On the regularization of the first kind integral equation with analytical kernel of logarithmic type

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Abstract

We study regularization methods for the integral equation of the first kind with analytical kernel of logarithmic type. The problem is severely ill-posed. In [1] a logarithmic type convergence rate for the Tikhonov regularized solution was proved. Here we are concerned with numerical aspects of the solution. First we consider the selfregularization of the problem by using projection methods in the sense of [9].Then we will see that the Tikhonov regularization of such methods is in accordance with a discretized version of the Tikhonov regularized solution in [1]. Finally, we describe numerical experiments being in a good agreement with the theoretical results.

1 Introduction

Many inverse problems from applications, such as tomography [8], geophysics [7], non-destructive detection [6], inverse contact problems [3], give rise to integral equations of the first kind with analytic kernels. In [2], [4], for certain integral equation of the first kind with analytic kernel, a conditional stability could be proved, provided some a-priori information about the solution is known. Since first kind integral equations with analytic kernels are severely ill-posed problems, their numerical solution is extremely difficult.

Let us consider the integral equation with logarithmic kernel

$$\int_0^1 \log(x-t) f(t) dt = g(x), \quad x \in [2,3]. \tag{1.1}$$

Since $[0,1] \cap [2,3] = \emptyset$, the kernel is analytic with respect to x, t. The integral equation (1.1) is severely ill–posed in Hadamard's sense. The purpose of this paper is to study fully discretized regularization methods for this problem.

In Section 2 we are engaged with projection methods in the sense of [9]. We describe the methods of least squares, dual least squares and collocation and investigate their properties. In each case the discretized problem is equivalent to a system of linear equations. As the matrix is ill-conditioned in the general case, for its numerical solution a combination with an additional regularization procedure is necessary.

The Section 3 is devoted to a discretized version of the Tikhonov regularized solution in the sense of [1]. This discretized version can be considered as the least squares method combined with additional Tikhonov regularization. In this section the results of [1] are used. The numerical experiments concern the two kinds of regularity assumptions: First, the solution is supposed to be H_0^1 on $[0,1]$ and second, the solution is supposed to be H^1 in a neighborhood of one point. In the first case the L^2-convergence of the approximating sequence to the solution is investigated, while in the second case the pointwise convergence is studied locally. Moreover, near a discontinuity point of the solution the sequence of approximated regularized solutions is growing unboundedly in the L^2-sense. We perform three experiments with synthetic data, confirming the theoretical results of [1].

2 Selfregularization by projection methods.

In this paper the problem

$$Af = g, \tag{2.1}$$

is considered, where the operator $A : L^2(0,1) \to L^2(2,3)$ is defined as

$$(Af)(x) = \int_0^1 log(x-t)f(t)dt, \quad x \in [2,3]. \tag{2.2}$$

Let $A^* : L^2(2,3) \to L^2(0,1)$ be the adjoint operator,

$$(A^*g)(t) = \int_2^3 log(x-t)g(x)dx, \quad t \in [0,1].$$

Projection methods.

First of all we are concerned with the definition and properties of abstract projection methods studied in [9].

Let X and Y be Hilbert spaces and A a uniquely invertible operator mapping X into Y with $\overline{R(A)} = Y$. Let Y' be the space of linear continuous functionals on Y (Y' can be identified with Y), let further $\|\cdot\|$ and $(\cdot,\cdot)$ be norm and scalar product in X and Y.

Consider finite dimensional subspaces $X_n \subset X$ (trial spaces) and $Y_n' \subset Y'$ (test spaces) and define the discretized problem:

Find $f_n \in X_n$ such that

$$\psi(Af_n) = \psi(g) \tag{2.3}$$

holds for all $\psi \in Y_n'$.

We assume that (2.3) is uniquely solvable for any $g \in Y$ and consider its solution as the approximate solution of the operator equation (2.1). In the case where we have instead of the exact right-hand side g only uncertain data g_ϵ, with $\|g - g_\epsilon\| < \epsilon$, at disposal we denote the solution of (2.3) by $f_{n,\epsilon}$.

Now, on the lines of [9] let us define the linear operators $P_n : X \to X_n$ and $Q_n : Y \to X_n$. Let f_n be the solution of (2.3) where $g = Af$ holds. Then

$$P_n f := f_n.$$

As $P_n u = u$ for $u \in X_n$ the operator P_n is a projector from X onto X_n. Let f_n be the solution of (2.3) where $g \in Y$ holds. Then

$$Q_n g := f_n.$$

It is clear that generally

$$P_n = Q_n A \tag{2.4}$$

holds.

Proposition 2.1 *If $P_n f \to f$ ($n \to \infty$) for each $f \in X$, then $\|P_n\| \le c$.*

If $\|P_n\| \le c$,

$$dist(f, X_n) := \inf_{u \in X_n} \|f - u\| \to 0,$$

then $P_n f \to f$.

Proof. The first assertion follows by the theorem of Banach-Steinhaus, the second one by

$$\|f - f_n\| = \|(I - P_n)(f - u)\| \le (1 + \|P_n\|) dist(f, X_n). \tag{2.5}$$

■

In the case of unexact data, using (2.5) we have the estimate

$$\|f - f_{n,\epsilon}\| \le (1 + \|P_n\|) dist(f, X_n) + \|Q_n\|\epsilon. \tag{2.6}$$

In the ill-posed case $\|Q_n\|$ will grow for growing n. To get a reasonable numerical procedure (2.3) we must

(i) be sure that $\|P_n\|$ is bounded,

(ii) estimate $\|Q_n\|$ from above,

(iii) choose n depending on the error level ϵ such that $\|Q_n\|\epsilon$ decreases for growing n with a rate similar to the rate of the first summand at the right-hand side of (2.6).

Generalized orthoprojectors.

Let S, Z be n-dimensional subspaces of the Hilbert space X,

$$S = span\{\sigma_1, \cdots, \sigma_n\},\ Z = span\{\zeta_1, \cdots, \zeta_n\}.$$

Let

$$Z^\perp = \{\zeta^\perp \in X,\ (\zeta^\perp, \zeta) = 0\ \forall \zeta \in Z\}.$$

Proposition 2.2 *If $S \cap Z^\perp = \{0\}$, then any $f \in X$ can be uniquely represented as*

$$f = \sigma + \zeta^\perp,\ \ \sigma \in S,\ \zeta^\perp \in Z^\perp. \tag{2.7}$$

Proof. From

$$(\sigma, \zeta) = (f, \zeta)\ \forall \zeta \in Z, \tag{2.8}$$

σ is uniquely determined: Let be $\sigma = \sum_1^n y_i \sigma_i$. Then the linear system

$$\sum_{i=1}^n y_i(\sigma_i, \zeta_j) = (f, \zeta_j),\ j = 1, \cdots, n,$$

is uniquely solvable as its matrix $((\sigma_i, \zeta_j))$ is invertible because of the assumption. ■

Now, let us define the generalized orthoprojector $\mathbf{P}_S^Z$ as

$$\mathbf{P}_S^Z f := \sigma,$$

where ϕ is uniquely determined by (2.7). Clearly, the generalized orthoprojector is a projector. Moreover, in the case $S = Z$

$$\mathbf{P}_S := \mathbf{P}_S^S$$

is the usual orthoprojector to S, where the assumption $S \cap S^\perp = \{0\}$ is trivially fulfilled.

Proposition 2.3 *Suppose $S \cap Z^\perp = \{0\}$. The generalized orthoprojector has the following properties:*

$$(\mathbf{P}_S^Z f, \zeta) = (f, \zeta)\ \ \forall \zeta \in Z, \tag{2.9}$$

$$\mathbf{P}_Z \mathbf{P}_S^Z = \mathbf{P}_Z, \tag{2.10}$$

$$\|\mathbf{P}_S^Z\| \le \|\mathbf{P}_Z^{-1}\|, \tag{2.11}$$

$$\|f - \mathbf{P}_S f\| \le \|f - \mathbf{P}_S^Z f\|.$$

where in (2.11) the restriction of $\mathbf{P}_Z$ to S is considered, being uniquely invertible.

Proof. $\mathbf{P}_Z\sigma = 0$ means $\sigma \in Z^\perp$, i.e. $\sigma = 0$ if $\sigma \in S$. Therefore, the restriction of $\mathbf{P}_Z$ to S is uniquely invertible, and (2.11) immediately follows from (2.10). The assertions (2.9) and (2.10) are immediate from (2.7). ∎

Example 2.1 *Let be $X_n = span\{\varphi_1, \cdots, \varphi_n\}$, $Y_n = span\{\psi_1, \cdots, \psi_n\}$, with the property $X_n \cap (A^*Y_n)^\perp = \{0\}$. Consider the projection method (cf.(2.3)): Find $f_n \in X_n$ such that*

$$(Af_n, v) = (g, v) \;\; \forall v \in Y_n. \tag{2.12}$$

Then $P_n = \mathbf{P}_S^Z$, where $S = X_n, Z = A^*Y_n$.

Proof. Write (2.12) as

$$(f_n, A^*v) = (f, A^*v) \;\; \forall v \in Y_n.$$

The assertion then follows from (2.8). ∎

We are going to study more concrete projection methods, where the assumption of Proposition 2.2 (necessary for the unique solvability) is fulfilled.

Method of least squares.

Let be $X_n = span\{\varphi_1, \cdots, \varphi_n\}, Y_n = AX_n$.

Find $f_n \in X_n$ such that

$$(Af_n, Au) = (g, Au) \;\; \forall u \in X_n. \tag{2.13}$$

Let $f_n = \sum_{i=1}^n x_i\varphi_i$. Then $\underline{x} = (x_1, \cdots, x_n)$ can be calculated from the linear system

$$\sum_{i=1}^n x_i(A\varphi_i, A\varphi_j) = (g, A\varphi_j), \; j = 1, \cdots, n. \tag{2.14}$$

It is clear that (2.14) is uniquely solvable.

Let $\mathbf{P}_n$ be the orthoprojector of Y to Y_n and denote the restriction of A to X_n by A_n. The operators P_n, Q_n, defined above, have the following properties.

Proposition 2.4

$$AQ_n = \mathbf{P}_n, \tag{2.15}$$

$$AP_n = \mathbf{P}_nA, \tag{2.16}$$

$$\|Q_n\| \le \|A_n^{-1}\|, \tag{2.17}$$

$$P_n = \mathbf{P}_S^Z, \;\; S = X_n, Z = A^*AX_n, \tag{2.18}$$

$$\|AQ_ng - g\| \le \|Au - g\| \;\; \forall u \in X_n. \tag{2.19}$$

Proposition 2.5 *If $\overline{R(A^*)} = X$, and $dist(f, X_n) \to 0$, then P_nf tends to f in the weak topology.*

Proof. Using (2.16), for each $y \in Y$ we have

$$(f - P_nf, A^*y) = (Af - AP_nf, y) = (Af - \mathbf{P}_nAf, y) \le \|Af - \mathbf{P}_nAf\|\|y\|.$$

As $\mathbf{P}_n$ is the orthoprojector of Y to AX_n we obtain

$$\|Af - \mathbf{P}_n Af\| \leq \|Af - A\mathbf{P}_{X_n} f\| \leq c\|f - \mathbf{P}_{X_n} f\|,$$

where $\mathbf{P}_{X_n}$ is the orthoprojector of X to X_n. Since $\|f - \mathbf{P}_{X_n} f\| \to 0$ the proof is complete. ∎

Method of least error.

This method is also called dual method of least squares. Here we choose

$$Y_n = span\{\psi_1, \cdots, \psi_n\},\ X_n = A^* Y_n.$$

Find $f_n \in X_n$, i.e. $f_n = A^* w_n, w_n \in Y_n$ such that

$$(A^* w_n, A^* v) = (g, v)\ \forall v \in Y_n. \tag{2.20}$$

Let $w_n = \sum_{i=1}^n x_i \psi_i$. Then $\underline{x} = (x_1, \cdots, x_n)$ can be calculated from the linear system

$$\sum_{i=1}^n x_i (A^* \psi_i, A^* \psi_j) = (g, \psi_j),\ j = 1, \cdots, n. \tag{2.21}$$

Again it is clear that (2.21) is uniquely solvable.

Let $\mathbf{P}_n$ be the orthoprojector of X to X_n and $f \in X$ arbitrary. Then P_n has the following properties:

Proposition 2.6

$$P_n = \mathbf{P}_n, \tag{2.22}$$

$$\|P_n f - f\| \leq \|u - f\|\ \ \forall u \in X_n, \tag{2.23}$$

$$\|P_n\| \leq c. \tag{2.24}$$

Collocation method.

Here we must suppose that X and Y are function spaces. Let

$$X = L^2(0,1),\ \ Y = L^2(2,3).$$

Suppose further, the data g is taken from $C[2,3]$. Given n collocation points

$$\tau_j \in (2,3),\ j = 1, \cdots, n,$$

let be

$$Y_n' := span\{\delta_1, \cdots, \delta_n\},$$

where δ_j is the point evaluation at τ_j,

$$\delta_j g = g(\tau_j),\ j = 1, \cdots, n.$$

Choose

$$X_n = span\{\varphi_1, \cdots, \varphi_n\}.$$

and find $f_n \in X_n$ such that

$$(Af_n)(\tau_j) = g(\tau_j),\ \ j = 1, \cdots, n. \tag{2.25}$$

Then, setting $f_n = \sum_{i=1}^n x_i \varphi_i$, the vector $\underline{x} = (x_1, \cdots, x_n)$ is to be calculated from the linear system

$$\sum_{i=1}^n x_i (A\varphi_i)(\tau_j) = g(\tau_j),\ \ j = 1, \cdots, n. \tag{2.26}$$

Proposition 2.7 *(i)The system (2.26) is uniquely solvable if the following is true:*

If $u \in X_n, (Au)(\tau_j) = 0, j = 1, ..., n$ *then* $u = 0$ *identically.*

(ii)Let (2.26) be uniquely solvable. For $f \in X$ *we obtain* $f_n = P_n f \to f$ *if the following is true:*

If for the sequence $\theta_n \in X$ *holds* $(A\theta_n)(\tau_j) = 0, j = 1, \cdots, n$ *then* $\theta_n \to 0$.

Proof. (i)The matrix $((A\varphi_i)(\tau_j))$ has full rank if its rows are linearly independent, i.e. $\sum_{i=1}^{n} \lambda_i (A\varphi_i)(\tau_j) = 0, j = 1, ..., n$ implies $\lambda_i = 0, i = 1, ..., n$. This means that $(Au)(\tau_j) = 0, j = 1, ..., n$, where $u := \sum_{i=1}^{n} \lambda_i \varphi_i$.

(ii)Take $\theta_n = f - f_n$, where $Ad_n(\tau_j) = 0$ follows from (2.25). ■

Now, using the concrete form (2.2) of the operator A we see that

$$(Af)(\tau_j) = (f, \beta_j), \quad j = 1, ..., n, \tag{2.27}$$

where for $t \in [0, 1]$

$$\beta_j(t) = \log(\tau_j - t), \quad j = 1, ..., n.$$

It can easily be proved that $\beta_j, j = 1, ..., n$, for different τ_j are linearly independent. Denote

$$B_n = span\{\beta_1, ..., \beta_n\}.$$

Proposition 2.8 *Assume* $X_n \cap B_n^{\perp} = \{0\}$. *Then*

(i)The matrix of (2.26)

$$((A\varphi_i)(\tau_j)) = ((\varphi_i, \beta_j)) \tag{2.28}$$

has full rank.

(ii) There holds

$$P_n = \mathbf{P}_{X_n}^{B_n}, \tag{2.29}$$

$$\|P_n\| \le \|\mathbf{P}_{B_n}^{-1}\|, \tag{2.30}$$

where in (2.30) $\mathbf{P}_{B_n}$ *is the restriction to* X_n *of the orthoprojector of* X *to* B_n.

Proof. (i) is clear from the assumption. (ii) follows writing (2.26) equivalently as

$$\sum x_i(\varphi_i, \beta_j) = (f, \beta_j), \; j = 1, ..., n,$$

or $(f_n, \beta) = (f, \beta) \forall \beta \in B_n$.

Estimation of $\|Q_n\|$.

In the case of the least squares method let

$$X_n = span\{e_1, ..., e_n\},$$

where $e_i = \chi_{[(i-1)/n, i/n]}$ are the characteristic functions. Clearly

$$(e_i, e_j) = \delta_{ij}/n.$$

For $g \in Y$ let $Q_n g = \sum x_i e_i$. Then $\|Q_n g\| = |\underline{x}|/\sqrt{n}$. From (2.14) we obtain

$$M\underline{x} = \underline{m},\ M = ((Ae_i, Ae_j)),\ \underline{m} = ((g, Ae_j)).$$

Then

$$\begin{aligned} \|Q_n g\| &= |\underline{x}|/\sqrt{n} = |M^{-1}\underline{m}|/\sqrt{n}, \\ |\underline{m}|^2 &= \sum_j (g, Ae_j)^2 \le c^2\|g\|^2. \end{aligned}$$

We obtain

$$\|Q_n\| \le c|M^{-1}|.$$

In the case of the dual least squares method (method of least error) let

$$Y_n = span\{e'_1, ..., e'_n\},$$

where $e'_i = \chi_{[2+(i-1)/n, 2+i/n]}$ again are the characteristic functions. Clearly

$$(e'_i, e'_j) = \delta_{ij}/n.$$

For $g \in Y$ let $Q_n g = \sum x_i A^* e'_i$. Then $Q_n g = A^* R_n g$. From (2.20) we obtain

$$(AQ_n g, R_n g) = (g, R_n g).$$

(2.21) gives

$$L\underline{x} = \underline{l},\ L = ((A^* e'_i, A^* e'_j)),\ \underline{l} = ((g, e'_j)).$$

Then

$$\|Q_n g\|^2 \le \|R_n g\|\|g\| = \|g\||\underline{x}|/\sqrt{n} \le |L^{-1}|\|g\|^2/\sqrt{n}.$$

We obtain

$$\|Q_n\| \le \left(\frac{|L^{-1}|}{\sqrt{n}}\right)^{1/2}.$$

Finally, in the case of the collocation method let

$$X_n = span\{e_1, ..., e_n\}.$$

For $g \in C[2,3]$ let $Q_n g = \sum x_i e_i$. From (2.26) we obtain

$$K\underline{x} = \underline{k},\ K = (Ae_i(\tau_j)),\ \underline{k} = (g(\tau_j)).$$

Then

$$\|Q_n g\|^2 = |\underline{x}|^2/n \le |K^{-1}|^2|\underline{k}|^2/n = |K^{-1}|^2\|g\|^2,$$

approximately, as $|\underline{k}|^2/n = \sum g(\tau_j)^2/n \approx \|g\|^2$. We obtain

$$\|Q_n\| \le |K^{-1}|.$$

3 Tikhonov regularization. A numerical treatment

Here we are engaged with the numerical solution of

$$\mathcal{A}f_0 = g,$$

where $\mathcal{A} : L^2(0,1) \to L^2(2,3)$ is defined in (2.2) and only an approximation g_δ of g is given,

$$\|g - g_\delta\|_{L^2(2,3)} \leq \delta.$$

We will find the numerical solution by discretization combined with Tikhonov regularization. To this end we are going to cite some results from [1]. Then we will give an overview over numerical experiments confirming the theoretical results.

Crucial for the numerical approximation is the a priori assumption on the solution f_0. Let us start our considerations with the

A priori assumption: $f_0 \in H_0^1(0,1)$.

Let $\delta > 0$ be fixed and $f \in H_0^1(0,1)$. Consider the functional

$$F_\alpha(f) = \|\mathcal{A}f - g_\delta\|^2_{L^2(2,3)} + \alpha\|f\|^2_{H_0^1(0,1)}, \tag{3.1}$$

where $\alpha > 0$. Define

$$\beta = \inf_{f \in H_0^1(0,1)} F_\alpha(f),$$

and the **regularized solution** f_α^δ such that

$$F_\alpha(f_\alpha^\delta) \leq \beta + \delta^2.$$

Proposition 3.1 *Suppose $f_0 \in H_0^1(0,1)$, $\alpha = \delta^2$. Then for $\delta \to 0$ the regularized solution converges to f_0 and*

$$\|f_\alpha^\delta - f_0\|_{L^2(0,1)} \leq C_1 \frac{1}{|\log \frac{1}{\delta}|},$$

where $C_1 > 0$ is a constant which depends on f_0.

Computation of a regularized solution.

We assume $f_0 \in H_0^1$.

Consider in the interval [0,1] the equidistant discretization

$$t_i = i/n, \ i = 1, \cdots, n-1.$$

Define

$$X_n = span\{\Lambda_i, i = 1, \cdots, n-1\},$$

where Λ_i is linear and continuous with $\Lambda_i(t_j) = 1$ for $j = i$ and $= 0$ for $j \neq i$, $i = 1, \cdots, n-1$.

It is known (cf.e.g. [5]), that for $\varphi \in H_0^1, \varphi_n = \sum_{i=1}^{n-1} \varphi(t_i)\Lambda_i$ will converge to φ for $n \to \infty$. If $\varphi \in H^{1+\nu}$

$$\|\varphi - \varphi_n\|_{H_0^1} \leq c \cdot n^{-\nu}\|\varphi\|_{H^{1+\nu}}.$$

Consider (3.1).

From the identity

$$\begin{aligned} F_\alpha\ (\ sf+(1-s)g) &= sF_\alpha(f)+(1-s)F_\alpha(g) \\ &-s(1-s)\{\|\mathcal{A}f-\mathcal{A}g\|^2_{L^2}+\alpha\|f-g\|^2_{H^1_0}\}, \end{aligned}$$

F_α is strongly convex, locally Lipschitz continuous and weakly lower semicontinuous.

There is a unique $f^* \in H^1_0$ with

$$F_\alpha(f^*) = \inf_{f\in H^1_0} F_\alpha(f),$$

and there is a unique $f^*_n \in X_n$ with

$$F_\alpha(f^*_n) = \inf_{f_n\in X_n} F_\alpha(f_n).$$

This element f^*_n can for $n > n_0$ serve as a regularized solution f^δ, since

$$F_\alpha(f^*_n) \longrightarrow F_\alpha(f^*)\ (n\to\infty).$$

This is clear by going to the limit in

$$F_\alpha(f^*) \le F_\alpha(f^*_n) \le F_\alpha(f_n),$$

where the sequence f_n approximates f^*.

Calculation of f^*_n: To minimize

$$\min_{f\in X_n}\{\|\mathcal{A}f-g_\delta\|^2_{L^2}+\alpha(\|f\|^2_{L^2}+\|f'\|^2_{L^2})\}$$

let us consider the equivalent problem: Set $f=\sum_{i=1}^{n-1}x_i\Lambda_i$, and solve

$$\min_{x\in\mathbb{R}^{n-1}}\{< W\underline{x}+\underline{y},\underline{x}> +b\},$$

where $\underline{x}=(x_i), W=((\mathcal{A}\Lambda_i,\mathcal{A}\Lambda_j)+\alpha((\Lambda_i,\Lambda_j)+(\Lambda'_i,\Lambda'_j)))$,
$\underline{y}=(-2(\mathcal{A}\Lambda_i,g_\delta)),\quad b=(g_\delta,g_\delta)$ and $<\cdot,\cdot>$ is the scalar product in $\mathbb{R}^{n-1}$. From the necessary (and sufficient) condition for a minimum we get

$$f^*_n=\sum_{i=1}^{n-1}x_{0i}\Lambda_i,$$

where $\underline{x}_0=W^{-1}\underline{v},\quad \underline{v}=-\underline{y}/2$.

Local regularization of a discontinuous solution

Let $x_0\in(0,1)$ and consider the neighborhood $O_r=O_r(x_0)$.
For $\alpha,\delta>0$ fixed and $f\in L^2(0,1)\cap H^1(O_r)$ define

$$G_\alpha(f) := \|\mathcal{A}f-g_\delta\|^2_{L^2(2,3)}+\alpha(\|f\|^2_{L^2(0,1)}+\|f\|^2_{H^1(O_r)}). \tag{3.2}$$

$$\beta_1 := \inf_{f\in L^2(0,1)\cap H^1(O_r)} G_\alpha(f).$$

and the **locally regularized solution** f^δ_α such that

$$G_\alpha(f^\delta_\alpha)\le\beta_1+\delta^2.$$

Proposition 3.2 *Suppose $\alpha = \delta^2$ and*

$$f_0 \in L^2(0,1) \cap H^1(O_r).$$

Then a locally regularized solution converges for $\delta \to 0$ to f_0 in some neighborhood of x_0, and

$$|f_\alpha^\delta(x) - f_0(x)| \leq C_1 \frac{1}{|\log \frac{1}{\delta}|^\gamma}, \qquad |x - x_0| \leq r_1 < r,$$

where $C_1 > 0$ depends on f_0, r and r_1.

Now, let us consider discontinuity points of the solution.

Proposition 3.3 *Suppose that the exact solution f_0 is a piecewise smooth function and x_0 is a discontinuity point such that*
$f_0 \in C^2((x_0 - \epsilon, x_0)$, $f_0 \in C^2(x_0, x_0 + \epsilon)$ and $f_0(x_0 + 0) \neq f_0(x_0 - 0)$. Let f_α^δ be a locally regularized solution. Then for $\alpha = \delta^2$

$$\lim_{\delta \to 0} ||f_\alpha^\delta||_{H^1(O_r(x_0))} = \infty.$$

Proposition 3.4 *Let O_r be an open subinterval of $[0,1]$. There is a discontinuity point of the solution f_0 in O_r if and only if for a locally regularized solution f_α^δ holds for $\alpha = \delta^2$: $||f_\alpha^\delta||_{H^1(O_r)}$ is unbounded for $\delta \to 0$.*

For proofs of Propositions 3.1 to 3.4 we refer to [1].

Our following numerical experiments concern 3 cases:

(i) The reconstruction of $f_0 \in H_0^1(0,1)$, where the approximating sequence converges in the sense of Proposition 3.1.

(ii) The reconstruction of $f_0 \in L^2(0,1) \cap H^1(O)$, where O is an open subinterval of $(0,1)$ and the approximating sequence converges inside O pointwise in the sense of Proposition 3.2.

(iii) Let O' be such that f_0 has a discontinuity point inside O'. Then the approximating sequence will grow according to Proposition 3.3.

The approximating sequence $f(n, \alpha, \delta)$ will depend on the discretization number n, the regularization parameter α and the noise level δ. It belongs to a finite-dimensional space U_n, that is a subspace of $H_0^1(0,1)$ in the case (i) and of $L^2(0,1) \cap H^1(O)$ in the cases (ii) and (iii). In the case (iii) the interval O' is not contained in O.

Now, let n be a fixed natural number and denote

$$f_\alpha^\delta = f(n, \alpha, \delta).$$

Let $O \subset [0,1]$ be such that its boundary points are points of the equidistant discretization $t_i = i/n, \ i = 0, ..., n,$

$$O = (i_0/n, i_1/n), \ i_0 < i_1.$$

Let $[0,1] = \cup \Delta_i, \ \Delta_i = [(i-1)/n, i/n]$ and define as basis functions $\phi_j, j = 1, ..., n+1,$ the functions

χ_{Δ_i} if $\Delta_i \cap O = \emptyset$, Λ_i if $i/n \in O$, $\Lambda_{i_0}^-$, $\Lambda_{i_1}^+$,
and numbering them according to the position of their support in [0,1]. Here χ_{Δ_i} is the characteristic function of Δ_i and

$$\Lambda_i^-(t) = \begin{cases} -nt+i+1 & \text{if } t \in \Delta_{i+1} \\ 0 & \text{else} \end{cases}, \quad \Lambda_i^+(t) = \begin{cases} nt-i+1 & \text{if } t \in \Delta_i \\ 0 & \text{else} \end{cases},$$

$$\Lambda_i(t) = \begin{cases} \Lambda_i^+(t) & \text{if } t \in \Delta_i \\ \Lambda_i^-(t) & \text{if } t \in \Delta_{i+1} \\ 0 & \text{else} \end{cases}$$

are the usual hat-functions. Now define

$$U_n = \begin{cases} span\{\phi_2, ..., \phi_n\} & \text{if } O = (0,1) \\ span\{\phi_1, ..., \phi_{n+1}\} & \text{if } O \subset (0,1) \end{cases}.$$

The solution f_α^δ of the minimum problems

$$\inf_{f \in U_n} \{\|\mathcal{A}f - g_\delta\|_{L^2(2,3)} + \alpha \|f\|_{H^1}^2\}$$

or

$$\inf_{f \in U_n} \{\|\mathcal{A}f - g_\delta\|_{L^2(2,3)} + \alpha \left(\|f\|_{L^2(0,1)}^2 + \|f\|_{H^1(O)}^2\right)\}$$

if $f_0 \in H_0^1(0,1)$ (case (i)) or $f_0 \in L^2(0,1) \cap H^1(O)$ (case (ii)), respectively, is gained by $f_\alpha^\delta = \sum x_i \phi_i$, where $\underline{x} = (x_i)$ is the solution of the linear system

$$W\underline{x} = \underline{u},$$

$W = ((\mathcal{A}\phi_i, \mathcal{A}\phi_j) + \alpha\{(\phi_i, \phi_j) + (\phi_i', \phi_j')\})$ in case (i),
$W = ((\mathcal{A}\phi_i, \mathcal{A}\phi_j) + \alpha\{(\phi_i, \phi_j) + \lambda_{ij}\})$ in case (ii), where $\lambda_{ij} = (\phi_i, \phi_j) + (\phi_i', \phi_j')$ if both ϕ_i, ϕ_j have support in O, and $= 0$ if not. Moreover,
$\underline{u} = ((\mathcal{A}\phi_i, g_\delta))$.

The scalar products in $L^2(2,3)$ are calculated by using Simpson's rule in an equidistant discretization s_i, $i = 1, ..., m$ of the interval [2,3]. The data g_δ are simulated in the following way.
Let $f_0 \in L^2(0,1)$ be given. Define

$$g_\delta(s_i) = (\mathcal{A}f_0)(s_i) + \delta \cdot z(s_i), \quad i = 1, ..., m,$$

where $z(s_i)$ is a random number, $|z(s_i)| \leq 1$.

The calculation was performed by using the LAPACK FORTRAN program system.

Let us descibe the numerical experiments. We put always $\alpha = \delta^2$, $n = 50$, $m = 200$.
Experiment 1 (case (i)). Here f_0 was taken linear with the properties $f_0(0.6) = 1, f_0(0) = f_0(1) = 0$.
Experiment 2 (case (ii)). Here $f_0 = 1$ in the interval (0.1,0.6) and $f_0 = 0$ else. We set $O = (0.2, 0.4)$ and calculated at the point $t_1 = 0.35 \in O$.
Experiment 3 (case(iii)). Here we took f_0 as in Experiment 2 and $O' = (0.5, 0.7)$.

The results are given in Table 3.1.

Table 3.1

	δ	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
1	$\|\|f_\alpha^\delta - f_0\|\|_{L^2}$	0.290	0.285	0.283	0.240	0.051	0.049	0.037
2	$\|(f_\alpha^\delta - f_0)(t_1)\|$	0.38	0.300	0.28	0.25	0.23	0.019	0.016
3	$\|\|f_\alpha^\delta\|\|_{H^1(O')}$	0.43	0.77	0.78	0.74	0.86	2.31	2.29

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