A CYCLICALLY CATALYTIC SUPER-BROWNIAN MOTION

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Abstract. In generalization of the mutually catalytic super-Brownian motion in R of Dawson/Perkins (1998) and Mytnik (1998), a function-valued cyclically catalytic model X is constructed as a strong Markov solution to a martingale problem. Starting with a finite population X_0, each pair of neighboring types will globally segregate in the long-term limit (non-coexistence of neighboring types). Also finer extinction/survival properties depending on X_0 are studied in the spirit of Mueller and Perkins (1999). In fact, X_0 can be chosen in such a way that all types survive for all finite times. On the other hand, sufficient conditions on X_0 are stated for the following situation: Given a type k and a positive time t, the k^{th} subpopulation X^k dies by time t with a large probability, provided that its initial value X_0^k was sufficiently small.

1. Introduction

1.1. **Background and motivation.** Recently Dawson and Perkins [11] and Mytnik [27] introduced and studied a mutually catalytic super-Brownian motion in R. This is a function-valued diffusion of two types of materials (species) where the small portions of mass ("particles") move chaotically in R but additionally branch (split or die) with a locally and temporally given rate proportional to the density of mass of the other type. Thus, each type serves as a catalyst for the other type's branching. This *true interaction* of types destroys the usual independence assumption in branching theory, in particular, this model is *not* a superprocess in its standard definition (for superprocesses, see, for instance, Dynkin [14]). For a recent survey on catalytic and mutually catalytic branching models, we refer to Dawson and Fleischmann [8, 9].

It is a natural desire to extend the mutually catalytic model to $K \geq 2$ types $A^0, ..., A^{K-1}$ of materials (as a rule, we write the index referring to the type as an upper index – please, do not misunderstand the index as a power). We restrict to a *cyclic* situation, as often met in epidemics (see, for instance, Mollison [25]), networks of neurons (see, e.g., Gravner and Griffeath [17]), or biological competition models (see, for instance, Durrett and Levin [13]):

$$A^k + A^{k+1} \longrightarrow A^k, \qquad k \in \mathsf{K}, \tag{1}$$

where $\mathsf{K} = \{0, ..., K-1\}$ denotes the cyclic group of size $K \geq 2$ (the additive group modulo K).

For treatments of cyclic reactions in terms of interacting particle systems, see Bramson and Griffeath [2], and Durrett [12], related to noise-induced transport phenomena, see Freund et al. [16], and in terms of deterministic equations, see

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Boerlijst and Hogeweg [1], Merino [22], Molina et al. [24], and Rujigrok and Rujigrok [30] (for instance).

1.2. **Rough description of the model.** A bit more precisely, we consider the following stochastic equation

$$\mathrm{d}X_t^k(a) \;=\; \frac{\sigma^2}{2}\Delta X_t^k(a)\,\mathrm{d}t \,+\, \sqrt{\gamma^k\, X_t^k(a)X_t^{k+1}(a)}\;\mathrm{d}W_t^k(a), \tag{2}$$

$t>0,\;\;(k,a)\in \mathsf{K}\times\mathsf{R}$. Here the one-dimensional Laplacian Δ acts on the real-valued variable a, and $\frac{\sigma^2}{2}\Delta$ reflects the chaotic motion of particles with diffusion constant $\sigma>0$. Moreover, the constants $\gamma^k>0$ are the interaction rates, and $\mathrm{d}W$ denotes a standard white noise on $\mathsf{R}_+\times\mathsf{K}\times\mathsf{R}$.

The quantity $X_t^k(a)$ can be interpreted as the *density* of mass of type k at time t at site a. *Intuitively,* the subpopulation X^k of X of type k evolves as a super-Brownian motion in R but with branching rate $\gamma^k\,X_t^{k+1}(a)$ changing with time t and site a. Hence, the subpopulation X^{k+1} serves as a catalyst for the branching of X^k, for each $k\in\mathsf{K}$. Recall again that by this cyclic interaction over *all* the types, the basic independence assumption in branching theory is violated, so that X is not a superprocess according to the usual definition.

Of course, in the special case $K=2$ we get the mutually catalytic branching model in R of [11] and [27]. (For further results on mutually catalytic models, see also Cox et al. [3] and [5], Cox and Klenke [4], Dawson et al. [6], [7], and [10], as well as Mueller and Perkins [26].)

Intuitively, a solution to equation (2) should be a (time-homogeneous) Markov process X. The first *purpose* of the paper is to establish that a weak solution X to equation (2) exists which is a strong Markov process (see Theorem 3 below). Unfortunately, uniqueness of solutions remains *open* at this stage. The main obstacle for this is that as opposed to the mutually catalytic model, for $K\geq 3$ a self-duality ([27]) does *not* hold, and we also have not been able to find any other dual (or approximate dual) process for X. Nevertheless, *each* strong Markov solution X to (2) we call a *cyclically catalytic super-Brownian motion* (SBM) in $\mathsf{K}\times\mathsf{R}$ (see also Definition 2 below). Besides the construction, we start the investigation of the survival/extinction behavior of cyclically catalytic SBMs in the case of *finite* populations (Theorems 4 and 5).

One expects that also a strong Markov Z^d–version of the cyclically catalytic model exists just as in the mutually catalytic case of [11]. The long-time results presented for the present cyclically catalytic SBMs in R should hold also for cyclically catalytic simple super-random walks in Z^d (for Theorem 4: restrict to $d\leq 2$). The existence of a R^2–version however remains *open* at this stage (note that for $K\geq 3$, moment dual processes or moment equations for $\varepsilon\mathsf{Z}^2$–approximations are much more complicated compared with the $K=2$ case, so that it is not clear how methods from [6, 7] could be extended).

2. Results

2.1. **Preliminaries: notations.** With c we always denote a positive constant which might vary from place to place. A c with some additional mark (as $\overline{c}$ or c_1) will however denote a specific constant. A constant of the form $c_{(\#)}$ means, this constant first occurred related to formula line (#).

For $\lambda \in \mathsf{R}$, introduce the *reference function* ϕ_λ :

$$\phi_\lambda(a) \;:=\; \mathrm{e}^{-\lambda|a|}, \qquad a \in \mathsf{R}, \tag{3}$$

(as usual, the colon attached to an equality sign "=" refers to the side of the introduced notation). For $f : \mathsf{K} \times \mathsf{R} \to \mathsf{R}$, put

$$|f|_\lambda \;:=\; \sup_{k\in\mathsf{K},\, a\in\mathsf{R}} |f^k(a)| \,/\, \phi_\lambda(a), \qquad \lambda \in \mathsf{R}. \tag{4}$$

(Note that compared with [11] we reversed the sign in the definition of ϕ_λ but we kept it in the definition of $|\cdot|_\lambda$, and, concerning this, we use the same conventions as in Shiga [31].)

At some places we will need also a smoothed version $\tilde{\phi}_\lambda$ of ϕ_λ. For this purpose, introduce the mollifier

$$\rho(a) \;:=\; c_{(5)}\, \mathbf{1}_{\{|a|<1\}}\, \exp\big[-1/(1-a^2)\big], \qquad a \in \mathsf{R}, \tag{5}$$

with $c_{(5)}$ the normalizing constant such that $\int \mathrm{d}a\, \rho(a) = 1$. For $\lambda \in \mathsf{R}$, set

$$\tilde{\phi}_\lambda(a) \;:=\; \int \mathrm{d}b\; \phi_\lambda(b)\, \rho(b-a), \qquad a \in \mathsf{R}. \tag{6}$$

Note that to each $\lambda \in \mathsf{R}$ and $m \geq 0$ there are positive constants $\underline{c}_{\lambda,m}$ and $\overline{c}_{\lambda,m}$ such that

$$\underline{c}_{\lambda,m}\, \phi_\lambda(a) \;\leq\; \left|\frac{\mathrm{d}^m}{\mathrm{d}a^m}\, \tilde{\phi}_\lambda\,(a)\right| \;\leq\; \overline{c}_{\lambda,m}\, \phi_\lambda(a), \qquad a \in \mathsf{R}, \tag{7}$$

(cf. Mitoma [23, (2.1)]).

For $\lambda \in \mathsf{R}$, let $\mathcal{C}_\lambda = \mathcal{C}_\lambda(\mathsf{K} \times \mathsf{R})$ denote the set of all continuous (real-valued) functions f on $\mathsf{K} \times \mathsf{R}$ such that $|f|_\lambda$ is finite, and such that $f^k(a)/\phi_\lambda(a)$ has a finite limit as $|a| \uparrow \infty$, for each $k \in \mathsf{K}$. Introduce the spaces

$$\mathcal{C}_{\mathrm{tem}} = \mathcal{C}_{\mathrm{tem}}(\mathsf{K} \times \mathsf{R}) := \bigcap_{\lambda>0} \mathcal{C}_{-\lambda}\,, \qquad \mathcal{C}_{\mathrm{rap}} = \mathcal{C}_{\mathrm{rap}}(\mathsf{K} \times \mathsf{R}) := \bigcap_{\lambda>0} \mathcal{C}_{\lambda} \tag{8}$$

of *tempered* and *rapidly decreasing* functions, respectively. (Roughly speaking, the functions in $\mathcal{C}_{\mathrm{tem}}$ are allowed to have a subexponential growth, whereas the ones in $\mathcal{C}_{\mathrm{rap}}$ decay faster than exponentially.) Write $\mathcal{C}_{\mathrm{rap}}^{(m)} = \mathcal{C}_{\mathrm{rap}}^{(m)}(\mathsf{K} \times \mathsf{R})$ if we additionally require that all partial derivatives $\frac{\partial^m}{\partial a^m}$ up to the order $m \geq 1$ belong to $\mathcal{C}_{\mathrm{rap}}$.

For each $\lambda \in \mathsf{R}$, the linear space $\mathcal{C}_\lambda$ equipped with the norm $|\cdot|_\lambda$ is a separable Banach space. The spaces $\mathcal{C}_{\mathrm{tem}}$ and $\mathcal{C}_{\mathrm{rap}}$ are topologized by the metrics

$$\mathrm{d}_{\mathrm{tem}}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n} \left(|f-g|_{-1/n} \,\wedge\, 1\right), \qquad f,g \in \mathcal{C}_{\mathrm{tem}}\,, \tag{9}$$

$$\mathrm{d}_{\mathrm{rap}}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n} \left(|f-g|_{1/n} \,\wedge\, 1\right), \qquad f,g \in \mathcal{C}_{\mathrm{rap}}\,, \tag{10}$$

making them to Polish spaces. Similarly, we also define in $\mathcal{C}_{\mathrm{rap}}^{(m)}$, $m \geq 1$, metrics in the obvious way to make them Polish.

Write $\Omega := \mathcal{C}(\mathsf{R}_+\,, \mathcal{C}_{\mathrm{tem}}^{+})$ for the set of all continuous paths $t \mapsto \omega_t \in \mathcal{C}_{\mathrm{tem}}$. Equipped with the metric

$$\mathrm{d}_\Omega(\omega,\omega') \;:=\; \sum_{n=1}^{\infty} 2^{-n} \Big(\sup_{0\leq t\leq n} \mathrm{d}_{\mathrm{tem}}\,(\omega_t\,,\omega'_t) \wedge 1\Big), \qquad \omega,\omega' \in \Omega, \tag{11}$$

Ω is a Polish space. The σ–field of all Borel subsets of Ω is denoted by $\mathcal{F}$.

If E is a topological space, a measure on E is meant to be a measure defined on the σ–field of all Borel subsets of E.

Let $\mathcal{P}$ denote the set of all probability measures on Ω. Endowed with the Prohorov metric $\mathrm{d}_{\mathcal{P}}$, we get a Polish space (Ethier and Kurtz [15, Theorem 3.1.7]). Write $\mathrm{com}(\mathcal{P})$ for the collection of all compact subsets of $\mathcal{P}$, equipped with the metric

$$\mathrm{d}_{\mathrm{com}}(K_1,K_2) \;:=\; \inf\{\varepsilon>0:\ K_1\subseteq K_2^{\varepsilon} \text{ and } K_2\subseteq K_1^{\varepsilon}\}, \qquad K_1,K_2\in\mathrm{com}(\mathcal{P}),$$

where K^{ε} is the ε–neighborhood of K (based on $\mathrm{d}_{\mathcal{P}}$). Then the metric space $\bigl(\mathrm{com}(\mathcal{P}),\mathrm{d}_{\mathrm{com}}\bigr)$ is separable (Stroock and Varadhan [32, Lemma 12.1.1]).

As a rule, the processes $X=\{X_t:\,t\geq 0\}$ considered in this paper are $\mathcal{C}_{\mathrm{tem}}^{+}$–valued, continuous, and presented in their canonical form. That is, we identify each process X with a probability law P on $\Omega=\mathcal{C}(\mathsf{R}_+,\mathcal{C}_{\mathrm{tem}}^{+})$, in other words, with the probability space $(\Omega,\mathcal{F},P)$. More precisely, we always consider $(\Omega,\mathcal{F},P)$ as a filtered probability space, using the usual filtration $\{\mathcal{F}_t:\,t\geq 0\}$. Write $\mathcal{F}^r$ for the sub–σ–field of $\mathcal{F}$ generated by the coordinate maps $\omega\mapsto\omega_t$, for $t\geq r$.

Let $\mathrm{d}k$ denote the counting measure (Haar measure) on the cyclic group K (that is, $\int_{\mathsf{K}}\mathrm{d}k\ f(k)=\sum_{k\in\mathsf{K}}f(k)$ for all functions $f:\mathsf{K}\to\mathsf{R}_+$). For functions f,g on $\mathsf{K}\times\mathsf{R}$ or R, we write $\langle f,g\rangle$ for the integral of $f\cdot g$ with respect to $\mathrm{d}k\,\mathrm{d}a$ or $\mathrm{d}a$, respectively, (if the integral makes sense). As opposed to the notation $|\cdot|_\lambda$ introduced in (4), for functions $f\geq 0$ on $\mathsf{K}\times\mathsf{R}$ or R we define

$$\|f\|_\lambda \;:=\; \bigl\langle f\cdot\tilde{\phi}_{-\lambda}\,,1\bigr\rangle, \qquad \lambda\in\mathsf{R}, \tag{12}$$

[with the smoothed reference function $\tilde{\phi}_{-\lambda}$ from (6)]. Set $\|f\|:=\|f\|_0$ for the "total mass" of the (density) function f.

Let p denote the heat kernel in R related to $\frac{\sigma^2}{2}\Delta$:

$$\mathrm{p}_t(a) \;:=\; (2\pi\sigma^2 t)^{-1/2}\exp\Bigl[-\frac{|a|^2}{2\sigma^2 t}\Bigr], \qquad t>0,\quad a\in\mathsf{R}, \tag{13}$$

and $\{S_t:\,t\geq 0\}$ the corresponding heat flow semigroup. Write $\xi=(\xi,\Pi_a)$ for the related Brownian motion in R, with Π_a denoting the law of ξ if $\xi_0=a\in\mathsf{R}$.

The (usually upper) index $+$ on a set of real-valued functions will refer to the collection of all non-negative members of this set, similarly to our notation $\mathsf{R}_+=[0,\infty)$. The Kronecker symbol is denoted by $\delta_{k,\ell}$.

2.2. Existence of X and basic properties of all solutions.

A more precise formulation of the stochastic equation (2) can be given in terms of the following martingale problem $\mathbf{MP}_x$. Recall that $K\geq 2$, $\sigma>0$, and $\gamma^k>0$, $k\in\mathsf{K}$.

Definition 1 (Martingale problem $\mathbf{MP}_x$)**.** Fix $x\in\mathcal{C}_{\mathrm{tem}}^{+}=\mathcal{C}_{\mathrm{tem}}^{+}(\mathsf{K}\times\mathsf{R})$. We say a stochastic process $X=\{X_t:\,t\geq 0\}$ with law P_x on $\Omega=\mathcal{C}(\mathsf{R}_+,\mathcal{C}_{\mathrm{tem}}^{+})$ is a *solution to the martingale problem* $\mathbf{MP}_x$ if $P_x(X_0=x)=1$ and, for test functions $\varphi\in\mathcal{C}_{\mathrm{rap}}^{(2)}=\mathcal{C}_{\mathrm{rap}}^{(2)}(\mathsf{K}\times\mathsf{R})$, setting

$$M_t^k(\varphi^k) \;:=\; \bigl\langle X_t^k,\varphi^k\bigr\rangle-\bigl\langle x^k,\varphi^k\bigr\rangle-\int_0^t\mathrm{d}s\,\Bigl\langle X_s^k,\,\frac{\sigma^2}{2}\Delta\varphi^k\Bigr\rangle, \tag{14}$$

$t\geq 0$, $k\in\mathsf{K}$, one has orthogonal continuous square-integrable martingales $M^k(\varphi^k)$, $k\in\mathsf{K}$, starting from $M_0^k(\varphi^k)\equiv 0$, and with square functions

$$\bigl\langle\!\bigl\langle M^k(\varphi^k)\bigr\rangle\!\bigr\rangle_t \;=\; \gamma^k\int_0^t\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}a\ X_s^k(a)X_s^{k+1}(a)\bigl[\varphi^k(a)\bigr]^2, \tag{15}$$

$t \geq 0, \;\; k \in \mathsf{K}.$ $\diamond$

Now the definition of our basic object of interest follows:

Definition 2 (Cyclically catalytic SBM X)**.** If $\bigl(X, P_x\,,\, x \in \mathcal{C}^{+}_{\text{tem}}\bigr)$ is a (time-homogeneous) strong Markov process such that (X, P_x) is a solution to the martingale problem $\mathbf{MP}_x$ of Definition 1, for each $x \in \mathcal{C}^{+}_{\text{tem}}$, then it is called a *cyclically catalytic super-Brownian motion (SBM)* in $\mathsf{K} \times \mathsf{R}$ with diffusion constant σ and interaction rate $\gamma = (\gamma^k)_{k \in \mathsf{K}}$. $\diamond$

Here is our first result:

Theorem 3 (Cyclically catalytic SBM X)**.**

(a) (Existence of X): *To each $K \geq 2$, $\sigma > 0$, and vector $\gamma > 0$, there exists a cyclically catalytic super-Brownian motion $\bigl(X, P_x\,,\, x \in \mathcal{C}^{+}_{\text{tem}}\bigr)$ in $\mathsf{K} \times \mathsf{R}$ with diffusion constant σ and interaction rate γ according to Definition 2.*

(b) (Finite moments): *Each cyclically catalytic SBM X has finite moments of all orders: For fixed $c_0\,, T, q > 0$ and $\lambda', \lambda \in \mathsf{R}$ with $q\lambda' < \lambda$,*

$$\sup_{x \in \mathcal{C}^{+}_{\text{tem}}\,,\; |x|_{-\lambda'} \leq c_0} P_x \sup_{0 \leq t \leq T} \sum_{k \in \mathsf{K}} \bigl\langle (X^k_t)^q, \phi_\lambda \bigr\rangle \;<\; \infty.$$

The expectation of X is given by

$$P_x\, X^k_t(a) \;=\; S_t x^k\,(a), \qquad x \in \mathcal{C}^{+}_{\text{tem}}\,, \quad (t,k,a) \in \mathsf{R}_+ \times \mathsf{K} \times \mathsf{R},$$

and the covariance by

$$\mathcal{C}\mathrm{ov}_x\Bigl(X^{k_1}_{t_1}(a_1),\, X^{k_2}_{t_2}(a_2)\Bigr) \;=\; \gamma^{k_1}\, \delta_{k_1,k_2} \int_0^{t_1 \wedge t_2} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b$$
$$\times\; S_s x^{k_1}(b)\; S_s x^{k_1+1}(b)\, \mathrm{p}_{t_1-s}(a_1 - b)\, \mathrm{p}_{t_2-s}(a_2 - b),$$

$x \in \mathcal{C}^{+}_{\text{tem}}\,,\;\; t_1, t_2 \geq 0,\;\; k_1, k_2 \in \mathsf{K},\;\; a_1, a_2 \in \mathsf{R}.$

Note that the covariance vanishes only if $x = 0$, $k_1 \neq k_2$, or $t_1 \wedge t_2 = 0$. In particular, the process X is *non-degenerate.*

Recall that the novelty of this theorem concerns the case $K \geq 3$, since $K = 2$ is due to [11], and that uniqueness remains unsolved if $K \geq 3$.

The proof of Theorem 3 will be provided in Section 3 below. There we will start from an approximating system of processes where on small time intervals we consider K conditionally independent catalytic super-Brownian motions with frozen, smoothed, and truncated branching rate functions (catalyst). Then tightness will be shown by an adoption of a method used in [11] which was based on [31]. This then yields the existence of solutions to the martingale problem $\mathbf{MP}_x$ of Definition 1. Note that the existence of a weak solution to the stochastic equation (2) (on an enlarged probability space) then follows from the standard martingale representation theorem (Walsh [35]). We mention also that the convolution form of (2) is given in equation (57) below.

Since uniqueness in the martingale problem is not established, some more efforts are needed to construct a *Markov* solution to the martingale problem. Moreover, since the topic of continuous dependence on the initial data of constructed solutions is also a delicate unsolved problem in the present model, we could not follow the usual route to deduce the strong Markov property from a Feller property. Nevertheless, by an adoption of methods developed in [32] for finite-dimensional diffusions, we succeeded in selecting a time-homogeneous *strong* Markov process from the set of *all* solutions of the family of martingale problems.

Part (a) of Theorem 3 is implied by Theorem 23 below, whereas (b) follows from Corollary 16.

2.3. **Global segregation of neighboring types.** Now we restrict our attention to any cyclically catalytic SBM X as introduced in Definition 2 (which exists by Theorem 3), and fix its initial state X_0.

In the mutually catalytic model (in R), the self-duality is a powerful tool not only for establishing the uniqueness in the martingale problem, but also to get results on the long-term behavior ([11]). In fact, the total mass process $t \mapsto \big(\|X_t^0\|, \|X_t^1\|\big)$ in the case of finite initial masses $\|X_0\| = X_0^0(\mathsf{R}) + X_0^1(\mathsf{R})$ is a non-negative martingale, and its a.s. limit $\big(\|X_\infty^0\|, \|X_\infty^1\|\big)$, say, can be identified in relatively simple terms. Indeed, it coincides in law with the state B_τ of a Brownian motion B in R_+^2 in its first hitting time τ of the boundary $\partial\mathsf{R}_+^2$ of R_+^2, if B was started from $B_0 = \big(\|X_0^0\|, \|X_0^1\|\big)$ (see the proof of Theorem 1.2 (a) in [11]). In particular, the limit population is non-degenerate and has full expectation (persistence).

Of course, the present cyclically catalytic model also has that convergence property:

$$\lim_{t\uparrow\infty} \big(\|X_t^0\|, ..., \|X_t^{K-1}\|\big) \ =: \ \big(\|X_\infty^0\|, ..., \|X_\infty^{K-1}\|\big) \quad \text{exists a.s.} \tag{16}$$

provided that $\|X_0\| < \infty$. But we have not been able to identify the limit (16). An obstacle is, that the random time change argument of [11] is not as powerful, since it leads to K Brownian motions which run with *different* clocks, as opposed to the $\mathsf{K} = 2$ case. In other words, in the terminology of Swart [33], the $K \geq 3$ case is an *anisotropic* situation, which is much more delicate than the isotropic $K = 2$ case. Nevertheless, we are able to verify the following "global segregation" (non-coexistence) of neighboring types in the limit, which in the $K = 2$ case is a simple consequence of a property of the hitting state B_τ, namely that $B_\tau^0 B_\tau^1 = 0$. Recall that $K \geq 2$.

Theorem 4 (Global segregation of neighboring types)**.** *Start any cyclically catalytic super-Brownian motion X with a finite initial mass $\|X_0\|$. Then, for each $k \in \mathsf{K}$,*

$$\lim_{t\uparrow\infty} \|X_t^k\| \cdot \|X_t^{k+1}\| \ = \ 0, \quad a.s. \tag{17}$$

Consequently, for each pair of neighboring types, only one of them has the chance to survive in the limit.

The proof of Theorem 4 will be given in Section 4. It will be based on a modification of arguments of [11], adapted to the aforementioned case of different clocks if $K \geq 3$. The strategy of proof is as follows. Set

$$Z_t \ := \ \sum_{k\in\mathsf{K}} \gamma^k \, \|X_t^k\| \cdot \|X_t^{k+1}\|, \qquad t \geq 0. \tag{18}$$

Note that by (15), Z is the square function of the non-negative, hence converging in R_+, martingale $t \mapsto \|X_t\|$. Assuming now that

$$\inf_{t\geq 0} Z_t \ > \ 0 \quad \text{with positive probability,} \tag{19}$$

our task is to construct stopping times $T_1, T_2, \ldots$ such that $Z_{T_n} \to 0$ as $n \uparrow \infty$ on the event in (19). But this is an obvious contradiction. Then the almost sure convergence of the martingales $t \mapsto \|X_t^k\|$, $k \in \mathsf{K}$, will yield the claim (17).

As opposed to the mutually catalytic case, Theorem 4 in particular leaves *open*, whether for $K \geq 3$ the limit $\left(\|X^0_\infty\|, ..., \|X^{K-1}_\infty\|\right)$ in (16) is non-degenerate, and whether it has full expectation (persistence).

2.4. **Finite time survival/extinction.** For the mutually catalytic model in Z^d (also established and investigated in [11]), the recent preprint [26] addresses the following questions: Is it possible that depending on the finite initial state both types survive all finite times a.s., or that one of the types dies in a given finite time with high probability? The following results on our cyclic model are in that spirit. Recall the reference function ϕ_λ introduced in (3).

Theorem 5 (Finite time behavior)**.** *Fix again any cyclically catalytic SBM X with $X_0 \in \mathcal{C}^+_{\mathrm{tem}}$ satisfying $\|X_0\| < \infty$.*

(a) (Finite time survival of all types): *Assume that $\prod_{k\in\mathsf{K}} \|X^k_0\| > 0$, and that there is a $T > 0$ such that*

$$\max_{k\in\mathsf{K}} \liminf_{|a|\uparrow\infty} \frac{S_t[X^k_0]^2(a)\; S_t[X^{k+1}_0]^2(a)}{\left[S_t X^k_0(a)\right]^4} \;=\; 0, \qquad t \geq T. \tag{20}$$

Then

$$\prod_{k\in\mathsf{K}} \|X^k_t\| \;>\; 0 \quad \textit{for all } t > 0, \quad \textit{a.s.} \tag{21}$$

(b) (Finite time extinction of a type with high probability): *Fix a type $k_0 \in \mathsf{K}$. For $i = 0, 1, 2$, consider positive constants $c_i\,, \lambda_i\,,$ and $c'_1\,, \lambda'_1$ with*

$$\lambda_0 > \lambda'_1 > \lambda_1\,, \quad 2\lambda'_1 < \lambda_1 + \lambda_2\,, \quad \textit{and} \quad c'_1 \leq c_1\,. \tag{22}$$

Then the following statement holds. For $\varepsilon \in (0, 1]$ and $T > 0$ fixed, c_0 can be chosen so small that if the initial state $X_0 = x \in \mathcal{C}^+_{\mathrm{tem}}$ is such that

$$x^{k_0} \;\leq\; c_0\, \phi_{\lambda_0} \tag{23}$$

as well as

$$|x|_{\lambda_1} \;\leq\; c_1\,, \quad x^{k_0+2} \;\leq\; c_2\, \phi_{\lambda_2}\,, \quad \textit{and} \quad x^{k_0+1} \;\geq\; c'_1\, \phi_{\lambda'_1}\,, \tag{24}$$

then

$$P_x\Big(X^{k_0}_t = 0 \ \textit{for } t \geq T\Big) \;\geq\; 1 - \varepsilon. \tag{25}$$

The proof of Theorem 5 in Section 5 below uses ideas of [26]. Of course, the condition (20) in Theorem 5 (a) looks a bit complicated, so we have to discuss it. Roughly speaking, it is for instance satisfied, if the initial states of each pair of neighboring types are separated in different half axes and have sufficiently large tails. This will now be made more precise in the following example.

Example 6 (Starting from separated neighbors with large tails)**.** Assume $K \geq 2$ is even and that

$$X^{2k}_0 \;:=\; \phi_1\, \mathbf{1}_{\mathsf{R}\setminus\mathsf{R}_+} \quad \text{and} \quad X^{2k+1}_0 \;:=\; \phi_1\, \mathbf{1}_{\mathsf{R}_+}\,, \qquad k \in \mathsf{K}, \tag{26}$$

[with the reference function ϕ_1 from (3)], ignoring the discontinuity at $0 \in \mathsf{R}$, which can simply be overcome by a smoothing procedure, for instance using the mollifier

ρ from (5). Then the *simultaneous finite time survival* as claimed in (21) holds. In fact,

$$S_t X_0^{2k}(a) \;=\; \mathcal{N}\Big(\frac{-a-t}{\sqrt{t}}\Big)\, \mathrm{e}^{a+t/2}, \tag{27a}$$

$$S_t [X_0^{2k}]^2(a) \;=\; \mathcal{N}\Big(\frac{-a-2t}{\sqrt{t}}\Big)\, \mathrm{e}^{2a+2t}, \tag{27b}$$

$$S_t [X_0^{2k+1}]^2(a) \;=\; \mathcal{N}\Big(\frac{a-2t}{\sqrt{t}}\Big)\, \mathrm{e}^{-2a+2t}, \tag{27c}$$

with $\mathcal{N}$ denoting the distribution function of the standard normal law on R. As $a \downarrow -\infty$, the $\mathcal{N}$–expressions in (27a) and (27b) tend to 1, for fixed $t>0$. Therefore, the ratio in assumption (20) with k replaced by $2k$ is of order $\mathcal{N}\big(a/\sqrt{t}\,\big)/\mathrm{e}^{4a}$ as $a \downarrow -\infty$, hence converges to zero by L'Hopital's rule. On the other hand, if we shift the type by one, then we get the same order of decay if $a \uparrow \infty$ instead. Altogether, assumption (20) is fulfilled, hence (21) holds. $\diamond$

The philosophy behind the proof of Theorem 5 (b) is as follows. Since 0 is an absorbing state for the subprocess $t \mapsto X_t^{k_0}$, it suffices to consider X on a possibly smaller time interval $[0,T]$. Moreover, because initially the catalyst X^{k_0+2} for X^{k_0+1} is not too large by assumption, X^{k_0+1} should not be very small on $[0,T]$, and since X^{k_0+1} serves as the catalyst for X^{k_0}, the latter should have some chance to die by time T. Actually, we want to bound $\|X_T^{k_0}\|_\lambda$ from above (for an appropriate λ and on a suitable new time scale) by a supercritical Feller's branching diffusion, which of course dies by a given time with a positive probability. Making then its initial state $\|x^{k_0}\|_\lambda$ sufficiently small, this extinction probability can be forced to be sufficiently close to one.

Remark 7 (Property of all solutions)**.** As can be seen from the proof in Section 5, Theorem 5 actually applies for *any* family $\big\{(X,P_x) : x \in \mathcal{C}_{\mathrm{tem}}^+\big\}$ of processes such that P_x solves $\mathbf{MP}_x$, $x \in \mathcal{C}_{\mathrm{tem}}^+$, that is, without requiring the Markov property. $\diamond$

Unfortunately, we do not have results on the long-time behavior of X for infinite initial populations (for $K \ge 3$). Note that the study of the long-term behavior of the mutually catalytic model in the case of infinite initial populations (see [3, 4, 5, 7, 11]) also relies heavily on the self-duality of the model. In fact, via self-duality, it is based on the long-term behavior of the finite mass system.

3. Construction (proof of Theorem 3)

We will start with proving the existence of a solution X to the martingale problem $\mathbf{MP}_x$ of Definition 1. Then a time-homogeneous strong Markov solution will be selected from the set of *all* solutions to the family of martingale problems $\mathbf{MP}_x$, $x \in \mathcal{C}_{\mathrm{tem}}^+$, as needed for Theorem 3.

3.1. **Construction of a solution to the martingale problem.** In this subsection, we want to verify that the martingale problem $\mathbf{MP}_x$ of Definition 1 has a solution. To this aim, we will start from some approximations (Definition 10), and will verify some properties of them (Lemmas 11 – 13), which turn out to be owned also by all solutions of the martingale problem $\mathbf{MP}_x$ (see Lemma 14 in Subsection 3.2).

To prepare for the selection of a strong Markov solution, a time-inhomogeneous point of view will be convenient to use: We start the process at times $r \ge 0$ (the model will still be time-homogeneous). On the other hand, for the sake of working with a single path space, we formally extend the paths backwards by assuming that they are constant in the interval $[0, r]$.

Definition 8 (Martingale problem $\mathbf{MP}_{r,x}$)**.** Fix $(r, x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\text{tem}}$. We say a stochastic process $X = \{X_t : t \geq 0\}$ with law $P_{r,x}$ on $\Omega = \mathcal{C}(\mathsf{R}_+, \mathcal{C}^+_{\text{tem}})$ is a *solution to the martingale problem* $\mathbf{MP}_{r,x}$ if the following three conditions hold.

(i) $$P_{r,x}(X_t = x \text{ for all } t \leq r) = 1.$$

(ii) For test functions $\varphi \in \mathcal{C}^{(2)}_{\text{rap}}$, setting

$$M^k_{r,t}(\varphi^k) := \left\langle X^k_t, \varphi^k \right\rangle - \left\langle x^k, \varphi^k \right\rangle - \int_r^t \mathrm{d}s \left\langle X^k_s, \frac{\sigma^2}{2}\Delta\varphi^k \right\rangle,$$

$t \geq r,\ k \in \mathsf{K}$, one has orthogonal continuous square-integrable martingales $t \mapsto M^k_{r,t}(\varphi^k),\ k \in \mathsf{K}$, (after time r) starting from $M^k_{r,r}(\varphi^k) \equiv 0$.

(iii) The square functions satisfy

$$\left\langle\!\left\langle M^k_{r,\cdot}(\varphi^k) \right\rangle\!\right\rangle_t = \gamma^k \int_r^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}a\, X^k_s(a) X^{k+1}_s(a) \left[\varphi^k(a)\right]^2,$$

$t \geq r,\ \ k \in \mathsf{K}$. ◇

Proposition 9 (Existence of a solution to the martingale problem $\mathbf{MP}_{r,x}$)**.** *For each $(r, x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\text{tem}}$, there exists a solution $(X, P_{r,x})$ to the martingale problem $\mathbf{MP}_{r,x}$ of Definition 8.*

For the verification of this proposition, we partly borrow ideas from the proof of Theorem 6.1 in [11] (starting from p.1133), which are in part based on [31, Appendix]. So first of all we will introduce in Definition 10 below an approximating sequence $\{{}^n\!X : n \geq 1\}$ of continuous $\mathcal{C}^+_{\text{tem}}$–valued processes. ${}^n\!X$ has the property that on small time periods $[\frac{i}{n}, \frac{i+1}{n}),\ i \geq 0$, given ${}^n\!X_{i/n}$ the single subpopulations ${}^n\!X^k$ behave as *independent* continuous catalytic super-Brownian motions in R with frozen, smoothed, and bounded branching rate function (catalyst) given by $\gamma^k \left(S_{1/n}{}^n\!X^{k+1}_{i/n} \wedge n\right)$ (see [11]). (The additional smoothing with $S_{1/n}$ – recall that S denotes the heat flow semigroup – will help us to make working a Gronwall's inequality argument in the proof of Lemma 12 below.). Then we pass to a pointwise stochastic equation (Lemma 11) and use it to derive some moment estimates (Lemmas 12 and 13). After these preparations, Proposition 9 then easily follows by an application of [31, Lemma 6.3 (ii)].

We start with introducing the system $\{{}^n\!X : n \geq 1\}$ of approximating $\mathcal{C}^+_{\text{tem}}$–valued processes:

Definition 10 (Martingale problem $\mathbf{MP}^n_{r,x}$)**.** For $n \geq 1$, $(r, x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\text{tem}}$, let $\left({}^n\!X, P^n_{r,x}\right)$ denote the *unique (in law) process* with the following two properties. First of all, ${}^n\!X_t \equiv x$ for $t \leq r$. On the other hand, for $\varphi \in \mathcal{C}^{(2)}_{\text{rap}}$, setting

$$ {}^n\!M^k_{r,t}(\varphi^k) := \left\langle {}^n\!X^k_t, \varphi^k \right\rangle - \left\langle x^k, \varphi^k \right\rangle - \int_r^t \mathrm{d}s \left\langle {}^n\!X^k_s, \frac{\sigma^2}{2}\Delta\varphi^k \right\rangle \tag{28}$$

$t \geq r,\ k \in \mathsf{K}$, one has orthogonal continuous square-integrable martingales $t \mapsto {}^n\!M^k_{r,t}(\varphi^k),\ k \in \mathsf{K}$, starting from ${}^n\!M^k_{r,r}(\varphi^k) = 0$ and with square functions

$$\left\langle\!\left\langle {}^n\!M^k_{r,\cdot}(\varphi^k) \right\rangle\!\right\rangle_t = \gamma^k \int_r^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\, {}^n\!X^k_s(b) \left(S_{1/n}{}^n\!X^{k+1}_{[ns]/n}(b) \wedge n\right) \left[\varphi^k(b)\right]^2, \tag{29}$$

$t \ge r,\ \ k \in \mathsf{K}.$ $\diamond$

Note that the uniqueness in law can be proved via log-Laplace representations of this system of "piecewise independent superprocesses".

The family (28) of martingales extends ([35]) to orthogonal square-integrable *martingale measures* ${}^{n}\!M_r^k = {}^{n}\!M_r^k\big(\mathrm{d}(s,b)\big)$ and to the usual class of predictable integrands. Moreover, for $t \ge r$, and $\varphi \in \mathcal{C}_{\mathrm{rap}}$, as well as $k \in \mathsf{K}$ fixed, the function $(s,a) \mapsto S_{t-s}\varphi^k(a)$ on $[r,t] \times \mathsf{R}$ can be included as integrand of the stochastic integrals. Then ${}^{n}\!X$ can be shown to satisfy the following stochastic equation:

$$\big\langle {}^{n}\!X_t^k, \varphi^k \big\rangle \;=\; \big\langle x^k, S_{t-r}\varphi^k \big\rangle + \int_{[r,t]\times\mathsf{R}} {}^{n}\!M_r^k\big(\mathrm{d}(s,b)\big)\, S_{t-s}\varphi^k(b), \tag{30}$$

$P_{r,x}^n$–a.s., for $t \ge r,\ k \in \mathsf{K},\ \varphi \in \mathcal{C}_{\mathrm{rap}}$. This, in particular, immediately implies the following moment formulas:

$$P_{r,x}^n\, {}^{n}\!X_t^k(a) \;=\; S_{t-r}x^k(a), \tag{31}$$

$$P_{r,x}^n\, {}^{n}\!X_t^k(a)\, {}^{n}\!X_t^\ell(a) \;=\; S_{t-r}x^k(a)\, S_{t-r}x^\ell(a), \tag{32}$$

for all $t \ge r,\ a \in \mathsf{R}$, and $k \ne \ell$. Moreover, replacing φ^k by $\mathrm{p}_\varepsilon(\,\cdot - a)$ in (30), where $0 < \varepsilon \le 1$, and $a \in \mathsf{R}$ are fixed, gives

$$S_\varepsilon {}^{n}\!X_t^k(a) \;=\; S_{\varepsilon+t-r}x^k(a) + \int_{[r,t]\times\mathsf{R}} {}^{n}\!M_r^k\big(\mathrm{d}(s,b)\big)\, \mathrm{p}_{\varepsilon+t-s}(b-a), \tag{33}$$

$P_{r,x}^n$–a.s. We want to let $\varepsilon \downarrow 0$:

Lemma 11 (Pointwise equation for ${}^{n}\!X$)**.** *For* $n \ge 1$, $(r,x) \in \mathsf{R}_+ \times \mathcal{C}_{\mathrm{tem}}^+$, $t \ge r$, *and* $(k,a) \in \mathsf{K} \times \mathsf{R}$ *fixed,*

$$ {}^{n}\!X_t^k(a) \;=\; S_{t-r}x^k(a) + \int_{[r,t]\times\mathsf{R}} {}^{n}\!M_r^k\big(\mathrm{d}(s,b)\big)\, \mathrm{p}_{t-s}(b-a), \quad P_{r,x}^n\text{–}a.s. \tag{34}$$

(*reading the integral term as* 0 *if* $t = r$).

Proof. Fix n, r, x, t, k, a as in the lemma. To check that the stochastic integrals in equation (33) converge in L^2 as $\varepsilon \downarrow 0$ to the one in (34), consider

$$\begin{aligned} &P_{r,x}^n\Big(\int_{[r,t]\times\mathsf{R}} {}^{n}\!M_r^k\big(\mathrm{d}(s,b)\big)\, \big[\mathrm{p}_{\varepsilon+t-s}(b-a) - \mathrm{p}_{t-s}(b-a)\big]\Big)^2 \\ &\quad = \gamma^k\, P_{r,x}^n \int_r^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\ {}^{n}\!X_s^k(b) \Big(S_{1/n}{}^{n}\!X_{[ns]/n}^{k+1}(b) \wedge n\Big) \\ &\qquad\qquad \times\ \big[\mathrm{p}_{\varepsilon+t-s}(b-a) - \mathrm{p}_{t-s}(b-a)\big]^2 \end{aligned} \tag{35}$$

(which holds by the well-known isometry properties of stochastic integration). By the mixed moment formula (32) we may continue with

$$\begin{aligned} &\le\ c \int_r^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\ S_{s-r}x^k(b)\, S_{[1+ns]/n\,-\,r}x^{k+1}(b) \\ &\qquad\qquad \times\ \big[\mathrm{p}_{\varepsilon+t-s}(b-a) - \mathrm{p}_{t-s}(b-a)\big]^2 . \end{aligned} \tag{36}$$

By definition, for fixed $x \in \mathcal{C}_{\mathrm{tem}}^+$ and $\lambda > 0$,

$$x^k \;\le\; c\,\phi_{-\lambda}\,. \tag{37}$$

On the other hand, for fixed $T>0$ and $\lambda\in\mathsf{R}$ there are constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}\,\phi_\lambda \;\le\; S_s\phi_\lambda \;\le\; \overline{c}\,\phi_\lambda, \qquad 0\le s\le T, \tag{38}$$

(cf. [31, Lemma 6.2 (ii)]). Hence, by Cauchy-Schwarz, the estimate (36) can be continued with

$$\begin{aligned} &\le\; c\,\Big(\int_0^t \mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\;\big[\mathrm{p}_{\varepsilon+t-s}(b-a)-\mathrm{p}_{t-s}(b-a)\big]^2\Big)^{1/2} \\ &\qquad\times\Big(\int_0^t \mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\;\big[\mathrm{p}^2_{\varepsilon+t-s}(b-a)+\mathrm{p}^2_{t-s}(b-a)\big]\,\phi_{-4\lambda}(b)\Big)^{1/2}. \end{aligned} \tag{39}$$

But the first term can be bounded by $c\,\varepsilon^{1/4}$ (cf. [31, Lemma 6.2 (i)]), whereas for the second term we use again (38) to get the bound

$$c\,\phi_{-2\lambda}(a)\Big(\int_0^t \mathrm{d}s\;\Big[(\varepsilon+t-s)^{-1/2}+(t-s)^{-1/2}\Big]\Big)^{1/2} \;\le\; c\,\phi_{-2\lambda}(a). \tag{40}$$

Altogether, (35) tends to zero as $\varepsilon\downarrow 0$, for fixed k,r,t,a and x, uniformly in n. Thus (33) implies (34), finishing the proof of the lemma. ■

Since for the later selection of a strong Markov solution we will need some measurable dependence on the initial data (r,x), in the construction we will already allow that the ${}^{n}\!X$ additionally depend on some varying initial data (r_m,x_m), and we will write ${}^{m,n}\!X$ instead of ${}^{n}\!X$.

To be more precise, consider $(r_m,x_m)\in\mathsf{R}_+\times\mathcal{C}^+_{\mathrm{tem}}$, $m\ge 1$. Fix now $m,n\ge 1$, hence $(r_m,x_m)\in\mathsf{R}_+\times\mathcal{C}^+_{\mathrm{tem}}$ for the moment. By definition, $({}^{m,n}\!X,P^n_{r_m,x_m})$ is the unique solution to the martingale problem $\mathbf{MP}^n_{r_m,x_m}$ of Definition 10, and for the related martingale measures we write now ${}^{m,n}\!M^k_{r_m}$ instead of ${}^{n}\!M^k_r$. Recall the notation $|\cdot|_\lambda$ from (4).

Lemma 12 (Uniformly bounded moments for ${}^{m,n}\!X$)**.** *For fixed $c_0,T,q>0$ and $\lambda',\lambda\in\mathsf{R}$ with $2q\lambda'<\lambda$,*

$$\sup_{\substack{m,n\ge 1,\;\; r_m,t\,\in[0,T]\\ x_m\in\mathcal{C}^+_{\mathrm{tem}},\;\; |x_m|_{-\lambda'}\le c_0}} \;\sum_{k\in\mathsf{K}} P^n_{r_m,x_m}\big\langle({}^{m,n}\!X^k_t)^{2q},\phi_\lambda\big\rangle \;<\;\infty. \tag{41}$$

Proof. Fix c_0,T,q,λ',λ as in the lemma, where without loss of generality we may assume that $q>5$. We may also restrict our attention to $r_m\le t$. In order to handle later the imposed time-partitioning in a Gronwall's inequality argument, we include now the approximating equation (33) in our consideration. Let $0\le\varepsilon\le 1$. Using equations (33) and (34) as well as Burkholder-Davis-Gundy's inequality applied to the martingale

$$t\;\mapsto\;\int_{[r_m,t]\times\mathsf{R}} {}^{m,n}\!M^k_{r_m}\big(\mathrm{d}(s,b)\big)\,\mathrm{p}_{\varepsilon+t'-s}(b-a), \qquad r_m\le t\le t', \tag{42}$$

gives the inequality

$$\begin{aligned} &P^n_{r_m,x_m}\big(S_\varepsilon\,{}^{m,n}\!X^k_t(a)\big)^{2q} \;\le\; c\,\big(S_{\varepsilon+t-r_m}x^k_m(a)\big)^{2q} \\ &\quad+c\,P^n_{r_m,x_m}\Big(\int_{r_m}^t\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\;\mathrm{p}^2_{\varepsilon+t-s}(b-a)\,{}^{m,n}\!X^k_s(b)\,\Big(S_{1/n}\,{}^{m,n}\!X^{k+1}_{[ns]/n}(b)\wedge n\Big)\Big)^q. \end{aligned} \tag{43}$$

Using the presupposed bound $c_0\,\phi_{-\lambda'}$ for x_m, and the heat flow estimate (38), the first term at the right hand side of (43) has the bound $c\,\phi_{-2q\lambda'}(a)$, which paired with ϕ_λ leads to a finite expression within (41), independent of m,n,k,r_m,t,x_m.

Hence it remains to deal with the remaining second term at the right hand side of (43). First of all, in the integrand of the double integral we may additionally introduce $\phi_{-\lambda/q}(b)\,\phi_{\lambda/q}(b) \equiv 1$. Moreover, we decompose the square term by using $2 = (2-\frac{2}{q}) + \frac{2}{q}$. Then by Hölder's inequality with p such that $\frac{1}{p}+\frac{1}{q}=1$, the q^{th} power of the double integral in (43) can be estimated from above by

$$\text{(44a)} \qquad \left(\int_{r_m}^{t} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\; \mathrm{p}^2_{\varepsilon+t-s}(b-a)\,\phi_{-\lambda p/q}(b)\right)^{q/p}$$

$$\text{(44b)} \qquad \times \int_{r_m}^{t} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\; \mathrm{p}^2_{\varepsilon+t-s}(b-a)\,\phi_\lambda(b) \left({}^{m,n}X^k_s(b)\big(S_{1/n}{}^{m,n}X^{k+1}_{[ns]/n}(b)\wedge n\big)\right)^q,$$

where we used that $(2-\frac{2}{q})p = 2$. In the double integral in (44a), we estimate one of the p–factors by $c\,(t-s)^{-1/2}$, and apply (38) in order to get for the $\frac{q}{p}$–th power of that integral the bound

$$\text{(45)} \qquad c\,\phi_{-\lambda}(a)\left(\int_0^t \mathrm{d}s\;(t-s)^{-1/2}\right)^{q/p} \;\le\; c\,\phi_{-\lambda}(a)$$

with c again independent of m,n,k,r_m,t,x_m, which cancels the $\phi_\lambda(a)$ in (41). On the other hand, in the double integral in (44b) we split p^2 as before, but use this time that the remaining p can be paired with 1 in (41). Altogether we found

$$\text{(46)} \qquad \begin{aligned} &\int_{\mathsf{R}} \mathrm{d}a\;\phi_\lambda(a)\,P^n_{r_m,x_m}\left(S_\varepsilon{}^{m,n}X^k_t(a)\right)^{2q} \;\le\; c \\ &+ c\int_{r_m}^{t} \mathrm{d}s\;(\varepsilon+t-s)^{-1/2} \int_{\mathsf{R}} \mathrm{d}b\;\phi_\lambda(b) P^n_{r_m,x_m}\left({}^{m,n}X^k_s(b)\,S_{1/n}{}^{m,n}X^{k+1}_{[ns]/n}(b)\right)^q \end{aligned}$$

with the constants c independent of m,n,k,r_m,t,x_m. The latter term can further be estimated by using the elementary inequality

$$\text{(47)} \qquad (uv)^q \;\le\; u^{2q}+v^{2q}, \qquad u,v\ge 0, \quad q>0.$$

Also, we may sum these inequalities over $k\in\mathsf{K}$. Setting (for fixed m,n,r_m,x_m)

$$\text{(48)} \qquad f_\varepsilon(t) \;:=\; \sum_{k\in\mathsf{K}} \int_{\mathsf{R}} \mathrm{d}a\;\phi_\lambda(a) P^n_{r_m,x_m}\left(S_\varepsilon{}^{m,n}X^k_t(a)\right)^{2q},$$

$0\le\varepsilon\le 1$, $r_m\le t\le T$, we thus obtained the following two estimates

$$f_0(t) \;\le\; \underline{c} + \underline{c}\int_{r_m}^{t} \mathrm{d}s\;(t-s)^{-1/2}\left[f_0(s)+f_{1/n}\big([ns]/n\big)\right],$$

$$f_{1/n}\big([nt]/n\big) \;\le\; \underline{c} + \underline{c}\int_{r_m}^{[nt]/n} \mathrm{d}s\;\big(1/n+[nt]/n-s\big)^{-1/2}\left[f_0(s)+f_{1/n}\big([ns]/n\big)\right],$$

with the constant $\underline{c}$ independent of m,n,r_m,t,x_m. Using in the latter integral first $1/n+[nt]/n \ge t$ and then $[nt]/n \le t$, we see that $g(t) := f_0(t)+f_{1/n}\big([nt]/n\big)$ satisfies

$$\text{(49)} \qquad g(t) \;\le\; \underline{c} + \underline{c}\int_{r_m}^{t} \mathrm{d}s\;(t-s)^{-1/2}\,g(s), \qquad r_m\le t\le T,$$

with $\underline{c}$ independent of m, n, r_m, t, x_m. Then Gronwall's inequality implies $g(t) \le c$, $0 \le r_m \le t \le T$, with c independent of m, n, r_m, t, x_m (see Kallianpur and Xiong [20, p.138]). Hence $f_0(t) \le g(t)$ gives the claim (41), finishing the proof. ■

Next we want to deal with moments of time increments of the integral part

$$ {}^{m,n}\mathrm{Y}_t^k(a) \;:=\; \int_{[r_m,t]\times\mathsf{R}} {}^{m,n}\!M_{r_m}^k\big(\mathrm{d}(s,b)\big)\,\mathrm{p}_{t-s}(b-a), \qquad t \ge r_m,\; k\in\mathsf{K},\; a\in\mathsf{R}, $$

in (34).

Lemma 13 (Moments of increments). *For constants $c_0, T, p, q > 0$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $q > 5$, and $\lambda', \lambda \in \mathsf{R}$ with $2q\lambda' < \lambda$, we have*

$$ \sup_{\substack{m,n\ge 1,\; r_m\in[0,T] \\ x_m\in\mathcal{C}_{\mathrm{tem}}^+,\; |x_m|_{-\lambda'}\le c_0}} \sum_{k\in\mathsf{K}} P_{r_m,x_m}^n \left| {}^{m,n}\mathrm{Y}_{t'}^k(a') - {}^{m,n}\mathrm{Y}_t^k(a)\right|^{2q} \tag{50} $$

$$ \le\; c\,\Big(|t'-t|^{1/2} + |a'-a|\Big)^{q/p}\;\phi_{-\lambda}(a), $$

whenever $t, t' \in [0,T]$, $a, a' \in \mathsf{R}$, and $|a-a'| \le 1$.

Proof. We may assume that $r_m \le t \le t'$. Let

$$ N_r \;:=\; \int_{[r_m,r]\times\mathsf{R}} {}^{m,n}\!M_{r_m}^k\big(\mathrm{d}(s,b)\big)\,\big(\mathrm{p}_{t'-s}(b-a') - \mathrm{p}_{t-s}(b-a)\big), \qquad r_m \le r \le t'. $$

Then $r \mapsto N_r$ is a martingale with square function

$$ \langle N\rangle_r \;=\; \int_{r_m}^{r}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\,\big[\mathrm{p}_{t'-s}(b-a') - \mathrm{p}_{t-s}(b-a)\big]^2\;{}^{m,n}\mathrm{X}_s^k(b)\,S_{1/n}\,{}^{m,n}\mathrm{X}_{[ns]/n}^{k+1}(b). $$

Note that

$$ {}^{m,n}\mathrm{Y}_{t'}^k(a') \;-\; {}^{m,n}\mathrm{Y}_t^k(a) \;=\; N_{t'}, \tag{51} $$

where we used the convention

$$ \mathrm{p}_s := 0 \quad\text{if}\quad s < 0. \tag{52} $$

Again by Burkholder-Davis-Gundy's inequality, we will deal with

$$ P_{r_m,x_m}^n\left(\int_{r_m}^{t'}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\,\big[\mathrm{p}_{t'-s}(b-a') - \mathrm{p}_{t-s}(b-a)\big]^2\;{}^{m,n}\mathrm{X}_s^k(b)\,S_{1/n}\,{}^{m,n}\mathrm{X}_{[ns]/n}^{k+1}(b)\right)^q, $$

As we derived (44a–44b), we get the bound

$$ c\left(\int_0^{t'}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\,\big[\mathrm{p}_{t'-s}(b-a') - \mathrm{p}_{t-s}(b-a)\big]^2\right)^{q/p} $$

$$ \times\int_{r_m}^{t'}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\,\big[\mathrm{p}_{t'-s}^2(b-a') + \mathrm{p}_{t-s}^2(b-a)\big]\,P_{r_m,x_m}^n\Big({}^{m,n}\mathrm{X}_s^k(b)\,S_{1/n}\,{}^{m,n}\mathrm{X}_{[ns]/n}^{k+1}(b)\Big)^q. $$

By [31, Lemma 6.2 (i)],

$$ \int_0^{t'}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b\,\big[\mathrm{p}_{t'-s}(b-a') - \mathrm{p}_{t-s}(b-a)\big]^2 \;\le\; c\,\Big(|t'-t|^{1/2} + |a'-a|\Big)^{q/p}, \tag{53} $$

$t, t' \ge 0$, $a, a' \in \mathsf{R}$. Therefore, the first double integral leads to the desired right hand side in (50), except for the $\phi_{-\lambda}(a)$. So it remains to show that the second double integral can uniformly be bounded by $c\,\phi_{-\lambda}(a)$. For this purpose we may assume that $t' = t$ [recall the convention (52)]. In the integrand we additionally introduce $\phi_{-\lambda}(b)\,\phi_\lambda(b) \equiv 1$, and apply the Cauchy-Schwarz inequality to the db–integral. This gives the bound

$$2\left(\int_{\mathsf{R}} \mathrm{d}b\ \mathrm{p}_{t-s}^4(b-a)\,\phi_{-2\lambda}(b)\right)^{1/2} \tag{54a}$$

$$\times\ \left(\int_{\mathsf{R}} \mathrm{d}b\ \phi_{2\lambda}(b) P_{r_m,x_m}^n\left({}^{m,n}\!X_s^k(b)\, S_{1/n}{}^{m,n}\!X_{[ns]/n}^{k+1}(b)\right)^{2q}\right)^{1/2} \tag{54b}$$

for the db–integral. The factor in (54b) is uniformly bounded. In fact, use once more (47) and (38), to get expressions of the type as in Lemma 12 with q, λ replaced by $2q, 2\lambda$. On the other hand, in the factor in (54a) we split $\mathrm{p}^4 = \mathrm{p}\,\mathrm{p}^3$, and use $\mathrm{p}_{t-s}^3(b-a) \leq c\,(t-s)^{-3/2}$ which, after taking the $1/2$ power, has a bounded ds–integral. In fact, the db–integral of the remaining quantities gives $\phi_{-\lambda}(a)$ by (38), uniformly in $(t-s)$ and r_m, x_m. Thus the proof of Lemma 13 is finished. ∎

Completion of the proof of Proposition 9. Fix (r_m, x_m) converging in $\mathsf{R}_+ \times \mathcal{C}_{\mathrm{tem}}^+$ to (r,x) as $m \uparrow \infty$, as well as $p, q > 0$ as in Lemma 13, implying $q/p > 4$. Since T, λ', λ in Lemma 13 are arbitrary, from (50) and Lemma 6.3 (ii) in [31] [1)], we see that the sequence of the laws of ${}^{m,n}Y$ with respect to P_{r_m,x_m}^n is tight in $\mathcal{P}$. So is that of the laws of ${}^{m,n}X$ [recall the definition of ${}^{m,n}Y$ before Lemma 13, and equation (34)]. Let X denote any limit point (in law) of the ${}^{m,n}X$ as $m \uparrow \infty$ and $n \uparrow \infty$. Since ${}^{m,n}X$ satisfies the martingale problem $\mathbf{MP}_{r_m,x_m}^n$ of Definition 8, for each (m,n), it follows from a standard limiting argument that X satisfies the martingale problem $\mathbf{MP}_{r,x}$. This finishes the proof of Proposition 9. ∎

3.2. **Some properties of all martingale problem solutions** $(X, P_{r,x})$**.** After we have constructed a solution to our basic martingale problem, we now want to collect some properties of *all* the solutions (that is, not only of the constructed ones).

For this purpose, we *redefine* $({}^{m,n}X, P_{r_m,x_m}^n)$ introduced before Lemma 12 as *any* solutions to the martingale problem $\mathbf{MP}_{r_m,x_m}$ of Definition 8 [instead of $\mathbf{MP}_{r_m,x_m}^n$], for each $n \geq 1$. In particular, in the case $(r_m, x_m) \equiv (r_1, x_1) =: (r,x)$, we have a whole sequence $\left\{{}^{1,n}X =: {}^{n}X : n \geq 1\right\}$ of solutions to $\mathbf{MP}_{r,x}$.

With that system $\left\{({}^{m,n}X, P_{r_m,x_m}^n) : m,n \geq 1\right\}$ we now repeat all the constructions in Subsection 3.1. Then, in particular, analogs of the Lemmas 11 − 13 are true, and once more by tightness, any limit point $(X, P_{r,x})$ of that new system again satisfies the martingale problem $\mathbf{MP}_{r,x}$ of Definition 8 :

Lemma 14 (Properties of all martingale problem solutions)**.** *For* $m,n \geq 1$, *take* $(r_m, x_m) \in \mathsf{R}_+ \times \mathcal{C}_{\mathrm{tem}}^+$ *and let* $({}^{m,n}X, P_{r_m,x_m}^n)$ *denote any solution to the martingale problem* $\mathbf{MP}_{r_m,x_m}$ *of Definition* 8*. Then the following statements hold:*

- **(a) (Pointwise equation):** *For* $t \geq r_m$, *and* $(k,a) \in \mathsf{K} \times \mathsf{R}$ *fixed,* P_{r_m,x_m}^n*–almost surely,*

$${}^{m,n}\!X_t^k(a) \;=\; S_{t-r_m} x_m^k\,(a) + \int_{[r_m,t]\times\mathsf{R}} {}^{m,n}\!M_{r_m}^k\big(\mathrm{d}(s,b)\big)\, \mathrm{p}_{t-s}(b-a)$$

[1)] Note that at the right hand side of the condition (6.5) in [31] the factor $\mathrm{e}^{\lambda|x|}$ has to be added, and that the laws of the initial states ${}^{n}X_0 \in \mathcal{C}_{\mathrm{tem}}$, $n \geq 1$, in Lemma 6.3 (ii) should be tight by assumption.

(with ${}^{m,n}\!M^k_{r_m}$ denoting the related martingale measure).

(b) (Uniformly bounded moments): *For fixed* $c_0, T, q > 0$ *and* $\lambda', \lambda \in \mathsf{R}$ *with* $2q\lambda' < \lambda$,

$$\sup_{\substack{m,n\geq 1,\; r_m,t\in[0,T] \\ x_m\in\mathcal{C}^+_{\mathrm{tem}},\; |x_m|_{-\lambda'}\leq c_0}} \sum_{k\in\mathsf{K}} P^n_{r_m,x_m} \bigl\langle ({}^{m,n}\!X^k_t)^{2q}, \phi_\lambda \bigr\rangle \;<\; \infty.$$

(c) (Moments of increments): *For constants* $c_0, T, p, q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $q > 5$, *and* $\lambda', \lambda \in \mathsf{R}$ *with* $2q\lambda' < \lambda$, *we have (with the notation* ${}^{m,n}Y$ *introduced before Lemma* 13*)*

$$\sup_{\substack{m,n\geq 1,\; r_m\in[0,T] \\ x_m\in\mathcal{C}^+_{\mathrm{tem}},\; |x_m|_{-\lambda'}\leq c_0}} \sum_{k\in\mathsf{K}} P^n_{r_m,x_m} \Bigl| {}^{m,n}Y^k_{t'}(a') - {}^{m,n}Y^k_t(a) \Bigr|^{2q}$$
$$\leq\; c\, \Bigl(|t'-t|^{1/2} + |a'-a| \Bigr)^{q/p} \;\phi_{-\lambda}(a),$$

whenever $t, t' \in [0,T]$, $a, a' \in \mathsf{R}$, *and* $|a - a'| \leq 1$.

(d) (Limit points): *Assume that* (r_m, x_m) *converges in* $\mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}$ *to* (r,x) *as* $m \uparrow \infty$. *Then any limit point* $(X, P_{r,x})$ *of* $\bigl\{({}^{m,n}\!X, P^n_{r_m,x_m}) : m,n \geq 1\bigr\}$ *as* $m \uparrow \infty$ *and* $n \uparrow \infty$ *satisfies the martingale problem* $\mathbf{MP}_{r,x}$ *of Definition* 8.

We also need the following property.

Corollary 15 (Uniformly bounded moments for each solution X). *Fix* $c_0, T, q > 0$ *and* λ', λ *in* R *with* $2q\lambda' < \lambda$. *Let* $(X, P_{r,x})$ *be any solution to the martingale problem* $\mathbf{MP}_{r,x}$ *of Definition* 8. *Then,*

$$\sup_{\substack{r\in[0,T],\; x\in\mathcal{C}^+_{\mathrm{tem}} \\ |x|_{-\lambda'}\leq c_0}} P_{r,x} \sup_{0\leq t\leq T} \sum_{k\in\mathsf{K}} \bigl\langle (X^k_t)^{2q}, \phi_\lambda \bigr\rangle \;<\; \infty. \tag{55}$$

Proof. We specialize in Lemma 14 (c) to ${}^{m,n}\!X \equiv {}^{1,1}\!X =: X$. Using the Banach space $L^{2q}\bigl(\mathsf{R}, \phi_\lambda(a)\mathrm{d}a\bigr)$ with q sufficiently large, from the proof of Theorem 1.2.1 in Revuz and Yor [28], and Lemma 14 (c) we get (55) with X replaced by $Y \equiv {}^{m,n}Y$. But by Lemma 14 (a) and the heat flow estimate (38), claim (55) also holds for X. Finally, (55) is then true for all $q > 0$, finishing the proof. ■

The special case ${}^{m,n}\!X \equiv X$ also gives the following result.

Corollary 16 (Pointwise equation for each solution). *For each solution* $(X, P_{r,x})$ *to the martingale problem* $\mathbf{MP}_{r,x}$ *of Definition* 8, *the family of martingales extends to orthogonal square-integrable martingale measures* $M^k_r = M^k_r\bigl(\mathrm{d}(s,b)\bigr)$ *such that, for the usual predictable functions* $f : [r,\infty] \times \mathsf{K} \times \mathsf{R} \times \Omega \to \mathsf{R}$ *in their domain,*

$$\begin{aligned} &\Bigl\langle\!\Bigl\langle \int_{[r,\,\cdot\,]\times\mathsf{R}} M^k_r\bigl(\mathrm{d}(s,b)\bigr)\, f^k_s(b) \Bigr\rangle\!\Bigr\rangle_t \\ &= \gamma^k \int_r^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}a\; X^k_s(a) X^{k+1}_s(a) \bigl[f^k_s(b)\bigr]^2, \qquad t \geq r, \quad k \in \mathsf{K}. \end{aligned} \tag{56}$$

Moreover, for $t \geq r$ *and* $(k,a) \in \mathsf{K} \times \mathsf{R}$ *fixed,*

$$X^k_t(a) \;=\; S_{t-r}x^k\,(a) + \int_{[r,t]\times\mathsf{R}} M^k_r\bigl(\mathrm{d}(s,b)\bigr)\, \mathrm{p}_{t-s}(b-a), \quad P_{r,x}\text{-}a.s. \tag{57}$$

In particular, the expectation formula

$$P_{r,x}\, X_t^k(a) \;=\; S_{t-r}x^k\,(a), \qquad t \geq r, \quad k \in \mathsf{K}, \quad a \in \mathsf{R}, \tag{58}$$

and the covariance formula

$$\begin{aligned} Cov_{r,x}\Big(X_{t_1}^{k_1}(a_1),\, X_{t_2}^{k_2}(a_2)\Big) \;&=\; \gamma_{k_1}\,\delta_{k_1,k_2}\int_r^{t_1\wedge t_2}\mathrm{d}s\int_{\mathsf{R}}\mathrm{d}b \\ &\quad\times\, S_{s-r}x^{k_1}(b)\; S_{s-r}x^{k_1+1}(b)\,\mathrm{p}_{t_1-s}(a_1-b)\,\mathrm{p}_{t_2-s}(a_2-b), \end{aligned} \tag{59}$$

$t_1, t_2 \geq r$, $k_1, k_2 \in \mathsf{K}$, $a_1, a_2 \in \mathsf{R}$, *are valid.*

Note that (55), (58), and (59) yield already the moment formulas in Theorem 3 (b).

3.3. **The mapping** $(r,x) \mapsto \mathcal{P}_{r,x}$**.** For $(r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}$, let $\mathcal{P}_{r,x} \subseteq \mathcal{P}$ denote the set of all solutions $P_{r,x}$ to the martingale problem $\mathbf{MP}_{r,x}$ of Definition 8. Note that $\mathcal{P}_{r,x} \neq \emptyset$ by Proposition 9. Recall the metric space $\big(\mathrm{com}(\mathcal{P}), \mathrm{d_{com}}\big)$ introduced in Subsection 2.1.

Lemma 17 (Set of all solutions)**.** $(r,x) \mapsto \mathcal{P}_{r,x} \neq \emptyset$ *is a measurable mapping of* $\mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}$ *into* $\mathrm{com}(\mathcal{P})$.

Proof. From the special case $(r_m, x_m) \equiv (r,x)$, that is ${}^{m,n}X \equiv {}^{1,n}X$, in Lemma 14 (d) we see that the set $\mathcal{P}_{r,x}$ of all solutions to the martingale problem $\mathbf{MP}_{r,x}$ of Definition 8 is compact. On the other hand, the special case ${}^{m,n}X \equiv {}^{m,1}X$ shows that the map $(r,x) \mapsto \mathcal{P}_{r,x}$ is measurable (see [32, Lemma 12.1.8]). This completes the proof. ∎

We continue with a time-homogeneity property of the family

$$\big\{\mathcal{P}_{r,x}:\ (r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}\big\} \tag{60}$$

of all solutions to the martingale problems of Definition 8. For this purpose, for $r \geq 0$, we introduce the *shift operator* Φ_r on $\Omega = \mathcal{C}(\mathsf{R}_+, \mathcal{C}^+_{\mathrm{tem}})$ by

$$\big(\Phi_r\omega\big)_t \;:=\; \omega_{(t-r)\vee 0}\,, \qquad \omega \in \Omega, \quad t \geq 0, \tag{61}$$

(producing a constant initial piece).

Lemma 18 (Time-homogeneity)**.** *The map* $(r,x) \mapsto \mathcal{P}_{r,x}$ *is time-homogeneous, that is,*

$$\mathcal{P}_{r,x} \;=\; \mathcal{P}_{0,x} \circ \Phi_r^{-1}, \qquad (r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}\,, \tag{62}$$

(with the obvious notation).

Proof. The proof is quite elementary and shows that there is a one-to-one correspondence between the solution to the martingale problems $\mathbf{MP}_{r,x}$ and $\mathbf{MP}_{0,x}$ (compare with [32, Lemma 6.5.1]). Fix $(r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}$ and $P \in \mathcal{P}$.

Step 1° (*constancy*). By the notation (61),

$$\big\{(\Phi_r X)_t = x,\ t \leq r\big\} \;=\; \{X_0 = x\}\,. \tag{63}$$

Hence, the process with law $P \circ \Phi_r^{-1}$ equals constantly x up to time r if and only if $P\,(X_0 = x) = 1$.

Step 2° (*martingale property*). For $\varphi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$, $t \geq r$, and $k \in \mathsf{K}$, again by definition of the shift operator,

$$
\begin{aligned}
(64)\qquad & \Big\langle (\Phi_r X)^k_t\,, \varphi^k \Big\rangle - \big\langle x^k, \varphi^k \big\rangle - \int_r^t \mathrm{d}s\, \Big\langle (\Phi_r X)^k_s\,, \frac{\sigma^2}{2}\Delta\varphi^k \Big\rangle \\
&= \big\langle X^k_{t-r}\,, \varphi^k \big\rangle - \big\langle x^k, \varphi^k \big\rangle - \int_0^{t-r} \mathrm{d}s\, \Big\langle X^k_s,\, \frac{\sigma^2}{2}\Delta\varphi^k \Big\rangle \\
&= M^k_{0,t-r}(\varphi^k).
\end{aligned}
$$

Thus, by the martingale problem $\mathbf{MP}_{r,x}$ of Definition 8, the map $t \mapsto M^k_{r,t}(\varphi^k)$ with respect to the law $P \circ \Phi_r^{-1}$ is a martingale after time r starting from 0 if and only if $t \mapsto M^k_{0,t-r}(\varphi^k)$ with respect to P is a martingale after 0 starting from 0.

Step 3° (*square function*). Similarly, by Definition 8 (iii),

$$
(65)\qquad t \mapsto \big[M^k_{r,t}(\varphi^k)\big]^2 - \gamma^k \int_r^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}a\; X^k_s(a) X^{k+1}_s(a) \big[\varphi^k(a)\big]^2 \;=:\; N^k_{r,t}(\varphi^k)
$$

with respect to $P \circ \Phi_r^{-1}$ is a martingale after time r if and only if the same is true for $r = 0$.

Step 4° (*conclusion*). Putting the steps 1°–3° together, the claim in the lemma follows. ■

We finish this section with an optional stopping argument which we need later for an integrability statement in the proof of selection of a strong Markov solution. By (7), for fixed $\lambda \in \mathsf{R}$, there is a constant $c_{(66)}$ such that for the smoothed reference function $\tilde{\phi}_\lambda$,

$$
(66)\qquad \Big|\frac{\sigma^2}{2}\Delta\tilde{\phi}_\lambda\Big| \;\leq\; c_{(66)}\, \tilde{\phi}_\lambda\,.
$$

Recall also the notation $\|\cdot\|_\lambda$ introduced in (12).

Lemma 19 (A conditional moment estimate)**.** *Let the law $P_{r,x}$ belong to $\mathcal{P}_{r,x}$, for $(r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}$. Consider $T \geq r$ and $[r,T]$–valued stopping times $\eta \leq \vartheta$. Then*

$$
(67)\qquad P_{r,x}\big\{\|X^k_\vartheta\|^n_{-\lambda} \,\big|\, \mathcal{F}_\eta\big\} \;\leq\; \mathrm{e}^{nc_{(66)}(T-\eta)}\, \|X^k_\eta\|^n_{-\lambda} \;<\; \infty, \;\; P_{r,x}\text{–a.s.},
$$

for all $k \in \mathsf{K}$, $n \geq 1$, $\lambda > 0$.

Proof. Fix $P_{r,x}$, k, n, and λ. From the martingale problem $\mathbf{MP}_{r,x}$ and Itô's formula, for $t \geq r$,

$$
\begin{aligned}
(68)\qquad \mathrm{d}\big(\mathrm{e}^{-nc_{(66)}t}\, \|X^k_t\|^n_{-\lambda}\big) \;&=\; n\,\mathrm{e}^{-nc_{(66)}t}\, \|X^k_t\|^{n-1}_{-\lambda} \Big\langle X^k_t\,, \Big(\frac{\sigma^2}{2}\Delta - c_{(66)}\Big)\tilde{\phi}_\lambda \Big\rangle\, \mathrm{d}t \\
&\quad +\; n\,\mathrm{e}^{-nc_{(66)}t}\, \|X^k_t\|^{n-1}_{-\lambda}\, \mathrm{d}M^k_{r,t}(\tilde{\phi}_\lambda).
\end{aligned}
$$

Hence, by the definition (66) of $c_{(66)}$, the process $t \mapsto \mathrm{e}^{-nc_{(66)}t}\, \|X^k_t\|^n_{-\lambda}$ is a $P_{r,x}$–supermartingale after time r. Then the claim (67) immediately follows from Jacod and Shiryaev [19, Theorem 1.1.39]. ■

3.4. **Selection of a strong Markov solution.** Here we now want to select a time-homogeneous strong Markov version $(r,x) \mapsto P_{r,x}$ from $(r,x) \mapsto \mathcal{P}_{r,x}$. The idea behind is an optimization procedure as in [32], which goes back to Krylov [21], and which uses an extremal property which is well-behaved under conditioning and weak convergence.

To make this precise, we first recall some notations taken from [32]. We stress the fact, that all those results we quote from [32] are valid also in our present case of the set $\Omega = \mathcal{C}(\mathsf{R}_+, \mathcal{C}_{\text{tem}}^+)$ of paths in an infinite-dimensional space. Recall that $\mathcal{F}^r$ denotes the σ–field generated by the coordinate process at times $t \geq r$.

Notation 20 (Composition I)**.** For fixed $\omega \in \Omega$, $r \geq 0$, and a law P on $(\Omega, \mathcal{F}^r)$ with the property that $P(X_r = \omega_r) = 1$, let $\delta_\omega \otimes_r P$ denote the *unique* law on Ω satisfying

$$\delta_\omega \otimes_r P \big(X_t = \omega_t \text{ for } t \leq r\big) \;=\; 1 \quad \text{and} \quad \delta_\omega \otimes_r P \;=\; P \quad \text{on } \mathcal{F}^r \tag{69}$$

(see [32, Lemma 6.1.1]). Roughly speaking, the irrelevant history of the process (X, P) up to time r is replaced by the one of ω yielding the process $(X,\, \delta_\omega \otimes_r P)$. For fixed $x \in \mathcal{C}_{\text{tem}}^+$, let the notation $\delta_x \otimes_r P$ however refer to the special case $\omega_t = x$ for $t \leq r$ (constant initial piece). $\diamond$

We also need the following notation.

Notation 21 (Composition II)**.** For a given probability measure P on Ω, a stopping time τ on Ω, and a mapping $\omega \mapsto Q_\omega$ of Ω into $\mathcal{P}$ satisfying

(a) $\omega \mapsto Q_\omega$ is $\mathcal{F}_\tau$–measurable,
(b) $Q_\omega \big(X_{\tau(\omega)} = \omega_{\tau(\omega)}\big) \;=\; 1$, for all $\omega \in \Omega$,

let $P \otimes_\tau Q$ denote the *unique* probability measure on Ω

(c) which equals P on $\mathcal{F}_\tau$,
(d) and such that $\omega \mapsto \delta_\omega \otimes_{\tau(\omega)} Q_\omega$ (recall Notation 20) is a regular conditional probability distribution of $P \otimes_\tau Q$ given $\mathcal{F}_\tau$ (see [32, Theorem 6.1.2]). $\diamond$

Roughly speaking, the process $(X,\, P \otimes_\tau Q)$ has the law P until the random time τ, and its conditional law after time τ is given by the family Q.

Definition 22 (Strong Markov solution)**.** $\big\{P_{r,x} :\ (r,x) \in \mathsf{R}_+ \times \mathcal{C}_{\text{tem}}^+\big\} \subseteq \mathcal{P}$ is said to be a time-homogeneous *strong Markov solution* to the family

$$\mathbf{MP} \;:=\; \big\{\mathbf{MP}_{r,x} :\ (r,x) \in \mathsf{R}_+ \times \mathcal{C}_{\text{tem}}^+\big\} \tag{70}$$

of martingale problems of Definition 8, if $(r,x) \mapsto P_{r,x}$ is a measurable map of $\mathsf{R}_+ \times \mathcal{C}_{\text{tem}}^+$ into $\mathcal{P}$, and if for each $(r,x) \in \mathsf{R}_+ \times \mathcal{C}_{\text{tem}}^+$,

(a) $P_{r,x} \in \mathcal{P}_{r,x}$,
(b) $P_{r,x} = P_{0,x} \circ \Phi_r^{-1}$ (time-homogeneity), and
(c) for each stopping time $\tau \geq r$ on Ω and each regular conditional probability distribution $\omega \mapsto P_\omega$ of $P_{r,x}$ given $\mathcal{F}_\tau$, there is a $P_{r,x}$–null set $\mathsf{N} \in \mathcal{F}_\tau$ such that

$$P_\omega \;=\; \delta_\omega \otimes_{\tau(\omega)} P_{\tau(\omega),\, \omega_{\tau(\omega)}}, \qquad \omega \notin \mathsf{N},$$

(recall Notation 20). $\diamond$

In other words, solutions $P_{r,x}$ of $\mathbf{MP}_{r,x}$ are selected in such a way that

$$\big(X, P_{r,x}, (r,x) \in \mathsf{R}_+ \times \mathcal{C}_{\text{tem}}^+\big) \tag{71}$$

is a time-homogeneous strong Markov process.

The existence statement on a cyclically catalytic SBM as claimed in Theorem 3 (a) can now be restated as follows.

Theorem 23 (Existence of a strong Markov solution)**.** *There exists a time-homogeneous strong Markov solution to the martingale problem* **MP** *according to Definition* 22.

The verification of this theorem (in the end of this subsection) needs some further preparation. Recall the second part of Notation 20, and Notation 21.

Lemma 24 (Compositions)**.** *Fix* $x \in \mathcal{C}_{\mathrm{tem}}^{+}$, $P \in \mathcal{P}_{0,x}$, *and a finite stopping time* τ *on* $\Omega = \mathcal{C}(\mathsf{R}_+, \mathcal{C}_{\mathrm{tem}}^{+})$.

(a) (composition I): *If* $\omega \mapsto P_\omega$ *is a regular conditional probability distribution of* P *given* $\mathcal{F}_\tau$, *then there is a* P*–null set* $\mathsf{N} \in \mathcal{F}_\tau$ *such that*

$$\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} P_\omega \in \mathcal{P}_{\tau(\omega),\omega_{\tau(\omega)}}, \qquad \omega \notin \mathsf{N}.$$

(b) (composition II): *If* $\omega \mapsto Q_\omega$ *is an* $\mathcal{F}_\tau$*–measurable map of* Ω *into* $\mathcal{P}$ *such that*

$$\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} Q_\omega \in \mathcal{P}_{\tau(\omega),\omega_{\tau(\omega)}}, \qquad \omega \in \Omega,$$

then $P \otimes_\tau Q$ *belongs to* $\mathcal{P}_{0,x}$.

Proof. Fix x, P, τ as in the lemma.

(a) Let $\omega \mapsto P_\omega$ be a regular conditional probability distribution of P given $\mathcal{F}_\tau$. Denote by $\mathcal{A}$ a countable dense subset of $\mathcal{C}_{\mathrm{rap}}^{(2)}$. Fix $\varphi \in \mathcal{A}$ and $k \in \mathsf{K}$ for a while. First note that by Definition 8 (ii),

$$M^k_{\tau(\omega),t}(\varphi^k) = M^k_{0,t}(\varphi^k) - M^k_{0,\tau(\omega)}(\varphi^k), \qquad t \ge \tau(\omega). \tag{72}$$

Hence, by the last part of Theorem 1.2.10 in [32] (applied to $\theta(t) = M^k_{0,t}(\varphi^k)$ and $s = 0$), there exists a P–null set $\mathsf{N}_\varphi \in \mathcal{F}_\tau$ such that for all $\omega \notin \mathsf{N}_\varphi$,

$$t \mapsto M^k_{\tau(\omega),t}(\varphi^k) \quad \text{is a } P_\omega\text{–martingale after time } \tau(\omega). \tag{73}$$

Recalling the notation $N^k_{r,t}(\varphi^k)$ introduced in (65), by Definition 8 (iii) the following identity holds:

$$N^k_{\tau(\omega),t}(\varphi^k) = N^k_{0,t}(\varphi^k) - N^k_{0,\tau(\omega)}(\varphi^k) - 2\, M^k_{\tau(\omega),t}(\varphi^k)\, M^k_{0,\tau(\omega)}(\varphi^k). \tag{74}$$

Appealing again to the same theorem in [32], and combining with (73), we may redefine the P–null set $\mathsf{N}_\varphi \in \mathcal{F}_\tau$ such that for $\omega \notin \mathsf{N}_\varphi$ additionally

$$t \mapsto N^k_{\tau(\omega),t}(\varphi^k) \quad \text{is a } P_\omega\text{–martingale after time } \tau(\omega). \tag{75}$$

Introduce the P–null set $\mathsf{N} := \bigcap_{\varphi \in \mathcal{A}} \mathsf{N}_\varphi \in \mathcal{F}_\tau$. We may additionally assume that N is independent of $k \in \mathsf{K}$. To $\varphi \in \mathcal{C}_{\mathrm{rap}}^{(2)}$ we now choose $\varphi_n \in \mathcal{A}$ converging in $\mathcal{C}_{\mathrm{rap}}^{(2)}$ to φ. Then, from (73) and (75) we conclude that for $\omega \notin \mathsf{N}$,

$$M^k_{\tau(\omega),\cdot}(\varphi^k) \text{ and } N^k_{\tau(\omega),\cdot}(\varphi^k) \text{ are } P_\omega\text{–martingales after time } \tau(\omega). \tag{76}$$

Since k and φ are arbitrary, and N does not depend on them, claim **(a)** is true.

(b) Let $\omega \mapsto Q_\omega$ be an $\mathcal{F}_\tau$–measurable map as presupposed in **(b)**. First of all, $P \otimes_\tau Q$ makes sense, according to Notation 21. Trivially, $P \otimes_\tau Q$ has the right initial state:

$$P \otimes_\tau Q\,(X_0 = x) \;=\; P\,(X_0 = x) \;=\; x. \tag{77}$$

Fix $\varphi \in \mathcal{C}_{\mathrm{rap}}^{(2)}$ and $k \in \mathsf{K}$. Next we want to show that

$$\bigl\{M_{0,t}^k(\varphi^k):\ t \geq 0\bigr\} \text{ is a } P \otimes_\tau Q\text{–martingale.} \tag{78}$$

By the last part of Theorem 6.1.2 in [32] (again with $\theta(t) = M_{0,t}^k(\varphi^k)$ and $s = 0$) is suffices to show that

$$M_{0,t}^k(\varphi^k) \text{ is } P \otimes_\tau Q\text{–integrable}, \qquad t \geq 0, \tag{79}$$

that

$$\bigl\{M_{0,t\wedge\tau}^k(\varphi^k):\ t \geq 0\bigr\} \text{ is a } P\text{–martingale,} \tag{80}$$

and that

$$\Bigl\{M_{0,t}^k(\varphi^k) - M_{0,t\wedge\tau(\omega)}^k(\varphi^k):\ t \geq 0\Bigr\} \text{ is a } Q_\omega\text{–martingale}, \quad \omega \in \Omega. \tag{81}$$

In order to check the integrability statement (79), we fix $T \geq t \vee 1$, and $\lambda > 0$, as well as a constant $c_{(82)} > 0$ such that

$$\bigl|\varphi^k\bigr| + \frac{\sigma^2}{2}\,\bigl|\Delta\varphi^k\bigr| \;\leq\; c_{(82)}\,\tilde{\phi}_\lambda \tag{82}$$

[recall (7)]. Then by the martingale definition (14),

$$P \otimes_\tau Q\,\bigl|M_{0,t}^k(\varphi^k)\bigr| \;\leq\; 2\,c_{(82)}\,T\,\sup_{s\leq T} P \otimes_\tau Q\,\|X_s^k\|_{-\lambda} \tag{83}$$

[recall notation (12)]. Conditioning on $\mathcal{F}_\tau$ in the latter expectation expression, by Notation 21 we get

$$P \otimes_\tau Q\,\|X_s^k\|_{-\lambda} \;=\; \int_\Omega P(\mathrm{d}\omega)\,\delta_\omega \otimes_{\tau(\omega)} Q_\omega\,\|X_s^k\|_{-\lambda}\,. \tag{84}$$

First we restrict in the latter integral additionally to $\tau(\omega) > s$. Then concerning the internal expectation, $\|X_s^k\|_{-\lambda}$ equals the deterministic value $\|\omega_s^k\|_{-\lambda}$, just by notation (69). Hence, for the considered first part of (84) we found the bound

$$\int_\Omega P(\mathrm{d}\omega)\,\|\omega_s^k\|_{-\lambda} \;=\; P\,\|X_s^k\|_{-\lambda} \;\leq\; \mathrm{e}^{c_{(66)}T}\,\|x^k\|_{-\lambda}\,, \tag{85}$$

where we used Lemma 19.

Under the restriction $\tau(\omega) \leq s$ however, again by notation (69),

$$\delta_\omega \otimes_{\tau(\omega)} Q_\omega \;=\; Q_\omega \;=\; \delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} Q_\omega \text{ on } \mathcal{F}^{\tau(\omega)}. \tag{86}$$

Since by assumption, $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} Q_\omega$ satisfies the martingale problem $\mathbf{MP}_{\tau(\omega),\omega_{\tau(\omega)}}$ (except for ω in the null set N_τ), we may apply Lemma 19 to get

$$\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} Q_\omega \|X_s^k\|_{-\lambda} \;\leq\; \mathrm{e}^{c_{(66)}T}\,\|\omega_{\tau(\omega)}^k\|_{-\lambda}\,. \tag{87}$$

Thus, for the part of (84) under consideration, we got the bound

$$\mathrm{e}^{c_{(66)}T} \int_\Omega P(\mathrm{d}\omega)\,\|\omega_{\tau(\omega)}^k\|_{-\lambda} \;=\; \mathrm{e}^{c_{(66)}T}\,P\,\|X_\tau^k\|_{-\lambda} \;\leq\; \mathrm{e}^{2c_{(66)}T}\,\|x^k\|_{-\lambda}\,, \tag{88}$$

where in the last step we exploited once more our conditional moment estimate in Lemma 19.

Altogether we obtained

$$\sup_{s\le T} P\otimes_\tau Q\,\|X^k_s\|_{-\lambda} \;\le\; 2\,\mathrm{e}^{2c_{(66)}T}\,\|x^k\|_{-\lambda} \;<\; \infty. \tag{89}$$

Thus, by (83), the integrability claim (79) is verified.

Statement (80) is immediately clear, and we turn to (81). Fix $\omega\in\Omega$. Since $\delta_{\omega_{\tau(\omega)}}\otimes_{\tau(\omega)}Q_\omega \in \mathcal{P}_{\tau(\omega),\omega_{\tau(\omega)}}$ by assumption,

$$t\mapsto M^k_{\tau(\omega),t}(\varphi^k) \text{ is a } \delta_{\omega_{\tau(\omega)}}\otimes_{\tau(\omega)}Q_\omega\text{–martingale after time } \tau(\omega), \tag{90}$$

by definition. But $M^k_{\tau(\omega),t}(\varphi^k)$ is $\mathcal{F}^{\tau(\omega)}$–measurable, thus in (90) we may replace $\delta_{\omega_{\tau(\omega)}}\otimes_{\tau(\omega)}Q_\omega$ by Q_ω. Hence, by the martingale identity (72),

$$t\mapsto M^k_{0,t}(\varphi^k) - M^k_{0,\tau(\omega)}(\varphi^k) \text{ is a } Q_\omega\text{–martingale after time } \tau(\omega), \tag{91}$$

and (81) follows.

Exploiting again [32, Theorem 6.1.2], analogously to (78) it can be shown that

$$\bigl\{N^k_{0,t}(\varphi^k):\; t\ge 0\bigr\} \text{ is a } P\otimes_\tau Q\text{–martingale.} \tag{92}$$

Together with (78), the claim follows, finishing the proof of Lemma 24. ■

Now it will be convenient for us to consider the map $(r,x)\mapsto\mathcal{P}_{r,x}$ introduced in the beginning of Subsection 3.3 also from a more general point of view:

Definition 25 (Nice family)**.** A family $\bigl\{\mathcal{P}_{r,x}\neq\emptyset:\;(r,x)\in\mathsf{R}_+\times\mathcal{C}^+_{\mathrm{tem}}\bigr\}$ of subsets of the set $\mathcal{P}$ of all probability laws on $\Omega=\mathcal{C}(\mathsf{R}_+,\mathcal{C}^+_{\mathrm{tem}})$ is said to be *nice,* if it is measurable of $\mathsf{R}_+\times\mathcal{C}^+_{\mathrm{tem}}$ into $\mathrm{com}(\mathcal{P})$ as in Lemma 17, time-homogeneous as in Lemma 18, and if it has the composition properties as in Lemma 24. ◇

Now we are ready to verify the existence Theorem 23.

Proof of Theorem 23. By the Lemmas 17, 18, and 24, we already know that our family $(r,x)\mapsto\mathcal{P}_{r,x}$ of all solutions to the martingale problem **MP** in (70) is nice according to the previous definition. By a successive optimization procedure, we would like to shrink down the sets $\mathcal{P}_{r,x}$ to single point sets $\{P_{r,x}\}$.

Let $\mathcal{A}_1,\mathcal{A}_2$, and $\mathcal{A}_3$ denote countable dense subsets of $(0,\infty)$, $\mathcal{C}_0$, and $\mathcal{C}_{\mathrm{rap}}$, respectively, where $\mathcal{C}_0 := \mathcal{C}_0(\mathsf{R})$ is the separable Banach space of all functions $f:\mathsf{R}\to\mathsf{R}$ vanishing at infinity, equipped with the supremum norm of uniform convergence. Let $\{(\theta_n,f_n,\varphi_n):n\ge 1\}$ denote an enumeration of $\mathcal{A}_1\times\mathcal{A}_2\times\mathcal{A}_3$.

Fix $(r,x)\in\mathsf{R}_+\times\mathcal{C}^+_{\mathrm{tem}}$ for the moment. For $P\in\mathcal{P}_{r,x}$, set

$$L^n_{r,x}(P) \;:=\; \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\theta_n t}\,Pf_n\bigl(\langle X_{r+t},\varphi_n\rangle\bigr), \qquad n\ge 1. \tag{93}$$

Define inductively

$$\mathcal{P}^{n+1}_{r,x} \;:=\; \Bigl\{P\in\mathcal{P}^n_{r,x}:\; L^n_{r,x}(P) = \sup_{P'\in\mathcal{P}^n_{r,x}} L^n_{r,x}(P')\Bigr\}, \qquad n\ge 1, \tag{94}$$

where $\mathcal{P}^1_{r,x} := \mathcal{P}_{r,x}$. Then, by [32, Lemma 12.2.2],

$$\mathcal{P}^n \;:=\; \bigl\{\mathcal{P}^n_{r,x}:\;(r,x)\in\mathsf{R}_+\times\mathcal{C}^+_{\mathrm{tem}}\bigr\} \tag{95}$$

is again a nice family, for each $n \ge 1$. Moreover, as in the proof of Theorem 12.2.3 in [32], also the monotone limits

$$\mathcal{P}^{\infty}_{r,x} \;:=\; \bigcap_{n\ge 1} \mathcal{P}^{n}_{r,x}\,, \qquad (r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}\,, \tag{96}$$

form a nice family.

Fix again $(r,x) \in \mathsf{R}_+ \times \mathcal{C}^+_{\mathrm{tem}}$, and consider $P, P' \in \mathcal{P}^{\infty}_{r,x}$. In order to finish the proof of the theorem, it remains to show that $P = P'$. By construction,

$$\int_0^\infty \mathrm{d}t\; \mathrm{e}^{-\theta t}\, Pf\big(\langle X_{r+t}\,, \varphi\rangle\big) \;=\; \int_0^\infty \mathrm{d}t\; \mathrm{e}^{-\theta t}\, P'f\big(\langle X_{r+t}\,, \varphi\rangle\big), \tag{97}$$

for all $(\theta, f, \varphi) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$. Since $\mathcal{A}_1$ is dense in $(0,\infty)$ by assumption, by the uniqueness theorem of Laplace transforms and by the integrands' continuity in t, we get

$$Pf\big(\langle X_{r+t}\,, \varphi\rangle\big) \;=\; P'f\big(\langle X_{r+t}\,, \varphi\rangle\big), \qquad t \ge 0, \quad (f,\varphi) \in \mathcal{A}_2 \times \mathcal{A}_3\,. \tag{98}$$

But also $\mathcal{A}_2$ and $\mathcal{A}_3$ are dense in $\mathcal{C}_0$ and $\mathcal{C}_{\mathrm{rap}}$, respectively, and we conclude that the martingale problem solutions (X,P) and (X,P') have the same one-dimensional distributions. Thus the laws P and P' coincide (cf. [15, Theorem 4.4.2]), finishing the proof of Theorem 23 ■

4. Global segregation of neighboring types (proof of Theorem 4)

Fix $X_0 = x \in \mathcal{C}^+_{\mathrm{tem}}$, and suppose it has a finite total mass $\|x\|$. In accordance with (18), set

$$Z_T \;=\; Z_T(x) \;:=\; \sum_{k\in\mathsf{K}} \gamma^k\, \|X^k_T\| \cdot \|X^{k+1}_T\|. \tag{99}$$

The strategy of the following proof of Theorem 4 is to construct a contradiction by assuming that

$$P_x\Big(\inf_{T\ge 0} Z_T \,>\, 0\Big) \;>\; 0. \tag{100}$$

Then $Z_T \to 0$ P_x–a.s. will follow [and therefore the claim (17)], since 0 is an absorbing state for the process Z, and since for the continuous non-negative martingales $T \mapsto \|X^k_T\|$ we have that

$$\lim_{T\uparrow\infty} \|X^k_T\| \;=:\; \|X^k_\infty\| \text{ exists in } \mathsf{R}_+\,, \qquad k \in \mathsf{K}. \tag{101}$$

The contradiction will arise when under the event in (100) we construct finite stopping times $T_n \uparrow \infty$ such that $Z_{T_n} \to 0$. This construction requires some preparation.

Step 1° First of all, for $T \ge 0$, we introduce the "global clock"

$$A_T \;:=\; \int_0^T \mathrm{d}t \sum_{k\in\mathsf{K}} \gamma^k \int_{\mathsf{R}} \mathrm{d}a\; X^k_t(a)\, X^{k+1}_t(a) \;\underset{T\uparrow\infty}{\nearrow}\; \text{some } A_\infty \;<\; \infty. \tag{102}$$

Indeed, A_∞ is finite P_x–a.s., since by formula (15) and orthogonality, A is the square function of the non-negative (hence convergent) martingale $T \mapsto \|X_T\|$. We want to decompose A_T by using the pointwise equation (57) (with $r=0$). For this purpose, put

$$ {}^0\!A_T \;:=\; \int_0^T \mathrm{d}t \sum_{k\in\mathsf{K}} \gamma^k \int_{\mathsf{R}} \mathrm{d}a\; S_t x^k(a)\, S_t x^{k+1}(a), \tag{103a} $$

$$ {}^1\!N_T(T) \;:=\; \int_0^T \mathrm{d}t \sum_{k\in\mathsf{K}} \gamma^k \int_{\mathsf{R}} \mathrm{d}a\; S_t x^k(a)\, N_t^{k+1}(t,a), \tag{103b} $$

$$ {}^2\!N_T(T) \;:=\; \int_0^T \mathrm{d}t \sum_{k\in\mathsf{K}} \gamma^k \int_{\mathsf{R}} \mathrm{d}a\; N_t^k(t,a)\, S_t x^{k+1}(a), \tag{103c} $$

$$ {}^3\!N_T(T) \;:=\; \int_0^T \mathrm{d}t \sum_{k\in\mathsf{K}} \gamma^k \int_{\mathsf{R}} \mathrm{d}a\; N_t^k(t,a)\, N_t^{k+1}(t,a), \tag{103d} $$

where, for $a\in\mathsf{R}$,

$$ N_r^k(t,a) \;:=\; \int_{[0,r]\times\mathsf{R}} M^k\bigl(\mathrm{d}(s,b)\bigr)\, \mathrm{p}_{t-s}(b-a), \qquad 0<r\le t, \tag{104} $$

with the martingale measures $M^k := M_0^k$ from Corollary 16. Note that all quantities make sense by the uniform moment estimates Corollary 15. Then, by our equation (57) (recall that there we have continuity in t,a), we get the decomposition

$$ A_T \;=\; {}^0\!A_T + {}^1\!N_T(T) + {}^2\!N_T(T) + {}^3\!N_T(T) \;=:\; {}^0\!A_T + N_T(T). \tag{105} $$

Step 2° In analyzing the fluctuating part $N_T(T)$ of A_T, a little care has to be taken since $T\mapsto N_T(T)$ [or the single terms $T\mapsto {}^i\!N_T(T)$] are *not* martingales.

Interchanging the order of integration in (103b) gives

$$ {}^1\!N_T(T) \;=\; \sum_k \gamma^k \int_{[0,T]\times\mathsf{R}} M^{k+1}\bigl(\mathrm{d}(s,b)\bigr) \int_s^T \mathrm{d}t\; S_{2t-s} x^k(b). \tag{106} $$

We generalize now the notation $T\mapsto {}^1\!N_T(T)$ by putting

$$ {}^1\!N_r(T) \;:=\; \sum_k \gamma^k \int_{[0,r]\times\mathsf{R}} M^{k+1}\bigl(\mathrm{d}(s,b)\bigr) \int_s^T \mathrm{d}t\; S_{2t-s} x^k(b), \qquad 0\le r\le T. \tag{107} $$

As opposed to $T\mapsto {}^1\!N_T(T)$, for fixed $T>0$, the process $r\mapsto {}^1\!N_r(T)$, $r\in[0,T]$, is a martingale. Analogously, we can define the martingale $r\mapsto {}^2\!N_r(T)$.

Integrating by parts in the third fluctuation term (103d) gives

$$ \begin{aligned} {}^3\!N_T(T) \;=\; \int_0^T \mathrm{d}t \sum_k \gamma^k \int_{\mathsf{R}} \mathrm{d}a\, \Bigl[& \int_{[0,t]\times\mathsf{R}} M^k\bigl(\mathrm{d}(s,b)\bigr)\, \mathrm{p}_{t-s}(b-a)\, N_s^{k+1}(t,a) \\ & + \int_{[0,t]\times\mathsf{R}} M^{k+1}\bigl(\mathrm{d}(s,b)\bigr)\, \mathrm{p}_{t-s}(b-a)\, N_s^k(t,a)\Bigr], \end{aligned} $$

which we write as ${}^{31}\!N_T(T) + {}^{32}\!N_T(T)$ in the obvious correspondence. Interchanging the order of integration yields

$$ {}^{31}\!N_T(T) \;=\; \sum_k \gamma^k \int_{[0,T]\times\mathsf{R}} M^k\bigl(\mathrm{d}(s,b)\bigr) \int_s^T \mathrm{d}t \int_{\mathsf{R}} \mathrm{d}a\; \mathrm{p}_{t-s}(b-a)\, N_s^{k+1}(t,a). \tag{108} $$

Put

$$(109)\quad {}^{31}N_r(T) \;:=\; \sum_k \gamma^k \int_{[0,r]\times\mathsf{R}} M^k\big(\mathrm{d}(s,b)\big) \int_s^T \mathrm{d}t \int_{\mathsf{R}} \mathrm{d}a\; \mathrm{p}_{t-s}(b-a)\, N_s^{k+1}(t,a),$$

$0 \le r \le T$, getting again a martingale $t \mapsto {}^{31}N_r(T)$. Similarly, we define the martingale $r \mapsto {}^{32}N_r(T)$.

Altogether, in generalization of the notations (103b) – (103d) and (105), for $T > 0$ fixed, we defined the martingales ${}^iN_\cdot(T)$, $i \in \{1, 2, 31, 32\}$, and

$$(110)\qquad r \;\mapsto\; N_r(T) := {}^1N_r(T) + {}^2N_r(T) + {}^{31}N_r(T) + {}^{32}N_r(T), \qquad 0 \le r \le T.$$

Step 3° Let us next mention the *idea* behind the following construction of a contradiction. It is relatively easy to see that for the deterministic part ${}^0A$ of A in the decomposition (105) we have

$$(111)\qquad {}^0A_T \;\ge\; c\, Z_0 \quad \text{for large } T$$

[see (127) below]. On the other hand, the martingale

$$(112)\qquad r \mapsto N_r(T) \text{ from (110) has a square function bounded by } A_T$$

[see (123) below]. Since $B_t \approx \sqrt{t}$ for Brownian motion in R, the martingale representation theorem "yields"

$$(113)\qquad \big|N_T(T)\big| \;\le\; \sup_{0\le r\le T} \big|N_r(T)\big| \;\le\; c\,\sqrt{A_T}\,,$$

hence

$$(114)\qquad N_T(T) \;\ge\; -c\,\sqrt{A_T} \quad \text{for large } T.$$

Combining with the decomposition (105) and the estimate (111) gives

$$(115)\qquad A_T \,+\, c\,\sqrt{A_T} \;\ge\; c\, Z_0 \quad \text{for a large } T_1\,.$$

Hence, there is a continuous function h with $h(0) = 0$ such that $h(A_{T_1}) \ge Z_0$. By our assumption (100), Z_{T_1} is different from 0 with positive probability. Starting at time T_1 anew, we will find $T_2 > T_1$ such that $h(A_{T_2} - A_{T_1}) \ge Z_{T_1}$, as so on. But $A_{T_{n+1}} - A_{T_n} \to 0$ as $n \uparrow \infty$ by (102) [provided that $T_n \uparrow \infty$], therefore $Z_{T_n} \to 0$, which contradicts (100), as desired.

Step 4° In order to make precise the previous ideas, we will control the random expressions ${}^iN_T(T)$ in terms of A_T, as needed for (112). By orthogonality, for the square function of the martingale ${}^1N(T)$ as defined in (107) we get

$$\big\langle\!\big\langle {}^1N(T) \big\rangle\!\big\rangle_r \;=\; \sum_k \gamma^k \int_0^r \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\;\, \gamma^{k+1}\, X_s^{k+1}(b)\, X_s^{k+2}(b) \Big[\int_s^T \mathrm{d}t\; S_{2t-s} x^k(b) \Big]^2.$$

Setting

$$(116)\qquad \mathrm{q}_t(a) \;:=\; \int_0^t \mathrm{d}s\; \mathrm{p}_s(a), \qquad t \ge 0, \quad a \in \mathsf{R},$$

for each constant $0 \le \theta \le 1$ we obtain

$$(117)\qquad \int_s^T \mathrm{d}t\; S_{2t-s-\theta s} X_{\theta s}^k(b) \;\le\; \frac{1}{2}\, \mathrm{q}_{2T}(0) \max_{k\in\mathsf{K},\; 0\le s\le T} \|X_s^k\|.$$

Applying this for $\theta = 0$, and denoting by $\overline{\gamma}$ the maximum of the γ^k, we find

$$\text{(118)} \qquad \langle\!\langle {}^1N(T) \rangle\!\rangle_r \;\le\; \frac{1}{4}\,\overline{\gamma}\,\mathrm{q}_{2T}^2(0) \max_{k\in\mathsf{K},\;0\le s\le T} \|X_s^k\|^2\,A_T\,, \qquad 0 \le r \le T.$$

The same estimate is true for $\langle\!\langle {}^2N(T) \rangle\!\rangle_r$.

The square function value $\langle\!\langle {}^{31}N(T) \rangle\!\rangle_r$ of the martingale ${}^{31}N(T)$ of (109) equals

$$\text{(119)} \quad \sum_k \gamma^k \int_0^r \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\;\gamma^k\,X_s^k(b)\,X_s^{k+1}(b) \left[\int_s^T \mathrm{d}t \int_{\mathsf{R}} \mathrm{d}a\;\mathrm{p}_{t-s}(b-a)\,N_s^{k+1}(t,a)\right]^2.$$

But

$$\text{(120)} \qquad N_s^{k+1}(t,a) \;=\; S_{t-s}N_s^{k+1}(s,\,\cdot\,)\,(a), \qquad 0 \le s \le t,$$

hence, by equation (57),

$$\text{(121)} \qquad \left|N_s^{k+1}(t,a)\right| \;\le\; S_{t-s}X_s^{k+1}(a) + S_tX_0^{k+1}(a).$$

Thus, the expression under the square brackets in (119) can in absolute value be bounded from above by

$$\text{(122)} \qquad \int_s^T \mathrm{d}t\,\left[S_{2(t-s)}X_s^{k+1}(b) + S_{2t-s}X_0^{k+1}(b)\right] \;\le\; \mathrm{q}_{2T}(0) \max_{k\in\mathsf{K},\;0\le s\le T} \|X_s^k\|,$$

where in the last step we used (117) (for $\theta = 1$ and $\theta = 0$). Hence, for $\langle\!\langle {}^{31}N(T) \rangle\!\rangle_r$ we get the same bound as in (118), except for the factor $\frac{1}{4}$. Moreover, for $\langle\!\langle {}^{32}N(T) \rangle\!\rangle_r$ we get the same bound as for $\langle\!\langle {}^{31}N(T) \rangle\!\rangle_r$.

Altogether, by the Kunita-Watanabe inequality, for the martingale $N(T)$ as defined in (110) we get − as announced in (112) − the square function estimate

$$\text{(123)} \qquad \langle\!\langle N(T) \rangle\!\rangle_r \;\le\; c_{(123)}\,\overline{\gamma}\,\mathrm{q}_{2T}^2(0)\,A_T \max_{k\in\mathsf{K},\;0\le s\le T} \|X_s^k\|^2, \qquad 0 \le r \le T,$$

where $c_{(123)}$ is a (universal) constant.

Step 5° Now we want to derive a lower estimate for the deterministic term ${}^0A_T$ from (103a), as announced in (111). For this purpose, for the fixed initial state x, choose a constant $L = L(x) \ge 1$ such that

$$\text{(124)} \qquad \left\langle x^k, \mathbf{1}_{[-L/2,\,L/2]}\right\rangle \;\ge\; \frac{1}{2}\,\|x^k\|, \qquad k \in \mathsf{K}.$$

Also, there is a (universal) constant $c_{(125)}$ such that

$$\text{(125)} \qquad \mathrm{q}_{2T}(L) \;\ge\; c_{(125)}\,\mathrm{q}_{2T}(0), \qquad T \ge L^2 =: T_1(x) \ge 1.$$

Then in the identity

$$\text{(126)} \qquad {}^0A_T \;=\; \frac{1}{2}\sum_k \gamma^k \int_{\mathsf{R}} \mathrm{d}a \int_{\mathsf{R}} \mathrm{d}b\;x^k(a)\,x^{k+1}(b)\,\mathrm{q}_{2T}(b-a)$$

we first restrict the integration domains in order to use (125), getting

$${}^0A_T \;\ge\; \frac{c_{(125)}}{2} \sum_k \gamma^k \int_{-L/2}^{L/2} \mathrm{d}a \int_{-L/2}^{L/2} \mathrm{d}b\;x^k(a)\,x^{k+1}(b)\,\mathrm{q}_{2T}(0), \qquad T \ge T_1(x).$$

Then (124) yields the estimate

$$\text{(127)} \qquad {}^0A_T \;\ge\; \frac{c_{(125)}}{8}\,\mathrm{q}_{2T}(0) \sum_k \gamma^k\,\|x^k\| \cdot \|x^{k+1}\| \;=\; \frac{c_{(125)}}{8}\,\mathrm{q}_{2T}(0)\,Z_0$$

for $T \geq T_1(x)$, with $Z_0 = Z_0(x)$ from (99).

Now we modify our definition of $T_1(x)$ from (125): If $Z_0 = 0$, we set $T_1(x) := \infty$. Otherwise we may enlarge $T_1(x)$ from the former definition (125) to a finite value by requiring that additionally

$$\frac{c_{(125)}}{8}\, \mathrm{q}_{2T}(0)\, Z_0 \geq 2, \qquad T \geq T_1(x). \tag{128}$$

Assume $T \in \bigl[T_1(x), \infty\bigr)$ for a while. From (127) and (128) we already know that

$$^0\!A_T \;\geq\; \frac{c_{(125)}}{8}\, \mathrm{q}_{2T}(0)\, Z_0 \;\geq 2. \tag{129}$$

Step 6° Next we want to bound below the probability $P_x\left(A_T \geq 1\right)$. Recall that $T_1(x) \leq T < \infty$. If we assume for the moment that $A_T < 1$, then (129) and (105) imply that

$$N_T(T) \;\leq\; -\frac{c_{(125)}}{16}\, \mathrm{q}_{2T}(0)\, Z_0\,. \tag{130}$$

Consequently,

$$P_x\left(A_T < 1\right) \;=\; P_x\Bigl(A_T < 1,\; N_T(T) \leq -\frac{c_{(125)}}{16}\, \mathrm{q}_{2T}(0)\, Z_0\Bigr). \tag{131}$$

Let

$$R \;=\; R(x) \;\geq\; \max_{k \in \mathsf{K}} \|x^k\|. \tag{132}$$

Distinguishing between

$$\max\bigl\{\|X_t^k\| : \, k \in \mathsf{K},\; 0 \leq t \leq T\bigr\} > 2R \tag{133}$$

and the opposite, identity (131) can be continued with

$$\begin{aligned} &\leq\; \sum_k P_x\Bigl(A_T \leq 1,\; \max_{0 \leq t \leq T} \|X_t^k\| \geq 2R\Bigr) \\ &\quad + P_x\Bigl(A_T \leq 1,\; N_T(T) \leq -\frac{c_{(125)}}{16}\, \mathrm{q}_{2T}(0)\, Z_0\,,\; \max_{\substack{k \in \mathsf{K} \\ 0 \leq t \leq T}} \|X_t^k\| \leq 2R\Bigr). \end{aligned} \tag{134}$$

For the first term in (134) we use that by (57) the process $t \mapsto \|X_t^k\| - \|x^k\|$ equals in law to a one-dimensional standard Brownian motion (B, Π_0) (starting from 0) running with a clock bounded by $t \mapsto A_t$. Hence, for the first term in (134) we get the bound

$$\sum_k \Pi_0\Bigl(\|x^k\| + \max_{0 \leq t \leq 1} B_t \geq 2R\Bigr) \;\leq\; K\, \Pi_0\Bigl(\max_{0 \leq t \leq 1} B_t \geq R\Bigr), \tag{135}$$

where in the last step we used the definition (132) of R. By the reflection principle of Brownian motion, and an elementary estimate for the normal law,

$$\Pi_0\Bigl(\max_{0 \leq t \leq 1} B_t \geq R\Bigr) \;\leq\; 2\, \Pi_0\Bigl(B_1 \geq R\Bigr) \;\leq\; \frac{2}{R}\, \mathrm{e}^{-R^2/2}. \tag{136}$$

Consequently, for the first term in (134) we got the bound $\frac{2K}{R}\, \mathrm{e}^{-R^2/2}$.

For the second term in (134) we use the square function estimate (123) to obtain the bound

$$P_x\Bigl(N_T(T) \leq -\frac{c_{(125)}}{16}\, \mathrm{q}_{2T}(0)\, Z_0\,,\; \bigl\langle\!\bigl\langle N(T) \bigr\rangle\!\bigr\rangle_T \leq 4\, c_{(123)}\, \overline{\gamma}\, \mathrm{q}_{2T}^2(0)\, R^2\Bigr). \tag{137}$$

But the law of $r \mapsto N_r(T)$ coincides with the distribution of B running with the clock $r \mapsto \langle\!\langle N(T) \rangle\!\rangle_r$ (for a finite time). Hence, (137) is bounded from above by

$$\begin{aligned}
&\Pi_0\Big(\min\Big\{B_t:\ 0 \le t \le 4\, c_{(123)}\, \overline{\gamma}\, \mathrm{q}^2_{2T}(0)\, R^2\Big\} \;\le\; -\frac{c_{(125)}}{16}\, \mathrm{q}_{2T}(0)\, Z_0\Big) \\
(138)\qquad &=\; \Pi_0\Big(\min\{B_t:\ 0 \le t \le 1\} \;\le\; -\frac{c_{(125)}\, Z_0}{32\sqrt{c_{(123)}\,\overline{\gamma}}\, R}\Big).
\end{aligned}$$

where in the last step we used Brownian scaling. Changing from B to $-B$, again by the first part of (136) we may continue with

$$(139)\qquad \le\; 1 - \Pi_0\Big(|B_1| \le \frac{c_{(125)}\, Z_0}{32\sqrt{c_{(123)}\,\overline{\gamma}}\, R}\Big) \;\le\; 1 - \frac{c_{(139)}\, Z_0}{R},$$

where the constant $c_{(139)}$ does not depend on x and T.

Altogether

$$(140)\qquad P_x\,(A_T \ge 1) \;\ge\; -\frac{2K}{R(x)}\,\mathrm{e}^{-R^2(x)/2} + \frac{c_{(139)}\, Z_0(x)}{R(x)} \;=:\; f(x),$$

provided that $T \in \big[T_1(x), \infty\big)$.

Step 7° Now we will make more precise our choice of $R(x)$ in (132). In fact, for the x considered in this proof, set

$$(141)\qquad R(x) \;:=\; \begin{cases} \sqrt{2\,\log 2K} \,\vee\, \max\limits_{k\in\mathsf{K}} \|x^k\|, & \text{if } c_{(139)}\, Z_0(x) \ge 2, \\[2ex] \sqrt{2\,\log \dfrac{4K}{c_{(139)}\, Z_0(x)}} \,\vee\, \max\limits_{k\in\mathsf{K}} \|x^k\|, & \text{otherwise.} \end{cases}$$

Note that then

$$(142)\qquad f(x) \;\ge\; \frac{c_{(139)}\, Z_0(x)}{2\, R(x)} \;\ge\; 0.$$

Moreover, setting

$$(143)\qquad V_{\delta,C} \;:=\; \Big\{x:\ Z_0(x) \ge \delta,\ \max_{k\in\mathsf{K}} \|x^k\| \le C\Big\}, \qquad 0 < \delta < C < \infty,$$

our choice (141) of R yields

$$(144)\qquad R\,(V_{\delta,C}) \text{ is a relatively compact subset of } (0,\infty), \qquad 0 < \delta < C < \infty.$$

Step 8° Setting $T_0 := 0$, and recalling the definition of T_1 around (128), define inductively the stopping times

$$(145)\qquad T_{n+1} \;:=\; \begin{cases} T_n + T_1\,(X_{T_n}), & \text{if } T_n < \infty, \\ \infty, & \text{otherwise,} \end{cases}$$

$n \ge 1$. Note that $T_n \ge n$ for all n. Recalling that almost surely

$$(146)\qquad 0 \;=\; A_0 \;\le\; A_t \uparrow A_\infty \;<\; \infty \quad \text{as} \quad t \uparrow \infty,$$

by the strong Markov property we have

$$\begin{aligned}
P_x\left\{A_{T_{n+1}} - A_{T_n} \ge 1 \,\middle|\, \mathcal{F}_{T_n}\right\} \;&=\; \mathbf{1}_{\{T_n<\infty\}}\, P_{X_{T_n}}(A_{T_1} \ge 1) \\
(147)\qquad &\ge\; \mathbf{1}_{\{T_n<\infty\}}\, f(X_{T_n})\,.
\end{aligned}$$

Hence, by the conditional version of Borel-Cantelli (see Williams [36, 12.15]),

$$\Big\{A_{T_{n+1}} - A_{T_n} \geq 1 \text{ infinitely often}\Big\} \supseteq \Big\{\sum_{n=1}^{\infty} \mathbf{1}_{\{T_n<\infty\}}\, f\left(X_{T_n}\right) = \infty\Big\}, \tag{148}$$

P_x–a.s. But by (146), the left hand side of (148) must be a null set. Hence,

$$\sum_{n=1}^{\infty} \mathbf{1}_{\{T_n<\infty\}}\, f\left(X_{T_n}\right) \;<\; \infty, \quad P_x\text{–a.s.} \tag{149}$$

Since $f \geq 0,$ on the set $\{T_n < \infty :\ n \geq 1\}$ we have (P_x–a.s.)

$$\lim_{n\uparrow\infty} f\left(X_{T_n}\right) \;=\; 0, \quad \text{hence} \quad \lim_{n\uparrow\infty} \frac{Z_{T_n}}{R(X_{T_n})} \;=\; 0, \tag{150}$$

the latter by (142).

Step 9° Suppose now that (100) is valid, and we want to derive a contradiction. By (100), there exist constants $\delta > 0$ and $\varepsilon \in \big(0, \frac{1}{2}\big)$ such that for our fixed $x,$

$$P_x\Big(\inf_{T\geq 0} Z_T \;\geq\; \delta\Big) \;\geq\; 2\varepsilon. \tag{151}$$

On the other hand, from the martingale convergence (101) we conclude for the existence of a constant $C > \delta$ such that

$$P_x\Big(\sup_{k\in\mathsf{K},\ T\geq 0} \|X_T^k\| \;\leq\; C\Big) \;\geq\; 1 - \varepsilon. \tag{152}$$

Introduce the event

$$\Omega_{\delta,C} \;:=\; \Big\{\omega :\ \inf_{T\geq 0} Z_T \;\geq\; \delta,\ \sup_{k\in\mathsf{K},\ T\geq 0} \|X_T^k\| \;\leq\; C\Big\}. \tag{153}$$

Then from (151) and (152),

$$P_x(\Omega_{\delta,C}) \;\geq\; \varepsilon. \tag{154}$$

Note that for $\omega \in \Omega_{\delta,C}$ we have $T_n < \infty$ for all $n,$ hence, by (150),

$$P_x\bigg(\omega \in \Omega_{\delta,C}\,,\ \lim_{n\uparrow\infty} \frac{Z_{T_n}}{R(X_{T_n})} \;=\; 0\bigg) \;\geq\; \varepsilon \;>\; 0. \tag{155}$$

But $X_{T_n} \in V_{\delta,C}$ for each n on the event $\Omega_{\delta,C}\,,$ implying $Z_{T_n} \to 0$ as $n \uparrow \infty$ by the relative compactness in (144), which contradicts $\inf_{T\geq 0} Z_T \;\geq\; \delta \;>\; 0$ in the definition of $\Omega_{\delta,C}\,.$ Therefore the statement (100) cannot be true, and the claim in Theorem 4 follows as already explained in the beginning of this subsection. This finishes the proof of Theorem 4. ■

5. Finite time behavior (proof of Theorem 5)

Finally, the finite time behavior Theorem 5 will be proved in the following two subsections.

5.1. **Finite time survival of all types [proof of (a)].** As a preparation for the proof, for convenience we give the following variance estimate.

Lemma 26 (Variance estimate). *For* $x \in \mathcal{C}_{\mathrm{tem}}^{+}$ *and* $(t,k,a) \in \mathsf{R}_{+} \times \mathsf{K} \times \mathsf{R}$,

$$\mathrm{Var}_x\, X_t^k(a) \;\le\; \gamma^k \sqrt{\frac{2t}{\pi}}\, \sqrt{S_t[x^k]^2(a)\, S_t[x^{k+1}]^2(a)}\,. \tag{156}$$

Proof. By the covariance formula in Theorem 3 (b), $X_t^k(a)$ has the following variance:

$$\mathrm{Var}_x\, X_t^k(a) \;=\; \gamma^k \int_0^t \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\; S_s x^k(b)\, S_s x^{k+1}(b)\, \mathrm{p}_{t-s}^2(a-b). \tag{157}$$

Estimate one of the p–factors by $\mathrm{p}_{t-s}(0) = 1/\sqrt{2\pi(t-s)}$, and use Cauchy-Schwarz to get the upper bound

$$\gamma^k \int_0^t \mathrm{d}s\, \frac{1}{\sqrt{2\pi(t-s)}} \prod_{i=0}^{1} \Big(\int_{\mathsf{R}} \mathrm{d}b\; \mathrm{p}_{t-s}(a-b)\, \big[S_s x^{k+i}(b)\big]^2 \Big)^{1/2}. \tag{158}$$

By Jensen's inequality, $\big[S_s x^k(b)\big]^2 \le S_s[x^k]^2(b)$, and altogether we get the desired variance estimate (156). ■

Completion of the proof of Theorem 5 (a). Fix $X_0 = x$, and $t \ge T > 0$ as in the theorem, and let P_x be any solution to the martingale problem $\mathbf{MP}_x$ (that is, the Markov property is not needed for this proof). Let $k \in \mathsf{K}$ and $\varepsilon > 0$. Then by our assumption (20), there is an $a \in \mathsf{R}$ such that

$$\frac{4\,\gamma^k \sqrt{2t}}{\sqrt{\pi}}\, \frac{\sqrt{S_t[x^k]^2(a)\; S_t[x^{k+1}]^2(a)}}{\big[S_t x^k(a)\big]^2} \;<\; \varepsilon. \tag{159}$$

Fix this a. By the continuity of states, $\|X_t^k\| = 0$ implies that $X_t^k(a) = 0$. Hence,

$$P_x\big(\|X_t^k\| = 0\big) \;\le\; P_x\Big(\big|S_t x^k(a) - X_t^k(a)\big| \;\ge\; \frac{1}{2}\, S_t x^k(a)\Big). \tag{160}$$

By the expectation formula in Theorem 3 (b), Chebychev's inequality, and the variance estimate in Lemma 26,

$$P_x\big(\|X_t^k\| = 0\big) \;\le\; \frac{4\,\gamma^k \sqrt{2t}}{\sqrt{\pi}}\, \frac{\sqrt{S_t[x^k]^2(a)\; S_t[x^{k+1}]^2(a)}}{\big[S_t x^k(a)\big]^2} \;<\; \varepsilon, \tag{161}$$

the latter by our choice (159) of a. Since ε is arbitrary, we arrive at

$$P_x\big(\|X_t^k\| = 0\big) = 0, \qquad t \ge T. \tag{162}$$

Denote by $\tau \le \infty$ the hitting time of 0 of the non-negative continuous martingale $t \mapsto \|X_t^k\|$. Assume for the moment that $P_x(\tau < \infty) > 0$ is true. Then also $P_x(\tau < t) > 0$ holds for some $t \ge T$. But the state 0 is a trap of that martingale, and we get $P_x\big(\|X_t^k\| = 0\big) > 0$ for that t, which contradicts (162). Hence, $P_x(\tau < \infty) = 0$, and since k is arbitrary, the claim (21) follows, finishing the proof of Theorem 5 (a). ■

5.2. **Finite time extinction of a type [proof of (b)].** Recall that $k_0 \in \mathsf{K}$ is fixed, where we may assume without loss of generality that $k_0 = 0$. Recall also that we have positive constants c_i and λ_i, $0 \le i \le 2$, as well as c_1', λ_1' which are related by assumption (22). Additionally, fix positive constants $\alpha, \beta', \beta, \gamma$ such that

$$\lambda_0 > \alpha > \lambda_1', \quad \lambda_1 > \beta, \quad \lambda_2 > \gamma, \quad \text{and} \quad 2\lambda_1' < 2\beta' < \beta + \gamma. \tag{163}$$

Once and for all, fix $\varepsilon \in (0,1]$, and a constant $p > 9$. Without loss of generality, we may consider a terminal time $T \in (0,1]$.

Let I_{c_0} denote the set of all initial states $x \in \mathcal{C}_{\mathrm{tem}}^{+}$ with $\|x\| < \infty$ and which satisfy (23) and (24). We may assume that $c_0 \le 1$, implying $I_{c_0} \subseteq I_1$.

Fix for the moment c_0 and $x \in I_{c_0}$. Let P_x be any solution to the martingale problem $\mathbf{MP}_x$ (that is, the Markov property will again not be used).

Recall the smooth reference functions $\tilde{\phi}_\lambda$ introduced in (6), and the related notation $\|\cdot\|_\lambda$ from (12). Introduce the stopping times

$$\tau_L^{k,p,\lambda} \;:=\; \inf\bigl\{t \ge 0:\ \bigl\|(X_t^k)^p\bigr\|_\lambda \ge L\bigr\}, \qquad L > 0, \quad \lambda > 0, \tag{164}$$

and

$$\tau_L(t) \;:=\; t \wedge \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma}, \qquad t \le T. \tag{165}$$

Recall the martingales $r \mapsto N_r^k(t,a)$, $r \le t$, from (104), for each $(k,a) \in \mathsf{K} \times \mathsf{R}$. Set

$$N_t^k(a) \;:=\; N_t^k(t,a). \tag{166}$$

Since the claim of Theorem 5 (b) highly depends on the interplay of "sizes" of the different types, some efforts are needed to control them. Here is our first result in this direction.

Lemma 27 (A moment estimate for some fluctuation increments)**.** *For* $0 \le t' \le t \le T$, $a, a' \in \mathsf{R}$, *and* $L > 0$, *there is a constant* $c_{(167)}$ *such that*

$$\begin{aligned}&\sup_{x \in I_1} P_x \bigl|N_t^1(a) - N_{t'}^1(a')\bigr|^{2p}\, \mathbf{1}\Bigl\{t \;\le\; \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma}\Bigr\}\\ &\quad\le\; c_{(167)}\, \sqrt{L}\, \bigl(|a-a'| + |t-t'|^{1/2}\bigr)^{p-1}\, T^{1/4}\, \bigl(\phi_{p(\beta+\gamma)}(a') + \phi_{p(\beta+\gamma)}(a)\bigr).\end{aligned} \tag{167}$$

Proof. By the definition (104) of $N_r^k(t,a)$, the moment expression in (167) can be written as

$$P_x \biggl| \int_{[0,t]\times\mathsf{R}} M^1\bigl(\mathrm{d}(s,b)\bigr)\bigl[\mathrm{p}_{t-s}(b-a) - \mathrm{p}_{t'-s}(b-a')\bigr]\, \mathbf{1}\Bigl\{t \;\le\; \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma}\Bigr\}\biggr|^{2p}.$$

Under the restriction as in the indicator, for the upper integration bound we may use that $t = \tau_L(t)$, by definition (165). Hence, the latter moment expression can be estimated from above by

$$P_x \biggl| \int_{[0,\tau_L(t)]\times\mathsf{R}} M^1\bigl(\mathrm{d}(s,b)\bigr)\, \bigl[\mathrm{p}_{t-s}(b-a) - \mathrm{p}_{t'-s}(b-a')\bigr]\biggr|^{2p}. \tag{168}$$

In virtue of the Burkholder-Davis-Gundy inequality, this can further be bounded by

$$c\, P_x \biggl| \int_0^{\tau_L(t)} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b\, \bigl[\mathrm{p}_{t-s}(b-a) - \mathrm{p}_{t'-s}(b-a')\bigr]^2\, X_s^1(b) X_s^2(b)\biggr|^{p}. \tag{169}$$

Writing $2 = (2-\frac{2}{p}) + \frac{2}{p}$, by Hölder's inequality the latter double integral can be estimated from above by

$$\left| \int_0^{\tau_L(t)} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b \, \left[\mathrm{p}_{t-s}(b-a) - \mathrm{p}_{t'-s}(b-a')\right]^2 \right|^{p-1} \tag{170a}$$

$$\times \int_0^{\tau_L(t)} \mathrm{d}s \int_{\mathsf{R}} \mathrm{d}b \, \left[\mathrm{p}_{t-s}(b-a) - \mathrm{p}_{t'-s}(b-a')\right]^2 \left[X_s^1(b)\right]^p \left[X_s^2(b)\right]^p. \tag{170b}$$

For the first factor (170a) we use $\tau_L(t) \le t$ and then the heat kernel estimate (53) to get the bound

$$c\left(|a-a'| + |t-t'|^{1/2}\right)^{p-1}. \tag{171}$$

On the other hand, in the second factor (170b) we introduce $\tilde{\phi}_{p(\beta+\gamma)}(b)\,\tilde{\phi}_{-p(\beta+\gamma)}(b)$ [which is bounded away from 0, recall (7)], and use Cauchy-Schwarz to get for the internal integral in (170b) the bound

$$\left| \int_{\mathsf{R}} \mathrm{d}b \, \left[\mathrm{p}_{t-s}(b-a) + \mathrm{p}_{t'-s}(b-a')\right]^4 \, \tilde{\phi}_{2p(\beta+\gamma)}(b) \right|^{1/2} \tag{172a}$$

$$\times \left| \int_{\mathsf{R}} \mathrm{d}b \, \left[X_s^1(b)\right]^{2p} \left[X_s^2(b)\right]^{2p} \, \tilde{\phi}_{-2p(\beta+\gamma)}(b) \right|^{1/2}. \tag{172b}$$

In the new first factor (172a), estimate three of the p–factor pairs by $2\,\mathrm{p}_{t'-s}(0)$, and use the heat kernel estimate (38) to get the bound

$$c\,\mathrm{p}_{t'-s}^{3/2}(0)\left[\phi_{p(\beta+\gamma)}(a) + \phi_{p(\beta+\gamma)}(a')\right] \tag{173}$$

for (172a). In the second new factor (172b) apply once more Cauchy-Schwarz to get

$$\le \left\langle \left[X_s^1\right]^{4p}, \phi_{-4p\beta}^1 \right\rangle^{1/4} \left\langle \left[X_s^2\right]^{4p}, \tilde{\phi}_{-4p\gamma} \right\rangle^{1/4} \le \sqrt{L}, \tag{174}$$

where in the last step we used $s \le \tau_L(t) \le \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma}$ [recall (165)]. Consequently, for (172a)/(172b) we have the bound

$$c\,\mathrm{p}_{t'-s}^{3/2}(0)\left[\phi_{p(\beta+\gamma)}(a) + \phi_{p(\beta+\gamma)}(a')\right]\sqrt{L}. \tag{175}$$

Inserting (171) and (175) into (170a)/(170b), gives the bound

$$c\sqrt{L}\left(|a-a'| + |t-t'|^{1/2}\right)^{p-1} \left[\phi_{p(\beta+\gamma)}(a) + \phi_{p(\beta+\gamma)}(a')\right] \int_0^t \mathrm{d}s \; \mathrm{p}_{t'-s}^{3/2}(0), \tag{176}$$

since $\tau_L(t) \le t$. This is clearly bounded by the right hand side of (167), finishing the proof of Lemma 27. ■

We continue with the proof of Theorem 5 (b). For the purpose of establishing a further control of the states of our process, for each $n \ge 1$, we consider the equidistant grid

$$G_n := \left\{(t_{n,i}, a_{n,j}): \; t_{n,i} := i2^{-n}T, \; a_{n,j} := j2^{-n}, \; 0 \le i \le 2^n, \; j \in \mathsf{Z}\right\} \tag{177}$$

partitioning $[0,T] \times \mathsf{R}$.

The *idea* is now to show that $X_t^1(a)$ is "not too small". As

$$X_t^1(a) = S_t x^1(a) + N_t^1(a) \tag{178}$$

[recall (57), (104), and (166)], we first will show that for the fluctuation part N^1 with a large probability

$$|N_t^1(a)| \ \leq \ \frac{1}{2}\, S_t x^1(a), \qquad 0 \leq t \leq T, \quad a \in \mathsf{R}, \tag{179}$$

[see (192) below]. In fact, using Lemma 27, we can estimate the increments of $N_t^1(a)$ for (t,a) in the union $G = \cup_n G_n$ of grids with a large probability. Thus, for any $(t,a) \in [0,T] \times \mathsf{R}$, we can approximate $N_t^1(a)$ by the sum of the aforementioned increments towards the boundary $\{0\} \times \mathsf{R}$ in order to obtain (179). Then by (178),

$$X_t^1(a) \ \geq \ \frac{1}{2}\, S_t x^1(a), \qquad 0 \leq t \leq T, \quad a \in \mathsf{R}. \tag{180}$$

with a large probability [see (193)].

Here are the details. Let $n \geq 1$. Two points $g = (t,b)$ and $g' = (t',b')$ in the grid G_n are called *neighboring* points, if one of their coordinates coincide and the other one are neighbors in the obvious meaning. For $0 < \varepsilon_1 \leq 1$, and neighboring points $g, g' \in G_n$ with $g \geq g'$, introduce the event

$$A_{\varepsilon_1,n}^{g,g'} \ := \ \Big\{ \big|N_t^1(a) - N_{t'}^1(a')\big| \geq 2^{-n/p} \varepsilon_1 \phi_{\beta'}(a), \ t \leq \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma} \Big\}, \tag{181}$$

and denote by A_{ε_1} the union of $A_{\varepsilon_1,n}^{g,g'}$, with all these $g, g' \in G_n$ and $n \geq 1$. By Markov's inequality and Lemma 27,

$$\begin{aligned}
&\sup_{x \in I_1} P_x\big(A_{\varepsilon_1,n}^{g,g'}\big) \\
&\leq \ 2^{2n} \varepsilon_1^{-2p} \phi_{-2p\beta'}(a) \sup_{x \in I_1} P_x \big|N_t^1(a) - N_{t'}^1(a')\big|^{2p} \, \mathbf{1}\Big\{ t \ \leq \ \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma} \Big\} \\
&\leq \ 2^{2n} \varepsilon_1^{-2p} \phi_{-2p\beta'}(a)\, c_{(167)} \sqrt{L}\, \big(2^{-n/2}\big)^{p-1} T^{1/4} \big(\phi_{p(\beta+\gamma)}(a') + \phi_{p(\beta+\gamma)}(a)\big) \\
&= \ c_{(167)} \sqrt{L}\, T^{1/4}\, 2^{-n(p-5)/2} \varepsilon_1^{-2p} \phi_{-2p\beta'}(a) \ \big(\phi_{p(\beta+\gamma)}(a') + \phi_{p(\beta+\gamma)}(a)\big).
\end{aligned}$$

Using our assumption (163) on β', β, γ, having in mind $a' = a - 2^{-n}$,

$$\sum_{a \in G_n(t)} \phi_{-2p\beta'}(a) \ \Big(\phi_{p(\beta+\gamma)}(a - 2^{-n}) + \phi_{p(\beta+\gamma)}(a)\Big) \ \leq \ c\, 2^n, \tag{182}$$

where $G_n(t)$ denotes the section of G_n with a fixed t from the grid. Hence, since there are $2^n + 1$ different t in G_n,

$$\begin{aligned}
\sup_{x \in I_1} P_x(A_{\varepsilon_1}) \ &\leq \ \sum_{n \geq 1} c \sqrt{L}\, T^{1/4}\, 2^{-n(p-5)/2} \varepsilon_1^{-2p} \,(2^n + 1)\, c\, 2^n \\
&\leq \ c_{(183)} \sqrt{L}\, T^{1/4}\, \varepsilon_1^{-2p},
\end{aligned} \tag{183}$$

since we assumed $p > 9$. Similarly to Tribe [34, p.295], for all $0 \leq t \leq T$ and $a \in \mathsf{R}$ we then obtain

$$\big|N_t^1(a)\big| \ \leq \ c_{(184)}\, \varepsilon_1 \,(1 + T)\, \phi_{\beta'}(a) \quad \text{on} \quad \Big\{ T \leq \tau_L^{1,4p,4p\beta} \wedge \tau_L^{2,4p,4p\gamma} \Big\} \cup A_{\varepsilon_1}^{\mathrm{c}} \tag{184}$$

for some constant $c_{(184)}$ only depending on p and β'. Recalling (24) and (38),

$$\frac{1}{2}\, S_t x^1 \ \geq \ \frac{1}{2}\, c_1'\, S_t \phi_{\lambda_1'} \ \geq \ c\, \phi_{\lambda_1'} \ \geq \ c_{(185)}\, \phi_{\beta'}\,. \tag{185}$$

Make $\varepsilon_1 > 0$ now so small that

$$c_{(185)} \ \geq \ c_{(184)}\, \varepsilon_1 \,(1 + T) \tag{186}$$

implying

$$\frac{1}{2}\,S_t x^1 \;\geq\; c_{(184)}\,\varepsilon_1\,(1+T)\,\phi_{\beta'}\,. \tag{187}$$

Then, by (184),

$$\begin{aligned}
&P_x\Big(\big|N^1_t(a)\big| \;\leq\; \frac{1}{2}\,S_t x^1(a),\;\; 0\leq t\leq T,\;\; a\in\mathsf{R}\Big)\\
&\geq\; P_x\Big(\big|N^1_t(a)\big| \;\leq\; c_{(184)}\,\varepsilon_1\,(1+T)\,\phi_{\beta'}(a),\;\; 0\leq t\leq T,\;\; a\in\mathsf{R}\Big) \\
&\geq\; 1 - P_x\Big(T>\tau_L^{1,4p,4p\beta}\Big) - P_x\Big(T>\tau_L^{2,4p,4p\gamma}\Big) - P_x\big(A_{\varepsilon_1}\big)\,.
\end{aligned} \tag{188}$$

By the definition (164) of $\tau_L^{1,4p,4p\beta}$,

$$P_x\Big(T>\tau_L^{1,4p,4p\beta}\Big) \;\leq\; L^{-1} \sup_{x\,\in\,\mathcal{C}^+_{\mathrm{tem}},\;\; |x|_{\lambda_1}\,\leq\,c_1} P_x \sup_{0\leq t\leq 1} \big\|(X^1_t)^{4p}\big\|_{-4p\beta} \tag{189}$$

since $T\leq 1$. Hence, because we assumed $\lambda_1>\beta$, by the uniform moment bounds in Corollary 15,

$$\sup_{x\in I_1} P_x\Big(T>\tau_L^{1,4p,4p\beta}\Big) \;\leq\; c\,L^{-1} \;\leq\; \frac{\varepsilon}{6}\,, \tag{190}$$

where for the latter estimate we made finally L sufficiently large (recall that we fixed ε in the beginning of the proof). Similarly, we may assume that also

$$\sup_{x\in I_1} P_x\Big(T>\tau_L^{2,4p,4p\gamma}\Big) \;\leq\; \frac{\varepsilon}{6}\,. \tag{191}$$

Now we further redefine our $T\in(0,1]$ by making it additionally so small that the right hand side in (183) gets smaller than $\varepsilon/6$. Then from the chain (188) of inequalities, from (190), (191), and (183) we obtain

$$\inf_{x\in I_1} P_x\Big(\,\big|N^1_t(a)\big| \;\leq\; \frac{1}{2}\,S_t x^1(a),\;\; 0\leq t\leq T,\;\; a\in\mathsf{R}\Big) \;\geq\; 1-\varepsilon/2 \tag{192}$$

[as announced in (179)]. Then by (178),

$$\inf_{x\in I_1} P_x\Big(X^1_t(a) \;\geq\; \frac{1}{2}\,S_t x^1(a),\;\; 0\leq t\leq T,\;\; a\in\mathsf{R}\Big) \;\geq\; 1-\varepsilon/2 \tag{193}$$

[as announced in (180)].

As X^1 is now seen to be not too small with a high probability, and since it is the catalyst for X^0, it will kill X^0 by time T with a large probability. This idea we want to make precise by comparing $t\mapsto\|X^0_t\|_\alpha\,,$ after an appropriate random time change, with a supercritical Feller's branching diffusion [see (216) below].

Let

$$\kappa \;:=\; \inf\Big\{t\geq 0:\; X^1_t(a) < \frac{1}{2}\,S_t x^1(a) \;\text{ for some }\; a\in\mathsf{R}\Big\}\,. \tag{194}$$

Then from (193) we already know that

$$\inf_{x\in I_1} P_x(\kappa\geq T) \;\geq\; 1-\varepsilon/2. \tag{195}$$

By Corollary 16 [recall our notation (12)], for $t\geq 0$,

$$\|X^0_t\|_\alpha \;=\; \|x^0\|_\alpha + \int_0^t \mathrm{d}s\,\Big\langle X^0_s, \frac{\sigma^2}{2}\Delta\tilde{\phi}_{-\alpha}\Big\rangle + M^0_t(\tilde{\phi}_{-\alpha}) \tag{196}$$

with the stochastic integral

$$M_t^0(\tilde{\phi}_{-\alpha}) \ := \ \int_{[0,t]\times\mathsf{R}} M_0^0\bigl(\mathrm{d}(s,b)\bigr)\,\tilde{\phi}_{-\alpha}(b) \tag{197}$$

satisfying

$$\mathrm{d}\bigl\langle\!\bigl\langle M^0_\cdot(\tilde{\phi}_{-\alpha})\bigr\rangle\!\bigr\rangle_t \ = \ \gamma^0\,\Bigl\langle X_t^0X_t^1,(\tilde{\phi}_{-\alpha})^2\Bigr\rangle\,\mathrm{d}t. \tag{198}$$

But for $t\le\kappa$ we have by the definition (194) of κ,

$$X_t^1 \ \ge \ \frac{1}{2}\,S_tx^1 \ \ge \ c\,\phi_{\lambda_1'} \tag{199}$$

[as in (185)]. Moreover,

$$\phi_{\lambda_1'}\,\tilde{\phi}_{-\alpha} \ \ge \ c \tag{200}$$

since $\alpha>\lambda_1'$ by assumption. Hence,

$$\mathrm{d}\bigl\langle\!\bigl\langle M^0_\cdot(\tilde{\phi}_{-\alpha})\bigr\rangle\!\bigr\rangle_t \ \ge \ c_{(201)}\,\|X_t^0\|_\alpha\,\mathrm{d}t \quad \text{on } [0,\kappa], \tag{201}$$

uniformly for $x\in I_1$.

Note that $\|X_t^0\|_\alpha=0$ if and only if $\|X_t^0\|=0$, and recall that the state 0 is absorbing for the continuous martingale $t\mapsto\|X_t^0\|$. Thus, for our further proof we may assume that $\|x^0\|_\alpha>0$.

Set

$$A_t \ := \ \int_0^t \mathrm{d}\bigl\langle\!\bigl\langle M^0_\cdot(\tilde{\phi}_{-\alpha})\bigr\rangle\!\bigr\rangle_s\,\frac{1}{\|X_t^0\|_\alpha} \ \ (\le\infty), \qquad t\in[0,\infty]. \tag{202}$$

We introduce the *new time scale*

$$t \ \mapsto \ \vartheta_t \ := \ \inf\bigl\{r\ge0:\ A_r>t\bigr\} \tag{203}$$

(on which A grows linearly), and the process

$$U_t \ := \ \bigl\|X^0_{\vartheta_t}\bigr\|_\alpha, \qquad t<A_\infty. \tag{204}$$

This U we want to bound by a supercritical Feller's branching diffusion.

By (196), we have

$$U_t \ = \ \|x^0\|_\alpha+\int_0^{\vartheta_t}\mathrm{d}s\,\Bigl\langle X_s^0,\frac{\sigma^2}{2}\Delta\tilde{\phi}_{-\alpha}\Bigr\rangle+M_t, \qquad t<A_\kappa, \tag{205}$$

where

$$t \ \mapsto \ M_t \ := \ M^0_{\vartheta_t}(\tilde{\phi}_{-\alpha}), \qquad t<A_\kappa, \tag{206}$$

is a continuous local martingale such that

$$\mathrm{d}\langle\!\langle M\rangle\!\rangle_t \ = \ U_t\,\mathrm{d}t. \tag{207}$$

In fact, from (202),

$$\mathrm{d}\bigl\langle\!\bigl\langle M^0_\cdot(\tilde{\phi}_{-\alpha})\bigr\rangle\!\bigr\rangle_t \ = \ \|X_t^0\|_\alpha\,\mathrm{d}A_t, \tag{208}$$

hence, by (206), and a change of variables (see, e.g., [28, Proposition (0.4.9)]),

$$\langle\!\langle M\rangle\!\rangle_t \ = \ \int_0^{\vartheta_t}\mathrm{d}A_s\,\|X_s^0\|_\alpha \ = \ \int_0^t\mathrm{d}s\,\bigl\|X^0_{\vartheta_s}\bigr\|_\alpha \ = \ \int_0^t\mathrm{d}s\,U_s. \tag{209}$$

By (205) – (207), and the martingale representation theorem (see, for instance, Ikeda and Watanabe [18, Theorem 2.7.1]), passing to an enlarged probability space $(\Omega', \mathcal{F}', \mathcal{P})$, there is a (standard) Brownian motion B in R such that

$$U_t \;=\; \|x^0\|_\alpha + \int_0^{\vartheta_t} \mathrm{d}s \left\langle X_s^0, \frac{\sigma^2}{2}\Delta\tilde{\phi}_{-\alpha}\right\rangle + \int_0^t \mathrm{d}B_s \sqrt{U_s}\,, \qquad t < A_\kappa\,. \tag{210}$$

Again by a change of variables,

$$\int_0^{\vartheta_t} \mathrm{d}s \left\langle X_s^0, \frac{\sigma^2}{2}\Delta\tilde{\phi}_{-\alpha}\right\rangle \;=\; \int_0^t \mathrm{d}\vartheta_s \left\langle X_{\vartheta_s}^0, \frac{\sigma^2}{2}\Delta\tilde{\phi}_{-\alpha}\right\rangle. \tag{211}$$

Recall from (66) that

$$\frac{\sigma^2}{2}\Delta\tilde{\phi}_{-\alpha} \;\le\; c\,\tilde{\phi}_{-\alpha}\,, \tag{212}$$

and from (201) and (202) that

$$\mathrm{d}A_t \;\ge\; c_{(201)}\,\mathrm{d}t \quad \text{on } [0,\kappa] \tag{213}$$

implying

$$\mathrm{d}\vartheta_s \;\le\; c\,\mathrm{d}s \quad \text{on } [0, A_\kappa]. \tag{214}$$

Inserting (211), (212), and (214) into (210), we get

$$U_t \;\le\; \|x^0\|_\alpha + c_{(215)} \int_0^t \mathrm{d}s\, U_s + \int_0^t \mathrm{d}B_s \sqrt{U_s}\,, \qquad 0 \le t \le A_\kappa\,, \tag{215}$$

with $c_{(215)}$ uniform in $x \in I_1$. Thus, by comparison (see Roger and Williams [29, V.43.1]),

$$U \;\le\; \widehat{U} \quad \text{on } [0, A_\kappa], \tag{216}$$

where $\widehat{U}$ is the pathwise unique solution to

$$\widehat{U}_t \;=\; \|x^0\|_\alpha + c_{(215)} \int_0^t \mathrm{d}s\, \widehat{U}_s + \int_0^t \mathrm{d}B_s \sqrt{\widehat{U}_s}\,, \qquad t \ge 0. \tag{217}$$

In other words, $\widehat{U}$ is a certain *supercritical Feller's branching diffusion.*

Now

$$P_x\big(X_t^0 = 0,\ t \ge T\big) \;\ge\; P_x\big(\|X_T^0\|_\alpha = 0,\ \kappa \ge T\big). \tag{218}$$

But $T = \vartheta_{A_T}$ by the definitions (202) and (203). Hence, by definition (204) of U, we may continue inequality (218) with

$$\ge\; P_x\big(U_{A_T} = 0,\ \kappa \ge T\big) \;\ge\; \mathcal{P}\left(\widehat{U}_{A_T} = 0 \;\middle|\; \widehat{U}_0 = \|x^0\|_\alpha\right) - P_x(\kappa < T), \tag{219}$$

where we also used (216). But $A_T \ge c_{(213)}\, T$ by (213), and by assumption (23),

$$\|x^0\|_\alpha \;\le\; c_0\, \|\phi_{\lambda_0}\|_\alpha \;=\; c_0\, c_{(220)} \tag{220}$$

[recall (163)]. Thus, by the branching property of Feller's branching diffusion and the estimate (195), we may continue (219) with

$$\ge\; \left(\mathcal{P}\left(\widehat{U}_{c_{(213)}\,T} = 0 \;\middle|\; \widehat{U}_0 = 1\right)\right)^{c_0\, c_{(220)}} - \;\varepsilon/2. \tag{221}$$

But the latter probability expression is positive, thus the right hand side in (221) can be made greater than or equal to $1 - \varepsilon$ by choosing $c_0 > 0$ sufficiently small. This completes the proof of Theorem 5 (b). ∎

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Contents

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