} d\varrho \right|^2 r^{m-1} dr + 2 \int_B |u|^2 dx.$$

Applying the Hölder inequality to the inner integral, we come to the estimate

$$|u(0)|^2 |S| m^{-1} \leq \frac{1}{\gamma} \int_{\partial B_\delta} dS \int_0^\delta \int_0^r |\nabla u|^2 \varrho^{\alpha+m-1} d\varrho r^{m-1+2\gamma} dr + 2 \int_B |u|^2 dx.$$

Putting δ instead of the upper bound of the inner integral we get the following

$$|u(0)|^2 |S| m^{-1} \delta^m \leq \frac{\delta^{m+2\gamma}}{\gamma(m+2\gamma)} \int_{B_\delta} |\nabla u|^2 r^{\alpha+m-1} dr + 2 \int_B |u|^2 dx.$$

Dividing by δ^m and taking into account that $m(m+2\gamma)^{-1} < 1$, we come to the inequality

$$|u(0)|^2 \leq \frac{\delta^{2\gamma}}{\gamma|S|} \int_{B_\delta} |\nabla u|^2 r^\alpha dx \quad + \frac{2m}{|S|\delta^m} \int_B |u|^2 dx.$$

Using the notation (2.6) we come to the inequalities (2.4) and (2.5). □

Corollary 2.1. *Let λ be the smallest absolute value of the eigenvalues for the operator Δ with condition* (2.2). *Then the inequalities*

$$|u(0)|^2 \leq \eta \int_B |Du|^2 r^\alpha dx + \frac{C_0(\eta)}{\lambda^2} \int_B |\Delta u|^2 dx \tag{2.7}$$

and

$$\sum_i^m |u_i(0)|^2 \leq \eta \int_B |D^2 u|^2 r^\alpha dx + \frac{C_0(\eta)}{\lambda} \int_B |\Delta u|^2 dx. \tag{2.8}$$

take place if u satisfies (2.2).

Proof. In fact the both second terms on the right hand side of (2.4) and (2.5) can be easily estimated by the integral of $|\Delta u|^2$.
Using the condition (2.2) and integrating by parts we'll have

$$\int_B |Du|^2 \, dx = -\int_B u\Delta u dx \leq \left(\int_B |u|^2 \, dx\right)^{\frac{1}{2}} \left(\int_B |\Delta u|^2 \, dx\right)^{\frac{1}{2}}.$$

Then

$$\int_B |u|^2 \, dx \leq \frac{1}{\lambda^2} \int_B |\Delta u|^2 \, dx$$

and from the previous inequality we have

$$\int_B |Du|^2 \, dx \leq \frac{1}{\lambda} \int_B |\Delta u|^2 \, dx$$

and so the corollary is proved. □

Lemma 2.2. *For $u \in W^{(2)}_{2,\alpha}(B)$, satisfying* (2.2), *the equality*

$$\int_B u_{ik} u_{ik} r^\alpha dx = \int_B |\Delta u|^2 r^\alpha dx + \alpha \int_B [u_i(x) - u_i(0)] \times$$
$$\times \left[u_{kk} \cos(x_i, r) - u_{ik} \cos(x_k, r)\right] r^{\alpha-1} dx - (m-1) \int_{\partial B} |u_r|^2 \, dS \qquad (2.9)$$

holds.

Proof. Integrating twice by parts we'll have

$$\int_B u_{ik} u_{ik} dx = \int_B [u_i - u_i(0)]_k \, [u_i - u_i(0)]_k \, dx =$$
$$= \int_B |\Delta u|^2 \, dx + \int_{\partial B} \{[u_i - u_i(0)] \, u_{ik} \cos(r, x_k) - [u_i - u_i(0)] \, u_{kk} \cos(r, x_i)\} \, dS.$$

Therefore

$$\int_{\partial B} \{[u_i - u_i(0)] \; u_{ik} \cos(r, x_k) - [u_i - u_i(0)] \, u_{kk} \cos(r, x_i) \; \} \, dS =$$
$$= \int_{\partial B} (\left|D^2 u\right|^2 - |\Delta u|^2). \qquad (2.10)$$

With the same kind of calculations (see for example [2] p. 142 etc.) we come to the identity

$$\int_B u_{ik} u_{ik} r^\alpha dx = \int_B |\Delta u|^2 \, |x|^\alpha \, dx +$$
$$+\alpha \int_B [u_i - u_i(0)] \, u_{kk} r^{\alpha-1} \cos(x_i, r) dx - \alpha \int_B [u_i - u_i(0)] \, u_{ik} r^{\alpha-1} \cos(x_k, r) dx +$$
$$+ \int_{\partial B} \{[u_i - u_i(0)] \, u_{ik} \cos(x_k, r) - [u_i - u_i(0)] \, \Delta u \cos(x_i, r)\} \, dS.$$

After applying (2.10) we come to

$$\int_B u_{ik} u_{ik} r^\alpha dx = \int_B |\Delta u|^2 r^\alpha dx + \alpha \int_B [u_i - u_i(0)] [u_{kk} \cos(x_i, r) -$$

$$- u_{ik} \cos(x_k, r)] r^{\alpha-1} dx + \int_B \left[u_{ik} u_{ik} - (\Delta u)^2 \right] dx.$$

Under condition (2.2) we have from (2.10) that

$$\int_B \left(\left| D^2 u \right|^2 - |\Delta u|^2 \right) dx = -(m-1) \int_{\partial B} |u_r|^2 dx$$

and we come to (2.9). □

Consider a function

$$v(x) = u(x) - u(0) - u_i(0) x_i, \tag{2.11}$$

which evidently satisfies the conditions

$$v(0) = v_i(0) = 0 \quad \text{and} \quad v_{ik} = u_{ik}.$$

Take a complete orthonormal set of spherical functions

$$\{Y_{j,l}(0)\} \, (j = 0, 1, 2, \ldots; l = 1, \ldots, k_j, \theta \in S)$$

and consider the expansion

$$v(x) = \sum_{j=0}^{+\infty} \sum_{l=0}^{k_j} v_{j,l}(r) Y_{j,l}(\theta). \tag{2.12}$$

Lemma 2.3. *For any $u \in W^{(2)}_{2,\alpha}(B)$, satisfying (2.2), the identity*

$$\int_B \left| D^2 u \right|^2 r^\alpha dx = \int_B |\Delta u|^2 r^\alpha dx - (m-1) \int_{\partial B} |u_r|^2 dS - \frac{\alpha}{2} \int_{\partial B} |\nabla v|^2 dS +$$

$$+ \frac{\alpha}{2} \int_{\partial B} |v_r|^2 dS - \frac{\alpha}{2} \sum_{j,l \geq 0} j(j+m-2) v_{j,e}^2(1) + \alpha \sum_{j,l \geq 0} \int_0^1 \left[(m-1) \left| v_{j,l}^1 \right|^2 + \right.$$

$$\left. + (\alpha + m - 3) \, j(j+m-2) \left| v_{j,l} \right|^2 r^{-2} \right] r^{\alpha+m-3} dr \tag{2.13}$$

takes place (by $j, l \geq 0$ we understand the summation in the same limits as in (2.12)).

Proof. We can write the identity (2.9) in the form

$$\int_B \left| D^2 u \right|^2 r^\alpha dx = \int_B |\Delta u|^2 r^\alpha dx - (m-1) \int_{\partial B} u_r^2 dS +$$

$$+ \alpha \int_B v_r \Delta v r^{\alpha-1} dx - \alpha \int_B v_i v_{ir} r^{\alpha-1} dx.$$

Using (2.11) we can integrate by parts in the last term on the right hand side

$$\int_B v_i v_{ir} r^{\alpha-1} dx = \frac{1}{2} \int_B \left(|\nabla v|^2\right)_r r^{\alpha-1} dx = \frac{1}{2} \int_{\partial B} dS \int_0^1 \left(|\nabla v|^2\right)_r r^{\alpha+m-2} dr =$$
$$= \frac{1}{2} \int_{\partial B} dS \left[|\nabla v|^2 r^{\alpha+m-2} \Big|_0^1 - (\alpha+m-2) \int_0^1 |\nabla v|^2 r^{\alpha+m-3} dr \right] =$$
$$= \frac{1}{2} \int_{\partial B} |\nabla v|^2 dS - \frac{\alpha+m-2}{2} \int_B |\nabla v|^2 r^{\alpha-2} dx.$$

So

$$\int_B \left|D^2 u\right|^2 r^\alpha dx = \int_B |\Delta u|^2 r^\alpha dx - (m-1) \int_{\partial B} |u_r|^2 dS + \alpha \int_B v_r \Delta v r^{\alpha-1} dx +$$
$$+ \frac{\alpha(\alpha+m-2)}{2} \int_B |\nabla v|^2 r^{\alpha-2} dx - \frac{\alpha}{2} \int_{\partial B} |\nabla v|^2 dS. \tag{2.14}$$

Integrating by parts we get

$$\int_B |\nabla v|^2 r^{\alpha-2} dx = \int_B v_i v_i r^{\alpha-2} dx = \int_B \left(v v_i r^{\alpha-2}\right)_i dx -$$
$$- \int_B v \Delta v r^{\alpha-2} dx - (\alpha-2) \int_B v v_i r^{\alpha-3} r_i dx = \int_{\partial B} v v_r dS - \int_B v \Delta v r^{\alpha-2} dx - (\alpha-2) \times$$
$$\times \int_{\partial B} dS \int_0^1 v v_r r^{\alpha+m-4} dr = \int_{\partial B} v v_r dS - \frac{\alpha-2}{2} \int_{\partial B} |v|^2 dS -$$
$$- \int_B v \Delta v r^{\alpha-2} dx + \frac{(\alpha-2)(\alpha+m-4)}{2} \int_0^1 |v|^2 r^{\alpha-4} dx.$$

Finally

$$\int_B |\nabla v|^2 r^{\alpha-2} dx = \int_{\partial B} \left(v v_r - \frac{\alpha-2}{2} |v|^2\right) dS - \int_B v \Delta v r^{\alpha-2} dx +$$
$$+ \frac{(\alpha-2)(\alpha+m-4)}{2} \int_B |v|^2 r^{\alpha-4} dx.$$

Substituting in (2.14) we come to

$$\int_B \left|D^2u\right|^2 r^\alpha dx = \int_B |\Delta u|^2 r^\alpha dx - (m-1)\int_{\partial B} |u_r|^2 dS -$$
$$-\frac{\alpha}{2}\int_{\partial B}\left[|\nabla v|^2 - (\alpha+m-2)(vv_r - \frac{\alpha-2}{2}|v|^2)\right] dS +$$
$$+\alpha\int_B v_r \Delta v r^{\alpha-1} dr - \frac{\alpha(\alpha+m-2)}{2}\int_B v\Delta v r^{\alpha-2} dx +$$
$$+\frac{\alpha(\alpha-2)(\alpha+m-2)(\alpha+m-4)}{4}\int_B |v|^2 r^{\alpha-4} dx. \tag{2.15}$$

Let us transform the last three terms on the right hand side of (2.15) with the help of the expansion (2.12).
Then

$$I_1 = \int_B v_r \Delta v r^{\alpha-1} dx = \sum_{j,l}\int_0^1 v'_{j,l}\left[\left(r^{m-1}v'_{j,l}\right)' r^{\alpha-m} - j(j+m-2)v'_{j,l}v_{j,l}r^{\alpha-3}\right] r^{m-1} dr =$$
$$=\sum_{j,l}\left\{r^{\alpha+m-2}v'_{j,l}v'_{j,l}\Big|_0^1 - \int_0^1 \left[v''_{j,l}v'_{j,l}r^{\alpha-m-2} + (\alpha-1)(v'_{j,l})^2 r^{\alpha+m-3}\right] dr -\right.$$
$$-j(j+m-2)\int_0^1 v'_{j,l}v_{j,l}r^{\alpha+m-4} dr = \int_{\partial B} v_r^2 dS - \sum_{j,l}\left\{\frac{1}{2}\left(v'_{j,l}\right)^2 r^{\alpha+m-2}\Big|_0^1 -\right.$$
$$-\frac{(\alpha+m-2)}{2}\int_0^1 \left(v'_{j,l}\right)^2 r^{\alpha+m-3} dr - j(j+m-2)\int_0^1 v'_{j,l}v_{j,l}r^{\alpha+m-4} dr -$$
$$-(\alpha-1)\int_B (v'_r)^2 r^{\alpha+m-3} dr = \frac{1}{2}\int_{\partial B}(v'_r)^2 dS + \frac{m-\alpha}{2}\int_B (v'_r)^2 r^{\alpha-2} dx -$$
$$-\sum_{j,l} j(j+m-2)\int_B v'_{j,l}v'_{j,l}r^{\alpha+m-4} dr = \frac{1}{2}\int_{\partial B}(v'_r)^2 dS - \frac{1}{2}\sum_{j,l} j(j+m-2)\times$$
$$\times v_{j,l}^2(1) + \frac{\alpha+m-4}{2}\sum_{j,l} j(j+m-2)\int_0^1 v_{j,l}^2 r^{\alpha+m-5} dr + \frac{m-\alpha}{2}\int_B (v'_r)^2 r^{\alpha-2} dx.$$

So

$$I_1 = \int_B v'_r \Delta v r^{\alpha-1} dx = \frac{1}{2}\int_{\partial B}|v'_r|^2 dS - \frac{1}{2}\sum_{j,l} j(j+m-2)v_{j,l}^2(1) +$$
$$+\frac{\alpha+m-4}{2}\sum_{j,l} j(j+m-2)\int_0^1 v_{j,l}^2 r^{\alpha+m-5} dr + \frac{m-\alpha}{2}\times$$
$$\times\int_B (v'_r)^2 r^{\alpha-2} dx. \tag{2.16}$$

Now

$$I_2 = \int_B v \Delta v r^{\alpha-2} dx = \sum_{j,l} \int_0^1 v_{j,l} \left[v''_{j,l} + (m-1) r^{-1} v'_{j,l} - \right.$$

$$\left. - j(j+m-2) r^{-2} v_{j,l} \right] r^{\alpha+m-3} dr = \sum_{j,l} \left[\int_0^1 v_{j,l} v''_{j,l} r^{\alpha+m-3} dr + (m-1) \times \right.$$

$$\left. \times \int_0^1 v_{j,l} v'_{j,l} r^{\alpha+m-4} dr - j(j+m-2) \int_0^1 v^2_{j,l} r^{\alpha+m-5} dr \right] = \sum_{j,l} \left[v_{j,l} v'_{j,l} r^{\alpha+m-3} \Big|_0^1 - \right.$$

$$- \int_0^1 \left(v'_{j,l} \right)^2 r^{\alpha+m-3} dr - (\alpha+m-3) \int_0^1 v_{j,l} v'_{j,l} r^{\alpha+m-4} dr + \frac{m-1}{2} v^2_{j,l} r^{\alpha+m-4} \Big|_0^1 -$$

$$\left. -(m-1) \frac{\alpha+m-4}{2} \int_0^1 v^2_{j,l} r^{\alpha+m-5} dr - j(j+m-2) \int_0^1 v^2_{j,l} r^{\alpha+m-5} dr \right].$$

Then

$$\begin{aligned} I_2 &= \int_B v \Delta v r^{\alpha-2} dx = \sum_{j,l} \left\{ v_{j,l}(1) v'_{j,l}(1) - \frac{\alpha-2}{2} v^2_{j,l}(1) + \right. \\ &+ \left[\frac{(\alpha+m-4)(\alpha-2)}{2} - j(j+m-2) \right] \times \\ &\times \int_0^1 v^2_{j,l} r^{\alpha+m-5} dr - \int_0^1 \left(v'_{j,l} \right)^2 r^{\alpha+m-3} dr. \end{aligned} \tag{2.17}$$

Combining (2.15), (2.16) and (2.17) we get (2.13) □

Lemma 2.4. *If $v \in W^{(2)}_{2,\alpha}(B)$ satisfies (2.2) and vanishes with all first derivatives at the center of B then the identity*

$$\begin{aligned} \int_B |\Delta v|^2 r^\alpha dx &= (m-1) \int_{\partial B} (v'_r)^2 dS - 2 \sum_{j,l} j(j+m-2) v'_{j,l}(1) v_{j,l}(1) + \\ &+ (\alpha-2) \sum_{j,l} j(j+m-2) v^2_{j,l}(1) + \sum_{j,l} \int_0^1 \left\{ \left| v''_{j,l} \right|^2 + \right. \\ &+ \left[(m-1)(1-\alpha) + 2j(j+m-2) \right] \left| v'_{j,l} \right|^2 r^{-2} + j(j+m-2) \times \\ &\times \left[j(j+m-2) + (2-\alpha)(\alpha+m-4) \right] \left| v_{j,l} \right|^2 r^{-4} \Big\} r^{\alpha+m-1} dr \end{aligned} \tag{2.18}$$

takes place.

Proof. Using the expansion (2.12) we get

$$\int_B |\Delta v|^2 r^\alpha dx = \sum_{j,l} \{ \int_0^1 \left[\left|v''_{j,l}\right|^2 + (m-1)^2 r^{-2} \left|v'_{j,l}\right|^2 + j^2(j+m-2)^2 r^{-4} \left|v_{j,l}\right|^2 \right] \times$$

$$\times r^{\alpha+m-1} dr + 2(m-1) \int_0^1 v''_{j,l} v'_{j,l} r^{\alpha+m-2} dr - 2j(j+m-2) \times$$

$$\times \int_0^1 v''_{j,l} v_{j,l} r^{\alpha+m-3} dr - 2(m-1)j(j+m-2) \int_0^1 v'_{j,l} v_{j,l} r^{\alpha+m-4} dr \} =$$

$$= \sum_{j,l} \left\{ \int_0^1 \left[\left|v''_{j,l}\right|^2 + (m-1)^2 \left|v'_{j,l}\right|^2 r^{-2} + j^2(j+m-2)^2 r^{-4} \times \right. \right.$$

$$\times \left|v_{j,l}\right|^2 \,] r^{\alpha+m-1} dr + 2(m-1) \left[\frac{|v'|^2}{2} r^{\alpha+m-2} \Big|_0^1 - \frac{\alpha+m-2}{2} \times \right.$$

$$\times \int_0^1 \left|v'_{j,l}\right|^2 r^{\alpha+m-3} dr \Big] - 2j(j+m-2) \left[v'_{j,l} v_{j,l} r^{\alpha+m-3} \Big|_0^1 - \int_0^1 \left|v'_{j,l}\right|^2 \times \right.$$

$$\times r^{\alpha+m-3} dr - (\alpha+m-3) \int_0^1 v'_{j,l} v_{j,l} r^{\alpha+m-4} dr - \frac{2(m-1)j(j+m-2)}{2} \times$$

$$\times \left[\left|v_{j,l}\right|^2 r^{\alpha+m-4} \Big|_0^1 - (\alpha+m-4) \int_0^1 \left|v_{j,l}\right|^2 r^{\alpha+m-5} dr \,\right] \Bigg\}.$$

Continuing this process we come to

$$\int_B |\Delta v|^2 r^\alpha dx = \sum_{j,l} \left\{ \int_0^1 \left[\left|v''_{j,l}\right|^2 + (m-1)^2 \left|v'_{j,l}\right|^2 r^{-2} + j^2(j+m-2)^2 \left|v_{j,l}\right|^2 r^{-4} \,\right] \times \right.$$

$$r^{\alpha+m-1} dr + (m-1) \left|v'_{j,l}(1)\right|^2 - (m-1)(\alpha+m-2) \int_0^1 \left|v'_{j,l}\right|^2 r^{\alpha+m-3} dr -$$

$$2j(j+m-2) v'_{j,l}(1) v_{j,l}(1) + 2j(j+m-2) \int_0^1 \left|v'_{j,l}\right|^2 r^{\alpha+m-3} dr + 2(\alpha+m-3) \times$$

$$j(j+m-2) \int_0^1 v'_{j,l} v_{j,l} r^{\alpha+m-4} dr - (m-1)j(j+m-2) \left|v_{j,l}(1)\right|^2 +$$

$$+(m-1)(\alpha+m-4)j(j+m-2) \int_0^1 \left|v_{j,l}\right|^2 r^{\alpha+m-5} dr \,\}.$$

In the same way we get

$$\int_B |\Delta v|^2 r^\alpha dx = \sum_{j,l} \int_0^1 \left[\left|v''_{j,l}\right|^2 + (m-1)^2 \left|v'_{j,l}\right|^2 r^{-2} + j^2(j+m-2)^2 \left|v_{j,l}\right|^2 r^{-4}\right] \times$$

$$\times r^{\alpha+m-1} dr + (m-1)\left|v'_{j,l}(1)\right|^2 - (m-1)(\alpha+m-2)\int_0^1 \left|v'_{j,l}\right|^2 \times$$

$$\times r^{\alpha+m-3} dr - 2j(j+m-2)v'_{j,l}(1)v_{j,l}(1) + 2j(j+m-2)\int_0^1 \left|v'_{j,l}\right|^2 \times$$

$$\times r^{\alpha+m-3} dr + 2(\alpha+m-3)j(j+m-2)\left[\frac{\left|v_{j,l}\right|^2}{2} r^{\alpha+m-4}\Big|_0^1 -\right.$$

$$\left.-\frac{(\alpha+m-4)}{2}\int_0^1 \left|v_{j,l}\right|^2 r^{\alpha+m-5} dr\right] - (m-1)j(j+m-2)\left|v_{j,l}(1)\right|^2 +$$

$$+(m-1)(\alpha+m-4)j(j+m-2)\int_0^1 \left|v_{j,l}\right|^2 r^{\alpha+m-5} dr.$$

After simple calculations we come to (2.18). □

Lemma 2.5. *For any $u \in W^{(2)}_{2,\alpha}(B)$ satisfying (2.2) the inequality*

$$\int_B \left|D^2 u\right|^2 r^\alpha dx \le \left(1+M_\gamma^2\right)\int_B |\Delta u|^2 r^\alpha dx - (m-1)\int_{\partial B} |u_r|^2 dS +$$
$$+\frac{(m-2+2\gamma)(m-1)+mM_\gamma^2}{m}\sum_{i=1}^m |u_i(0)|^2 |S| + \left[(m+1+2\gamma)M_\gamma^2 +\right.$$
$$\left.+\frac{m-2+2\gamma}{2}\right](m-1)|u(0)|^2 |S|, \tag{2.19}$$

where

$$M_\gamma^2 = \frac{(m-2+2\gamma)\{1+\gamma)^2 + [2-(1-\gamma)^2]m\}}{(m+1+\gamma)^2(1-\gamma)^2} \tag{2.20}$$

holds.

Proof. From (2.18) we have

$$\sum_{j,l}\int_0^1\left\{\left|v''_{j,l}\right|^2+[(m-1)(1-\alpha)+2j(j+m-2)]\left|v'_{j,l}\right|^2 r^{-2}+j(j+m-2)\times\right.$$

$$\left.\times[j(j+m-2)+(2-\alpha)(\alpha+m-4)]\left|v_{j,l}\right|^2 r^{-4}\right\}r^{\alpha+m-1}dr=$$

$$=\int_B|\Delta v|^2 r^\alpha dx-(m-1)\int_{\partial B}|v'_r|^2\,dS+2\sum_{j,l}j(j+m-2)v'_{j,l}(1)v_{j,l}(1)-$$

$$-(\alpha-2)\sum_{j,l}j(j+m-2)v^2_{j,l}(1).$$

According to [2] (p. 51 and p. 54) for $\alpha=2-m-2\gamma(0<\gamma<1)$ we have

$$\alpha\sum_{j,l}\int_0^1\left[(m-1)\left|v'_{j,l}\right|^2+(\alpha+m-3)j(j+m-2)\left|v'_{j,l}\right|^2 r^{-2}\right]r^{\alpha+m-3}dr\le$$

$$\le M_\gamma^2\sum_{j,l}\int_0^1\{\left|v''_{j,l}\right|^2+[(m-1)(1-\alpha)+2j(j+m-2)\,]\left|v'_{j,l}\right|^2 r^{-2}+$$

$$+j(j+m-2)\,[j(j+m-2)+(2-\alpha)\,(\alpha+m-4)\,]\left|v_{j,l}\right|^2 r^{-4}\,\}\,r^{\alpha+m-1}dr.$$

The fact that in this case v and ∇v can differ from zero on ∂B plays no role. Then from (2.13) and (2.18) we have

$$\int_B\left(\left|D^2u\right|-|\Delta u|^2\right)r^\alpha dx\le-(m-1)\int_{\partial B}|u_r|^2\,dS+\frac{\alpha}{2}\int_{\partial B}|v'_r|^2\,dS-$$

$$-\frac{\alpha}{2}\int_{\partial B}|\nabla v|^2\,dS+M_\gamma^2\left[\int_B|\Delta v|^2\,r^\alpha dx-(m-1)\int_{\partial B}|v'_r|^2\,dS+\right.$$

$$\left.+2\sum_{j,l}j(j+m-2)v'_{j,l}(1)v_{j,l}(1)-(\alpha-2)\sum_{j,l}j(j+m-2)v^2_{j,l}(1)\right]-$$

$$-\frac{\alpha}{2}\sum_{j,l}j(j+m-2)\left|v_{j,l}(1)\right|^2.$$

Since u vanishes on ∂B and, according to (2.11), v is a linear function on ∂B we have that $v_{j,l}=0$ on ∂B for $j>1$. Therefore

$$\left|\sum_{j,l}j(j+m-2)v'_{j,l}(1)v_{j,l}(1)\right|=(m-1)\left|\sum_{l=1}^{k_1}v'_{1,l}(1)v_{1,l}(1)\right|\le$$

$$\le\left(\sum_{l=1}^{k_1}|v'_{1,l}(1)|^2\right)^{\frac12}\left(\sum_{l=1}^{k_1}|v_{1,l}(1)|^2\right)^{\frac12}(m-1)\le\left(\int_{\partial B}|v'_r|^2\,dS\right)^{\frac12}\left(\int_{\partial B}|v|^2dS\right)^{\frac12}(m-1).$$

So

$$\left|\sum_{j,l} j(j+m-2)v'_{j,l}(1)v_{j,l}(1)\right| \le (m-1)\left(\int\limits_{\partial B} |v'_r|^2\,dS\right)^{\frac{1}{2}}\left(\int\limits_{\partial B} |v|^2\,dS\right)^{\frac{1}{2}}. \quad (2.21)$$

In the same way we come to the inequality

$$\sum_{j,l} j(j+m-2)\,|v_{j,l}(1)|^2 \le (m-1)\int\limits_{\partial B} |v|^2 dS. \quad (2.22)$$

With the help of (2.21) and (2.22) we get

$$\int\limits_B \left|D^2u\right|^2 r^\alpha dx \le \left(1+M_\gamma^2\right)\int\limits_B |\Delta v|^2 r^\alpha dx - (m-1)\int\limits_{\partial B} |u'_r|^2\,dS - \frac{\alpha}{2}\int\limits_{\partial B} |\nabla v|^2 dS +$$

$$+\frac{\alpha}{2}\int\limits_{\partial B} |v'_r|^2\,dS + (m-1)M_\gamma^2\,[\,-\int\limits_{\partial B} |v'_r|^2\,dS + 2\left(\int\limits_{\partial B} |v'_r|^2\,dS\right)^{\frac{1}{2}} \times$$

$$\times\left(\int\limits_{\partial B} |v|^2\,dS\right)^{\frac{1}{2}} - (\alpha-2)\int\limits_{\partial B} |v|^2 dS\,] - \frac{\alpha}{2}(m-1)\int\limits_{\partial B} |v|^2 dS. \quad (2.23)$$

Let us estimate now the right-hand side of (2.23). Evidently from (2.11) we have

$$\frac{\alpha}{2}\int\limits_{\partial B} \left(|v'_r|^2 - |\nabla v|^2\right) dS = \frac{\alpha}{2}\int\limits_{\partial B} \left\{\left[u'_r - \left(\sum_{i=1}^m u_i(0)x_i\right)'_r\right]^2 - \sum_{i=1}^m [u_i - u_i(0)]^2\right\} dS =$$

$$= \frac{\alpha}{2}\int\limits_{\partial B} \left[|u'_r|^2 - 2u'_r \sum_{i=1}^m u_i(0)\cos(r,x_i) + \left(\sum_{i=1}^m u_i(0)\cos(r,x_i)\right)^2 -\right.$$

$$\left. - \sum_{i=1}^m u_i^2 + 2\sum_{i=1}^m u_i u_i(0) - \sum_{i=1}^m u_i^2(0)\right] dS.$$

Taking in account that on ∂B $\quad |\nabla u|^2 = u_r^2$ and $u_i = u_r\cos(r,x_i)$, after cancelling some terms we come to the equality

$$\frac{\alpha}{2}\int\limits_{\partial B} \left(|v'_r|^2 - |\nabla v|^2\,\right) dS = \frac{\alpha}{2}\int\limits_{\partial B} \left\{\sum_{i=1}^m u_i^2(0)\,\left[\cos^2(r,x_i) - 1\right] +\right.$$

$$\left. +2\sum_{i<k} u_i(0)u_k(0)\cos(r,x_i)\cos(r,x_k)\,\right\} dS.$$

After easy calculations we have

$$\frac{\alpha}{2}\int\limits_{\partial B} \left(|v'_r|^2 - |\nabla v|^2\,\right) dS = \frac{\alpha(1-m)}{2m}\sum_{i=1}^m u_i(0)|S|. \quad (2.24)$$

Applying the inequality

$$2ab < a^2 + b^2$$

to the middle term in quadratic brackets on the right-hand side of (2.23) and taking into account that $\alpha - 3 = -(m+1+2\gamma)$, we'll have

$$\int_B \left|D^2u\right| r^\alpha dx \le (1+M_\gamma^2)\int_B |\Delta u|^2 r^\alpha dx - (m-1)\int_{\partial B} |u_r'|^2\, dS +$$
$$+\frac{(m-2+2\gamma)(m-1)}{2m}\sum_{i=1}^m u_i^2(0)|S| + (m+1+2\gamma)(m-1)M_\gamma^2\int_{\partial B} |v|^2 dS -$$
$$-\alpha\frac{(m-1)}{2}\int_{\partial B} |v|^2 dS.$$

Since

$$\int_{\partial B} |v|^2 dS = u^2(0)|S| + \sum_{i=1}^m u_i^2(0)|S|m^{-1}$$

we come to (2.19). □

Lemma 2.6. *If $u \in W_{2,\alpha}^{(2)}(B)$ satisfies (2.2), then for any $\eta > 0$ the inequality*

$$\int_B |D^2u|^2 r^\alpha dx \left\{1 - \eta|S|\left[\frac{(m-2+2\gamma)(m-1)+mM_\gamma^2}{m}+\right.\right.$$
$$\left.\left.+\frac{2(m+1+2\gamma)M_\gamma^2+m-2+2\gamma}{(1-\gamma)^2}(m-1)\right]\right\} \le$$
$$\le (1+M_\gamma^2)\int_B |\Delta u|^2 r^\alpha dx + C_0(\eta)|S|\left\{\frac{(m-2+2\gamma)(m-1)+mM_\gamma^2}{m}\times\right.$$
$$\left.\times\int_B |\nabla u|^2 dx + (m-1)\left[(m+1+2\gamma)M_\gamma^2 + (m-2+2\gamma)/2\right]\int_B |u|^2 dx\right\} +$$
$$+(m-1)\left[|S|\frac{(m+1+2\gamma)M_\gamma^2+(m-2+2\gamma)/2}{1-\gamma}\eta - 1\right]\int_{\partial B} |u_r'|^2 dS \qquad (2.25)$$

takes place. Here $\alpha = 2-m-2\gamma$ $(0<\gamma<1), C_0(\eta)$ and M_γ^2 are determined by (2.6) and (2.20). The value $|S|$ (the area of the unit sphere in R^m) is determined by the formula

$$|S| = 2\pi^{\frac{m}{2}}\Gamma^{-1}(\frac{m}{2}).$$

Proof. From the identity

$$u_i = u_i\,|_{r=1} + \int_1^r (u_i)_\rho'\, d\rho$$

follows the inequality

$$\int_B u_i^2 r^\alpha dx \le 2\int_B (u_i\,|_{r=1})^2 r^\alpha dx + 2\int_B \left(\int_1^r u_{i\rho} d\rho\right)^2 r^\alpha dx. \qquad (2.26)$$

Evidently

$$\int_B (u_i|_{r=1})^2 r^\alpha dx = \frac{1}{2(1-\gamma)} \int_{\partial B} (u_i|_{r=1})^2 dS.$$

Since u satisfies the condition (2.2) we have

$$\sum_{i=1}^m \int_B (u_i|_{r=1})^2 r^\alpha dx = \frac{1}{2(1-\gamma)} \int_{\partial B} (u_r')^2 dS. \tag{2.27}$$

From the inequality of Hardy follows

$$\int_B \left(\int_1^r u_{i\rho} d\rho \right)^2 r^\alpha dx \le \frac{1}{(1-\gamma)^2} \int_B |u_{ir}|^2 r^{\alpha+2} dx.$$

So, taking in account that $r \le 1$ we come to the inequality

$$\sum_{i=1}^m \int_B \left(\int_1^r u_{i\rho} d\rho \right)^2 r^\alpha dx \le \frac{1}{(1-\gamma)^2} \int_B \left|D^2 u\right|^2 r^\alpha dx.$$

Applying (2.26) and (2.27) we get

$$\int_B |Du|^2 r^\alpha dx \le \frac{1}{1-\gamma} \int_{\partial B} |u_r'|^2 dS + \frac{2}{(1-\gamma)^2} \int_B \left|D^2 u\right|^2 r^\alpha dx. \tag{2.28}$$

Now combining (2.4), (2.5), (2.19) and (2.28) we come to the inequality (2.25). □

Corollary 2.2. *We can also apply the inequalities* (2.4) *and* (2.5). *Then we come to the relation*

$$\int_B \left|D^2 u\right|^2 r^\alpha dx \left\{ 1 - \eta |S| \left[\frac{(m-2+2\gamma)(m-1) + mM_\gamma^2}{m} + \right.\right.$$
$$\left.\left. + \frac{2(m+1+2\gamma)M_\gamma^2 + m - 2 + 2\gamma}{(1-\gamma)^2}(m-1) \right] \right\} \le$$
$$\le \left(1 + M_\gamma^2\right) \int_B |\Delta u|^2 r^\alpha dx + \frac{C_0(\eta)|S|}{\lambda} \left\{ \frac{(m-2+2\gamma)(m-1) + mM_\gamma^2}{m} + \right.$$
$$\left. + \frac{(m-1)\left[(m+1+2\gamma)M_\gamma^2 + (m-2+2\gamma)/2\right]}{\lambda} \right\} \int_B |\Delta u|^2 dx +$$
$$+ (m-1)\left[|S| \frac{(m+1+2\gamma)M_\gamma^2 + (m-2+2\gamma)/2}{1-\gamma} \eta - 1 \right] \int_{\partial B} |u_r'|^2 dS. \tag{2.29}$$

□

Corollary 2.3. *Taking in account that $r \le 1$, we can write*

$$\int_B |\Delta u|^2 dx \le \int_B |\Delta u|^2 r^\alpha dx$$

and from (2.29) *then follows*

$$\int_B |D^2 u|^2 r^\alpha dx \left\{ 1 - \eta |S| \left[\frac{(m-2+2\gamma)(m-1) + m M_\gamma^2}{m} + \right.\right.$$
$$\left.\left. + \frac{2(m+1+2\gamma)M_\gamma^2 + m - 2 + 2\gamma}{(1-\gamma)^2}(m-1) \right] \right\} \le$$
$$\le \left(1 + M_\gamma^2 + \frac{C_0(\eta)|S|}{\lambda} \left\{ \frac{(m-2+2\gamma)(m-1) + m M_\gamma^2}{m} + \right.\right.$$
$$\left.\left. + \frac{(m-1)\left[(m+1+2\gamma)M_\gamma^2 + (m-2+2\gamma)/2\right]}{\lambda} \right\} \right) \int_B |\Delta u|^2 r^\alpha dx +$$
$$+ (m-1) \left[|S| \frac{(m+1+2\gamma)M_\gamma^2 + (m-2+2\gamma)/2}{1-\gamma} \eta - 1 \right] \int_{\partial B} |u_r'|^2 dS. \quad (2.30)$$

It is easy to see, that if

$$1 - \eta |S| \left[\frac{(m-2+2\gamma)(m-1) + m M_\gamma^2}{m} + \frac{2(m+1+2\gamma)M_\gamma^2 + m - 2 + 2\gamma}{(1-\gamma)^2}(m-1) \right]$$

is nonnegative, then the expression

$$|S| \frac{(m+1+2\gamma)M_\gamma^2 + (m-2+2\gamma)/2}{1-\gamma} \eta - 1 \quad (2.31)$$

is nonpositive. □

After rescaling *in x* we come to the following

Theorem 2.1. *Let $u \in W_{2,\alpha}^{(2)}(B_R)$ satisfies the condition* (2.2). *Let also the inequality*

$$E \equiv 1 - \eta |S| \left[\frac{(m-2+2\gamma)(m-1) + m M_\gamma^2}{m} + \right.$$
$$\left. + \frac{2(m+1+2\gamma)M_\gamma^2 + m - 2 + 2\gamma}{(1-\gamma)^2}(m-1) \right] > 0 \quad (2.32)$$

holds. Then the following estimates

$$\int_{B_R} |D^2u|^2 r^\alpha dx \le \frac{1}{E}\left\{(1+M_\gamma^2)\int_{B_R} |\Delta u|^2 r^\alpha dx + C_0(\eta)|S|\times\right.$$

$$\times\left[\frac{(m-2+2\gamma)(m-1)+mM_\gamma^2}{m}R^{\alpha-2}\int_{B_R}|\nabla u|^2dx+\right.$$

$$\left.\left.+\ (m-1)\left[(m+1+2\gamma)M_\gamma^2+(m-2+2\gamma)/2\right]R^{\alpha-4}\int_{B_R}|u|^2dx\right]\right\}, \qquad (2.33)$$

$$\int_{B_R} |D^2u|^2 r^\alpha dx \le \frac{1}{E}\left\{(1+M_\gamma^2)\int_{B_R} |\Delta u|^2 r^\alpha dx + \frac{C_0(\eta)|S|}{\lambda}R^\alpha\times\right.$$

$$\times\left[\frac{(m-2+2\gamma)(m-1)+mM_\gamma^2}{m}+\right.$$

$$\left.\left.+\frac{(m-1)\left((m+1+2\gamma)M_\gamma^2+(m-2+2\gamma)/2\right)}{\lambda}\right]\int_{B_R}|\Delta u|^2dx\right\} \qquad (2.34)$$

and

$$\int_{B_R} |D^2u|^2 r^\alpha dx \le \frac{1}{E}\left\{1+M_\gamma^2+\frac{C_0(\eta)|S|}{\lambda}\left[\frac{(m-2+2\gamma)(m-1)+mM_\gamma^2}{m}+\right.\right.$$

$$\left.\left.+\frac{(m-1)\left((m+1+2\gamma)M_\gamma^2+(m-2+2\gamma)/2\right)}{\lambda}\right]\right\}\int_{B_R}|\Delta u|^2 r^\alpha dx \qquad (2.35)$$

take place.
Let us remind, that $\alpha = 2-m-2\gamma$ $(0<\gamma<1)$, M_γ and $C_0(\eta)$ are correspondingly defined by (2.20) and (2.6), λ is least absolute value of the eigenvalues for the operator Δ in B with condition (2.2) and

$$|S| = 2\pi^{m/2}\Gamma^{-1}(m/2)$$

is the area of the unit sphere in R^m.
Consider now the cylinder $Q_R = (0,T)\times B_R (Q_1 = Q)$ with boundary conditions

$$u|_{\partial B_R} = u|_{t=0} = 0 \qquad (2.36)$$

for a function $u(t,x)$ given in Q. Denote $\beta = -\alpha$.
For $m>2$ the inequality

$$\int_Q |\Delta u|^2 r^\beta dxdt \le \frac{m}{m-\beta}\int_Q |\varepsilon\dot{u}-\Delta u|^2 r^\beta dxdt \qquad (2.37)$$

was proven in [3] (ε is an arbitrary nonnegative value).

Lemma 2.7. *Let $m = 2$ and therefore $\beta = 2\gamma$ $(0 < \gamma < 1)$. Suppose that u satisfies (2.36) and $u \in L_2\left\{(0,T); W^{(2)}_{2,\beta}(B)\right\}$. Then the inequality*

$$\int\limits_Q |\Delta u|^2 r^\beta dx dt \leq \left[1 + \frac{\beta}{2-\beta} + \frac{2-\beta}{(1-\frac{\beta}{4})^2\beta}\right] \int\limits_Q |\varepsilon\dot{u} - \Delta u|^2 r^\beta dx dt \qquad (2.38)$$

holds, where ε is an arbitrary nonnegative constant.

Proof. Denote

$$\varepsilon\dot{u} - \Delta u = f, \qquad (2.39)$$

multiply this equality by $\Delta u \cdot r^\beta$ and integrate by parts on the left–hand side. Then according to lemma 2 in our paper ([3], (2.38)) we get

$$\frac{\varepsilon}{2}\int\limits_B |\nabla u|^2 r^\beta dx\,|_{t=T} + \beta\int\limits_Q \Delta u\, u'_r dx dt + \int\limits_B |\Delta u|^2 r^\beta dx dt = \int\limits_Q f(\Delta u + \beta u'_r r^{-1}) r^\beta dx dt.$$

After using for $u(x,t)$ the expansion, analogous to (2.12), according to the same lemma in [3] ((2.39)), we come to the inequality

$$\int\limits_Q |\Delta u|^2 r^\beta dx dt + \frac{\beta(2-\beta)}{2}\sum_{s,k}\int\limits_0^T\int\limits_0^1 |u'_{s,k}|^2 r^{\beta-1} dr dt \leq$$

$$\leq \int\limits_Q f(\Delta u + \beta u'_r r^{-1}) r^\beta dx dt + \frac{\beta(2-\beta)}{2}\sum_{s,k} s^2 \int\limits_0^T\int\limits_0^1 |u_{s,k}|^2 r^{\beta-3} dr. \qquad (2.40)$$

Now we have to estimate the right–hand side term of (2.40). Like in [3] we multiply (2.39) by $u_{s,k} r^{\beta-2} (s \geq 1)$. Integrating by parts we come to the inequality ([3], (3.30) etc.)

$$\left[(1-\frac{\beta}{4})\beta + s^2 - 1\right]\int\limits_0^T\int\limits_0^1 |u_{s,k}|^2 r^{\beta-3} dr dt \leq \Big|\int\limits_0^T\int\limits_0^1 f_{s,k} u_{s,k} r^{\beta-1} dr dt\Big|.$$

It is clear that

$$\sum_{s\geq 1} s^2 \int\limits_0^T\int\limits_0^1 |u_{s,k}|^2 r^{\beta-3} dr dt \leq \frac{1}{(1-\frac{\beta}{4})\beta}\sum_{s\geq 1}\left[(1-\frac{\beta}{4})\beta + s^2 - 1\right]\int\limits_0^T\int\limits_0^1 |u_{s,k}|^2 r^{\beta-3} dr dt.$$

So, we have

$$\sum_{s\geq 1} s^2 \int\limits_0^T\int\limits_0^1 |u_{s,k}|^2 r^{\beta-3} dr dt \leq \frac{1}{(1-\frac{\beta}{4})\beta}\left|\int\limits_0^T\int\limits_0^1 f_{s,k} u_{s,k} r^{\beta-1} dr dt\right|.$$

After applying the Hölder's inequality we get

$$\sum_{s\geq 1} s^2 \int_0^T \int_0^1 |u_{s,k}|^2 r^{\beta-3} dr dt \leq \frac{1}{(1-\frac{\beta}{4})^2\beta^2} \int_Q |f|^2 r^\beta dx.$$

With the help of (2.40) we come to the inequality

$$\int_Q |\Delta u|^2 r^\beta dx dt + \frac{\beta(2-\beta)}{2} \sum_{s,k} \int_0^T \int_0^1 \left|u'_{s,k}\right|^2 r^{\beta-1} dr dt \leq$$

$$\leq \frac{2-\beta}{2(1-\frac{\beta}{4})^2\beta} \int_Q |f|^2 r^\beta dx dt + \int_Q f\left(\Delta u + \beta u'_r r^{-1}\right) r^\beta dx dt. \tag{2.41}$$

Applying the well known inequalities we get

$$\int_Q f\left(\Delta u + \beta u'_r r^{-1}\right) r^\beta dx dt \leq \left|\int_Q f \Delta u r^\beta dx dt\right| + \eta\beta \int_Q |u'_r|^2 r^{\beta-2} dx dt + \frac{\beta}{4\eta} \int_Q |f|^2 r^\beta dx dt.$$

According to the expansion (2.12)

$$\int_Q |u'_r|^2 r^{\beta-2} dx dt = \sum_{s,k} \int_0^T \int_0^1 \left|u'_{s,k}\right|^2 r^{\beta-1} dr dt$$

and we can write

$$\left|\int_Q f(\Delta u + \beta u'_r r^{-1}) r^\beta dx dt\right| \leq \left|\int_Q f \Delta u r^\beta dx dt\right| +$$

$$+\eta\beta \sum_{s,k} \int_0^T \int_0^1 \left|u'_{s,k}\right|^2 r^{\beta-1} dr dt + \frac{\beta}{4\eta} \int_Q |f|^2 r^\beta dx dt.$$

So, from (2.41) we come to

$$\int_Q |\Delta u|^2 r^\beta dx dt + \frac{\beta(2-\beta)}{2} \sum_{s,k} \int_0^T \int_0^1 |u'_{s,k}|^2 r^{\beta-1} dr dt \leq \left|\int_Q f \Delta u r^\beta dx dt\right| +$$

$$+\eta\beta \sum_{s,k} \int_0^T \int_0^1 \left|u'_{s,k}\right|^2 r^{\beta-1} dr dt + \frac{\beta}{4\eta} \int_Q |f|^2 r^\beta dx dt + \frac{2-\beta}{2(1-\frac{\beta}{4})^2\beta} \int_Q |f|^2 r^\beta dx dt.$$

Taking $\eta = \frac{2-\beta}{2}$ and applying the inequality

$$\left|\int_Q f \Delta u r^\beta dx dt\right| \leq \frac{1}{2} \int_Q |f|^2 r^\beta dx dt + \frac{1}{2} \int_Q |\Delta u|^2 r^\beta dx dt,$$

we get (2.38). □

Denote

$$A^2_{\alpha,m} = \begin{cases} 1 - \frac{\alpha}{2+\alpha} - \frac{2+\alpha}{(1+\frac{\alpha}{4})^2\alpha} & m = 2 \\ \frac{m}{m+\alpha} & m > 2. \end{cases} \tag{2.42}$$

Theorem 2.2. *Suppose $u \in L_2\left\{(0,T); W^{(2)}_{2,\alpha}(B_R)\right\}$ and satisfies the boundary conditions (2.36). Then the following estimates*

$$\int\limits_{Q_R} |D^2u|^2 r^\alpha \zeta dxdt \le \frac{1}{E}\{A^2_{\alpha,m}(1+M^2_\gamma)\int\limits_{Q_R} |\varepsilon\dot{u}-\Delta u|^2 r^\alpha\zeta dxdt + C_0(\eta)|S| \times$$
$$\times \left[\frac{(m-2+2\gamma)(m-1)+mM^2_\gamma}{m} R^{\alpha-2}\int\limits_{Q_R} |\nabla u|^2 dxdt + \right.$$
$$+(m-1)[(m+1+2\gamma)M^2_\gamma + (m-2+2\gamma)/2]R^{\alpha-4}\times$$
$$\times \int\limits_{Q_R} |u|^2 dxdt] \} + CR^\alpha \int\limits_{Q_R} |\varepsilon\dot{u}-\Delta u|^2\, dxdt, \tag{2.43}$$

$$\int\limits_{Q_R} |D^2u|^2 r^\alpha \zeta dxdt \le \frac{1}{E}\left\{A^2_{\alpha,m}(1+M^2_\gamma)\int\limits_{Q_R} |\varepsilon\dot{u}-\Delta u|^2 r^\alpha\zeta dxdt + \frac{C_0(\eta)|S|}{\lambda}R^\alpha \times\right.$$
$$\times\left[\frac{(m-2+2\gamma)(m-1)+mM^2_\gamma}{m} + \frac{(m-1)\left((m-1+2\gamma)M^2_\gamma+(m-2+2\gamma)/2\right)}{\lambda}\times\right]$$
$$\left.\times\int\limits_{Q_R} |\Delta u|^2 dxdt\right\} + CR^\alpha\int\limits_{Q_R} |\varepsilon\dot{u}-\Delta u|^2 dxdt \tag{2.44}$$

and

$$\int\limits_{Q_R} |D^2u|^2 r^\alpha \zeta dxdt \le \frac{1}{E}\left\{A^2_{\alpha,m}(1+M^2_\gamma) + \frac{C_0(\eta)|S|}{\lambda}\times\right.$$
$$\left.\times\left[\frac{(m-2+2\gamma)(m-1)+mM^2_\gamma}{m} + \frac{(m-1)\left((m+1+2\gamma)M^2_\gamma+(m-2+2\gamma)/2\right)}{\lambda}\right]\right\}$$
$$\int\limits_{Q_R} |\varepsilon\dot{u}-\Delta u|^2 r^\alpha\zeta dxdt + CR^\alpha\int\limits_{Q_R} |\varepsilon\dot{u}-\Delta u|^2 dxdt \tag{2.45}$$

hold where ε and T are arbitrary positive values, and all other constants are defined at the end of the formulation of theorem (2.1) and by (2.42).(C doesn't depend on ε and R).
The function ζ is a smooth monotone cut–off function, defined by the relation

$$\zeta(r) = \begin{cases} 1 & 0 \le r \tfrac{1}{2} R \\ smooth & \tfrac{1}{2}R \le r \le \tfrac{3}{4}R \\ 0 & \tfrac{3}{4}R \le r\ . \end{cases} \tag{2.46}$$

Proof. We can assume at first that u is as smooth as we wish. Let $w(t,x)$ be a solution in Q of the follwing boundary value problem

$$\varepsilon\dot{w} + \Delta w = -\Delta u r^\alpha \zeta, \tag{2.47}$$

$$w|_{t=T} = w|_{\partial B_R} = 0.$$

Multiply the equation (2.47) by $\Delta u\zeta$ and integrate once by parts with respect to t and twice with respect to x. Then we shall get

$$\int_Q (\varepsilon\dot{u} - \Delta u)\Delta w\zeta dxdt = \int_Q |\Delta u|^2 r^\alpha \zeta^2 dxdt + \dots,$$

where the nonwritten terms contain the derivatives of ζ under the sign of the integrals. Applying the Hölder inequality, we get

$$\begin{aligned}\left(\int_Q |\Delta u|^2 r^\alpha \zeta^2 dxdt\right)^2 &\le \left(\int_Q |\varepsilon\dot{u} - \Delta u|^2 r^\alpha \zeta^2 dxdt\right)^{\frac{1}{2}} \left(\int_Q |\Delta w|^2 r^{-\alpha} dxdt\right)^{\frac{1}{2}} + \\ &+ C\int_Q |\varepsilon\dot{u} - \Delta u|^2 dxdt. \qquad (2.48)\end{aligned}$$

It is trivial that w also satisfies the inequalities (2.37) and (2.38) (you should only change t for $T-t$). Therefore

$$\begin{aligned}\int_Q |\Delta w|^2 r^{-\alpha} dxdt &\le A^2_{\alpha,m} \int_Q |\varepsilon\dot{w} + \Delta w|^2 r^{-\alpha} dxdt = \\ &= A^2_{\alpha,m} \int_Q |\Delta u|^2 r^\alpha \zeta^2 dxdt.\end{aligned}$$

Now from (2.48) after rescaling we get the results of the theorem, if we take into account that

$$\int_Q |\Delta u|^2 r^\alpha dxdt \ge \int_Q |\Delta u|^2 r^\alpha \zeta^2 dxdt - C\int_Q |\varepsilon\dot{u} - \Delta u|^2 dxdt.$$

□

Let us return now to the inequalities (2.4) and (2.5) of *lemma*(2.1). Since the power of integrals on the right hand side of these inequalities is equal to one they belong to the so called class of the linear inequalities. But in some problems there is important to have the so called multiplicative inequalities. We shall obtain them now.

Lemma 2.8. *If $u \in W^{(2)}_{2,\alpha}(B_R)$ and (2.2) takes place, then the inequalities*

$$\begin{aligned}|u(0)|^2 &\le C\left(\int_{B_R} |Du|^2 r^\alpha dx\right)^{\frac{m}{m+2\gamma}} \left(\int_{B_R} |u|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}}, \\ \sum_{i=0}^m |u_i(0)|^2 &\le C\left(\int_B \left|D^2 u\right|^2 r^\alpha dx\right)^{\frac{m}{m+2\gamma}} \left(\int_B |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} \qquad (2.49)\end{aligned}$$

hold.

Proof. Evidently it's enough to prove only the first of the inequalities (2.49). Substitute in (2.4) the expression (2.6). We get

$$|u(0)|^2 < \eta \int_{B_R} |Du|^2 r^\alpha dx + C\eta^{-\frac{m}{2\gamma}} \int_{B_R} |u|^2 dx.$$

Take now

$$\eta = (\int_{B_R} |Du|^2 r^\alpha dx)^{-2\gamma/(m+2\gamma)} (\int_{B_R} |u|^2 dx)^{2\gamma/(m+2\gamma)}$$

and we come to (2.49) (if $|Du| = 0$ then $u \equiv 0$ and (2.49) is trivial.) $\square$

Remark. Under assumption of the lemma the inequality

$$|u(0)|^2 \leq C(\int_{B_R} |D^2 u|^2 r^\alpha dx)^{\frac{m}{m+2\gamma}} (\int_{B_R} |Du|^2 dx)^{\frac{2\gamma}{m+2\gamma}} \tag{2.50}$$

holds.
In fact

$$\begin{aligned} \int_B |Du|^2 r^\alpha dx &\leq 2(\int_{B_R} |Du - Du|_0|^2 r^\alpha dx) + 2\int_{B_R} |Du|_0|^2 r^\alpha dx \\ &\leq C[\int_{B_R} |Du - Du|_0|^2 r^{\alpha-2} dx + |Du|_0|^2]. \end{aligned}$$

From the very well known Hardy's inequality and from (2.5) follows

$$\int_{B_R} |Du|^2 r^\alpha dx \leq C(\int_{B_R} |D^2 u|^2 r^\alpha dx + \int_{B_R} |Du|^2 dx).$$

Then from (2.2), $r \leq 1$ and $\alpha < 0$ we get

$$\begin{aligned} \int_B |Du|^2 r^\alpha dx &\leq C(\int_{B_R} |D^2 u|^2 r^\alpha dx + \int_{B_R} |D^2 u|^2 dx) \\ &\leq C \int_{B_R} |D^2 u|^2 r^\alpha dx. \end{aligned}$$

Applying (2.49) and (2.50) to (2.19) we come to

Theorem 2.3. *Let $u \in W^{(2)}_{2,\alpha}(B_R)$ and satisfy (2.2). Then the inequality*

$$\begin{aligned} \int_{B_R} |D^2 u|^2 r^\alpha dx &\leq (1 + M_\gamma^2) \int_{B_R} |\Delta u|^2 r^\alpha dx + C(\int_{B_R} |D^2 u|^2 r^\alpha dx)^{\frac{m}{m+2\gamma}} \times \\ &\times (\int_{B_R} |Du|^2 dx)^{\frac{2\gamma}{m+2\gamma}} \end{aligned} \tag{2.51}$$

takes place.

The proof follows immediately after applying the estimates (2.49) and (2.50) . We get

$$\int\limits_{B_R} |D^2u|^2 r^\alpha dx \le (1+M_\gamma^2)\int\limits_{B_R} |\Delta u|^2 r^\alpha dx + C\left[\left(\int\limits_{B_R} |D^2u|^2 r^\alpha dx\right)^{\frac{m}{m+2\gamma}} + \right.$$

$$\left. + \left(\int\limits_{B_R} |Du|^2 r^\alpha dx\right)^{\frac{m}{m+2\gamma}}\right] \left(\int\limits_{B_R} |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}},$$

From (2.51) and

$$\int\limits_{B_R} |u|^2 dx \le \int\limits_{B_R} |Du|^2 dx$$

we come to the result. □

Suppose now that the condition (2.2) is not satisfied. How will change in this case the estimate (2.51).

Theorem 2.4. *Let $u \in W_{2,\alpha}^{(2)}(B_R)$. Then the inequality*

$$\begin{aligned}\int\limits_{B_R} |D^2u|^2 r^\alpha \zeta dx &\le (1+M_\gamma^2+\eta)\int\limits_{B_R} |\Delta u|^2 r^\alpha \zeta dx + \\ &+ C\Bigg\{\left(\int\limits_{B_R} |D^2u|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left[\int\limits_{B_R} (|Du|^2+|u|^2) dx\right]^{\frac{2\gamma}{m+2\gamma}} + \\ &+ \int\limits_{B_R} \left(|Du|^2+|u|^2\right) dx\Bigg\}\end{aligned} \tag{2.52}$$

takes place, where ζ is defined by (2.46) and η is an arbitrary small positive number.

□

The result follows immediately if you substitute in (2.51) instead of u the function $u\zeta$.

Theorem 2.5. *Let $u \in L_2\{(o,T)\ W_{2,\alpha}^{(2)}(B_R)\}$ and satisfies only the second of the condition (2.36) $u = 0$ when $t = 0$. Then the estimate*

$$\begin{aligned}\int\limits_{Q_R} \left|D^2u\right|^2 r^\alpha \zeta dxdt &\le \left(1+M_\gamma^2+\eta\right) A_{\alpha,m}^2 \int\limits_{Q_R} |\varepsilon \dot{u} - \Delta u|^2 r^\alpha \zeta dxdt + \\ &+ C\Bigg\{\left(\int\limits_{Q_R} \left|D^2u\right|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left[\int\limits_{Q_R} \left(|Du|^2+|u|^2\right) dxdt\right]^{\frac{2\gamma}{m+2\gamma}} + \\ &+ \int\limits_{Q_R} \left(|Du|^2+|u|^2\right) dxdt\Bigg\}\end{aligned} \tag{2.53}$$

holds, where C doesn't depend on ε.

Proof. Take a function w, which satisfies the equation (2.47) with the same conditions and multiply both sides of the differential equation by $\Delta u \cdot \zeta$. After integration over B_R we come to

$$\int_{Q_R} (\varepsilon \dot{w} + \Delta w) \Delta u \zeta^2 dx dt = - \int_{Q_R} |\Delta u|^2 r^\alpha \zeta dx dt.$$

After integrating on the left–hand side two times by x, we get

$$\begin{aligned} \int_{Q_R} [\varepsilon \Delta \dot{w} \zeta u + \Delta w \Delta u \zeta] \, dx dt &= \int_{Q_R} |\Delta u|^2 r^\alpha \zeta^2 dx dt - \\ -2\varepsilon \int_{Q_R} \nabla \dot{w} \nabla \zeta \cdot u dx dt &- \varepsilon \int_{Q_R} \dot{w} \Delta \zeta u dx dt. \end{aligned}$$

Integrating on the left–hand side once by t and on the right–hand side in the second integral once by x, we get

$$\begin{aligned} \int_{Q_R} (\varepsilon \dot{u} - \Delta u) \Delta w \zeta dx dt &= \int_{Q_R} |\Delta u|^2 r^\alpha \zeta^2 dx dt + \\ +\varepsilon \int_{Q_R} \dot{w} u \cdot \Delta \zeta dx dt &- \varepsilon \int_{Q_R} \nabla \zeta \nabla u dx dt. \end{aligned}$$

So

$$\begin{aligned} \int_{Q_R} |\Delta u|^2 r^\alpha \zeta^2 dx dt &= \int_{Q_R} (\varepsilon \dot{u} - \Delta u) \Delta w \zeta dx dt - \\ - \int_{Q_R} (\varepsilon \dot{w} - \Delta w) u \Delta \zeta dx dt &+ \int_{Q_R} (\varepsilon \dot{w} - \Delta w) \nabla \zeta \nabla u dx dt - \\ - \int_{Q_R} u \Delta w \Delta \zeta dx dt &+ \int_{Q_R} \Delta w \nabla u \cdot \nabla \zeta dx dt. \end{aligned}$$

Let us now estimate the integrals on the right–hand side. After applying an elementary inequality we come to the following relations

$$1)\quad \left|\int_{Q_R}(\varepsilon\dot{u}-\Delta u)\Delta w\zeta dxdt\right| \le \frac{1}{4\eta}\int_{Q_R}|\varepsilon\dot{u}-\Delta u|^2 r^\alpha\zeta^2 dxdt + +\eta\int_{Q_R}|\Delta w|^2 r^{-\alpha}dxdt;$$

$$2)\quad \left|\int_{Q_R}(\varepsilon\dot{w}-\Delta w)u\Delta\zeta dxdt\right| \le \eta\int_{Q_R}|\varepsilon\dot{w}-\Delta w|^2 r^{-\alpha}dxdt + +\frac{1}{4\eta}\int_{Q_R}|u|^2|\Delta\zeta|^2 r^\alpha dxdt \le \eta_1\int_{Q_R}|\Delta u|^2 r^\alpha\zeta^2 dxdt + +C\int_{Q_R}|u|^2 dxdt(\eta_1>0- \text{ arbitrary});$$

$$3)\quad \left|\int_{Q_R}(\varepsilon\dot{w}-\Delta w)\nabla\zeta\nabla u dxdt\right| \le \eta_2\int_{Q_R}|\Delta u|^2 r^\alpha\zeta^2 dxdt + C\int_{Q_R}|\nabla u|^2 dxdt(\eta_2>0- \text{ arbitrary}).$$

Then after (2.37) we get

$$\begin{aligned}\int_{Q_R}|\Delta u|^2 r^\alpha\zeta^2 dxdt &\le \eta A^2_{\alpha,m}\int_{Q_R}|\Delta u|^2 r^\alpha\zeta^2 dxdt + \\ +\frac{1}{4\eta}\int_{Q_R}|\varepsilon\dot{u}-\Delta u|^2 r^\alpha\zeta^2 dxdt &+ \eta_1\int_{Q_R}|\Delta u|^2 r^\alpha\zeta^2 dxdt \\ &+ C\int_{Q_R}\left(|Du|^2+|u|^2\right)dxdt.\end{aligned}$$

Taking $\eta = A^2_{\alpha,m}/(2m)$ we receive the inequality

$$\begin{aligned}\int_{Q_R}|\Delta u|^2 r^\alpha\zeta^2 dxdt &\le \left(A^2_{\alpha,m}+\eta\right)\int_{Q_R}|\varepsilon\dot{u}-\Delta u|^2 r^\alpha\zeta^2 dxdt \\ &+ C\int_{Q_R}\left(|Du|^2+|u|^2\right)dxdt.\end{aligned}$$

After using Theorem 2.4. the proof of the theorem is concluded. □

3. Coercieve estimates for the Stoke's system in weighted spaces

Consider now at first the stationary Stoke's system

$$\left.\begin{aligned} \Delta u + \nabla p &= f \\ \operatorname{div} u &= 0 \end{aligned}\right\} \tag{3.1}$$

with normed p and boundary condition

$$u|_{\partial\Omega} = 0. \tag{3.2}$$

Using the inequalities (2.52) and (2.53) we shall derive some estimates with explicit constants for the solution of the problem (3.1), (3.2). The Stoke's system was very intensively discussed in many books and papers. We refer here only to the papers of V. Solonnikov [5]. From the results of these papers in particular follows that if $f \in W_q^{(k)}(\Omega)(q > 1)$ then the second derivatives of u and the first derivatives of p also belong to this space. The analogous result for the nonstationary system is also included there.

Suppose that $\Omega \subset \mathbb{R}^m$ is a bounded domain and $\partial\Omega$ is sufficiently smooth.

Theorem 3.1. *If $f \in L_{2,\alpha}(\Omega)$ with $\alpha = 2 - m - 2\gamma (0 < \gamma < 1)$ then the weak solution of the system* (3.1) *with the boundary condition* (3.2) *satisfies the inequalities*

$$\begin{aligned} \int_{B_R} |\nabla p|^2 r^\alpha \zeta dx \leq & \left[1 - \frac{\alpha(m-2)}{m-1} + \eta\right] \left[1 - \frac{\alpha(\alpha + m - 2)}{2(m-1)}\right]^{-2} \int_{B_R} |f|^2 r^\alpha \zeta dx + \\ & + C \int_{B_R} |p|^2 dx, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \int_{B_R} |D^2 u|^2 r^\alpha \zeta dx \leq & \left\{1 + \left[1 - \frac{\alpha(m-2)}{m-1}\right]^{\frac{1}{2}} \left[1 - \frac{\alpha(\alpha+m-2)}{2(m-1)}\right]^{-1} + \eta\right\}^2 \times \\ & \times (1 + M_\gamma^2) \int_{B_R} |f|^2 r^\alpha \zeta dx + C \left[\left(\int_{B_R} |D^2 u|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int_{B_R} |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \right. \\ & \left. + \int_{B_R} |p^2|^2 dx + \int_{B_R} (|Du|^2 + |u|^2) dx \right], \end{aligned} \tag{3.4}$$

where x_0 is an arbitrary point inside $\Omega, R <$ dist $(x_0, \partial\Omega), \eta =$ const > 0 is arbitrary and M_γ is defined by (2.20).

The proof of this theorem is analogous to the proof, which was given in [2] for the solution of Poisson equation. Let us sketch this proof. According to the above mentioned result of V. Solonnikov we can assume at first that both f and the solution u, p are as smooth as we wish. Take a point $x_0 \in \Omega$ and consider a ball $B_R(x_0)$ with $R <$ dist $(x_0, \partial\Omega)$. After rescaling we can consider only the ball $B_1(0) = B$.

Let $Y_{s,k}(\Theta)(\Theta \in S)$ be a complete orthonormal set of spherical functions and let

$$p(x) = \sum_{s=0}^{+q\infty} \sum_{k=1}^{ks} p_{s,k}(r) Y_{s,k}(\Theta). \tag{3.5}$$

Construct the function

$$v(x) = \sum_{s=0}^{+\infty} \sum_{k=1}^{ks} v_{s,k}(r) Y_{s,k}(\Theta), \qquad (3.6)$$

where

$$\begin{aligned} v_0(r) &= -\int_r^1 p_0'(\varrho)\varrho^\alpha d\varrho, \\ v_{s,k}(r) &= p_{s,k}(r) r^\alpha \quad (s \geq 1). \end{aligned}$$

Take the function

$$w(x) = v(x)\zeta(r), \qquad (3.7)$$

where the cut-off function $\zeta(r)$ is determined by (2.46).
Multiplying the Stoke's system (3.1) with ∇w and taking in account that

$$\int_\Omega \Delta u \nabla w dx = -\int_\Omega \Delta(divu) w dx = 0,$$

we come to the equality

$$\int_\Omega \nabla p \nabla w dx = \int_\Omega f \nabla w dx.$$

Integrating by parts and substituting the expansions (3.5), (3.6) for p and w we'll have

$$\int_\Omega \nabla p \nabla w dx = \int_0^1 p_0'^2 r^{\alpha+m-1} dr + \sum_{s\geq 1,k} \int_0^1 \Big\{ |p_{s,k}'|^2 + \\ + \Big[s(s+m-2) - \tfrac{\alpha(\alpha+m-2)}{2} \Big] |p_{s,k}|^2 r^{-2} \Big\} r^{\alpha+m-1} \zeta dr + \dots$$

where the unwritten terms contain only integrals without singularity.
From this immediately follows

$$\int_\Omega \nabla p \nabla w dx \geq \int_0^1 |p_0'|^2 r^{\alpha+m-1} dr + \sum_{s\geq 1,k} \int_0^1 \Big[|p_{s,k}'|^2 + \\ s(s+m-2) \min_{s\geq 1} \tfrac{s(s+m-2)-\alpha(\alpha+m-2)/2}{s(s+m-2)} |p_{s,k}|^2 r^{-2} \Big] r^{\alpha+m-1} \zeta dr + \dots$$

Finally

$$\int_\Omega \nabla p \nabla w dx \geq \left[1 - \frac{\alpha(\alpha+m-2)/2}{m-1} \right] \int_\Omega |\nabla p|^2 r^\alpha \zeta dx - c \int_\Omega |p|^2 dx.$$

From the other side

$$\int_\Omega \nabla p \nabla w dx = \int_\Omega f \nabla w dx \leq \Big(\int_\Omega |f|^2 r^\alpha \zeta dx \Big)^{1/2} \Big(\int_\Omega |\nabla w|^2 r^{-\alpha} \zeta^{-1} dx \Big)^{1/2}.$$

Comparing the last two relations we come to the inequality

$$\left[1-\frac{\alpha(\alpha+m-2)}{2(m-1)}\right]\int\limits_\Omega |\nabla p|^2 r^\alpha \zeta dx \le (\int\limits_\Omega |f|^2 r^\alpha \zeta dx)^{1/2}(\int\limits_\Omega |\nabla w|^2 r^{-\alpha}\zeta^{-1}dx)^{1/2} + \\ + C\int\limits_\Omega |p|^2 dx.$$

In our book ([2], p. 120 see also [8]) by the same method was shown that

$$\int\limits_\Omega |\nabla w|^2 r^{-\alpha}\zeta^{-1}dx \le \left[1-\frac{\alpha(m-2)}{m-1}\right]\int\limits_\Omega |\nabla p|^2 r^\alpha \zeta dx + \\ C\int\limits_\Omega |p|^2 dx.$$

So one of the statements of the theorem is proved.
Take now in Stoke's system (3.1) ∇p to the right hand side and apply the inequality (2.52). After small calculations you come to the inequality (3.4). □
Consider now the nonstationary Stoke's system

$$\left.\begin{array}{c}\dot{u}-\nu\Delta u+\nabla p=f \quad (\nu=\text{ const. }>0)\\ \operatorname{div} u=0\end{array}\right\} \tag{3.8}$$

with boundary conditions

$$u|_{\partial\Omega}=u|_{t=0}=0 \tag{3.9}$$

At first we'll also consider the inner estimates.
Suppose $f\in L_2\{(0,T);L_{2,\alpha}(\Omega)\}$ and $Q_R=(0,T)\times B_R$, where $R<$ dist $(x_0,\partial\Omega)$.
It is trivial to see that the estimate (3.3) holds if we change B_R for Q_R.
Then dividing the first equation of (3.8) by ν and applying (2.53) we come to

Theorem 3.2. *The solution of the problem* (3.8), (3.9) *satisfies the inequalities*

$$\int\limits_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt \le \left[1-\frac{\alpha(m-2)}{m-1}+\eta\right]\left[1-\frac{\alpha(\alpha+m-2)}{2(m-1)}\right]^{-2}\times \\ \times\int\limits_{Q_R}|f|^2 r^\alpha\zeta dxdt + C\int\limits_{Q_R}|p|^2 dxdt, \tag{3.10}$$

$$\int\limits_{Q_R}|D^2u|^2 r^\alpha\zeta dxdt \le \frac{1}{\nu^2}\left\{1+\left[1-\frac{\alpha(m-2)}{m-1}\right]^{\frac12}\left[1-\frac{\alpha(\alpha+m-2)}{2(m-1)}\right]^{-1}+\eta\right\}^2\times \\ \times(1+M_\gamma^2)A^2_{\alpha,m}\int\limits_{Q_R}|f|^2 r^\alpha\zeta dxdt + C\Bigg[\int\limits_{Q_R}|p|^2 dxdt+ \\ +(\int\limits_{Q_R}|D^2u|^2 r^\alpha\zeta dxdt)^{\frac{m}{m+2\gamma}}(\int\limits_{Q_R}|Du|^2 dxdt)^{\frac{2\gamma}{m+2\gamma}}+\int\limits_{Q_R}(|Du|^2+|u|^2)dxdt\Bigg]. \tag{3.11}$$

Here C doesn't depend on ν, M_γ and $A_{\alpha,m}$ are correspondingly defined by (2.20) and (2.42).
For small $\gamma > 0$ we have

Theorem 3.3. *Let the conditions of the th* 3.1. *be satisfied and* $\gamma > 0$ *is small. Then the following estimates for the solutions of the system* (3.1)

$$\int_{B_R} |\nabla p|^2 r^\alpha \zeta dx \leq \left[1 + \frac{(m-2)^2}{m-1} + 0(\gamma)\right] \int_{B_R} |f|^2 r^\alpha \zeta dx + C \int_{B_R} |p|^2 dx,$$

$$\begin{aligned}\int_{B_R} |D^2 u|^2 r^\alpha \zeta dx \quad \leq \quad & \left\{1 + \left[1 + \frac{(m-2)^2}{m-1} + 0(\gamma)\right]^{1/2}\right\}^2 \left(1 + \frac{m-2}{m+1}\right) \int_{B_R} |f|^2 r^\alpha \zeta dx + \\ & + C\left[\left(\int_{B_R} |D^2 u|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int_{B_R} |Du|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \right. \\ & \left. + \int_{B_R} (|Du|^2 + |u|^2) dx + \int_{B_R} |p|^2 dx\right] \qquad (3.12)\end{aligned}$$

are true.

Theorem 3.4. *Let the conditions of th 3.2 be satisfied and* $\gamma > 0$ *is small. Then the following estimates for the solutions of the system* (2.8)

$$\begin{aligned}&\int_{B_R} |\nabla p|^2 r^\alpha \zeta dx dt \leq \left[1 + \frac{(m-2)^2}{m-1} + 0(\gamma)\right] \int_{Q_R} |f|^2 r^\alpha \zeta dx dt + \\ &+ C \int_{Q_R} |p|^2 dx, \qquad (3.13)\end{aligned}$$

$$\begin{aligned}&\int_{Q_R} |D^2 u|^2 r^\alpha \zeta dx dt \leq \frac{1}{\nu^2 \gamma}[2 + 0(\gamma)] \int_{Q_R} |f|^2 r^\alpha \zeta dx dt + \\ &+ C\left[\int_{Q_R} |p|^2 dx dt + \left(\int_{Q_R} |D^2 u|^2 r^\alpha \zeta dx dt\right)^{\frac{1}{1+\gamma}} \times \right. \qquad (3.14) \\ &\times \left(\int_{Q_R} |Du|^2 dx dt\right)^{\frac{\gamma}{1+\gamma}} + \left.\int_{Q_R} (|Du|^2 + |u|^2) dx dt\right], (m = 2)\end{aligned}$$

$$\int_{Q_R} |D^2u|^2 r^\alpha \zeta dx dt \le \frac{m}{2\nu^2}\left\{1+\left[1+\frac{(m-2)^2}{m-1}+0(\gamma)\right]^{1/2}\right\}\times$$
$$\times\left(1+\frac{m-2}{m+1}\right)\int_{Q_R}|f|^2 r^\alpha\zeta dx dt+$$
$$+C\left[\int_{Q_R}|p|^2 dx dt+\left(\int_{Q_R}|D^2u|^2 r^\alpha\zeta dx dt\right)^{\frac{m}{m+2\gamma}}\times\right.$$
$$\left.\times\left(\int_{Q_R}|Du|^2 dx dt\right)^{\frac{2\gamma}{m+2\gamma}}+\int_{Q_R}\left(|Du|^2+|u|^2\right)dx dt\right]\quad (m\ge 3) \tag{3.15}$$

take place, where C doesn't depend on ν.
It is necessary now for the problem (3.1) (3.2), to get the estimates in the neighbourhood of the boundary $\partial\Omega$. For this purpose suppose that a piece of the boundary is flat and has the equation $x_m = 0$. So, in the neighbourhood the domain Ω lies in the half space $x_m < 0$. Take a point $M_0(x_1^{(0)},\ldots,x_m^{(0)})$ in Ω and consider the ball $B_{R_0}(M_0)$ such that

$$R_0 > |x_m^{(0)}|. \tag{3.16}$$

Consider also a parallelelepiped Π_-

$$\left.\begin{array}{lll} x_k^{(0)} - R \le x_k \le x_k^{(0)} + R & & (k=1,\ldots,m-1) \\ x_m^{(0)} - R \le x_m \le 0, & (R > R_0) & \end{array}\right\} \tag{3.17}$$

which contains $B_{R_0}(M_0)$. Suppose for a moment that $x_k^{(0)} = 0$ $(k=1,\ldots, m-1)$ and $R=\pi$. The principal part of the estimates doesn't depend on these assumptions. Without loss of generality we can assume also that $f=0$ for $x_m=0$. Expand f, u and p in Π_- in the following Fourier series

$$f^{(k)}(x) = \sum_{n'} f_{n'}^{(k)}(x_m)e^{i(n',x')}\ (k=1,\ldots,m), \tag{3.18}$$

$$\left.\begin{array}{rcl} p(x) &=& \sum_{n'} p_{n'}(x_m)e^{i(n',x')}, \\ u^{(k)}(x) &=& \sum_{n'} u_{n'}^{(k)}(x_m)e^{i(n',x')}\ (k=1,\ldots,m), \end{array}\right\} \tag{3.19}$$

where $n' = (n,\ldots,n_{m-1})$, all integers n_i $(i=1,\ldots,m-1)$ run from $-\infty$ to $+\infty$ and $x' = (x_1,\ldots,x_{m-1})$. Substituting (3.18) and (3.19) in (3.1) and (3.2), we get

$$\left.\begin{array}{rcl} \ddot{u}_{n'}^{(k)} - |n'|^2 u_{n'}^{(k)} + i n_k p &=& f_{n'}^{(k)}, (k=1,\ldots,m-1) \\ \ddot{u}_{n'}^{(k)} - |n'|^2 u_{n'}^{(m)} + \dot{p}_{n'} &=& f_{n'}^{(k)}, \\ \dot{u}_{n'}^{(k)} + i\sum_{k=1}^{m-1} n_k u_{n'}^{(k)} &=& 0, \end{array}\right\} \tag{3.20}$$

$$u_{n'}^{(k)}(0) = 0. \tag{3.21}$$

Here

$$|n'|^2 = \sum_{s=1}^{m-1} n_s^2$$

and the point over $u_{n'}^{(k)}$ denotes the derivative with respect to x_m. Multiply the first $m-1$ equations of (3.20) correspondingly by in_k. After summation and using the last equation (3.20) we shall have

$$\dddot{u}_{n'}^{(m)} - |n'|^2 \dot{u}_{n'}^{(m)} + |n'|^2 p_{n'} = -\sum_{k=1}^{m-1} (in_k) f_{n'}^{(k)}. \qquad (3.22)$$

If we differentiate now the second equation of (3.20) with respact to x_m and subtract from the result the relation (3.22) we shall have

$$\ddot{p}_{n'} - |n'|^2 p_{n'} = \dot{f}_{n'}^{(m)} + \sum_{k=1}^{m-1} (in_k) f_{n'}^{(k)} \equiv F_{n'}^{-}(x_m). \qquad (3.23)$$

The bounded solution of this equation for $x_m < 0$ is the following one

$$p_n = p_n^{-}(x_m) = -\frac{1}{2|n'|} \int_{-\infty}^{x_m} F_{n'}^{-}(\xi_m) e^{|n'|(\xi_m - x_m)} d\xi_m +$$

$$+\frac{1}{2|n'|} \int_{0}^{x_m} F_{n'}^{-}(\xi_m) e^{|n'|(x_m - \xi_m)} d\xi + C_- e^{|n'|x_m}. \qquad (3.24)$$

Here $F_{n'}^{-}(x_m)$ is a function, which coincides with $F_{n'}(x_m)$ on $x_m > -\pi$ and is expanded on $x_m < -\pi$ continuously for $k = 1, \ldots, m$. We suppose also that the functions will be absolutely summable on $(-\infty, 0]$.
Let us also consider the equation (3.23) in $x_m > 0$ with such suitable right–hand side $F_{n'}^{+}(x_m)$ that $p_n(x_m)$ is continuous and absolutely summable on the whole strip $-\infty < x_m < +\infty$. The solution for $x_m > 0$ will be the following

$$p_n = p_n^{+}(x_m) = -\frac{1}{2|n'|} \int_{0}^{x_m} F_{n'}^{+}(\xi_m) e^{|n'|(\xi_m - x_m)} d\xi_m +$$

$$+\frac{1}{2|n'|} \int_{+\infty}^{x_m} F_{n'}^{+}(\xi_m) e^{|n'|(x_m - \xi_m)} d\xi_m + C_+ e^{-|n'|x_m}. \qquad (3.25)$$

Take $x_m = 0$ in (3.24) and (3.25) $c_- = c_+$ and

$$\int_{-\infty}^{0} F_{n'}^{-}(\xi_m) e^{|n'|\xi_m} d\xi_m = \int_{0}^{+\infty} F_{n'}^{+}(\xi_m) e^{-|n'|\xi_m} d\xi_m.$$

We see that $F_{n'}^{-}$ should be expanded on $x_m > 0$ symmetrically. From the right–hand side of (3.24) it is easy to see that the function $f_{n'}^{(m)}(x_m)$ should be expanded in antisymmetric way and $f_{n'}^{(k)}(x_m)(k = 1, \ldots, m-1)$ symmetrically. Integrating once by part we come to

$$p_{n'}=p_{n'}(x_m)=\frac{1}{2}\int\limits_{-\infty}^{x_m} f_{n'}^{(m)}(\xi_m)e^{|n'|(\xi_m-x_m)}d\xi_m+$$

$$+\frac{1}{2}\int\limits_{0}^{x_m} f_{n'}^{(m)}(\xi_m)e^{|n'|(x_m-\xi_m)}d\xi_m+\frac{1}{2|n'|}\sum_{k=1}^{m-1}(in_k)\left[\int\limits_{0}^{x_m} f_{n'}^{(k)}(\xi_m)e^{|n'|(x_m-\xi_m)}d\xi_m-\right.$$

$$\left.-\int\limits_{-\infty}^{x_m} f_{n'}^{(k)}(\xi_m)\,e^{|n'|(\xi_m-x_m)}d\xi_m\right]+ce^{-|n'||x_m|}\ (x_m<0). \tag{3.26}$$

The analogons formula, following from (3.25) will take place for $x_m>0$.
Denote by $\tilde{f}_{n'}^{(k)}(\lambda)$ the Fourier transform of the functions $f_{n'}^{(k)}(x_m)$ $(k=1,\ldots,m)$ on $-\infty<x_m<+\infty$ (the functions are correspondingly expanded in symmetric and antisymmetric ways).
We have

$$f_{n'}^{(k)}(x_m)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\tilde{f}_{n'}^{(k)}(\lambda)e^{i\lambda x_m}d\lambda.$$

Substituting in (3.26), changing the order of integration and taking a suitable C we get

$$\begin{aligned} p_n(x_m) &= \frac{i}{\sqrt{\pi}}\left[\int\limits_{-\infty}^{+\infty}\tilde{f}_{n'}^{(m)}(\lambda)\frac{\lambda}{\lambda^2+|n'|^2}e^{i\lambda x_m}d\lambda+\right.\\ &\left.+\sum_{k=1}^{m-1}\int\limits_{-\infty}^{+\infty}\tilde{f}_{n'}^{(k)}(\lambda)\frac{n_k}{\lambda^2+|n'|^2}e^{i\lambda x_m}d\lambda\right]. \end{aligned} \tag{3.27}$$

Consider now the symmetric with respect to Π_- parallelepiped Π_+

$$\left.\begin{array}{rcl} x_k^{(0)}-R & \le & x_k\le x_k^{(0)}+R\ \ (k=1,\ldots,m-1)\\ -x_m^{(0)}+R & \ge & x_m\ge 0. \end{array}\right\}$$

The parallelepiped $\Pi_-\cup\Pi_+$ gives a parallelepiped Π in which the function $p_{n'}(x_m)$ is expanded on $x_m>0$ in a symmetric continuous way. In this parallelepiped for $x_m<0$

$$p(x)=\frac{i}{\sqrt{\pi}}\sum_{n'}\int\limits_{-\infty}^{+\infty}\left[\tilde{f}_{n'}^{(m)}(\lambda)\frac{\lambda}{\lambda^2+|n'|^2}+\sum_{k=1}^{m-1}\tilde{f}_{n'}^{(k)}(\lambda)\frac{n_k}{\lambda^2+|n'|^2}\right]e^{i\lambda x_m}d\lambda e^{i(n',x')}$$

and the analogons formula for $x_m>0$. The function $p(x)$ as we have seen is continuous for all $x\in\Pi$. Then there exist the first derivatives for p in the Soboleff's sense and, for example

$$\frac{\partial p}{\partial x_m}=-\frac{1}{\sqrt{\pi}}\sum_{n'}\int\limits_{-\infty}^{+\infty}\left[\tilde{f}_{n'}^{(m)}(\lambda)\frac{\lambda^2}{\lambda^2+|n'|^2}+\sum\tilde{f}_{n'}^{(k)}(\lambda)\frac{n_k\lambda}{\lambda^2+|n'|^2}\right]e^{i\lambda x_m}d\lambda e^{i(n',x')}. \tag{3.28}$$

Take in account that each term on the right–hand side of (3.28) represents the second derivative of a solution of the Poisson equation

$$\Delta Z = f^{(k)} \ \ (k = 1, \dots, m)$$

in Π with boundary conditions of the first or the second boundary value problems. Then according to (2.52) theorem (2.4)

$$\int\limits_{B_{R_0}} \left|\frac{\partial p}{\partial x_m}\right|^2 r^\alpha \zeta dx \le (1 + M_\gamma^2 + \eta) m \int\limits_{B_{R_0}} |f|^2 r^\alpha \zeta dx +$$
$$+C\left[\left(\int\limits_{B_{R_0}} \left|\frac{\partial p}{\partial x_m}\right|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{B_{R_0}} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{B_{R_0}} |p|^2 dx\right].$$

Differentiating $p(x)$ with respect to $x_k (k = 1, \dots, m-1)$ we get the same estimates. Then

$$\int\limits_{B_{R_0}} |\nabla p|^2 r^\alpha \zeta dx \le m^2 (1 + M_\gamma^2 + \eta) \int\limits_{B_{R_0}} |f|^2 r^\alpha \zeta dx +$$
$$+C\left[\left(\int\limits_{B_{R_0}} |\nabla p|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{B_{R_0}} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{B_{R_0}} |p|^2 dx\right]. \quad (3.29)$$

Denote

$$B_{R_0}^+ = B_{R_0} \cap (x_m > 0), \ \ B_{R_0}^- = B_{R_0} \cap (x_m < 0). \quad (3.30)$$

Suppose that $\zeta(r)$, defined by (2.46), is monotone. As far as $x_0 \in B_{R_0}^-$ then for $x \in B_{R_0}^+$

$$|\bar{x} - x_0| \le |x - x_0|,$$

where $\bar{x}$ is symmetric to x with respect to $x_m = 0$. For $\alpha = 2 - m - 2\gamma < 0$ we have

$$|\bar{x} - x_0|^\alpha \ge |x - x_0|^\alpha$$

and from the monotonicity of ζ (2.46) follows

$$\zeta(|\bar{x} - x_0|) \ge \zeta(|x - x_0|).$$

Since f and p are expanded on B_{R_0} in symmetric and antisymmetric ways we have

$$\int\limits_{B_{R_0}} |f|^2 r^\alpha \zeta dx = \int\limits_{B_{R_0}^+} |f|^2 r^\alpha \zeta dx + \int\limits_{B_{R_0}^-} |f|^2 r^\alpha \zeta dx \le 2 \int\limits_{B_{R_0}^-} |f|^2 r^\alpha \zeta dx.$$

Taking in account that

$$\int\limits_{B_{R_0}} |\nabla p|^2 r^\alpha \zeta dx \ge \int\limits_{B_{R_0}^-} |\nabla p|^2 r^\alpha \zeta dx$$

we come with the help of (3.29) to

$$\int\limits_{B^-_{R_0}} |\nabla p|^2 r^\alpha \zeta dx \le 2m^2(1+M_\gamma^2+\eta) \int\limits_{B^-_{R_0}} |f|^2 r^\alpha \zeta dx +$$

$$+C\left[\left(\int\limits_{B^-_{R_0}} |\nabla p|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{B^-_{R_0}} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{B^-_{R_0}} |p|^2 dx\right]. \quad (3.31)$$

Theorem 3.5. *If $f \in L_{2,\alpha}(\Omega)$ with $\alpha = 2-m-2\gamma$ $(0<\gamma<1)$ then the solution of the boundary value problem (3.1), (3.2) satisfies the inequalities*

$$\int\limits_{\Omega_R} |\nabla p|^2 r^\alpha \zeta dx \le 2m^2(1+M_\gamma^2+\eta) \int\limits_{\Omega_R} |f|^2 r^\alpha \zeta dx +$$

$$+C\left[\left(\int\limits_{\Omega_R} |\nabla p|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{\Omega_R} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{\Omega_R} |p|^2 dx\right] \quad (3.32)$$

and

$$\int\limits_{\Omega_R} |D^2u|^2 r^\alpha \zeta dx \le 2\left(1+M_\gamma^2+\eta\right)\left(1+\sqrt{2}m\sqrt{1+M_\gamma^2}\right)^2 \int\limits_{\Omega_R} |f|^2 r^\alpha \zeta dx +$$

$$+C\left[\left(\int\limits_{\Omega_R} |D^2u|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{\Omega_R} |D^2u|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{\Omega_R} (|Du|^2+|u|^2)dx + \right.$$

$$\left. + \left(\int\limits_{\Omega_R} |\nabla p|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int\limits_{\Omega_R} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int\limits_{\Omega_R} |p|^2 dx\right], \quad (3.33)$$

where $\Omega_R = \Omega \cap B_R(x_0)$, R sufficiently small, $r = |x-x_0|$, $x_0 \in \bar{\Omega}$, C doesn't depend on x_0 and η is an arbitrary small positive number.

Proof. The inequality (3.33) follows after comparing the estimates (3.3) and (3.32). In fact it is enough to compare the coefficients before $\int |f|^2 r^\alpha \zeta dx$. Since

$$1 - \frac{\alpha(m-2)}{m-1} < 2m^2(1+M_\gamma^2)$$

for $m \ge 2$ we come to (3.33). Of course the technique of the apriori estimates of J. Shauder should be applied (see for example [6]). To get (3.34) the system (3.1) should be written in the form

$$\Delta u = f - \nabla p$$

and the boundary conditions (3.2) used.

In the interior domain the inequality follows from the estimate (2.52). In the boundary strip the solution should be continued in an antisymmetric way. The proof of the theorem is then completed after very simple calculations. □

Consider now the nonstationary system (3.8) with the condition (3.9).

Theorem 3.6. *If $f \in L_2\{(0,T); L_{2,\alpha}(\Omega)\}$ with $\alpha = 2-m-2\gamma (0<\gamma<1)$, then the solution of the system* (3.8) *with the boundary condition* (3.9) *satisfies the estimates*

$$\int_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt \leq 2m^2(1+M_\gamma^2+\eta) \int_{Q_R} |f|^2 r^\alpha \zeta dxdt +$$
$$+C\left[\left(\int_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left(\int_{Q_R} |p|^2 dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{Q_R} |p|^2 dxdt\right] \quad (3.34)$$

and

$$\int_{Q_R} |D^2 u|^2 r^\alpha \zeta dxdt \leq \frac{2A^2_{\alpha,m}}{\nu^2}(1+M_\gamma^2+\eta)(1+\sqrt{2}m\sqrt{1+M_\gamma^2})^2 \int_{Q_R} |f|^2 r^\alpha \zeta dxdt +$$
$$+C\left[\left(\int_{Q_R} |D^2 u|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left(\int_{Q_R} |Du|^2 dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + \left(\int_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \times\right.$$
$$\left.\times \left(\int_{Q_R} |p|^2 r^\alpha \zeta dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{Q_R} |p|^2 dxdt\right]. \quad (3.35)$$

Here $r = |x-x_0|$ and $Q_R = (0,T) \times B_R(x_0) \cap \Omega$ with sufficiently small R. The constant C doesn't depend on x_0 and ν.

The proof is absoloutely analogous to the proof of the previous theorem. The only difference is that the references should be made to the estimate (2.53). □

For small $\gamma > 0$ the last two theorems can be formulated in a more explicit way.

Theorem 3.7. *If the conditions of theorem* 3.5 *are satisfied then for the solution* *of the problem* (3.1), (3.2) *the following estimates*

$$\int_{\Omega_R} |\nabla p|^2 r^\alpha \zeta dx \le 2m^2 \left[1+\frac{m-2}{m+1}+0(\gamma)\right] \int_{\Omega_R} |f|^2 r^\alpha \zeta dx +$$

$$+C\left[\left(\int_{\Omega_R} |\nabla p|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int_{\Omega_R} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{\Omega_R} |p|^2 dx\right],$$

$$\int_{\Omega_R} |D^2 u|^2 r^\alpha \zeta dx \le 2\left[1+\frac{m-2}{m+1}+0(\gamma)\right]\left[1+\sqrt{2}m(1+\frac{m-2}{m+1})^{1/2}\right]^2 \int_{\Omega_R} |f|^2 r^\alpha \zeta dx +$$

$$+C\left[\left(\int_{\Omega_R} |D^2 u|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int_{\Omega_R} |u|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{\Omega_R} (|Du|^2+|u|^2) dx +\right.$$

$$\left. + \left(\int_{\Omega_R} |p|^2 r^\alpha \zeta dx\right)^{\frac{m}{m+2\gamma}} \left(\int_{\Omega_R} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{\Omega_R} |p|^2 dx\right]$$

hold.

Theorem 3.8. *If the conditions of theorem* 3.6 *are satisfied then the solution of the problem* (3.8), (3.9) *satisfies the following inequalities*

$$\int_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt \le 2m^2 \left[1+\frac{m-2}{m+1}+0(\gamma)\right] \int_{Q_R} |f|^2 r^\alpha \zeta dxdt +$$

$$+C\left[\left(\int_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left(\int_{Q_R} |p|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{Q_R} |p|^2 dxdt\right],$$

$$\int_{Q_R} |D^2 u|^2 r^\alpha \zeta dxdt \le \frac{2m}{\nu^2}\left[1+\frac{m-2}{m+1}+0(\gamma)\right]\left[1+\sqrt{2}m(1+\frac{m-2}{m+1})^{1/2}\right]^2 \int_{Q_R} |f|^2 r^\alpha \zeta dxdt +$$

$$+C\left[\left(\int_{Q_R} |D^2 u|^2 r^\alpha \zeta dxdt\right)^{\frac{m}{m+2\gamma}} \left(\int_{Q_R} |Du|^2 dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + \left(\int_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt\right)^{\frac{2m}{m+2\gamma}} \times\right.$$

$$\left. \times \left(\int_{Q_R} |p|^2 dxdt\right)^{\frac{2\gamma}{m+2\gamma}} + \int_{Q_R} |p|^2 dxdt\right], \quad (m \ge 3),$$

$$\int\limits_{Q_R} |D^2u|^2 r^\alpha \zeta dxdt \le \frac{2(1+2\sqrt{2})^2}{\nu^2\gamma}\,[1+0(\gamma)] \int\limits_{Q_R} |f|^2 r^\alpha \zeta dxdt +$$

$$C\left[\left(\int\limits_{Q_R} |D^2u|^2 r^\alpha \zeta dxdt\right)^{\frac{1}{1+\gamma}} \left(\int\limits_{Q_R} |D^2u|^2 dxdt\right)^{\frac{\gamma}{1+\gamma}} + \int\limits_{Q_R} (|D^2u|^2 + |u|^2) dxdt + \right.$$

$$\left. + \left(\int\limits_{Q_R} |\nabla p|^2 r^\alpha \zeta dxdt\right)^{\frac{1}{1+\gamma}} \left(\int\limits_{Q_R} |p|^2 dxdt\right)^{\frac{\gamma}{1+\gamma}} + \int\limits_{Q_R} |p|^2 dxdt\right] \quad (m=2).$$

4. Regularity of solutions for degenerated elliptic systems

In a bounded domain $\Omega \subset R^m \quad (m \ge 2)$ consider a system

$$L(u) \equiv \sum_{i=0}^{m} D_i a_i(x, Du) = 0, \tag{4.1}$$

where u and $a_i(x,p)(i=0,1,\dots,m)$ are N-dimensional vector functions with components $u^{(k)}(x), a_i^{(k)}(x,p)(k=1,\dots,N)$,

$$Du = (D_0u, D_1u, \dots, D_mu) D_i = \frac{\partial}{\partial x_i}(i=1,\dots,m) \text{ and } D_0 = -I,$$

(now, different from §, we include u in Du). About the functions $a_i(x,p)$ we assume, that they satisfy some of the following conditions:

(1) All $a_i(x,p)$ satisfy the Caratheodory conditions and are differentiable with respect to variables p;

(2) The $(m+1)N \times (m+1)N$ matrix

$$A = \left\{ \frac{\partial a_i^{(k)}}{\partial p_j^{(l)}} \right\} \tag{4.2}$$

is symmetric and the eigenvalues of this matrix satisfy the inequalities

$$\frac{\lambda}{1+|p|^s} \le \lambda_j(x,p) \le \frac{\Lambda}{1+|p|^s} \tag{4.3}$$

where $\lambda, \Lambda =$ const. > 0 and $0 \le s < 1$;

(3) For arbitrary $u \in W_q^{(1)}(\Omega)(q>1)$ the result of the substitution $a_i(x, Du(x))(i = 0,\dots,m)$ will belong to $L_{q/(1-s)}(\Omega)$;

(4) The inequality

$$\left|\frac{\partial a_i}{\partial x_k}\right| \leq C(1+|p|) \quad (i=1,\ldots,m) \tag{4.4}$$

holds;

(5) For $\forall u \in W_q^{(2)}(\Omega)$ the result of substitution in $L(u)$ belongs to $L_q(\Omega)$.

Consider the solution of (4.1) with the boundary condition

$$u|_{\partial\Omega} = 0. \tag{4.5}$$

In [2] (ch 1, § 4) was proved that the universal iterational process

$$\Delta u_{n+1} - u_{n+1} = \Delta u_n - u_n - \Lambda^{-1} L(u_n), \qquad u_n|_{\partial\Omega} = 0 (n=0,1,\ldots,) \tag{4.6}$$

converges in $W_{2-s}^{(1)}(\Omega)$ to the weak solution u of (4.1), (4.5) if $u \in W_2^{(1)}(\Omega)$.
Consider also ,the process (4.6) with a penalty

$$\Delta u_{n+1} - u_{n+1} = \Delta u_n - u_n - \Lambda^{-1}[\delta \Delta u_n + L(u_n)] \quad (\delta \geq 0) \tag{4.7}$$

with the same condition (4.5).
In [2] was also shown that a subsequence of the iterations of the process (4.7) converges weekly to the solution. So, if we want to show that the solution has the Hölder continuous first derivatives it is enough to show that the iterations of(4.6) or (4.7) satisfy the inequality

$$\int_{\Omega_R} |D^2 u_n|^2 r^\alpha dx \leq C, \tag{4.8}$$

where $\Omega_R = B_R(x_0) \cap \Omega, \quad x_0 \in \overline{\Omega}, \quad \alpha = 2 - m - 2\gamma (0 < \gamma < 1), \quad r = |x - x_0|$ and C doesn't depend on x_0 and n. It is also assumed that R is sufficiently small and fixed.

Lemma 4.1. *If the conditions 1) - 3) are satisfied and $u_0(x) \in \dot{W}_2^{(1)}(\Omega)$ then*

$$\left(\int_\Omega |Du_{n+1}|^2 dx\right)^{1/2} \leq \left(1 - \frac{\lambda\Lambda^{-1}}{1 + [\max\{\sup_\Omega |Du_n| \sup_\Omega |Du_{n+1}\}]^s}\right)^{1/2} \times$$
$$\times \left(\int_\Omega |Du_n|^2 dx\right)^{1/2} + \Lambda^{-1}|a|, \tag{4.9}$$

holds, where

$$|a|^2 = \int_\Omega \sum_{i=0}^m |a_i(x,0)|^2 dx. \tag{4.10}$$

Proof. Multiply both sides of the system (4.6) by u_{n+1} and integrate once by part. Then

$$\int_\Omega Du_{n+1}Du_{n+1}dx = \int_\Omega [D_iu_n - \Lambda^{-1}a_i(x, Du_n)]D_iu_{n+1}dx,$$

where summation as always runs over repeated indices.
Adding and subtracting $a_i(x,0)$ under the square brackets on the right hand side, we get

$$\begin{aligned}\int_\Omega Du_{n+1}Du_{n+1}dx &= \int_\Omega [D_iu_n - \Lambda^{-1}(a_i(x, Du_n) - a_i(x,0))]D_iu_{n+1}, dx - \\ &\quad - \int_\Omega a_i(x,0)D_iu_{n+1}dx.\end{aligned}$$

Applying the mean value theorem we come to

$$\begin{aligned}\int_\Omega |Du_{n+1}Du_{n+1}|dx &= \int_\Omega (I - \Lambda^{-1}\overline{A})Du_n \cdot Du_{n+1}dx - \\ &\quad - \int_\Omega a_i(x,0)D_iu_{n+1}dx,\end{aligned}$$

where $\bar{A}$ denotes the matrix A with intermediate values of variables.
The Hölder inequality gives

$$\left(\int_\Omega |Du_{n+1}|^2dx\right)^{1/2} \le \sup_\Omega ||(I - \Lambda^{-1}\overline{A}|| \left(\int_\Omega |Du_n|^2dx\right)^{1/2} + |a|,$$

It can be easily proved (see for example [3] p. 58, (2.29)), that

$$\sup_\Omega ||I - \Lambda^{-1}\overline{A}||^2 \le \sup_{i,\Omega} |1 - \Lambda^{-1}\overline{\lambda}_i|^2. \tag{4.11}$$

Using the right side of the inequalities (4.3) we get

$$||I - \Lambda^{-1}\overline{A}|| \;\le\; 1 - \frac{\lambda\Lambda^{-1}}{1 + \left[\max\{\sup_\Omega |Du_n| \sup_\Omega |Du_{n+1}\}\right]^s}. \tag{4.12}$$

□

Suppose that the cut-off function $\zeta(r)$(2.46) satisfies in addition the inequality

$$|\zeta'||\zeta|^{-1/2} < C \tag{4.13}$$

Assume now that the boundary of Ω belongs to $C^{(1,æ)}$ ($æ > 0$). If the condition 4) and 5) are satisfied and u_0 (the initial iteration of (4.6) or (4.7) belongs to $W_q^{(2)}(\Omega) \cap \dot{W}_q^{(1)}(\Omega)(q > 1)$ then all the iterations belong to the same space. The iterations can be expanded outside the domain Ω in a sufficiently narrow strip preserving the class. This can be made with the help of the well known procedure, we have used in the previous paragraph. First you consider a plane peace of the boundary and expand all of the u_n in an antisymmetric way. This gives you the same class of $W_q^{(2)}(\Omega \cup \Omega_R)$ for balls $B_R(x_0)$ which don't lie completely in Ω. As we have shown in [2] (ch. 4, § 3) all the conditions 1) - 5) don't change and the values, s, λ and Λ will be the same. This gives also us the possibility to consider only the case when $\Omega_R = \Omega \cap B_R(x_0) = B_R(x_0)$ and this gives the fixed small R_0.

Lemma 4.2. *If the conditions 1)-5) are satisfied and* $u_0 \in W_q^{(2)}(\Omega) \cap \dot{W}_q^{(1)}(\Omega)(q > 2)$ *then the iterations* (4.6) *or* (4.7) *satisfy the inequality*

$$\int\limits_{\Omega_R} |D^2 u_{n+1}|^2 \zeta dx dt \le \left[1 - \frac{\lambda \Lambda^{-1}}{1 + [\max\{\sup\limits_{\Omega} |Du_n|, \sup\limits_{\Omega} |Du_{n+1}|\}]^s} + \eta \right] \times$$

$$\times \int\limits_{\Omega_R} D^2 u_n|^2 \zeta dx dt + C|a|^2, \tag{4.14}$$

where C *doesn't depend on* $x_0 \in \overline{\Omega}, n$ *and in the case of* (4.7) *on* δ.

Proof.
According to our previous consideration we can suppose that $\Omega_R = B_R(x_0)$. Multiply (4.6) (or 4.7) by $\Delta u_{n+1}\zeta$ and integrate by parts as we have done it in lemma 1.2 or lemma 1.5 for $\alpha = 0$. In [3] (theorem 1) it is shown that if ζ satisfies (4.13) then

$$\begin{aligned} \int\limits_{B_R} |Du_{n+1}|^2 \zeta dx \;&\le\; \sum_{k=1}^{m} \int\limits_{B} (I - \Lambda^{-1}\overline{A}) Du_{n,k} \cdot Du_{n+1,k} \zeta dx \\ &+ C|a| \Big(\int\limits_{B_R} |Du_{n+1}|^2 \zeta dx \Big)^{1/2}. \end{aligned} \tag{4.15}$$

From this immediately follows (4.14). □

Let $w_k(x)$ satisfy the equation

$$\Delta w_k = \Delta u_{n+1} \cdot r^{\alpha_k} \zeta \tag{4.16}$$

and the boundary condition

$$w_k|_{\partial B_R} = 0 (k = 1, 2, \ldots, M),$$

where M is a positive integer and α_k is monotone and satisfy the following relations

$$\left.\begin{array}{lcl} \alpha_1 & = & -m/2 + \eta, \\ 0 & < & \alpha_{k-1} - 2\alpha_k < m, \\ \alpha_k & \notin & [2 - m, 3 - m], \alpha_{M-1} > 2 - m \\ \alpha_M & = & \alpha = 2 - m - 2\gamma (0 < \gamma < 1). \end{array}\right\} \tag{4.17}$$

According to the results of E.M. Stein [4] and V.A. Kondratjev [7] the inequality

$$\int\limits_{B_R} (|D^2 w|^2 + |Dw|^2 + |w|^2) r^\beta dx \le C \int\limits_{B_R} |\Delta w|^2 r^\beta dx \tag{4.18}$$

holds, if $-m < \beta < m$ and $w = 0$ on ∂B_R.

Multiply (4.6) or (4.7) by $\Delta w_k \zeta$ and integrate twice by parts. It is obvious that ζ^2 also satisfies (2.46) and (4.13). Then we get

$$\begin{aligned}\int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_k}\zeta^2 dx &= \int\limits_{B_R} \{u_{n,i,j} - \Lambda^{-1}[a_i(x, Du_n)]_j\} w_{k,i,j}\zeta^2 dx + \\ &+ \int\limits_{B_R} [u_{n,i} - \Lambda^{-1}a_i(x, Du_n)] w_{k,ij}(\zeta^2)_j dx - \int\limits_{B_R} [u_{n,i} - \\ &-\Lambda^{-1}a_i(x, Du_n)] w_{k,ij}(\zeta^2)_i dx + \cdots = I_1 + I_2 + +I_3\end{aligned} \tag{4.19}$$

(the nonwritten terms contains only the weaker ones). Let us estimate at first the integral I_1. It is easy to see that

$$\begin{aligned} I_1 = \int\limits_{B_R} \{u_{n,ij} - \Lambda^{-1}[a_i(x, Du_n)]_j\} w_{k,ij}\zeta^2 dx \le \sup_{\bar\Omega} ||I - \Lambda^{-1}\overline{A}|| \times \\ \times \left(\int\limits_{B_R} |D^2 u_n|^2 r^{\alpha_k}\zeta^2 dx \right)^{\frac{1}{2}} \left(\int\limits_{B_R} |D^2 w_k|^2 r^{-\alpha_k}\zeta^2 dx \right)^{\frac{1}{2}} + \ldots \end{aligned} \tag{4.20}$$

Further

$$\begin{aligned}\int\limits_{B_R} |D^2 w_k|^2 r^{-\alpha_k}\zeta^2 dx &= \sum_{i,j=1}^{m} \int\limits_{B_R} w_{k,ij}^2 r^{-\alpha_k}\zeta^2 dx = \\ &= \sum_{i,j=1}^{m} \int\limits_{B_R} [(w_k\zeta)_{ij} - (w_{k,j}\zeta_i + w_{k,i}\zeta_j) - w_k\zeta_{ij}]^2 r^{-\alpha_k} dx \le \\ &\le \int\limits_{B_R} |D^2(w_k\zeta)|^2 r^{-\alpha_k} dx \cdot (1+\eta) + C\int\limits_{B_R} (|Dw_k|^2 + |w_k|^2) r^{-\alpha_k} |D^2\zeta|^2 dx.\end{aligned}$$

According to the inequality of S. Chelkak ([8], p. 28, Lemma 1.2), we have

$$\int\limits_{B_R} |D^2(w_k\zeta)|^2 r^{\alpha_k} dx \le \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2}\right] \int\limits_{B_R} |\Delta(w_k\zeta)|^2 r^{-\alpha_k} dx$$

Then, from (4.16) and the fact that $D\zeta \equiv 0$ for $r \le R/2$ follow that

$$\begin{aligned}\int\limits_{B_R} |D^2 w_k|^2 r^{-\alpha_k}\zeta^2 dx \quad &\le \quad \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2}\right] \int\limits_{B_R} |\Delta w_k|^2 r^{-\alpha_k}\zeta^2 dx + \\ &+C\int\limits_{B_R} (|Dw_k|^2 + |w_k|^2) r^{-\alpha_k} |D^2\zeta|^2 dx \le \\ &\le \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2}\right] \int\limits_{B_R} |\Delta u_{n+1}|^2 r^{-\alpha_k}\zeta^2 dx + \\ &+C\int\limits_{B_R} (|Dw_k|^2 + |w_k|^2) r^{-2\alpha_k+\alpha_{k-1}} dx.\end{aligned}$$

As far as α_k satisfy (4.17) we can apply (4.18) and we come to the inequality

$$\int\limits_{B_R} |D^2 w_k|^2 r^{-\alpha_k} \zeta^2 dx \leq \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2}\right] \int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_k} \zeta^2 dx +$$

$$+C \left(\int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_k - 1} \zeta^2 dx + \int\limits_{B_R} |Du_n|^2 dx \right).$$

After the same considerations for I_2 and I_3 (4.12), (4.19) and (4.20) gives us the relation

$$\int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_k} \zeta^2 dx \leq \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2} + \eta\right] \times$$

$$\times \left(1 - \frac{\lambda \Lambda^{-1}}{1 + [\max\{\sup\limits_{\Omega} |Du_n|, \sup\limits_{\Omega} |Du_{n+1}|\}]^s}\right) \int\limits_{B_R} |D^2 u_n|^2 r^{\alpha_k} \zeta^2 dx +$$

$$+C \left(|a|^2 + \int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_k - 1} \zeta^2 dx + \int\limits_{B_R} |Du_n|^2 dx \right).$$

The inequality (2.52) (th. 2.4) gives for $k = M$

$$\int\limits_{B_R} |D^2 u_{n+1}|^2 r^{\alpha} \zeta^2 dx \leq (1 + M_\gamma^2) \left[1 - \frac{4\alpha(m-1)}{(\alpha+m)^2} + \eta\right] \times$$

$$\times \left(1 - \frac{\lambda \Lambda^{-1}}{1 + [\max\{\sup\limits_{\Omega} |Du_n|, \sup\limits_{\Omega} |Du_{n+1}|\}]^s}\right) \int\limits_{B_R} |D^2 u_n|^2 r^{\alpha} \zeta^2 dx +$$

$$+C \left[|a|^2 + \int\limits_{B_R} |Du_n|^2 dx + \int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_{M-1}} \zeta^2 dx + \right. \tag{4.21}$$

$$\left. + \left(\int\limits_{B_R} |D^2 u_{n+1}|^2 r^{\alpha} \zeta^2 dx \right)^{\frac{m}{m+2\gamma}} \cdot \left(\int\limits_{B_R} |Du_n|^2 dx \right)^{\frac{2\gamma}{m+2\gamma}} \right].$$

For $k < M$ according to [2] (p. 51, lemma 2.2)(see also [8])

$$\int\limits_{B_R} |D^2 u_{n+1}|^2 r^{\alpha_k} \zeta^2 dx \leq \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2} + \eta\right] \times$$

$$\times \left(1 - \frac{\lambda \Lambda^{-1}}{1 + [\max\{\sup\limits_{\Omega} |Du_n|, \sup\limits_{\Omega} |Du_{n+1}|\}]^s}\right) \int\limits_{B_R} |D^2 u_n|^2 r^{\alpha_k} \zeta^2 dx +$$

$$+C \left(\int\limits_{B_R} |Du_n|^2 dx + |a|^2 + \int\limits_{B_R} |\Delta u_{n+1}|^2 r^{\alpha_{k-1}} \zeta^2 dx \right). \tag{4.22}$$

Theorem 4.1. *Suppose the condition 1) - 5) are satisfied and the inequalities*

$$\left.\begin{array}{rcl} \int\limits_{\Omega} |Du_0|^2 dx & < & \eta_0^2, \\ \int\limits_{\Omega} |D^2 u_0|^2 r^{\alpha_k} dx & < & \eta_k^2 (k = 1, \dots, M-1), \\ |a|^2 + \sum_{s=1}^{k-1} \eta_s^2 & < & \varepsilon \eta_k^2 \end{array}\right\} \tag{4.23}$$

take place ($\varepsilon, a, \eta_k =$ *const.* > 0 *are suficiently small numbers).*
If the relation

$$\frac{\Lambda}{\lambda} \frac{(1 + M_\gamma^2)\left[1 - \frac{4\alpha(m-1)}{(\alpha+m)^2}\right] - 1}{(1 + M_\gamma^2)\left[1 - \frac{4\alpha(m-1)}{(\alpha+m)^2}\right]} < 1 \tag{4.24}$$

holds, then the solution of the problem (4.1), (4.5) *belongs to* $C^{(1,\gamma)}(\overline{\Omega})$ *with* $\gamma = -(\alpha + m - 2)/2$. *The process* (4.6), (4.7) *converge to solution in* $W^{(1)}_{2-s}$.

Proof. Consider at first the case $m \geq 4$. As we have mentioned before it is sufficient to prove the inequality (4.8). Suppose that $u \in W^{(2)}_q(\Omega)$ with $q > m(m+\alpha)^{-1}$. Then $\forall u_n \in W^{(2)}_q(\Omega)$. From this follows that $\forall u_n \in W^{(2)}_{2,\alpha}(\Omega)$.
If we write (2.49) for the functions $u_s \zeta$, we get

$$|u_{s,i}(x_0)|^2 \leq C \left(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2\right)^{\frac{2\gamma}{m+2\gamma}} \left[\left(\int\limits_{B_R} |D^2 u_0|^2 r^\alpha \zeta^2 dx\right)^{\frac{m}{m+2\gamma}} + \right.$$

$$\left. + \left(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2\right)^{\frac{m}{m+2\gamma}}\right] \quad (s = 0, 1; i = 1, \dots, m-1). \tag{4.25}$$

In fact from (2.49) and $u = u_s \zeta$ we have after some calculations

$$|u_{s,i}(x_0)|^2 \leq C \left(\int\limits_{\Omega} |Du_s|^2 dx\right)^{\frac{2\gamma}{m+2\gamma}} \times$$

$$\times \left[\left(\int\limits_{B_R} |D^2 u_s|^2 r^\alpha \zeta^2 dx\right)^{\frac{m}{m+2\gamma}} + \left(\int\limits_{\Omega} |Du_s|^2 dx\right)^{\frac{m}{m+2\gamma}}\right] (s = 0, 1).$$

Now (4.25) follows from (4.23) for $s = 0$. Applying (4.9) we have

$$\int\limits_{\Omega} |Du_1|^2 dx \leq 2 \left(\int\limits_{\Omega} |Du_0|^2 dx + |a|^2\right) \leq 2(\eta_0^2 + |a|^2 \Lambda^2).$$

After using (4.21) and (4.22) the inequalitiy $|ab| < \eta a^2 + 4^{-1}\eta^{-1} b^2$ the estimates give the relation

$$\int\limits_{B_R(x_0)} |D^2 u_1|^2 r^\alpha \zeta^2 dx \leq C \left[\int\limits_{B_R(x_0)} |D^2 u_0|^2 r^\alpha \zeta^2 dx + |a|^2 + \sum_{k=1}^{M-1} \eta_k^2\right].$$

Therefore

$$|u_{1,i}(x_0)|^2 \le C(\eta_0^2+|a|^2)^{\frac{2\gamma}{m+2\gamma}} \left[\left(\int\limits_{B_R(x_0)} |D^2u_0|^2 r^\alpha \zeta dx \right)^{\frac{m}{m+2\gamma}} + \left(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2 \right)^{\frac{m}{m+2\gamma}} \right].$$

From (4.25) follows, that for $s = 0, 1$

$$\sup_{\Omega} |Du_s|^2 \le C \left(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2 \right)^{\frac{2\gamma}{m+2\gamma}} \times$$

$$\times \left[\left(\sup_{x_0 \in \overline{\Omega}} \int\limits_{B_R(x_0)} |D^2u_0|^2 r^\alpha \zeta dx \right)^{\frac{m}{m+2\gamma}} + \left(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2 \right)^{\frac{m}{m+2\gamma}} \right].$$

Take $|a|^2 + \sum_{k=1}^{M-1} \eta_k^2$ so small that $C(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2)^{\frac{2\gamma}{m+2\gamma}} < 1$. Then (4.21) gives

$$\sup_{x_0 \in \Omega} \int\limits_{B_R} |D^2u_1|^2 r^\alpha \zeta dx \le (1 + M_\gamma^2) \left[1 - \frac{4\alpha(m-1)}{(\alpha+m)^2} + \eta \right] \times$$

$$\times \left\{ 1 - \frac{\lambda\Lambda^{-1}}{1 + [(\sup\limits_{x_0 \in \overline{\Omega}} \int_{B_R(x_0)} |D^2u_0|^2 r^\alpha \zeta dx)^{\frac{m}{2(m+2\gamma)}} + C(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2)^{\frac{m}{2(m+2\gamma)}}]^s} \right\} \times$$

$$\times \sup_{x_0 \in \overline{\Omega}} \int\limits_{B_R} |D^2u_0|^2 r^\alpha \zeta dx + C \left(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2 \right). \tag{4.26}$$

Denote

$$\left.\begin{aligned} X_i &= \sup_{x_0 \in \Omega} \int_{B_R} |D^2u_i|^2 r^\alpha \zeta dx, (i = 0, 1) \\ Q &= (1 + M_\gamma^2) \left[1 - \tfrac{4\alpha(m-1)}{(\alpha+m)^2} + \eta \right], \\ H &= C(|a|^2 + \textstyle\sum_{k=1}^{M-1} \eta_k^2). \end{aligned}\right\} \tag{4.27}$$

The inequality (4.26) turns now to

$$X_1 \le Q \left(1 - \frac{\lambda\Lambda^{-1}}{1 + X_0^{\frac{ms}{2(m+2\gamma)}} + H^{\frac{ms}{2(m+2\gamma)}}} \right) X_0 + H.$$

It can be written in the form

$$X_1 \;\; \le \;\; X_0 + (Q-1) \left\{ \left[1 - \frac{Q\lambda\Lambda^{-1}}{(Q-1)(1 + X_0^{\frac{ms}{2(m+2\gamma)}} + H^{\frac{ms}{2(m+2\gamma)}}} \right] X_0 + \frac{H}{Q-1} \right\}$$

After elementary consideration it can be proved that for sufficiently small H and

$$Q\lambda\Lambda^{-1}(Q-1)^{-1} > 1 \tag{4.28}$$

both X_0 and X_1 satisfy the estimate

$$X_i < [Q\lambda\Lambda^{-1}(Q-1)^{-1} - H^{\frac{ms}{2(m+2\gamma)}} - 1]^{\frac{2(m+2\gamma)}{ms}}. \tag{4.29}$$

Let us return now to (4.22). From (4.25) and $C(|a|^2 + \sum_{k=1}^{M-1} \eta_k^2)^{\frac{2\gamma}{m+2\gamma}} < 1$ we get

$$\int |D^2u|^2 r^{\alpha_k}\zeta^2 dx \leq \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2} + \eta\right] \times$$

$$\times \left(1 - \frac{\lambda\Lambda^{-1}}{1 + [(\sup\limits_{x_0\in\Omega B_R} \int |D^2u_0|^2 r^{\alpha}\zeta^2 dx)^{\frac{m}{2(m+2\gamma)}} + C(|a|^2 + \sum_{k=1}^{M-1}\eta_k^2)^{\frac{m}{2(m+2\gamma)}}]^s}\right) \times$$

$$\times \int\limits_{B_R} |D^2u_0|^2 r^{\alpha_k}\zeta^2 dx + C\left[\int\limits_{B_R} |Du_n|^2 dx + |a|^2 + \int |\Delta u_{n+1}|^2 r^{\alpha_{k-1}}\zeta^2 dx\right].$$

With the help of (4.27) and (4.29) we have for $k < M$

$$\int\limits_{B_R(x_0)} |D^2u_1|^2 r^{\alpha_k}\zeta dx \leq \left[1 - \frac{4\alpha_k(m-1)}{(\alpha_k+m)^2} + \eta\right]\left[1 - \frac{4\alpha(m-1)}{(\alpha+m)^2}\right]^{-1} \times$$

$$\times(1+M_\gamma^2)^{-1} \int\limits_{B_R(x_0)} |D^2u_0|^2 r^{\alpha_k}\zeta dx + C(|a|^2 + \sum_{s=1}^{k-1}\eta_s^2).$$

All α_k are negative and decreasing. Then from the last inequality we have

$$\int\limits_{B_R(x_0)} |D^2u_1|^2 r^{\alpha_k}\zeta dx \leq (1+M_\gamma^2)^{-1}\eta_k^2 + C(|a|^2 + \sum_{s=1}^{k-1}\eta_s^2).$$

From (4.23) follows that

$$\int\limits_{B_R(x_0)} |D^2u_1|^2 r^{\alpha_k}\zeta dx < \eta_k^2$$

and therefore for u_1 all conditions of the theorem are satisfied . So the inequality (4.8) and the theorem are proved for $m \leq 4$.
For $m = 2$ and $m = 3$ let us remark, that if we take $\alpha_1 = -\frac{m}{2} + \eta$ then the condition

$$-\frac{m}{2} + \eta < 2 - m - 2\gamma$$

can be satisfied at least for small γ and all consideration are simplified. □

Remark. If $\gamma > 0$ is small then the condition (4.24) gives

$$\frac{\Lambda}{\lambda}\frac{(1+\frac{m-2}{m+1})[1+(m-2)(m-1)] - 1}{(1+\frac{m-2}{m+1})[1+(m-2)(m-1)]} < 1$$

For $m = 2$ this inequality doesn't restrict the dispersion of the spectrum for the matrix of ellipticity.

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24. P. Knabner, I. Kögel–Knabner, K.U. Totsche: The modeling of reactive solute transport with sorption to mobile and immobile sorbents.

25. S. Seifarth: The discrete spectrum of the Dirac operators on certain symmetric spaces.

26. J. Schmeling: Hölder continuity of the holonomy maps for hyperbolic basic sets II.

27. P. Mathé: On optimal random nets.

28. W. Wagner: Stochastic systems of particles with weights and approximation of the Boltzmann equation. The Markov process in the spatially homogeneous case.

29. A. Glitzky, K. Gröger, R. Hünlich: Existence and uniqueness results for equations modelling transport of dopants in semiconductors.

30. J. Elschner: The *h-p*–version of spline approximation methods for Mellin convolution equations.

31. R. Schlundt: Iterative Verfahren für lineare Gleichungssysteme mit schwach besetzten Koeffizientenmatrizen.

32. G. Hebermehl: Zur direkten Lösung linearer Gleichungssysteme auf Shared und Distributed Memory Systemen.

33. G.N. Milstein, E. Platen, H. Schurz: Balanced implicit methods for stiff stochastic systems: An introduction and numerical experiments.

34. M.H. Neumann: Pointwise confidence intervals in nonparametric regression with heteroscedastic error structure.

35. M. Nussbaum: Asymptotic equivalence of density estimation and white noise.

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36. B. Kleemann, A. Rathsfeld: Nyström's method and iterative solvers for the solution of the double layer potential equation over polyhedral boundaries.

37. W. Dahmen, S. Prössdorf, R. Schneider: Wavelet approximation methods for pseudodifferential equations II: matrix compression and fast solution.

38. N. Hofmann, E. Platen, M. Schweizer: Option pricing under incompleteness and stochastic volatility.

39. N. Hofmann: Stability of numerical schemes for stochastic differential equations with multiplicative noise.

40. E. Platen, R. Rebolledo: On bond price dynamics.

41. E. Platen: An approach to bond pricing.

42. E. Platen, R. Rebolledo: Pricing via anticipative stochastic calculus.

43. P.E. Kloeden, E. Platen: Numerical methods for stochastic differential equations.

44. L. Brehmer, A. Liemant, I. Müller: Ladungstransport und Oberflächenpotentialkinetik in ungeordneten dünnen Schichten.

45. A. Bovier, C. Külske: A rigorous renormalization group method for interfaces in random media.

46. G. Bruckner: On the regularization of the ill–posed logarithmic kernel integral equation of the first kind.

47. H. Schurz: Asymptotical mean stability of numerical solutions with multiplicative noise.

48. J.W. Barrett, P. Knabner: Finite element approximation of transport of reactive solutes in porous media. Part I: Error estimates for non–equilibrium adsorption processes.

49. M. Pulvirenti, W. Wagner, M.B. Zavelani Rossi: Convergence of particle schemes for the Boltzmann equation.

50. J. Schmeling: Most β shifts have bad ergodic properties.

51. J. Schmeling: Self normal numbers.

52. D.A. Dawson, K. Fleischmann: Super–Brownian motions in higher dimensions with absolutely continuous measure states.