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Domain separation by means of sign changing eigenfunctions of p-Laplacians, II

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Abstract. We are interested in algorithms for constructing surfaces Γ of possibly small measure that separate a given domain Ω into two regions of equal measure. Using the integral formula for the total gradient variation, we show that such separators can be constructed approximatively by means of sign changing eigenfunctions of the p-Laplacians, $p \to 1$, under homogeneous Neumann boundary conditions. These eigenfunctions are proven to be limits of a steepest descent methods applied to suitable norm quotients. Finally we use these ideas for the construction of separators on simplex grids.

1 Introduction

Let $\Omega \subset I\!R^n$, $n \geq 1$, be an open, bounded, connected Lipschitzian domain. We denote by C_0^1, L^p, $H^{1,p}$ and $(H^{1,p})' = H^{-1,p'}$, $1 \leq p \leq 2$, $p' = \frac{p}{p-1}$, the usual spaces of functions defined on Ω (comp [13]); $(\cdot,\cdot)$ means the pairing between spaces and their duals, $||\cdot||_p$ is the norm in L^p. Further BV denotes the space of functions with bounded variation on Ω [10] and

$$\int_\Omega |Du| = \sup_g \left(\int_\Omega u \nabla \cdot g \; dx \right), \; g \in C_0^1(\Omega, I\!R^n), \; |g(x)| \leq 1, \; x \in \Omega.$$

(Note that $\int_\Omega |Du| = \|\nabla u\|_1$, provided $u \in H^{1,1}$). Let finally

$$V_p = \left\{ \begin{array}{ll} \{u \in H^{1,p}, \;\; \int_\Omega \; |u|^{p-2} u \; dx = 0\}, & \text{if } p > 1, \\ \{u \in BV, \;\; \int_\Omega \; \text{sign u dx} = 0\} \;\;\; , & \text{if } p = 1. \end{array} \right\}$$

There is a practical interest [11], [12] in algorithms for constructing surfaces Γ of possibly small measure $|\Gamma|$ which separate Ω into two regions of equal measure, i. e. , in solving the minimum problem

$$\varphi_1(E) = 2 \, \frac{P_\Omega(E)}{|E|} \to \min, \;\; E \subset \Omega, \;\; |E| = \frac{|\Omega|}{2}, \tag{1}$$

where $P_\Omega(E) = |\Gamma|$ is the perimeter of E relative to Ω and $|E|$ is the measure of E. This paper aims to solve the geometrical problem (1) by analytical tools. Roughly speaking, we look for approximative solutions of the form $E = \{x \in \Omega, \; u(x) > 0\}$, where u minimizes

$$F_1(u) = \frac{\int_\Omega |Du|}{||u||_1} \to \min, \;\; u \in V_1. \tag{2}$$

The key idea for this approach is Federer's observation (comp. [5]), that the infimum of the functional

$$\varphi(E) = \frac{P_\Omega(E)}{\min(|E|^{\frac{1}{p^*}}, |\Omega \setminus E|^{\frac{1}{p^*}})} \to \min, \quad E \subset \Omega, \quad p^* = \frac{n}{n-1}, \tag{3}$$

coincides with that of

$$\phi(u) = \frac{\int_\Omega |Du|}{||u - t_0(u)||_{p^*}} \to \min, \quad u \in BV, \tag{4}$$

where the functional t_0 is defined by

$$t_0(u) = \sup\{t : |E_t| \geq |\Omega \setminus E_t|\}, E_t = \{x \in \Omega, u(x) > t\}. \tag{5}$$

To specify the connection between (3) and (4) we quote some basic facts from [5], [6]:

(i) Let u be locally integrable on Ω. Then

$$\int_\Omega |Du| = \int_{-\infty}^{\infty} P_\Omega(E_t) dt. \tag{6}$$

(ii) Let $\Omega \subset I\!R^n$ be an open, bounded and connected Lipschitzian domain. Then Ω satisfies a relative isoperimetric inequality, i. e., there exists a constant $Q = Q(\Omega)$, such that

$$\min(|E|^{\frac{1}{p^*}}, |\Omega - E|^{\frac{1}{p^*}}) \leq Q P_\Omega(E). \tag{7}$$

(iii) Let Ω, Q be as in (ii) and let u be as in (i). Then

$$||u - t_0(u)||_{p^*} \leq Q \int_\Omega |Du|. \tag{8}$$

A special case of (i) is

$$\int_\Omega |D\chi_E| = P_\Omega(E), \tag{9}$$

where χ is the characteristic function. Hence the map $E \to \chi_E - \chi_{\Omega\setminus E}$ directly connects (1) and (2). The inverse direction may be indicated by the map $u \to E_u$ with

$$E_u = \{x \in \Omega, \; u(x) > 0\}.$$

The functional F_1 still is unpleasant from the algorithmical point of view. Therefore we shall approximate F_1 by (apart from zero) differentiable functionals

$$F_p(u) = \frac{||\nabla u||_p^p}{||u||_p^p}, \quad 0 \neq u \in V_p, p \in (1, 2]. \tag{10}$$

The next section clarifies the relation between φ, φ_1 and F_1. In Section 3 we prove convergence of minimizers of F_p, $p \to 1$, to minimizers of F_1. Section 4 is devoted the convergence proof of a steepest descent method for F_p. Here each iteration $u_{p,i}$ has to be calculated as (unique) solution of a nonlinear elliptic boundary value problem under homogeneous Neumann conditions. It is shown that $F_p(u_{p,i})$ for $i \to \infty$ tends monotonously decreasing to $F_p(u_p)$, where u_p is a sign changing eigenfunction of the p-Laplacien. Finally we consider a numerical example related to graph partitioning.

2 Relations between φ_1 and F_1

In this Section we want to justify the transition from (1) to (2). We start with an adaption of inequality (8), being more convenient for our purposes.

Lemma 1 *Let Q be the relative isoperimetric constant from (7). Then*

$$
\begin{aligned}
||u||_1 &\leq \left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q \int_\Omega |Du| \ , \quad u \in V_1, \\
||u||_p &\leq 2^{\frac{p-1}{p}} \left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q||\nabla u||_p \ , \quad u \in H^{1,p} \cap V_1, \quad p \in [1, p^* = \frac{n}{n-1}].
\end{aligned}
\tag{11}
$$

PROOF:
Following the proof of (8) in [6], let

$$
A_t = \left\{ \begin{array}{rl} E_t, & \text{if } t > 0, \\ \{\Omega \setminus E_t\}, & \text{if } t \leq 0. \end{array} \right\}
$$

Since $u \in V_1$, we have $|A_t| \leq |\{\Omega \setminus A_t\}|$ for all t and for all $x \in \Omega$ ([10])

$$
u(x) = \int_{-\infty}^{\infty} \text{sign t } \chi_{\text{A}_\text{t}}(\text{x}) \text{ dt}
$$

and hence (comp. [6])

$$
\|u\|_p \leq \int_{-\infty}^{\infty} \|\chi_{A_t}\|_p \ dt.
$$

Now, Hölder's inequality, (6) and (7) yield

$$
\begin{aligned}
\|u\|_p &\leq \int_{-\infty}^{\infty} \|\chi_{A_t}\|_p \ dx \leq \left(\frac{|\Omega|}{2}\right)^{\frac{1}{p}-\frac{1}{p^*}} \int_{-\infty}^{\infty} \|\chi_{A_t}\|_{p^*} \ dt \\
&\leq \left(\frac{|\Omega|}{2}\right)^{\frac{1}{p}-\frac{1}{p^*}} Q \int_{-\infty}^{\infty} P_\Omega(A_t) \ dt = \left(\frac{|\Omega|}{2}\right)^{\frac{1}{p}-\frac{1}{p^*}} Q \int_\Omega |Du| \\
&= \left(\frac{|\Omega|}{2}\right)^{\frac{1}{p}-\frac{1}{p^*}} Q\|\nabla u\|_1 \leq 2^{\frac{p-1}{p}} \left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q\|\nabla u\|_p.
\end{aligned}
$$

□

Remark 1 *The inequality (11) specifies the constant in Poincaré's inequality. For $p = 1$, (11) is sharp. Indeed, suppose equality is attained in (7) for a set E with $|E| = \frac{|\Omega|}{2}$, as for example in the case of convex domains $\Omega \subset I\!R^2$ (comp. [2]). Then $u = \chi_E - \chi_{\Omega \setminus E} \in V_1$ and*

$$
\begin{aligned}
||u||_1 &= |\Omega| = 2\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} \left(\frac{|\Omega|}{2}\right)^{\frac{1}{p^*}} = 2\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} QP_\Omega(E) \\
&= 2^0 \left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q \int_\Omega |Du|.
\end{aligned}
$$

For convex domains Ω another specification is well known [9]

$$||u||_p \le \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1-n}{n}} d^n ||\nabla u||_p, \quad u \in H^{1,p}, \quad \int_\Omega u \; dx = 0,$$

where ω_n is the volume of the unit sphere in $I\!R^n$ and d is the diameter of Ω.

The minimum problems (3) and (2) are equivalent in the following sense:

Proposition 1 *A set $E_1 \subset \Omega$ with $|E_1| = \frac{|\Omega|}{2}$ is minimizer of φ if and only if $u_1 = \chi_{E_1} - \chi_{\Omega \setminus E_1} \in V_1$ is minimizer of F_1.*

PROOF:
($\rightarrow$) By (11) we find for arbitrary $u \in V_1$

$$F_1(u) \ge \frac{\left(\frac{|\Omega|}{2}\right)^{-\frac{1}{n}}}{Q} = \left(\frac{|\Omega|}{2}\right)^{-\frac{1}{n}} \frac{2P_\Omega(E_1)}{|E_1|^{\frac{1}{p^*}}} = \frac{2P_\Omega(E_1)}{|\Omega|} = F_1(u_1).$$

($\leftarrow$) Let $E \subset \Omega$ be any set with $P_\Omega(E) < \infty$. Then (11) and (9) imply

$$\varphi(E) \ge \frac{1}{Q} = \left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} F_1(u_1) = \left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} \frac{2P_\Omega(E_1)}{|\Omega|} = \frac{P_\Omega(E_1)}{|E_1|^{\frac{1}{p^*}}} = \varphi(E_1).$$

$\square$

Remark 2 *Evidently, each minimizer E of φ with $|E| = \frac{|\Omega|}{2}$ is solution of the minimum problem (1). For convex domains $\Omega \subset I\!R^2$ the existence of such minimizers is proved in [2].*

On the basis of the next result we shall replace (1) by (2).

Theorem 1 *(i) Let $u_1 \in V_1$ be minimizer of F_1 and $E_1 = \{x \in \Omega,\; u_1(x) > 0\}$. Then*

$$\varphi_1(E_1) \le \varphi_1(E) \quad \textit{for all } E \subset \Omega \textit{ with } |E| = \frac{|\Omega|}{2}. \tag{12}$$

(ii) Let in addition $|\{x \in \Omega,\; u_1(x) = 0\}| = 0$. Then u_1 is solution of (1).

PROOF:
(i) By (11) and (9) we get

$$\varphi_1(E) = \frac{2P_\Omega(E)}{|\Omega|} = \left(\frac{|\Omega|}{2}\right)^{-\frac{1}{n}} \frac{P_\Omega(E)}{|E|^{\frac{1}{p^*}}} \ge \frac{\left(\frac{|\Omega|}{2}\right)^{-\frac{1}{n}}}{Q} = F_1(u_1). \tag{13}$$

Let for $\varepsilon > 0$

$$w_\varepsilon(x) = \tanh\left(\frac{u_1(x)}{\varepsilon}\right).$$

Since u_1 is minimizer of F_1 and $w_\varepsilon \in BV$, we have

$$\frac{1}{||u_1||_1}(||w_\varepsilon||_{BV} - F_1(u_1)||w_\varepsilon||_1) = \frac{d}{dt}F_1(u_1 + tw_\varepsilon)|_{t=0} = 0.$$

Passing $\varepsilon \to 0$, the lower semicontinuity of the BV-norm [10] and Lebesgue's dominated convergence theorem imply

$$\varphi_1(E_1) = \frac{2P_\Omega(E_1)}{|\Omega|} \leq F_1(u_1). \tag{14}$$

Putting this together with (13), we get (12).
(ii) $u_1 \in V_1$ along with $|\{x \in \Omega,\ u_1(x) = 0\}| = 0$ imply $|E_1| = \frac{|\Omega|}{2}$. Thus (ii) is a consequence of (i).

□

3 The functionals F_p and the limit $p \to 1$

In this Section we will justify the transition from the minimum problem (2) to the regularized minimum problems

$$F_p(u) = \frac{||\nabla u||_p^p}{||u||_p^p} \to min, \quad 0 \neq u \in V_p, \quad p \in (1, 2]. \tag{15}$$

Remark 3 *Since F_p is homogeneous, (15) is equivalent with*

$$G_p(u) = ||\nabla u||_p \to min, \quad u \in V_p, \quad \|u\|_p = 1, \quad p \in (1, 2].$$

Proposition 2 *Let*

$$d = \inf_{u \in V_p} F_p(u).$$

Then there exists a (minimizer) $u \in V_p$ such that $F_p(u) = d$.

PROOF:
Let $(v_i) \subset V_p$ be a minimal sequence, i.e., $v_i \neq 0, \quad F_p(v_i) \to d$. In view of Remark 3 we set $u_i = v_i/\|v_i\|_p$. Because of the reflexity of $H^{1,p}$, its compact imbedding into L^p and the continuity of the operator $u \to |u|^{p-2}u \in (L^p \to L^{\frac{p}{p-1}})$, there are a subsequence $(u_j) \subset (u_i)$ and a $u \in V_p$ such that

$$u_j \to u \ \ in \ \ L^p, \quad \|u_j\|_p = 1, \quad u_j \rightharpoonup u \ \ in \ \ H^{1,p}, \quad F_p(u_j) \to d.$$

Since $v \to \|\nabla v\|_p$ is weakly lower semicontinuous, this implies $F_p(u) = d$. □

Minimizers of $u \in H^{1,p}$ satisfy necessarily the Euler Lagrange equations, i. e., the nonlinear eigenvalue problem (comp.[4])

$$A_p u = F_p(u) B_p u, \tag{16}$$

where the operators $A_p, B_p \in (H^{1,p} \to (H^{-1,p'})$ are defined by

$$\begin{aligned}(A_p u, h) &= (|\nabla u|^{p-2} \nabla u, \nabla h), \;\; \forall h \in H^{1,p}, \\ (B_p u, h) &= (b_p(u), h), \; b_p(u) = |u|^{p-2} u.\end{aligned}$$

(17)

Remark 4 *(16), (17) can be seen as weak formulation (comp. [8]) of the nonlinear eigenvalue problem*

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = F_p(u) |u|^{p-2} u \text{ in } \Omega, \quad \nu \cdot \nabla u = 0 \text{ on } \partial\Omega,$$

where ν is the outer unit normal on $\partial\Omega$.

The minimum problem (15) approximates (2) in the following sense:

Theorem 2 *Let $u_p \in H^{1,p}$ $1 < p \le 2$, be minimizer for F_p in (15), such that*

$$||u_p||_p = 1. \tag{18}$$

Then:
(i) $u_p \in V_p$;
(ii) A sequence $p_i \to 1$ and a function $u \in BV$ exist such that

$$u_i := u_{p_i} \to u \;\; in \; L^q, \;\; q \in (1, p^*), \quad F_{p_i}(u_i) \to \lambda \ge F_1(u);$$

(iii) u is minimizer of F_1;
(iv)

$$B_i u_i \rightharpoonup z \; in \; L^q, \;\; z \in Su, \;\; \int_\Omega z \, dx = 0,$$

where S is the maximal monotone operator generated by the (multivalued) function

$$\text{Sign } s = \left\{ \begin{array}{ll} \text{sign s}, & \text{ifif } s \neq 0, \\ [-1, 1], & \text{if } s = 0. \end{array} \right\}$$

PROOF:
(i) Testing (16) with $h = 1$ yields $(B_p u_p, 1) = 0$, i. e., $u_p \in V_p$.
(ii) Let $w \in H^1$ be fixed. Using that u_p is minimizer and (18), we find

$$|\Omega|^{1-p} ||\nabla u_p||_1^p \le ||\nabla u_p||_p^p = F_p(u_p) ||u_p||_p^p \le F_p(w) ||u||_p^p \le c.$$

Since $H^{1,1}$ is compactly imbedded into L^q, $q < p^*$, a sequence $p_i \to 1$ and $u \in BV$ exist such that

$$\begin{aligned} u_i := u_{p_i} &\to u \text{ in } L^q \quad \text{and a. e. in } \Omega, &(19)\\ F_{p_i}(u_i) &\to \lambda. &(20)\end{aligned}$$

Using the lower semicontinuity of the BV-norm, Hölder's and Young's inequalities, we get from (16), setting $p = p_i$, $r = \frac{p*(p-1)}{p*-1}$ temporaryly,

$$\begin{aligned} \int_\Omega |Du| &\le \liminf \int_\Omega |Du_i| = \liminf ||\nabla u_i||_1 \\ &\le \liminf(|\Omega|^{\frac{p-1}{p}}||\nabla u_i||_p) \le \liminf(\frac{p-1}{p}|\Omega| + \frac{1}{p}||\nabla u_i||_p^p) \\ &\le \liminf ||\nabla u_i||_p^p = \liminf(F_p(u_i)||u_i||_p^p) \\ &= \lambda \liminf ||u_i||_p^p \le \lambda \liminf(||u_i||_1^{p-r}||u_i||_{p*}^r) \\ &\le \lambda \liminf((p-r)||u_i||_1 + (1+r-p)||u_i||^r) = \lambda ||u||_1 \end{aligned}$$

and hence

$$F_1(u) \le \lambda. \tag{21}$$

(iii) Let $v \in BV$, $v \ne 0$. We want to show that $F_1(u) \le F_1(v)$. To this end let $(v_j) \subset C^\infty$ be a sequence (comp. [10]) such that

$$v_j \to v \text{ in } L^1, \quad \int_\Omega |Dv_j| \to \int_\Omega |Dv|. \tag{22}$$

We have

$$\begin{aligned} F_1(v) &= F_1(v_j) + F_1(v) - F_1(v_j) = F_p(v_j) + F_1(v_j) - F_p(v_j) + F_1(v) - F_1(v_j) \\ &\ge F_p(u_p) - |F_1(v_j) - F_p(v_j)| - |F_1(v) - F_1(v_j)|. \end{aligned}$$

(23)

By (22) we can choose j such that for given $\varepsilon > 0$

$$|F_1(v) - F_1(v_j)| < \varepsilon.$$

Further we have

$$\begin{aligned} ||\nabla v_j||_p^p &\le ||\nabla v_j||_1 ||\nabla v_j||_\infty^{p-1} \le \frac{1}{p}||\nabla v_j||_1^p + \frac{p-1}{p}||\nabla v_j||_\infty^p \\ &\le ||\nabla v_j||_1(1 + |\frac{1}{p}||\nabla v_j||_1^{p-1} - 1|) + \frac{p-1}{p}||\nabla v_j||_\infty^p \end{aligned}$$

and

$$||v_j||_p^p \le \frac{1}{p}||v_j||_1^p + \frac{p-1}{p}||v_j||_\infty^p \le ||v_j||_1(1 + |\frac{1}{p}||v_j||_1^{p-1} - 1|) + \frac{p-1}{p}||v_j||_\infty^p.$$

Consequently, we can choose $p_i = p_i(j)$ such that

$$|F_1(v_j) - F_{p_i}(v_j)| < \varepsilon.$$

Thus, using (19) and (21), we get from (23)

$$F_1(u) \le F_{p_i}(u_i) \le F_1(v) + 2\varepsilon.$$

Passing to $\varepsilon \to 0$, we see that u is minimizer of F_1.
(iv) Since

$$\|B_i u_i\|_n = \||u_i|^{p-1}\|_n \le \|u_i\|_{p^*}^{p-1} |\Omega|^{\frac{1}{n} - \frac{p-1}{p^*}} \le c,$$

we can assume that

$$B_i u_i \rightharpoonup z \text{ in } L^n.$$

Then

$$(z, 1) = \lim_{i\to\infty} (B_i u_i, 1) = 0,$$

and for any $v \in L^q$

$$\begin{aligned}
(z - Sv, u - v) &= \lim_{i\to\infty} (B_i u_i - Sv, u - v) \ge \lim_{i\to\infty} (B_i u_i - \mathrm{sign\ v, u - v}) \quad (24)\\
&= \lim_{i\to\infty} (B_i u_i - B_i v, u - v) \ge 0. \quad (25)
\end{aligned}$$

Thus the maximal monotonicity of S implies $z = Su$. □

4 Steepest descent method

Due to the Theorems 1,2 the original minimum problem (1) is approximatively reduced to construction of minimizers u_p of the functional F_p for suitable p near 1. In this section we fix $p \in (1, 2]$ and establish a steepest descent method for solving iteratively the corresponding Euler Lagrange equations, i. e., the nonlinear eigenvalue problems (16).

$$B_p u_i + \tau A_p u_i = B_p u_{i-1} + \tau F_p(u_{i-1}) B_p u_i, \quad , i = 1, 2, ..., \ u_0 \ \in V_p, \ , u_0 \ne 0, \quad (26)$$

where τ is a relaxation parameter, which may be interpreted as time step.

Theorem 3 *Let $\tau p' F_p(u_0) < 1$, $p' = \frac{p}{p-1}$. Then:*
(i) for each i (26) has a unique solution $u_i \in V_p$;
(ii) the sequence $(F_p(u_i))$ is decreasing, $F_p(u_i) \to \lambda > 0$;
(iii) the sequence $(||u_i||_p)$ is bounded, moreover

$$\|u_0\|_p^p \le \|u_i\|_p^p \le c := \frac{1}{1 - \tau p' F(u_0)} \|u_0\|_p^p, \ \|B_p u_i - B_p u_{i-1}\|_1 \to 0;$$

(iv) there exist a subsequence $(u_{i_j}) \subset (u_i)$ and a function $u \in V_p$ such that u is nontrivial solution of the nonlinear eigenvalue problem (16) and

$$u_{i_j} \to u \ \text{ in } \ H^{1,p}, \ F_p(u) = \lambda, \ \int_\Omega B_p u \, dx = 0.$$

PROOF:
For simplifying the notation during the proof we drop the index p at A, B, F and V.
(i) The operator $A+B \in (H^{1,p} \to H^{-1,p'})$ is continuous ([8], [13]). The inequalities (comp. [3])

$$0 \le (|y|^{p-1} - |z|^{p-1})(|y|-|z|) \le (|y|^{p-2}y - |z|^{p-2}z, y-z) \le c(p)|y-z|^p, \ y,z \in I\!R^n, \tag{27}$$

imply strict monotonicity and coercitivity of $A+B$. Thus the Browder-Minty theorem ensures existence of a unique solution $u_i \in H^{1,p}$ for given $u_{i-1} \in H^{1,p}$.
(ii) Testing (26) by $h = u_i - u_{i-1}$ and using the inequalities of Hölder and Young, we get

$$\begin{aligned}
(\frac{1}{\tau} - F(u_{i-1}))(Bu_i - Bu_{i-1}, u_i - u_{i-1}) + (Au_i, u_i)& \\
= \ & (Au_i, u_{i-1)}) + F(u_{i-1})(Bu_{i-1}, u_i - u_{i-1}) \\
\le \ & ||\nabla u_i||_p^{p-1}||\nabla u_{i-1}||_p + F(u_{i-1})(||u_{i-1}||_p^{p-1}||u_i||_p - ||u_{i-1}||_p^p) \\
\le \ & \frac{1}{p}[(p-1)||\nabla u_i||_p^p + ||\nabla u_{i-1}||_p^p + F(u_{i-1})(||u_i||_p^p - ||u_{i-1}||_p^p)] \\
= \ & \frac{||u_i||_p^p}{p}((p-1)F(u_i) + F(u_{i-1})),
\end{aligned}$$

and hence

$$(\frac{1}{\tau} - F(u_{i-1}))(Bu_i - Bu_{i-1}, u_i - u_{i-1}) + ||u_i||_p^p(F(u_i) - F(u_{i-1})) \le 0. \tag{28}$$

Since $\tau F(u_0) < 1$ and B is monotone, this means

$$F(u_i) \le F(u_{i-1}), \quad i = 1, 2, .., \quad F(u_i) \to \lambda > 0 \tag{29}$$

and

$$0 \le (Bu_i - Bu_{i-1}, u_i - u_{i-1}) \to 0. \tag{30}$$

(iii) Testing (26) by $h = u_{i-1}$, using Hölder's inequality and (ii) yield

$$\begin{aligned}
\|u_{i-1}\|_p^p &= (1 - \tau F(u_{i-1}))(Bu_i, u_{i-1}) + \tau(Au_i, u_{i-1}) \\
&\le (1 - \tau F(u_{i-1}))||u_i||_p^{p-1}\|u_{i-1}\|_p + \tau||\nabla u_i||_p^{p-1}\|\nabla u_{i-1}\|_p \\
&\le (1 - \tau F(u_{i-1}))||u_i||_p^{p-1}\|u_{i-1}\|_p + \tau F(u_i)^{\frac{p-1}{p}}||u_i||_p^{p-1}F(u_{i-1})^{\frac{1}{p}}\|u_{i-1}\|_p \\
&\le (1 - \tau F(u_{i-1}))||u_i||_p^{p-1}\|u_{i-1}\|_p + \tau F(u_{i-1})^{\frac{p-1}{p}}||u_i||_p^{p-1}F(u_{i-1})^{\frac{1}{p}}\|u_{i-1}|_p \\
&= \|u_i\|_p^{p-1}\|u_{i-1}\|_p
\end{aligned} \tag{31}$$

and hence

$$\|u_{i-1}\|_p \le \|u_i\|_p.$$

Now, in order to show $\|u_i\|_p^p \le c$ we test (26) by $h = u_i$ and apply Young's inequality to get

$$\frac{1}{\tau p'}(||u_i||_p^p - \|u_{i-1}\|_p^p) + (Au_i, u_i) \le \frac{1}{\tau}(||u_i||_p^p - (Bu_{i-1}, u_i)) + (Au_i, u_i) = F(u_{i-1})\|u_i\|_p^p$$

and thus

$$\frac{1}{\tau p'}(||u_i||_p^p - \|u_{i-1}||_p^p) \leq ||u_i||_p^p(F(u_{i-1}) - F(u_i)) \leq \max_i\{||u_i||_p^p\}(F(u_{i-1}) - F(u_i)) \tag{32}$$

Summing up over $i = 1, k$ and using (29) and (32), we get

$$||u_k||_p^p \leq ||u_0||_p^p + \tau p \max_i\{||u_i||_p^p\}(F(u_0) - F(u_k)) \leq ||u_0||_p^p + \tau p' \max_i\{||u_i|||_p^p\}F(u_0))$$

Since this holds for all k and $\tau p' F(u_0) < 1$, we conclude

$$||u_i||_p^p \leq c, \quad \|\nabla u_p\| = F(u_p)\|u_p\|_p^p \leq cF(u_0). \tag{33}$$

This along with (30) and Lemma 1.8 from [1] imply

$$||Bu_i - Bu_{i-1}||_1 \to 0. \tag{34}$$

(iv) Because of (29), (32), (33) and the compactness of the imbedding of $H^{1,p}$ into L^p there exist a subsequence $(u_{i_j}) \subset (u_i)$ and a function $u \in H^{1,p}$, $\|u\|_p \geq \|u_0\|_p$ such that

$$u_{i_j} \rightharpoonup u \ \text{ in } H^{1,p}, \quad u_{i_j} \to u \text{ in } L^p, \quad F(u_{i_j}) \to \lambda. \tag{35}$$

Hence we get

$$(Au_{i_j}, u_{i_j}) = F(u_{i_j})||u_{i_j}||_p^p \to \lambda||u||_p^p = \lambda(Bu, u). \tag{36}$$

Further, using the continuity of B, (34) and (35), we find for arbitrary $h \in H^{1,p} \cap L^\infty$

$$\begin{aligned}
|(Au_{i_j} - \lambda Bu, h)| &= |(F(u_{i_j-1})Bu_{i_j} - \lambda Bu, h) + \frac{1}{\tau}(Bu_{i_j-1} - Bu_{i_j}, h)| \\
&\leq |F(u_{i_j-1}) - \lambda|\|u_{i_j}\|_p^{p-1}\|h\|_p + \lambda|(Bu_{i_j} - Bu, h)| \\
&\quad + \frac{1}{\tau}\|Bu_{i_j} - Bu_{i_j-1}\|_1||h||_\infty \to 0.
\end{aligned}$$

Since Au_{i_j} is bounded in $H^{-1,p'}$ and $H^{1,p} \cap L^\infty$ lies densely in $H^{1,p}$ this means

$$Au_{i_j} \rightharpoonup \lambda Bu \ \text{ in } H^{-1,p'}. \tag{37}$$

In view of (36) and (37) we can apply the usual monotonicity argument ([8], [13]) in order to verify that

$$Au = \lambda Bu.$$

Testing this equation with $h = u$, we find $F(u) = \lambda$ and by (35)

$$||\nabla u_{i_j}||_p \to ||\nabla u||_p.$$

This along with the weak convergence and the uniform convexity of $H^{1,p}$ ensure the strong convergence of u_{i_j} to u in $H^{1,p}$. Finally, the continuity of B and $u_{i_j} \in V$ imply

$$\int_\Omega Bu \ dx = \lim_{j\to\infty} \int_\Omega Bu_{i_j} \ dx \ = \ 0.$$

□

Reinserting the index p and using that (16) is homogeneous we get

Corollary 1 *The nonlinear eigenvalue problem (16) has a solution $u_p \in H^{1,p}$ for $p \in (1,2]$ such that*

$$\|u_p\|_p = 1, \quad \int_\Omega |u_p|^{p-2} u_p \; dx \; = 0.$$

u_p is in $H^{1,p}$ strong limes of the iteration sequence $u_{p,i}$ defined by (26). Moreover,

$$F_p(u_{p,i}) \downarrow_{i\to\infty} F_p(u_p).$$

5 Construction of separators on simplex grids

In this section we want to apply our results to partition discretized domains. For this purpose let us assume that we are given a simplex discretization Ω_h of Ω, as it is commonly used for numerically solving partial differential equations. To give an example, Figure 1 shows a triangulation of a two dimensional section through an electronic device to be simulated.

Let $p_i \in \Omega_h$ be a grid point and let

$$V_i = \{x \in \mathbb{R}^n : \|x - p_i\| < \|x - p_j\|, \; \forall \; p_j \in \Omega_h\}$$

denote the corresponding Voronoi volume with the Voronoi surface ∂V_i. The Voronoi volume element V_{S_i} of the vertex i with respect to the simplex $S \subset \Omega_h$ is the intersection of V_i and S. The discrete gradient of a piecewise linear function u on a simplex S is given by (comp. [7])

$$\nabla u|_S = \mu_S G_S \mathbf{u}, \;\; G_S := \mu_S \tilde{G}_S, \;\; \mathbf{u}^T = (u_{p_i}), \;\; p_i \in S.$$

with suitable matrices $\tilde{G}_S$ and μ_S. In the two dimensional special case of triangles S with vertices $p_i = (x_i, y_i)$ (indices have to be understood as the cyclic extension, if necessary) we have

$$\tilde{G}_S = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

The symmetric positive definite matrix μ_S represents the underlying metric. For a triangle S under Euclidian metric it holds

$$\begin{aligned} \mu_{ii}^2 &= \frac{\sigma_i}{l_i}, \;\; \mu_{ij} = 0 \;\text{ if }\; i \neq j, \\ l_i^2 &= (x_j - x_k)^2 + (y_j - y_k)^2, \;\; \sigma_i = \frac{l_i}{8|S|}(l_j^2 + l_k^2 - l_i^2). \end{aligned}$$

In view of efficient parallelization procedures one is interested to partition Ω_h into two parts Ω_{hi}, $i = 1, 2$, containing equal numbers of Voronoi volumes and minimal numbers of cut edges, for instance. Consequently we replace the usual Euclidian metric by the graph metric assigning the length $l = 1$ to each simplex edge. For a triangle S that implies: $\sigma_i = \frac{1}{2\sqrt{3}}, \;\; \forall i, \;\; |S| = \frac{\sqrt{3}}{4}$.

The discrete L^p-norm of u on a simplex S is defined by

$$||u||_{S,p}^p = \sum_{i \in S} V_{S_i} |u_i|^p,$$

and the discrete L_p-norm of the modulus of the gradient of u on S is defined by

$$||\nabla u||_{S,p}^p = |S| s^p, \quad s^2 = |\nabla u|_S^2 = \mathbf{u}_S^T G_S^T G_S \mathbf{u}_S / |S|.$$

Accordingly, we define discrete norms on Ω_h by:

$$||u||_p^p = \sum_i ||u||_{S_i,p}^p \ , \quad ||\nabla u||_p^p = \sum_i ||\nabla u||_{S_i,p}^p \ ,$$

and herewith the discrete counter part of (10)

$$F_p(u) = \frac{||\nabla u||_p^p}{||u||_p^p}, \quad u \in V_p, \tag{38}$$

$$V_p = \left\{ u : u \neq 0, \quad \mathbf{V}^T \mathrm{diag}(|u_i|^{p-2}) \mathbf{u} = 0, \quad \mathbf{u}^T = (u(p_i)), \quad p_i \in \Omega_h, \quad 1 < p \leq 2 \right\}. \tag{39}$$

Differentiation of the discrete functional yields the discrete Euler Lagrange equations:

$$A_p(|\nabla u|)\mathbf{u} = F_p(u) B_p(|u|)\mathbf{u},$$

and

$$A_p = \sum_S A_S, \quad A_S = G_S^T |\nabla u|_S^{p-2} G_S, \quad B_p(|u|) = \mathrm{diag}(V_i |u_i|^{p-2}).$$

The steepest descent scheme preserves its properties independent of the special choice of the matrix μ_S: Using Hölder inequalities with weights ($\alpha_i > 0$) ($|\sum_i \alpha_i u_i v_i| = |\sum_i (\alpha_i^{1/p'} u_i)(\alpha_i^{1/p} v_i)| \leq (|\sum_i \alpha_i |u_i|^{p'})^{1/p'} (|\sum_i \alpha_i |u_i|^p)^{1/p}$) the proof of Theorem 3 can be repeated. Monotonicity can be shown for $A_p\mathbf{u}$ per simplex, for $B_p\mathbf{u}$ per vertex.

The steepest descent equations for constructing the unique solutions $\mathbf{u}_i \in V_p$ are solved by Newtons method. The modulus is regularized by $|s|_\epsilon^2 = s^2 + \epsilon$, $|s|'_\epsilon = s/|s|_\epsilon$ and the Jacobian matrices (of $A_p\mathbf{u}$, $B_p\mathbf{u}$) related to powers ϵ^0 degenerate proportional to $p-1$ (in gradient direction $A_{p,J}$, $B_{p,J}$ per node).
The initial value is constructed by solving the linear eigenvalue problem for $p = 2$ ($A_2\tilde{\mathbf{u}} = \lambda V \tilde{\mathbf{u}}$). The constraint $\mathbf{u}_0 \in V_p$ is fulfilled by the ansatz $\mathbf{u}_0 := \tilde{\mathbf{u}} + c(p)$ such that the constant $c(p)$ satisfies $\mathbf{1}^T B(|\tilde{\mathbf{u}} + c(p)|)(\tilde{\mathbf{u}} + c(p)) = 0$ ($p = 1.05$ in the example presented).
(Partioning the domain accordingly to the signs of the components of $\mathbf{u}_0$ (with $c(p)$, $p \to 1$) is equivalent to the approach of sorting the vector $\tilde{\mathbf{u}}$, ($\tilde{u}_{p_i} \leq \tilde{u}_{p_{i+1}}$), and assigning the points p_i related to the first half of components to the first subdomain, compare [11]. Hence this algorithm is understood from the presented point of view as using a linear approximation $\tilde{\mathbf{u}}$ and fulfilling the constraint afterwards by sorting and counting.)
Due to rounding errors the transformation of variables $\mathbf{z} := B(|\mathbf{u}|)\mathbf{u}$ is introduced – the transformation back to $\mathbf{u}$ has to be evaluated to compute the functional. Figure 1 shows the eigenvector $\tilde{\mathbf{u}}$ and the stationary solution $\mathbf{z}$ on the domain Ω_h. $\mathbf{z}$ changes its sign along the rather steep jump, the 'separator surface'.

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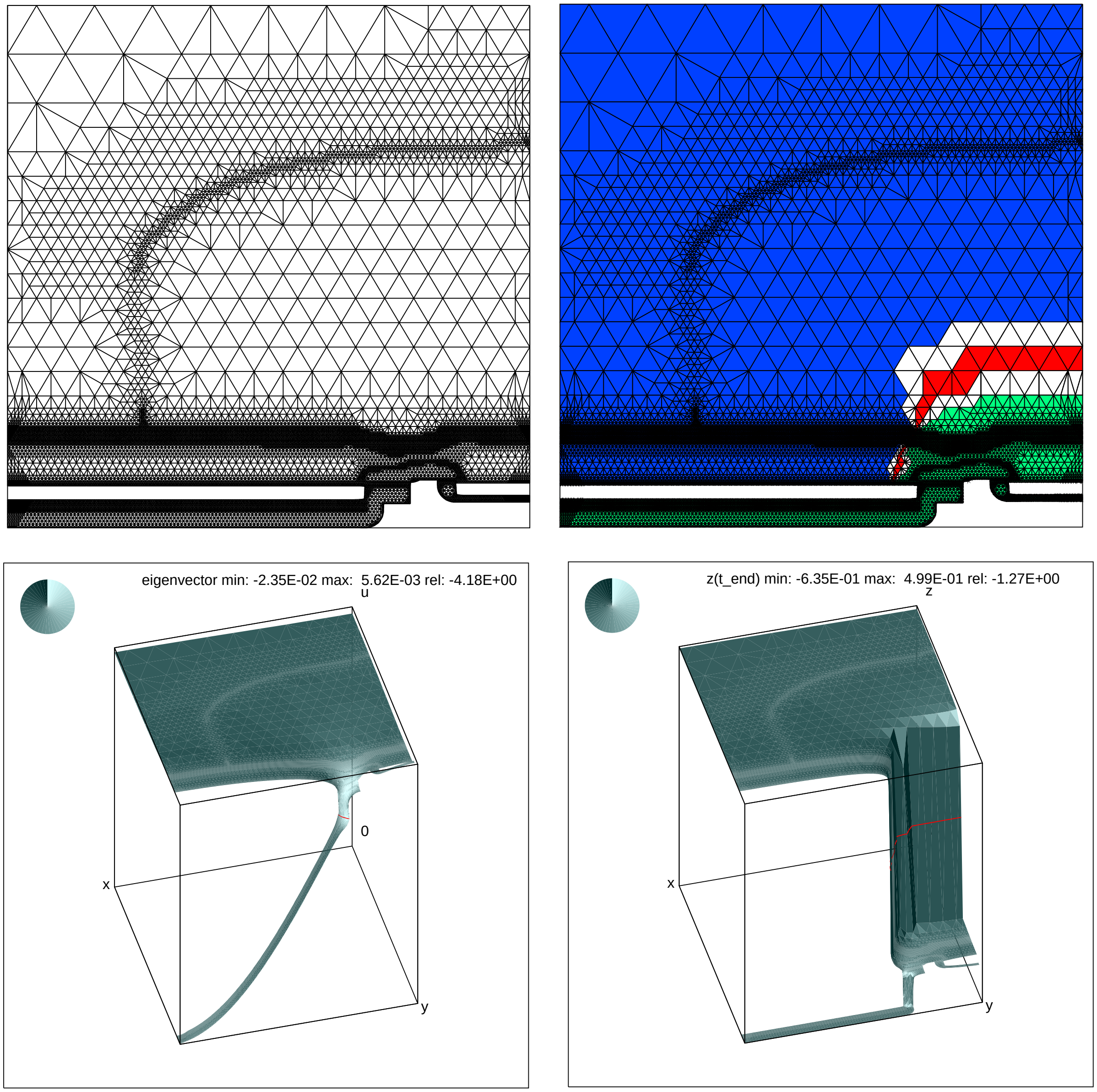

Figure 1: Upper left: the triangulation of Ω_h; upper right: triangle sign pattern related to the solution (triangles with sign change red (grey), neighbours white, negative part green (light-grey), positive part blue (dark)); lower left: eigenvector $\tilde{\mathbf{u}}$, $p = 2$; lower right: the stationary solution $\mathbf{z}$, $p = 1.05$; ($\hat{f}$: linear interpolant of $\mathbf{f}$, the level lines $\hat{\tilde{u}} = 0$, $\hat{z} = 0$ are indicated in red (black respectively white)).