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Asymptotic behaviour for a phase-field system with hysteresis

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Abstract

The method of hysteresis operators in modelling phase transitions is applied here to the problem of asymptotic stabilization of solutions to a phase-field system with hysteresis. While it is known that a unique global strong solution exists for every initial data and that the evolution process described by this system is thermodynamically consistent in the sense that the absolute temperature remains positive for all times and the Clausius-Duhem inequality holds almost everywhere, we study here the asymptotic behaviour of the solution as $t \to \infty$.

1 Introduction

This paper is devoted to the study of asymptotic behaviour of the solution to the problem

$$\mu(\theta)\, w_t \,+\, f_1[w] \,+\, \theta\, f_2[w] \;=\; 0\,, \tag{1.1}$$

$$(\theta \,+\, F_1[w])_t \,-\, \Delta\theta \;=\; 0\,, \tag{1.2}$$

in $\Omega\times\,]0,\infty[$, subject to the initial and boundary conditions

$$\theta(x,0) \,=\, \theta^0(x)\,,\; w(x,0) \,=\, w^0(x) \;\text{ in } \Omega\,, \quad \frac{\partial\theta}{\partial n} \,=\, 0 \;\text{ on } \partial\Omega\times\,]0,\infty[\,, \tag{1.3}$$

where $\Omega \subset I\!\!R^N$ is an open bounded Lipschitzian domain, w^0, θ^0, μ are given functions and f_1, f_2, F_1 denote hysteresis operators fulfilling the hypotheses **H3** – **H5** below.

Systems of the form (1.1) – (1.2) arise as phase-field equations from the mathematical study of phase transitions with hysteresis (see [KS1]–[KS3]) that include among others the relaxed Stefan problem introduced in [FV] and hysteretic analogues of the models due to Caginalp [C] and Penrose-Fife [PF] for nonconserved order parameters with zero interfacial energy.

It was shown in [KS1] – [KS3] in detail how the energy dissipation properties in hypothesis **H5** ensure on the one hand the unique global solvability of the above system, and, on the other hand, the thermodynamic consistency of the model, that is, the positivity of the temperature field θ and the validity of the Clausius-Duhem inequality. Our objective here is to study the asymptotic behaviour of the solution to Eqs. (1.1) – (1.3) as $t \to \infty$, see Theorem 2.3 below.

Long time behaviour of solutions to phase-field systems has been studied in the literature, see e.g. [EZ], [BZ], [BCH], [L] for the Caginalp model, and [Z], [ShZ2], [KK] and the references cited there for the Penrose-Fife model. However, all these papers mentioned above deal with the case where Eq. (1.1) contains a term of the form $-\kappa\,\Delta w$ with $\kappa > 0$ and no hysteresis operators are involved.

The result proved in this paper could be expected from the combined effects of dissipation due to thermal conductivity, relaxation and hysteresis; it turns out, however, that the result requires an additional concept related to the hysteresis dissipation, namely the 'second order energy inequality' introduced in [K1] and reformulated here as Proposition 5.2 below, which has been systematically used as the main tool for solving *hyperbolic* equations with hysteresis, see [K].

The paper is organized as follows. In Section 2 we recall the notion of hysteresis operators and state the main result (Theorem 2.3). In Section 3 we use a variant of the Moser iteration scheme for proving a global L^∞-bound for solutions to a general nonlinear heat equation (Theorem 3.1) which is substantially used in the proof of Theorem 2.3 given in the subsequent Section 4. Finally, since hysteresis always implies lack of smoothness, we devote Section 5 to results from nonsmooth analysis that are used throughout the paper.

2 Preliminaries and statement of the main result

We first introduce some notation. By $|\cdot|_p$, $1 \le p \le \infty$, we denote the norm in $L^p(\Omega)$. Since we are interested in the large time behavior of solutions to the system (1.1) – (1.3), we define the spaces

$$C_{\rm loc} \;=\; \{u : [0,\infty[\, \to I\!R\,;\; u|_{[0,T]} \in C[0,T] \;\; \forall T>0\}\,, \tag{2.1}$$

$$W^{k,p}_{loc} \;=\; \{u \in C_{\rm loc}\,;\; u|_{[0,T]} \in W^{k,p}(0,T) \;\; \forall T>0\}\,, \tag{2.2}$$

for $1 \le p \le \infty$, $k \in I\!N$, endowed with the system of seminorms

$$\|u\|_{[0,t]} \;:=\; \max_{0\le\tau\le t} |u(\tau)|\,, \tag{2.3}$$

$$|u|_{W^{k,p}(0,t)} \;:=\; \sum_{j=0}^{k-1} |u^{(j)}(0)| + |u^{(k)}|_{L^p(0,t)}\,, \tag{2.4}$$

respectively. We further define the spaces

$$L^p_{\Omega,\rm loc} \;:=\; \{u : \Omega\times\,]0,\infty[\, \to I\!R\,;\; u|_{\Omega\times\,]0,T[} \in L^p(\Omega\times\,]0,T[\,) \;\; \forall T>0\}\,. \tag{2.5}$$

We consider hysteresis operators acting in the spaces $C_{\rm loc}$ and $W^{1,1}_{\rm loc}$. Recall that a mapping $f : C_{\rm loc} \to C_{\rm loc}$ is called a *hysteresis operator* if it is

causal , that is, if the implication

$$u(\tau) = v(\tau) \quad \forall\, \tau \in [0,t] \;\Rightarrow\; f[u](t) = f[v](t) \tag{2.6}$$

holds for every $t \ge 0$, and

rate-independent , that is, if for every $u \in C_{\rm loc}$, every $t>0$ and every nondecreasing mapping α of $[0,t]$ onto $[0,t]$ we have

$$f[u\circ\alpha](\tau) = f[u](\alpha(\tau)) \quad \forall \tau \in [0,t]\,. \tag{2.7}$$

The following easy result has been proved in [K], Proposition II.4.14.

Lemma 2.1 *Let $g : C_{\rm loc} \to C_{\rm loc}$ be a rate-independent operator and let $u \in C_{\rm loc}$ be a function which is monotone (nondecreasing or nonincreasing) in an interval $[t_1, t_2] \subset$*

$[0,\infty[\,,\ u(t_i) = u_i\,,\ i=1,2$. *Then there exists a continuous function* $\Gamma : \mathrm{Conv}\,\{u_1,u_2\} \to \mathbb{R}$ *such that for every function* $v \in C_{\mathrm{loc}}$ *which is monotone in* $[t_1,t_2]$ *and* $v(t) = u(t)$ *for every* $t \in [0,\infty[\,\setminus\,]t_1,t_2[$ *we have*

$$g[v](t) \;=\; \Gamma(v(t)) \quad \forall t \in [t_1,t_2]\,. \tag{2.8}$$

Lemma 2.1 states that every rate-independent operator can be locally represented in each interval of monotonicity of the input function by a superposition operator. The functions Γ are called *shape functions* of the operator g.

Definition 2.2 *A rate-independent operator* $g : C_{\mathrm{loc}} \to C_{\mathrm{loc}}$ *is said to be* clockwise convex, *if all shape functions* Γ *corresponding to nondecreasing inputs are concave and all shape functions corresponding to nonincreasing inputs are convex.*

Clockwise convex rate-independent operators play a particular role in the theory (see the monograph [K]), since they admit an 'energy-type' inequality of the form

$$w_{tt}\, g[w]_t \;-\; \mathcal{V}[w]_t \geq 0 \quad \text{in the sense of distributions} \tag{2.9}$$

for every input function $w \in W^{2,1}_{\mathrm{loc}}$, where

$$\mathcal{V}[w](t) \;:=\; \frac{1}{2} w_t(t)\, g[w]_t(t)\,. \tag{2.10}$$

According to (2.9), $\mathcal{V}[w]$ can be interpreted as 'potential energy' corresponding to the 'power' $w_{tt}\, g[w]_t$.
Inequality (2.9) will be rigorously reformulated and proved below in Proposition 5.2. We now state the main result of this paper under the following hypotheses.

Hypotheses

H1. The initial data are given in such a way that

$$\text{(i)} \qquad w^0 \in L^\infty(\Omega)\,,\ \theta^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)\,, \tag{2.11}$$

$$\text{(ii)} \qquad \exists \delta > 0 \,:\, \theta^0(x) \geq \delta \quad \text{for a.e. } x \in \Omega\,.$$

H2. The function $\mu : [0,\infty[\,\to\,]0,\infty[$ is continuously differentiable, $\mu(\theta) \geq \mu_0 > 0$ for every $\theta \geq 0$.

H3. The operators f_1, f_2 map C_{loc} into C_{loc}, $W^{1,1}_{\mathrm{loc}}$ into $W^{1,1}_{\mathrm{loc}}$, and there exist constants $K_1 > 0$, $K_2 > 0$ such that the implications

$$w_1, w_2 \in C_{\mathrm{loc}} \;\Rightarrow\; |f_i[w_1](t) \,-\, f_i[w_2](t)| \,\leq\, K_1\, |w_1 \,-\, w_2|_{[0,t]}\,, \tag{2.12}$$

$$w \in W^{1,1}_{\mathrm{loc}} \;\Rightarrow\; |f_i[w]_t(t)| \,\leq\, K_1\, |w_t(t)|\,, \tag{2.13}$$

$$w \in C_{\mathrm{loc}} \;\Rightarrow\; |f_i[w](t)| \,\leq\, K_2 \tag{2.14}$$

hold for $i = 1,2$ and a.e. $t > 0$.

H4. The operator F_1 maps $W^{1,1}_{\mathrm{loc}}$ into $W^{1,1}_{\mathrm{loc}}$, and there exists a constant $K_3 > 0$ such that

$$w \in W^{1,1}_{\mathrm{loc}} \quad\Rightarrow\quad |F_1[w]_t\,(t)| \,\leq\, K_3 |w_t(t)| \ \text{ a.e.} \tag{2.15}$$

We moreover assume that the following implication holds for every $t > 0$:

$$\forall R > 0 \;\; \exists M_R > 0 \quad : \quad w_1, w_2 \in W^{1,1}_{\mathrm{loc}}\,, \;\; |w_i|_{W^{1,1}(0,t)} \,\le\, R\,,\; i \,=\, 1,2, \qquad \Rightarrow \;\; |F_1[w_1](t) \,-\, F_1[w_2](t)| \,\le\, M_R\, |w_1 \,-\, w_2|_{W^{1,1}(0,t)}\,. \tag{2.16}$$

H5. There exist hysteresis operators $F_2, g : C_{\mathrm{loc}} \to C_{\mathrm{loc}}$ which map $W^{1,1}_{\mathrm{loc}}$ into $W^{1,1}_{\mathrm{loc}}$ such that g is clockwise convex, and there exists a constant $K_4 > 0$ such that for all $w \in W^{1,1}_{\mathrm{loc}}$ we have

$$\tfrac{1}{K_4}\,(g[w]_t)^2 \,\le\, w_t\, g[w]_t \,\le\, K_4\, w_t^2 \qquad \text{a. e.}\,, \tag{2.17}$$

$$F_i\,[w]_t \,\le g\,[w]_t\, f_i[w] \qquad \text{a. e.}\,, \quad i \,=\, 1,2\,, \tag{2.18}$$

$$F_i[w](t) \,\ge\, 0 \qquad \forall t \in [0,T]\,, \quad i \,=\, 1,2\,. \tag{2.19}$$

In applications, we typically have either $g[w] = w$, or $g[w] = \mathrm{s}[w]$, where s is the stop operator with thresholds 0 and 1. In both cases, the operator g fulfils the conditions of hypothesis **H5** whenever f_i, F_i are of the form $f_i[w] = \tilde{f}_i[g[w]]$, $F_i[w] = \tilde{F}_i[g[w]]$, $i = 1,2$, where, according to the terminology introduced in [BS], $\tilde{f}_i$ are clockwise admissible hysteresis operators with potentials $\tilde{F}_i$.

According to Theorem 2.2 of [KS3], system (1.1) – (1.3) admits a unique solution $(w,\theta) \in L^{\infty}_{\Omega,\mathrm{loc}} \times L^{\infty}_{\Omega,\mathrm{loc}}$ such that $w_t \in L^{\infty}_{\Omega,\mathrm{loc}}$, $\theta_t, \Delta\theta \in L^{2}_{\Omega,\mathrm{loc}}$, Eqs. (1.1) – (1.2) are satisfied almost everywhere and there exists $\kappa > 0$ such that $\theta(x,t) \ge \delta\, e^{-\kappa t}$.

The main objective of this paper is to prove the following result.

Theorem 2.3 *Let the hypotheses* **H1** – **H5** *hold. Then there exists a constant* $\hat{C} > 0$ *such that the solution* (w,θ) *to the system (1.1) – (1.3) satisfies the conditions*

$$0 \,<\, \theta(x,t) \,\le\, \hat{C}\,, \quad |w_t(x,t)| \,\le\, \hat{C} \quad \textit{a. e. in } \;\Omega \times\,]0,\infty[\;. \tag{2.20}$$

Moreover, if for $t > 0$ *we put*

$$\begin{cases} E_1(t) &:= \;\tfrac{1}{2}\int_\Omega |\nabla\theta(x,t)|^2 dx\,, \\ E_2(t) &:= \;\tfrac{1}{2}\int_\Omega w_t(x,t)\, g[w]_t(x,t) dx\,, \\ E(t) &:= \;E_1(t) + E_2(t)\,, \end{cases} \tag{2.21}$$

then we have

$$\int_0^\infty E(t)\, dt \;\le\; \hat{C}\,, \quad \lim_{t\to\infty} E_1(t) \;=\; 0\,, \tag{2.22}$$

and there exists a function $E_2^* : [0,\infty[\, \to [0,\infty[$ *such that*

$$E_2(t) \;=\; E_2^*(t) \quad \textit{a. e.}\;, \quad \lim_{t\to\infty} E_2^*(t) \;=\; 0\,, \quad \operatorname*{Var}_{[0,\infty[} \left((E_1 + E_2^*)^2\right) \;\le\; \hat{C}\,. \tag{2.23}$$

In particular, the function E_2 *satisfies the condition*

$$\lim_{t\to\infty} \;\operatorname{sup\,ess}\,\{E_2(s)\,;\; s > t\} \;=\; 0\,. \tag{2.24}$$

3 An L^∞-estimate

We use here a variant of the Moser iteration scheme, see e.g. [LSU], to prove Theorem 3.1 below which will be substantially exploited in the next section. We first recall the well-known interpolation inequality

$$|v|_2 \;\le\; A\left(\delta\,|\nabla v|_2 + \delta^{-N/2}\,|v|_1\right), \tag{3.1}$$

which holds for every $v \in W^{1,2}(\Omega)$ and every $\delta \in]0,1[$ with a constant $A > 0$ independent of v and δ. Let us note that it is equivalent to Gagliardo-Nirenberg's inequality

$$|v|_2 \;\le\; A^*\left(|v|_1 + |v|_1^{\frac{2}{N+2}}\,|\nabla v|_2^{\frac{N}{N+2}}\right). \tag{3.2}$$

Theorem 3.1 *Let $\mathcal{H} : L^\infty_{\Omega,\mathrm{loc}} \to L^\infty_{\Omega,\mathrm{loc}}$ be a mapping with the following property:*

$$\exists B > 0\ \forall u \in L^\infty_{\Omega,\mathrm{loc}} : \quad |\mathcal{H}[u](x,t)| \;\le\; B\,(1 + |u(x,t)|) \quad a.\,e. \tag{3.3}$$

Let $u^0 \in L^\infty(\Omega)$ be a given function and let $u \in L^\infty_{\Omega,\mathrm{loc}} \bigcap L^2(0,T;H^1(\Omega))$ for any $T > 0$ be a solution of the problem

$$u_t - \Delta u \;=\; \mathcal{H}[u] \quad in\ \ \Omega\times]0,\infty[\,, \tag{3.4}$$

$$u(x,0) \;=\; u^0(x) \ \ in\ \Omega\,, \quad \frac{\partial u}{\partial n} \;=\; 0 \ \ on\ \partial\Omega\times]0,\infty[\,, \tag{3.5}$$

such that

$$\exists E > 0\ \forall t \ge 0 : \ |u(t)|_1 \le E\,. \tag{3.6}$$

Then there exists a constant $R > 0$ depending only on A, B, E, N, $|\Omega| := \mathrm{meas}\,\Omega$ and $U := |u^0|_\infty$ such that

$$|u(t)|_\infty \;\le\; R \quad for\ a.\,e.\ \ t > 0\,. \tag{3.7}$$

Proof. For $k \in I\!N$ put

$$\psi_k \;:=\; u\,|u|^{2^{k-1}-1} \tag{3.8}$$

and test Eq. (3.4) with $u\,|u|^{2^k-2}$. For a.e. $t > 0$ this yields

$$2^{-k}\frac{d}{dt}\int_\Omega |u|^{2^k}\,dx \,+\, (2^k-1)\int_\Omega |\nabla u|^2\,|u|^{2^k-2}\,dx \;\le\; B\int_\Omega\left(|u|^{2^k-1} + |u|^{2^k}\right)dx\,, \tag{3.9}$$

where

$$\int_\Omega |u|^{2^k}(x,t)\,dx \;=\; |\psi_k(t)|_2^2\,, \tag{3.10}$$

$$\begin{aligned}\int_\Omega |u|^{2^k-1}(x,t)\,dx \;&\le\; |\Omega|^{2^{-k}}\left(\int_\Omega |u|^{2^k}(x,t)\,dx\right)^{1-2^{-k}} \\ &\le\; 2^{-k}\,|\Omega| \,+\, (1-2^{-k})\,|\psi_k(t)|_2^2\,,\end{aligned} \tag{3.11}$$

$$\int_\Omega |\nabla u|^2\,|u|^{2^k-2}(x,t)\,dx \;=\; \int_\Omega\left(|\nabla u|\,|u|^{2^{k-1}-1}\right)^2(x,t)\,dx \;=\; 2^{2-2k}\,|\nabla\psi_k(t)|_2^2\,. \tag{3.12}$$

Inequality (3.9) can therefore be written as

$$\frac{d}{dt}\,|\psi_k(t)|_2^2 \;+\; 4\,(1-2^{-k})\,|\nabla\psi_k(t)|_2^2 \;\le\; B\,\Big(|\Omega| \;+\; (2^{k+1}-1)\,|\psi_k(t)|_2^2\Big) \tag{3.13}$$

for every $k \in I\!N$ and a. e. $t>0$. The interpolation inequality (3.1) for $v=\psi_k(t)$ yields

$$|\psi_1(t)|_2 \;\le\; A\,\Big(\delta\,|\nabla\psi_1(t)|_2 \;+\; \delta^{-N/2}\,E\Big)\,, \tag{3.14}$$

$$|\psi_k(t)|_2 \;\le\; A\,\Big(\delta\,|\nabla\psi_k(t)|_2 \;+\; \delta^{-N/2}\,|\psi_{k-1}(t)|_2^2\Big) \tag{3.15}$$

for every $k\ge 2$, $\delta \in]0,1[$ and a. e. $t>0$.

We now estimate the right-hand side of Ineq. (3.13) using (3.14) and (3.15) with an appropriately chosen δ.

Throughout this section, we denote by $C_1, C_2, \ldots$ any positive constants depending exclusively on A, B, E, N, U and $|\Omega|$. We may indeed assume that $\min\{A,B\}\ge 1$.

Putting $\delta := (2^{1+(k/2)}\,A\,\sqrt{B})^{-1}$ in (3.14), (3.15), we infer from (3.13) that the inequalities

$$\frac{d}{dt}\,|\psi_1(t)|_2^2 \;+\; |\nabla\psi_1(t)|_2^2 \;\le\; C_1\,, \tag{3.16}$$

$$\frac{d}{dt}\,|\psi_k(t)|_2^2 \;+\; |\nabla\psi_k(t)|_2^2 \;\le\; C_1\,\Big(1+2^{\beta k}\,|\psi_{k-1}(t)|_2^4\Big) \tag{3.17}$$

hold for every $k\ge 2$ and a. e. $t>0$, with $\beta := 1+(N/2)$. Using again Ineqs. (3.14), (3.15) for, say, $\delta := 1/(2A)$, we obtain from (3.16), (3.17) the inequalities

$$\frac{d}{dt}\,|\psi_1(t)|_2^2 \;+\; |\psi_1(t)|_2^2 \;\le\; C_2\,, \tag{3.18}$$

$$\frac{d}{dt}\,|\psi_k(t)|_2^2 \;+\; |\psi_k(t)|_2^2 \;\le\; C_2\,\Big(1+2^{\beta k}\,|\psi_{k-1}(t)|_2^4\Big) \tag{3.19}$$

for $k\ge 2$ and a. e. $t>0$, hence

$$|\psi_1(t)|_2^2 \;\le\; \max\Big\{|\psi_1(0)|_2^2\,,\,C_2\Big\}\,, \tag{3.20}$$

$$|\psi_k(t)|_2^2 \;\le\; \max\Big\{|\psi_k(0)|_2^2\,,\,C_2\,\Big(1+2^{\beta k}\,\max_{0\le\tau\le t}|\psi_{k-1}(\tau)|_2^4\Big)\Big\} \tag{3.21}$$

for all $k\ge 2$ and $t\ge 0$.

We have by definition $|\psi_k(t)|_2^2 = |u(t)|_{2^k}^{2^k}$ for every $k\in I\!N$ and $t\ge 0$, hence in particular

$$|\psi_k(0)|_2^2 = |u^0|_{2^k}^{2^k} \le |\Omega|\,U^{2^k} \quad \forall k\in I\!N\,. \tag{3.22}$$

Let us introduce the auxiliary functions

$$z_k(t) \;:=\; \max_{0\le\tau\le t}\,|u(\tau)|_{2^k} \tag{3.23}$$

for $k\in I\!N$ and $t\ge 0$. Ineqs. (3.20), (3.21) then read

$$z_1(t) \;\le\; C_3\,, \tag{3.24}$$

$$z_k^{2^k}(t) \;\le\; C_3\,\max\Big\{U^{2^k},\,1+2^{\beta k}\,z_{k-1}^{2^k}(t)\Big\} \tag{3.25}$$

$$\le\; C_3\,\Big(1+2^{\beta k}\Big)\,\Big(\max\{U,\,1,\,z_{k-1}(t)\}\Big)^{2^k}$$

for $k \geq 2$ and $t \geq 0$. Let us now fix any $t \geq 0$ and for $k \in I\!N$ put $y_k := \max\{U,\, 1,\, z_k(t)\}$. Then

$$y_1 \;\leq\; C_4\,, \tag{3.26}$$

$$y_k \;\leq\; \left(C_3\,\left(1+2^{\beta\,k}\right)\right)^{2^{-k}}\, y_{k-1} \quad \text{for} \quad k \geq 2\,, \tag{3.27}$$

and we easily obtain

$$y_k \leq C_4\, \exp\left(\sum_{j=2}^{k} 2^{-j}\, \log\left(C_3\,\left(1+2^{\beta\,j}\right)\right)\right) \;\leq\; C_5 \tag{3.28}$$

independently of k and t. We thus conclude that

$$\sup_{t\geq 0,\, k\in I\!N} |u(t)|_{2^k} \;\leq\; C_5\,, \tag{3.29}$$

and to complete the proof of Theorem 3.1, it suffices to put $R := C_5$. □

4 Proof of Theorem 2.3

The proof of Theorem 2.3 is based on a series of estimates. Similarly as in the previous section, we denote by $C_1, C_2, \ldots$ any positive constant independent of x and t.

Estimate 1.
Integrating Eq. (1.2) over Ω, we obtain

$$|\theta(t)|_1 + |F_1[w](t)|_1 \;\leq\; C_1\,. \tag{4.1}$$

From Eq. (1.1) and Hypothesis **H3** it follows that

$$|w_t(x,t)| \;\leq\; C_2\,(1+\theta(x,t)) \quad \text{a.e.}\;\;, \tag{4.2}$$

and Hypothesis **H4** yields that

$$|F_1[w]_t(x,t)| \;\leq\; C_3\,(1+\theta(x,t)) \quad \text{a.e.} \tag{4.3}$$

Theorem 3.1 enables us to conclude that

$$\left.\begin{array}{lcl} \theta(x,t) & \leq & C_4 \\ |w_t(x,t)| & \leq & C_5 \end{array}\right\} \quad \text{a.e.} \tag{4.4}$$

Estimate 2.
Put $\lambda(x,t) := \log\, \theta(x,t)$. Then for a.e. $(x,t) \in \Omega\times\,]0,\infty[$ we have

$$\lambda_t - \Delta\,\lambda \;=\; \frac{1}{\theta}(\theta_t - \Delta\,\theta) + \left|\frac{\nabla\theta}{\theta}\right|^2\,, \tag{4.5}$$

where

$$\theta_t - \Delta\theta \;=\; -F_1[w]_t \;\geq\; -\; f_1[w]\,g[w]_t \;=\; \mu(\theta)\,w_t\,g[w]_t \;+\; \theta\,f_2[w]\,g[w]_t \qquad (4.6)$$
$$\geq\; \mu(\theta)\,w_t\,g[w]_t \;+\; \theta\,F_2[w]_t\,,$$

hence

$$\lambda_t - \Delta\lambda \geq\; F_2[w]_t \;+\; \frac{\mu(\theta)}{\theta}\,w_t\,g[w]_t \;+\; \left|\frac{\nabla\theta}{\theta}\right|^2 \qquad (4.7)$$

a.e. in $\Omega\times\,]0,\infty[$. Integrating Ineq. (4.7) with respect to x and t, we obtain for every $t>0$ that

$$\int_0^t\int_\Omega\left(\frac{\mu(\theta)}{\theta}\,w_t\,g[w]_t \;+\; \left|\frac{\nabla\theta}{\theta}\right|^2\right)(x,\tau)\,dx\,d\tau \qquad (4.8)$$
$$\leq\; \int_\Omega\Big(\log\theta(x,t)-\log\theta^0(x)+F_2[w](x,0)\Big)\;dx\;\leq\;C_6\,,$$

and from (4.4) it follows that

$$\int_0^t E(\tau)\,d\tau \;\leq\; C_7 \qquad (4.9)$$

for every $t>0$.

Estimate 3.
Test Eq. (1.2) with θ_t. This yields for a. e. t that

$$|\theta_t(t)|_2^2 + \frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|_2^2 \;\leq\; C_8\,(1+|\theta_t(t)|_1) \;\leq\; \frac{1}{2}\left(|\theta_t(t)|_2^2 + C_9\right)\,, \qquad (4.10)$$

hence

$$|\theta_t(t)|_2^2 + \frac{d}{dt}|\nabla\theta(t)|_2^2 \;\leq\; C_9 \quad \text{a. e.} \qquad (4.11)$$

Thus, combining (4.11) with (4.9) and applying Lemma 3.1 of [ShZ1] yields that $E_1(t) = \int_\Omega |\nabla\theta|^2(x,t)\,dx \to 0$ as $t\to\infty$.

Estimate 4.
We differentiate the equation

$$w_t + \frac{1}{\mu(\theta)}\,f_1[w] + \frac{\theta}{\mu(\theta)}\,f_2[w] \;=\; 0 \qquad (4.12)$$

with respect to t and test with $g[w]_t$. This yields

$$(w_{tt}\,g[w]_t)(x,t) \;\leq\; C_{10}\,(1+|\theta_t(x,t)|) \quad \text{a. e.}\,, \qquad (4.13)$$

hence

$$\int_\Omega (w_{tt}\,g[w]_t)(x,t)\,dx \;\leq\; \frac{1}{2}\left(|\theta_t(t)|_2^2 + C_{11}\right) \qquad (4.14)$$

for a. e. $t>0$. Combining (4.11) with (4.14) we obtain

$$\int_\Omega (w_{tt}\,g[w]_t)(x,t)\,dx \;+\; \frac{1}{2}\,\frac{d}{dt}|\nabla\theta(t)|_2^2 \;\leq\; C_{12} \quad \text{a. e.} \qquad (4.15)$$

For $t > 0$ put

$$q(t) \;:=\; C_{12} t \;-\; E(t)\,. \tag{4.16}$$

We claim that for every $T > 0$ and every $\phi \in \overset{\circ}{W}{}^{1,1}(0,T)$ such that $\phi(t) \ge 0$ for every $t \in [0,T]$ we have

$$\int_0^T q(t)\,\phi_t(t)\,dt \;\le\; 0\,. \tag{4.17}$$

Indeed, let $T > 0$ and $\phi \in \overset{\circ}{W}{}^{1,1}(0,T)$ such that $\phi(t) \ge 0$ for every $t \in [0,T]$ be given. Then Ineq. (4.15) together with the Fubini theorem yield

$$\begin{aligned}
\int_0^T q(t)\,\phi_t(t)\,dt \;&=\; -\int_0^T (C_{12}\,\phi(t) + E(t)\,\phi_t(t))\,dt \\
&\le\; -\int_0^T \left(\frac{d}{dt}\left(\frac{1}{2}\,\phi(t)\,\int_\Omega |\nabla\theta(x,t)|^2\,dx\right)\right. \\
&\quad + \phi(t)\,\int_\Omega (w_{tt}\,g[w]_t)(x,t)\,dx + \frac{1}{2}\,\phi_t(t)\,\int_\Omega (w_t\,g[w]_t)(x,t)\,dx\Big)\,dt \\
&=\; -\int_\Omega\int_0^T \left(\phi(t)\,(w_{tt}\,g[w]_t)(x,t) + \frac{1}{2}\,\phi_t(t)\,(w_t\,g[w]_t)(x,t)\right)\,dt\,dx\,.
\end{aligned} \tag{4.18}$$

By Proposition 5.2 we have for a. e. $x \in \Omega$ that

$$\int_0^T \left(\phi(t)\,(w_{tt}\,g[w]_t)(x,t) + \frac{1}{2}\,\phi_t(t)\,(w_t\,g[w]_t)(x,t)\right)\,dt \;\ge\; 0\,, \tag{4.19}$$

and Ineq. (4.17) follows. From Lemma 5.1 we conclude that there exists a nondecreasing function $q_* : [0,\infty[\, \to I\!R$ such that $q(t) = q_*(t)$ a. e. For $t \ge 0$ it now suffices to put $E_*(t) := C_{12}t - q_*(t)$. Proposition 5.4 with $y = E_*$, $Y = C_7$, $h \equiv 0$, $f(u) \equiv C_{12}$ entails that the function $(E_*)^2$ has bounded variation in $[0,\infty[$ and $\lim_{t\to\infty} E_*(t) = 0$. It now suffices to put $E_2^* := E_* - E_1$ and the assertion follows from Ineqs. (4.4) and (4.9).

5 Monotonicity and convexity

The easy Lemma 5.1 below plays a crucial role in our analysis. For the reader's convenience, we give here the proof taken from [K], Lemma II.4.16.

Lemma 5.1 *Let $[a,b] \subset I\!R$ be a compact interval and let $u \in L^\infty(a,b)$ be a given function. Then the following two conditions are equivalent.*

(i) *There exists a nondecreasing function $u_* : [a,b] \to I\!R$ such that $u(t) = u_*(t)$ a. e.*

(ii) *For every function $\phi \in \overset{\circ}{W}{}^{1,1}(a,b)$ such that $\phi(t) \ge 0$ for every $t \in [a,b]$ we have*

$$\int_a^b u(t)\,\phi_t(t)\,dt \;\le\; 0\,. \tag{5.1}$$

Proof.
(i) $\Rightarrow$ (ii): Let ϕ with the above properties be given. For an arbitrary partition $a = t_0 < t_1 < \ldots < t_n = b$ we define the piecewise linear approximation

$$\hat{u}(t) := u_*(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(u_*(t_j) - u_*(t_{j-1})) \quad \text{for} \quad t \in [t_{j-1}, t_j[\,,\; j = 1, \ldots, n\,.$$

Obviously, Ineq. (5.1) holds for $\hat{u}(t)$, and refining the partition we pass to the limit.

(ii) $\Rightarrow$ (i): Let (5.1) hold and let $r, s \in]a,b[\,,\; r < s$ be arbitrary Lebesgue points of u. For $0 < \varepsilon < \min\{(s-r)/2, b-s, r-a\}$ put

$$\phi_t(t) \;:=\; \begin{cases} \frac{1}{2\varepsilon} & \text{for} \;\; t \in]r-\varepsilon, r+\varepsilon[\;, \\ \frac{-1}{2\varepsilon} & \text{for} \;\; t \in]s-\varepsilon, s+\varepsilon[\;, \\ 0 & \text{otherwise}\;. \end{cases}$$

Then Ineq. (5.1) yields

$$\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} u(t)\,dt \;\leq\; \frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} u(t)\,dt\,,$$

hence $u(r) \leq u(s)$. The set $\Lambda(u)$ of Lebesgue points of u has full measure, hence we may put $u_*(t) := u(t)$ for $t \in \Lambda(u)$, $u_*(t) := \sup\{u(s)\,;\, s \in]a,t[\cap\Lambda(u)\}$ for $t \in]a,b] \setminus \Lambda(u)$, continuously extended to $t = a$, and the proof is complete. □

The 'energy inequality' (2.9) is stated in the following form.

Proposition 5.2 *Let $g : C_{\text{loc}} \to C_{\text{loc}}$ be a clockwise convex rate-independent operator, and let $w \in W^{2,1}_{\text{loc}}$ be a given function such that $g[w] \in W^{1,\infty}_{\text{loc}}$. Then for every $T > 0$ and every $\phi \in \overset{\circ}{W}{}^{1,1}(0,T)$ such that $\phi(t) \geq 0$ for every $t \in [0,T]$ we have*

$$\int_0^T \left(\phi(t)\, w_{tt}(t)\, g[w]_t(t) \;+\; \frac{1}{2}\,\phi_t(t)\, w_t(t)\, g[w]_t(t)\right) dt \;\geq\; 0\,. \tag{5.2}$$

According to Lemma 5.1, Proposition 5.2 says that the function

$$P(t) \;:=\; \int_0^t w_{tt}(\tau)\, g[w]_t(\tau)\, d\tau \;-\; \frac{1}{2} w_t(t)\, g[w]_t(t) \tag{5.3}$$

is equal to a nondecreasing function a. e. in $]0,\infty[$.

Proof. Let $T > 0$ and $\phi \in \overset{\circ}{W}{}^{1,1}(0,T)$ such that $\phi(t) \geq 0$ for every $t \in [0,T]$ be given, and put $A_0 := \{t \in [0,T]\,;\, w_t(t) = 0\}$, $A_1 :=]0,T[\setminus A_0$. Then $g[w]_t(t) = 0$ for a. e. $t \in A_0$. By hypothesis, the function w_t is continuous, hence the set A_1 is open and can be written as a countable union of disjoint open intervals, that is,

$$A_1 \;=\; \sum_{k=1}^{\infty}]a_k, b_k[\;, \qquad]a_k, b_k[\,\cap\,]a_\ell, b_\ell[\, = \emptyset \quad \text{for} \quad k \neq \ell\,. \tag{5.4}$$

Let $K \subset I\!N$ be the set of all indices k such that the interval $]a_k, b_k[$ is nonempty. For each $k \in K$, the function w is strictly monotone in $[a_k, b_k]$; let Γ_k denote the shape function of g in the interval $[a_k, b_k]$. We then have

$$\int_0^T \Big(\phi(t)\, w_{tt}(t)\, g[w]_t(t) \,+\, \frac{1}{2}\, \phi_t(t)\, w_t(t)\, g[w]_t(t)\Big)\; dt \tag{5.5}$$
$$= \sum_{k\in K} \int_{a_k}^{b_k} \Gamma_k'(w(t))\, \frac{d}{dt}\, \Big(\frac{1}{2}\phi(t)\, w_t^2(t)\Big)\; dt\,.$$

For each $k \in K$, we evaluate the above integral over $[a_k, b_k]$ by substituting $s := w(t)$. This yields

$$\int_{a_k}^{b_k} \Gamma_k'(w(t))\, \frac{d}{dt}\, \Big(\frac{1}{2}\, \phi(t)\, w_t^2(t)\Big)\; dt \tag{5.6}$$
$$= \begin{cases} \displaystyle\int_{w(a_k)}^{w(b_k)} \Gamma_k'(s)\, \frac{d}{ds}\, \Big(\frac{1}{2}\, \phi(w^{-1}(s))\, w_t^2(w^{-1}(s))\Big)\, ds & \text{if } w \text{ is increasing in } [a_k, b_k]\,, \\ \displaystyle -\int_{w(b_k)}^{w(a_k)} \Gamma_k'(s)\, \frac{d}{ds}\, \Big(\frac{1}{2}\, \phi(w^{-1}(s))\, w_t^2(w^{-1}(s))\Big)\, ds & \text{if } w \text{ is decreasing in } [a_k, b_k]\,. \end{cases}$$

Put $\phi_k(s) := (1/2)\, \phi(w^{-1}(s))\, w_t^2(w^{-1}(s))$ for $s \in I_k := \mathrm{Conv}\,\{w(a_k), w(b_k)\}$. We have $w_t(a_k) = w_t(b_k) = 0$ for every $a_k > 0$, $b_k < T$, and $\phi(0) = \phi(T) = 0$, hence $\phi_k \in \overset{\circ}{W}{}^{1,1}(I_k)$ for every $k \in K$, $\phi_k(s) \ge 0$ for every $s \in I_k$.

At this point, we make use of the convexity of the operator g: the function Γ_k' is nonincreasing in I_k if w is increasing and nondecreasing if w is decreasing. We can therefore apply Lemma 5.1 to each of the integrals on the right-hand side of Eq. (5.6) and conclude that

$$\int_{a_k}^{b_k} \Gamma_k'(w(t))\, \frac{d}{dt}\, \Big(\frac{1}{2}\, \phi(t)\, w_t^2(t)\Big)\; dt \;\ge\; 0 \tag{5.7}$$

for every $k \in K$. The assertion now follows from Eq. (5.5). Proposition 5.2 is proved.

□

The rest of this section is devoted to a generalization of Lemma 3.1 of [ShZ1] and Theorem 9 of [KS4] which have been stated in the framework of continuous functions. Proposition 5.4 below extends them to the discontinuous case which is needed here. Its proof is based on the following auxiliary result.

Lemma 5.3 *Let $T > 0$, $g \in C_{\mathrm{loc}}$, $p \in L^1(0,T)$ and $y \in BV(0,T)$ be given such that*

(i) *$g(u) \ge 0$, $p(t) \ge 0$, $y(t) \ge 0$ for every $u \ge 0$ and (almost) every $t \in [0,T]$,*

(ii) *the function $q(t) \;:=\; \int_0^t p(\tau)\, d\tau \;-\; y(t)$ is nondecreasing in $[0,T]$.*

For $v \ge 0$ put $G(u) := \int_0^u g(v)\, dv$. Then the function

$$Q(t) \;:=\; \int_0^t g(y(\tau))\, p(\tau)\, d\tau \;-\; G(y(t)) \tag{5.8}$$

is nondecreasing in $[0,T]$.

Proof. For any $n \in \mathbb{N}$ we construct the equidistant partition $0 = s_0 < s_1 < \ldots < s_n = T$ of the interval $[0,T]$, $s_k := Tk/n$ for $k = 0,1,\ldots,n$. We approximate the functions p, y by piecewise constant and piecewise linear interpolates, respectively, that is,

$$\begin{aligned} p_n(t) &:= \frac{n}{T}\int_{s_{k-1}}^{s_k} p(\tau)\,d\tau\,, \\ y_n(t) &:= y(s_{k-1}) + \frac{n}{T}(t - s_{k-1})\,(y(s_k) - y(s_{k-1})) \end{aligned}$$

for $t \in [s_{k-1}, s_k[$, $k = 1,\ldots,n$, continuously extended to $t = T$. Let $Q_n : [0,T] \to \mathbb{R}$ be the function

$$Q_n(t) := \int_0^t g(y_n(\tau))\,p_n(\tau)\,d\tau \;-\; G(y_n(t))\,.$$

By hypothesis (ii), we have for $t \in]s_{k-1}, s_k[$

$$\dot{y}_n(t) = \frac{n}{T}\,(y(s_k) - y(s_{k-1})) \le \frac{n}{T}\int_{s_{k-1}}^{s_k} p(\tau)\,d\tau = p_n(t)\,,$$

hence $\dot{Q}_n(t) = g(y_n(t))\,(p_n(t) - \dot{y}_n(t)) \ge 0$. We have $p_n \to p$ strongly in $L^1(0,T)$ as $n \to \infty$, $y_n(t) \to y(t)$ a. e., hence $Q_n(t) \to Q(t)$ a. e. Since Q_n are nondecreasing for every n, the function Q is also nondecreasing and Lemma 5.3 is proved. □

Proposition 5.4 *Let $f \in C_{\mathrm{loc}}$, $h \in L^1(0,\infty)$ and $y \in BV_{\mathrm{loc}}(0,\infty) \cap L^1(0,\infty)$ be given such that*

(i) $f(u) \ge 0$, $h(t) \ge 0$, $y(t) \ge 0$ *for every* $u \ge 0$ *and (almost) every* $t \ge 0$,

(ii) $\int_0^\infty h(t)\,dt =: H$, $\int_0^\infty y(t)\,dt =: Y$,

(iii) *the function* $q_1(t) := \int_0^t (f(y(\tau)) + h(\tau))\,d\tau - y(t)$ *is nondecreasing in* $[0,\infty[$.

Then we have

$$\lim_{t\to\infty} y(t) = 0\,. \tag{5.9}$$

If moreover the function $F(u) := \int_0^u (\max\{1, f(v)/v\})^{-1}\,dv$ *for* $u \ge 0$ *satisfies the condition*

$$\lim_{u\to\infty} F(u) = \infty\,, \tag{5.10}$$

then

$$y(t) \le \bar{Y} := F^{-1}(F(y(0)) + Y + H) \qquad \forall t \ge 0\,, \tag{5.11}$$

$$y(t) \le F^{-1}(2Y + H) \qquad \forall t \ge 1\,, \tag{5.12}$$

$$\operatorname*{Var}_{[0,\infty[}(y^2) \le y^2(0) + 4\left(Y\,\|f\|_{[0,\bar{Y}]} + H\,\bar{Y}\right)\,. \tag{5.13}$$

Remark. If we drop the condition (5.10), then we have no a priori bound for $y(t)$ anymore. It suffices to consider any continuous function $f(u) \ge \max\{1,u\}$ such that $F_\infty := \int_0^\infty v/f(v)\,dv < \infty$, and to put $\Phi(u) := \int_0^u 1/f(v)\,dv$. We then have

$$\Phi_\infty := \int_0^\infty 1/f(v)\,dv \le \int_0^1 1/f(v)\,dv + \int_1^\infty v/f(v)\,dv \le 1 + F_\infty - F(1)\,.$$

For an arbitrary $\varepsilon \in]0, \Phi_\infty[$ we can find some $R_\varepsilon > 0$ such that $\Phi_\infty - \Phi(R_\varepsilon) = \int_{R_\varepsilon}^\infty 1/f(v)\,dv = \varepsilon$ and define the function

$$y_\varepsilon(t) := \begin{cases} \Phi^{-1}(t) & \text{for } t \in [0, \Phi_\infty - \varepsilon]\,, \\ 0 & \text{for } t > \Phi_\infty - \varepsilon\,. \end{cases}$$

Then $y_\varepsilon(\Phi_\infty - \varepsilon) = R_\varepsilon$ and $\dot{y}_\varepsilon(t) = f(y_\varepsilon(t))$ for $t \in]0, \Phi_\infty - \varepsilon[$, hence $\int_0^\infty y_\varepsilon(t)\,dt = \int_0^{\Phi_\infty - \varepsilon} \dot{y}_\varepsilon(t)\,y_\varepsilon(t)/f(y_\varepsilon(t))\,dt = F(R_\varepsilon) \le F_\infty$ and

$$\int_0^t f(y_\varepsilon(\tau))\,d\tau \;-\; y_\varepsilon(t) \;=\; \begin{cases} 0 & \text{for } t \in [0, \Phi_\infty - \varepsilon]\,, \\ R_\varepsilon + t\,f(0) & \text{for } t > \Phi_\infty - \varepsilon\,. \end{cases}$$

Hence, the hypotheses (i) – (iii) of Proposition 5.4 hold with $h \equiv 0$ and $Y = F_\infty$ independently of ε, while $y_\varepsilon(\Phi_\infty - \varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof of Proposition 5.4. Assume first that condition (5.10) holds, and for $t > 0$ put $p(t) := f(y(t)) + h(t)$. The hypotheses of Lemma 5.3 are satisfied for every $T > 0$, and we can conclude that the function

$$Q_1(t) \;:=\; \int_0^t \frac{y(\tau)\,(f(y(\tau)) + h(\tau))}{\max\{y(\tau), f(y(\tau))\}}\,d\tau \;-\; F(y(t)) \tag{5.14}$$

is nondecreasing in $[0, \infty[$. In particular, we have for every $t \ge s \ge 0$

$$\begin{aligned} F(y(t)) - F(y(s)) \;&\le\; \int_s^t \frac{y(\tau)\,f(y(\tau)) + y(\tau)\,h(\tau)}{\max\{y(\tau), f(y(\tau))\}}\,d\tau \\ &\le\; \int_s^t (y(\tau) \;+ h(\tau))\,d\tau \;\le\; Y + H\,, \end{aligned}$$

and Ineq. (5.11) follows for $s = 0$. Assuming $t \ge 1$, we integrate $\int_{t-1}^t ds$ the inequality

$$F(y(t)) \;\le\; F(y(s)) + Y + H \;\le\; y(s) + Y + H\,,$$

and we obtain precisely Ineq. (5.12).

We further put

$$q_2(t) \;:=\; \int_0^t (\|f\|_{[0,\bar{Y}]} + h(\tau))\,d\tau \;-\; y(t)\,. \tag{5.15}$$

By assumption (iii) and Ineq. (5.11), the function q_2 is nondecreasing, and Lemma 5.3 for $g(u) = u$ and arbitrary $T > 0$ implies that the function

$$Q_2(t) \;:=\; \int_0^t y(\tau)(\|f\|_{[0,\bar{Y}]} + h(\tau))\,d\tau \;-\; \frac{1}{2}\,y^2(t) \tag{5.16}$$

is nondecreasing in $[0, \infty[$.

Let $S \;:=\; \{t_j\}_{j=0}^n$, $0 = t_0 < t_1 < \ldots < t_n$ be an arbitrary sequence, and put

$$V(S) \;:=\; \sum_{j=1}^n \left|y^2(t_j) - y^2(t_{j-1})\right|\,. \tag{5.17}$$

In the above definition of $V(S)$, we first eliminate monotone parts of the sequence $\{y^2(t_j)\}$. We put $j_0 := 0$, and for $k \ge 1$ we define by induction the sets M_k of all indices

$i \geq j_{k-1}$ such that the sequence $\{y^2(t_j)\}_{j=j_{k-1}}^{i}$ is monotone. We then put $j_k := \max M_k$ until $j_k = n$ for some $k = n'$. Then the sequence $\{y^2(t_{j_k})\}_{k=0}^{n'}$ is alternating, that is,

$$\left(y^2(t_{j_{k+1}}) - y^2(t_{j_k})\right)\left(y^2(t_{j_k}) - y^2(t_{j_{k-1}})\right) \;<\; 0 \quad \forall\, k = 1, \dots, n'-1\,, \tag{5.18}$$

and the identity

$$V(S) \;=\; \sum_{k=1}^{n'} \left|y^2(t_{j_k}) - y^2(t_{j_{k-1}})\right| \tag{5.19}$$

holds. We now have for every $k = 1, \dots, n'$ either $(-1)^k(y^2(t_{j_k}) - y^2(t_{j_{k-1}})) > 0$ and

$$V(S) \;=\; y^2(0) - y^2(t_n) + 2\sum_{i=1}^{k'}\left(y^2(t_{j_{2i}}) - y^2(t_{j_{2i-1}})\right)\,, \quad k' = \left[\frac{n'}{2}\right],$$

or $(-1)^k(y^2(t_{j_k}) - y^2(t_{j_{k-1}})) < 0$ and

$$V(S) \;=\; y^2(0) - y^2(t_n) + 2\sum_{i=0}^{k''}\left(y^2(t_{j_{2i+1}}) - y^2(t_{j_{2i}})\right)\,, \quad k'' = \left[\frac{n'-1}{2}\right].$$

Using the fact that the function (5.16) is nondecreasing, we obtain in both cases

$$V(S) \;\leq\; y^2(0) + 4\int_0^{t_n} y(\tau)(\|f\|_{[0,\bar{Y}]} + h(\tau))\,d\tau \;\leq\; y^2(0) + 4\left(Y\,\|f\|_{[0,\bar{Y}]} + H\,\bar{Y}\right). \tag{5.20}$$

Since the sequence S was arbitrary, the estimate (5.13) follows. Especially, the function $y^2(t)$ tends to a finite limit as $t \to \infty$. Since y is integrable, this limit must be zero.
Let now f be an arbitrary nonnegative continuous function. For $u \geq 0$ put

$$g(u) \;:=\; \max\{0,\, \min\{1,\, 2-u\}\}\,, \qquad G(u) \;:=\; \int_0^u g(v)\,dv\,.$$

By Lemma 5.3, the function

$$Q(t) \;:=\; \int_0^t g(y(\tau))\,(f(y(\tau)) + h(\tau))\,d\tau \;-\; G(y(t))$$

is nondecreasing in $[0,\infty[$. For $t \geq 0$ put $y^*(t) := G(y(t)) \leq y(t)$, $q_1^*(t) := \int_0^t (F^* + h(\tau))\,d\tau - y^*(t)$, where $F^* := \|f\|_{[0,2]}$. Then for every $t > s > 0$ we have $q_1^*(t) - q_1^*(s) \geq Q(t) - Q(s) \geq 0$, hence q_1^* is nondecreasing.
We are now in the previous situation, with $y^*(t)$, $q_1^*(t)$, $f^*(u) \equiv F^*$ instead of $y(t)$, $q_1(t)$, $f(u)$, respectively. This enables us to conclude that $\lim_{t\to\infty} y^*(t) = 0$. In particular, there exists $T > 0$ such that $y^*(t) \leq 1$ for $t \geq T$, hence $y(t) = y^*(t)$ for $t \geq T$ and Proposition 5.4 is proved. □

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