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# Super–Brownian motions in higher dimensions with absolutely continuous measure states


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# SUPER-BROWNIAN MOTIONS IN HIGHER DIMENSIONS WITH ABSOLUTELY CONTINUOUS MEASURE STATES

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**Abstract.** Continuous super-Brownian motions in two and higher dimensions are known to have singular measure states. However, by weakening the branching mechanism in an irregular way they can be forced to have absolutely continuous states. The sufficient conditions we impose are identified in a couple of examples with irregularities in only one coordinate. This includes the case of branching restricted to some densely situated ensemble of hyperplanes.

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# 1. INTRODUCTION

## 1.1. Motivation

Consider a $D$-dimensional *super-Brownian motion* $\mathcal{X} = \{\mathcal{X}_t;t\geq 0\}$ with constant *branching rate* $\rho>0$, related (via Laplace transition functionals) to the equation

$$(1.1.1) \qquad (\partial/\partial t)v = \Delta v - \rho v^2 \quad \text{on} \quad \mathbb{R}_+\times\mathbb{R}^D.$$

It is well-known that in dimensions $D\geq 2$, the states $\mathcal{X}_t$ of $\mathcal{X}$ are *singular* measures (Dawson and Hochberg (1979)), whereas in the one-dimensional case they are absolutely continuous, and a corresponding density field can even be chosen is such a way that it is jointly continuous and satisfies a stochastic equation (see Konno and Shiga (1988) or Reimers (1989)).

The *purpose of this paper* is to show that, by changing to a sufficiently *irregular* branching rate $\rho$, even in higher dimensions super-Brownian motions can be forced to have *absolutely continuous* states.

The idea is to restrict the branching effect to a fractal set of space points *(fractal catalytic medium)*. Then the heat flow can more effectively smear out the population mass possibly resulting in absolutely continuous measure states. From this point of view, the only problem is to guarantee that the catalytic set is not too diffuse, i.e. that it can be hit by the "underlying" motion (think of an approximating particle branching Brownian motion), in other words, that the motion component "will feel" the catalytic set.

## 1.2. Some Review of the One-dimensional Case

It might be useful to discuss at this point what is known in the *one-dimensional case* $D=1$. Here Brownian particles have a positive occupation density *(Brownian local time)* on a single point set, say $\{c\}$. Hence, it is actually possible to restrict the branching effect to this "extremely thin" set $\{c\}$, more precisely, to describe the branching rate $\rho$ by the Dirac $\delta$-function $\delta_c$. Consequently, branching is allowed only at $c$ and there with an unbounded intensity, whereas outside $c$ only the heat flow acts.

The resulting *super-Brownian motion with a single point catalyst* is actually non-degenerate and even lives (excluding the initial time point) on the set of absolutely continuous states. Moreover, it has a series of interesting properties around the catalyst, significantly different from the ones in the case of a regular branching rate; see Dawson and Fleischmann (1993), Dawson et al. (1993), or the recent survey Fleischmann (1993).

More generally, to *any* finite measure $\rho(dy)$ on the real line $\mathbb{R}$ there exists a (one-dimensional) super-Brownian motion with branching rate *formally* described by the *generalized* Radon-Nikodym derivative $\frac{\rho(dy)}{dy}$ with respect to the Lebesgue measure $dy$. Moreover, $\rho$ may even be *time-dependent*, that is, a fairly general *kernel* $\rho(t,dy)$ from the half line $\mathbb{R}_+$ into the set of all tempered measures on $\mathbb{R}$; see Dawson and Fleischmann (1991, 1992).

Nevertheless, under not too restrictive conditions, the resulting super-Brownian motion $\mathcal{X}=\{\mathcal{X}_t;t\geq 0\}$ with *branching rate kernel* $\rho$, or even (one-dimensional) superprocesses with a more general motion law and branching mechanism, may have absolutely continuous states; see Dawson et al. (1991).

### 1.3. Additive Functional Approach

If $\rho$ be a branching rate kernel as just discussed, and $W = \{W_t;t\geq 0\}$ a one-dimensional (continuous) *Brownian motion*, then by

$$A_\rho(dt) := dt \int \rho(t,dy)\, \delta_y(W_t) \tag{1.3.1}$$

we can formally associate a continuous additive (non-negative) functional $A_\rho$ of $W$, interpreted as the *collision local time* $\mathbf{L}[w,\rho]$ of $W$ with the deterministic path $\rho$ (*"occupation density of $W$ at $\rho$"*). For instance, if $\rho\equiv\delta_c$ as in the single point-catalytic super-Brownian motion above, then $A_\rho$ is nothing else than the *Brownian local time* $\mathbf{L}^c(\mathrm{d}t) = \delta_c(W_t)\,\mathrm{d}t$ at the point c.

In general, the additive functional $A_\rho(\mathrm{d}t)$ provides a more sophisticated way to think of the branching rate in the model. In fact, $A_\rho(\mathrm{d}t)$ can be interpreted as the rate of branching at time $t$ at $W_t$, the (random) position of an "infinitely small particle hidden in the cloud" $\mathcal{X}_t$.

In contrast to the other papers [Hf,Eq,At,Va,Li] mentioned above, in this note *Dynkin's additive functional approach* to superprocesses is followed. That is, for the description of the $D$-dimensional super-Brownian motion $\mathcal{X}$ we use a *continuous additive functional* $A$ of the $D$-dimensional Brownian motion $W$ instead of a (deterministic) branching rate kernel $\rho$. As above, $A(\mathrm{d}t)$ is interpreted as the *rate of branching* at time $t$ at $W_t$, the location of an infinitesimal small particle hidden in the cloud $\mathcal{X}_t$.

Our approach in the present paper devoted to the higher-dimensional case is to impose sufficiently strong *technical conditions* on the functional $A$ (see the Definitions 2.6.1 and 2.4.7 below), which guarantee that a super-Brownian motion with $A$ as branching rate functional has absolutely continuous states.

That such conditions are meaningful at all will be demonstrated by discussing a couple of *examples* with the formal structure (1.3.1) (but now in $D$ dimensions), see Section 4 below. To mention at this point only one of them, think of a branching rate kernel $\rho$ of the form

$$\rho(t,\mathrm{d}y) = \rho_d(t,y_d)\mathrm{d}y_d\ \rho_1(\mathrm{d}y_1), \qquad y = [y_d,y_1] \in \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^D,$$

where $\rho_d$ is a bounded measurable function whereas $\rho_1(\mathrm{d}y_1)$ is a realization of a *stable random measure* $\sum_{i=1}^{\infty} \alpha_i \delta_{x(i)}$ on $\mathbb{R}$ of sufficiently small index. In other words, the branching effect is restricted to a *densely situated collection of randomly selected hyperplanes;* see Example 4.4.4 below. In particular, if $\rho_d \equiv 1$, given $\rho_1$, the corresponding time-homogeneous branching rate functional $A_\rho$ is just the Brownian local time at this weighted ensemble of hyperplanes.

To be honest, we emphasize at this point that all of our examples have *irregularities in at most one coordinate.* From a technical point of view, this is a type of reduction to the one-dimensional case. Examples concerning truly higher-dimensional irregularities would involve a more general class of higher-dimensional collision local times which is a more delicate problem (we refer to Barlow et al. (1991)). Consequently, the present note is only a first step in the study of higher-dimensional superprocesses with absolutely continuous states. Examples with irregularities in more than one dimension and a deeper understanding of conditions which guarantee absolutely continuous states seem to need tools which are beyond the scope of the present note.

Concerning the construction of a super-Brownian motion $\mathcal{X}$, we completely rely on Dynkin (1991). There some *moment conditions* on the additive functional $A$ are imposed which guarantee the existence of $\mathcal{X}$. Applied to the special case $D=1$, $A=A_\rho$ as in (1.3.1), such conditions are stronger than those used in [Hf,Va]. On the other hand, the additional assumptions imposed in the present paper to guarantee absolutely continuous states when specialized to $D=1$ and $A=A_\rho$, cover the results of [Va] (applied to a Brownian motion law and a branching mechanism with finite second moment).

As in [Va], the absolute continuity of the states of $\mathcal{X}$ is shown via the construction of *fundamental solutions* of the related non-linear integral equation *(cumulant equation)* by a regularization procedure. In [Va] purely analytic methods were used (contraction principle related to an $L^1$-space) for the construction of fundamental solutions. In contrast, in this note we exploit some *Grönwall inequality type techniques* which are adapted to the additive functional approach.

### 1.4. Outline

The structure of the paper is as follows. In Section 2 we state the results, the main point is Theorem 2.6.2. Proofs concerning the fundamental solutions of the cumulant equation and the absolute continuity of the measure states follow in Section 3. The final section is devoted to examples.

We assume that the reader is familiar with the basic notion and properties of super-Brownian motions; see Dawson (1992) for a recent survey.

## 2. RESULTS

### 2.1. Preliminaries: Some Terminology

Start by introducing some terminology. We call $\underline{\varepsilon} = \{\varepsilon_n; n \geq 1\}$ a *zero sequence* if $0 < \varepsilon_n \leq 1$, $n \geq 1$, and $\varepsilon_n \underset{n \to \infty}{\longrightarrow} 0$, whereas we set $\underline{\varepsilon} = 0$ if $\varepsilon_n \equiv 0$. To avoid double indices, sometimes we write also $\varepsilon(n)$ instead of $\varepsilon_n$, for instance. Integrals $\int \nu(dx)\, f(x)$ are often written as $(\nu, f)$. A lower index + on a symbol of a set will always refer to the subset of all of its non-negative members.

Fix a dimension $D \geq 1$. Let $I$ be a (non-empty) *finite* subinterval of $\mathbb{R}_+ := [0, \infty)$, and write $[L,T]$ for the *smallest closed* interval which covers $I$. Denote by $\Phi$ and $\Phi^I$ the set of all *bounded measurable* functions $\varphi: \mathbb{R}^D \mapsto \mathbb{R}$ and $u: I \times \mathbb{R}^D \mapsto \mathbb{R}$, respectively. We endow $\Phi$ and $\Phi^I$ with the topology of *bounded pointwise convergence*. We will use the symbol $\xrightarrow{\text{bp}}$ to denote this convergence. (Recall that functions converge boundedly pointwise if they are uniformly bounded and converge pointwise.) Then $\Phi$ and $\Phi^I$ are *Banach algebras* with respect to the pointwise product of functions.

Write $\mathcal{M}_f$ for the set of all *finite* measures defined on $\mathbb{R}^D$, endowed with the topology of *weak* convergence.

Let $W := [W, \Pi_{s,a}, s \geq 0, a \in \mathbb{R}^D]$ denote the canonical (continuous) *Brownian motion* in $\mathbb{R}^D$ with generator $\kappa\Delta$, where the *diffusion constant* $\kappa > 0$ is fixed once and for all. Write

$$p(t,y) := (4\pi\kappa t)^{-D/2} \exp[-|y|^2/4\kappa t], \qquad t>0,\ y \in \mathbb{R}^D,$$

for the corresponding *Brownian transition density function*, and let $\{S_t; t \geq 0\}$ denote the related *Brownian contraction semi-group* on $\Phi$. Set

$$(S^I\varphi)(s,a) := S_{T-s}\varphi(a) = \Pi_{s,a}\varphi(W_T), \qquad \varphi \in \Phi,\ s \in I,\ a \in \mathbb{R}^D,$$

for the heat flow on $I$ with *"terminal condition"* $\varphi$. Note that $S^I$ is a (linear) *contraction operator* of $\Phi$ into $\Phi^I$.

## 2.2. Branching Rate Functionals

Let $A(\mathrm{d}t)$ always denote a (non-negative) *continuous additive functional* of the Brownian motion $W$. Consequently, given $W$, the measure $A(\mathrm{d}t)$ on $\mathbb{R}_+$ is locally finite and does not carry mass at any single point set. On the other hand, if $(s,t)$ is an open subinterval of $\mathbb{R}_+$, then $A(s,t) := A((s,t))$ is assumed to be measurable with respect to the universal completion of the $\sigma$-field generated by $\{W_r; s<r<t\}$.

Superprocesses where the branching is governed by certain additive functionals $A$ of the underlying motion Markov process had been introduced by Dynkin (1991). The conditions on $A$ imposed there are stronger than needed. However the construction of superprocesses with much more general additive functionals $A$ will be provided in Dynkin (1993). For our purpose, we leave open the problem how such conditions should look like, but to be on a firm base, we introduce the following definition.

**Definition** 2.2.1 **(branching rate functionals).** A (non-negative) continuous additive functional $A$ of $W$ is called a *branching rate functional* if there exists a time-inhomogeneous measure-valued Markov process $\mathcal{X} = [\mathcal{X},\mathbb{P}_{s,\mu},s\geq 0,\mu\in\mathcal{M}_f]$ with *Laplace transition functional*

(2.2.2) $$\mathbb{P}_{s,\mu}\exp\langle\mathcal{X}_t,-\varphi\rangle = \exp\langle\mu,\ -u_\varphi(s,\bullet,t)\rangle, \quad 0\leq s\leq t,\ \mu\in\mathcal{M}_f,\ \varphi\in\Phi_+.$$

where $u_\varphi\geq 0$ is (uniquely) determined as (bounded) solution of the *cumulant equation*

(2.2.3) $$u(s,a,t) = S_{t-s}\varphi(a) - \Pi_{s,a}\int_s^t A(\mathrm{d}r)\ u^2(r,W_r), \quad 0\leq s\leq t,\ a\in\mathbb{R}^D.$$

In this case, $\mathcal{X}$ is called a *super-Brownian motion with branching rate functional* $A$. ■

## 2.3. Existence of Super-Brownian Motions

In this subsection we review the existence of super-Brownian motions under the following (exponential) moment assumptions taken from condition 1.2.C in Dynkin (1991).

**Definition** 2.3.1 **(exponential moment assumptions).** Let $\mathfrak{A}^I$ denote the set of all those continuous additive functionals $A$ of the Brownian motion $W$ satisfying the following moment conditions:

(2.3.2) $$\Pi_{s,a}\exp[\lambda A(s,T)] < \infty, \quad s\in I,\ a\in\mathbb{R}^D,\ \lambda>0,$$

(2.3.3) $$\sup\{\Pi_{s,a}A(s,T);\ s\in I,\ a\in\mathbb{R}^D\} < \infty.$$

Write $A\in\mathfrak{A}$ if $A$ belongs to $\mathfrak{A}^I$ for all (finite) intervals $I$. ■

*Examples* of such branching rate functionals will be discussed in Subsection 4.2 below.

The existence of superprocesses with branching rate functional in $\mathfrak{A}$ is due to Dynkin (1991), Theorem 1.1; we apply it to Brownian motion and continuous branching mechanism:

**Lemma** 2.3.4 **(existence of super-Brownian motions with branching rate functionals in $\mathfrak{A}$).** *To each functional $A$ in $\mathfrak{A}$ there exists a super-Brownian* *motion $\mathcal{X}$ with $A$ as branching rate functional. $\mathcal{X}$ has the following* **expectation** *and* **variance** *expressions: For $0 \le s \le t$, $\mu \in \mathcal{M}_f$, and $\varphi \in \Phi_+$,*

$$\mathbb{P}_{s,\mu}(\mathcal{X}_t, \varphi) = \int \mu(da)\ S_{t-s}\varphi(a), \tag{2.3.5}$$

$$\mathbb{V}\mathbf{ar}_{s,\mu}(\mathcal{X}_t, \varphi) = 2 \int \mu(da)\ \Pi_{s,a} \int_s^t A(dr)\ (S_{t-r}\varphi)^2(W_r). \tag{2.3.6}$$

Note that for non-vanishing $A, \mu, \varphi$ and $s<t$, the variance is strictly positive, i.e. that $\mathcal{X}$ is really random.

## 2.4. $\vartheta$-Regular Branching Rate Functionals

In this subsection we introduce the technical conditions on a branching rate functional $A$ which we will later show guarantee the existence of fundamental solutions needed for our approach to absolute continuity.

From now on assume that $I$ is a *halfopen* interval $[L,T)$, $0 \le L < T$. Write

$$(S^I\vartheta_\varepsilon)(r,y) := \int \vartheta(dz)\ p(\varepsilon+T-r, y-z), \quad \vartheta \in \mathcal{M}_f,\ \varepsilon>0,\ r \in I,\ y \in \mathbb{R}^D, \tag{2.4.1}$$

for the heat flow $S^I\vartheta_\varepsilon$ on $I$ terminated at time $T$ by the *regularized measure*

$$\vartheta_\varepsilon := \vartheta S_\varepsilon, \quad \vartheta \in \mathcal{M}_f,\ \varepsilon>0, \tag{2.4.2}$$

with density function (denoted by the same symbol $\vartheta_\varepsilon$ by an abuse of notation):

$$\vartheta_\varepsilon(y) = \int \vartheta(dz)\ p(\varepsilon, y-z) =: \vartheta * p(\varepsilon)(y), \quad \vartheta \in \mathcal{M}_f,\ \varepsilon>0,\ y \in \mathbb{R}^D. \tag{2.4.3}$$

Set $S^I\vartheta := S^I\vartheta_0$ in the formal boundary case $\varepsilon=0$ in (2.4.1). Note that

$$S^I\vartheta_\varepsilon(r,\bullet) \xrightarrow[\varepsilon \downarrow 0]{bp} S^I\vartheta(r,\bullet), \quad \vartheta \in \mathcal{M}_f,\ r \in I, \tag{2.4.4}$$

by bounded convergence.

**Notation** 2.4.5. Recalling our convention in the beginning of Subsection 2.1, let $\underline{\varepsilon}$ be a zero sequence or $\underline{\varepsilon}=0$. Moreover, let $A$ be a branching rate functional, $\vartheta \in \mathcal{M}_f$, $s \in I=[L,T]$, and $a \in \mathbb{R}^d$. For $k \ge 0$, $t \in [s,T]$, and $m,n \ge 1$, set

$$H^{k,t}_{m,n} := \Pi_{s,a} \int_t^T A(d\tau)\ [S^I\vartheta_{\varepsilon_m}]^2(\tau, W_\tau) \left(\int_s^\tau A(dr)\ [S^I\vartheta_{\varepsilon_n}](r, W_r)\right)^k. \quad \blacksquare \tag{2.4.6}$$

**Definition** 2.4.7 (**$\vartheta$-regular branching rate functionals**). Recall that $I=[L,T)$, $0 \le L < T$, is fixed. Let $\vartheta$ belong to $\mathcal{M}_f$ and $A$ be a branching rate functional. $A$

is called $\vartheta$-*regular*, if there is a zero sequence $\underline{\varepsilon}$ (depending on $I,\vartheta,A$), called $\vartheta$-*admissible* sequence, with the following properties. For each fixed $s\in I$ and $a\in\mathbb{R}^d$,

(2.4.8) $$\limsup_{n\to\infty} \Pi_{s,a} \left[\int_s^t A(\mathrm{d}r)\ S^I\vartheta_{\varepsilon(n)}(r,W_r)\right]^k < \infty, \quad s\le t<T,\ k\ge 1,$$

(2.4.9) $$\limsup_{m,n\to\infty} H^{k,t}_{m,n} \xrightarrow[t\uparrow T]{} 0, \quad k\ge 0,$$

(2.4.10) $$(\lambda^k/k!)\ \limsup_{m,n\to\infty} H^{k,s}_{m,n} \xrightarrow[k\to\infty]{} 0, \quad \lambda>0.$$

If $A$ is $\delta_{\mathfrak{z}}$-regular for some $\mathfrak{z}\in\mathbb{R}^D$, then we say that $A$ is *regular at* $\mathfrak{z}$. ∎

Formally speaking, a branching rate functional $A$ is $\vartheta$-regular, if some higher moment conditions concerning certain functionals of $A$ are fulfilled. To clarify these conditions, in Subsection 4.3 below we will exhibit a couple of *examples* of branching rate functionals $A$ which are regular at $\mathfrak{z}$. But let us sketch it at this point.

All these examples have the formal structure

(2.4.11) $$A_\xi(\mathrm{d}r) = \mathrm{d}r \int \xi(r,\mathrm{d}y)\ \delta_y(W_r)$$

where $\xi$ is a *kernel*. As above we interpret $A_\xi$ as *collision local time* of the $D$-dimensional Brownian motion $W$ and the deterministic path $\xi$. An enriched version of $A_\xi$ is the random measure $\mathbf{L}[W,\xi]$ on the product space $\mathbb{R}_+\times\mathbb{R}^D$ defined by

(2.4.12) $$\int_s^T A_\xi(dr)\ f(r,W_r) = \int_s^T\!\!\int \mathbf{L}[W,\xi](\mathrm{d}r,\mathrm{d}y)\ f(r,y), \qquad f\in\Phi^I_+.$$

(For a rigorous development of collision local times we again refer to Barlow et al. (1991) or to Evans and Perkins (1993).)

Specialized to the single point measure $\vartheta=\delta_{\mathfrak{z}}$ (aiming at a fundamental solution with terminal condition $\delta_{\mathfrak{z}}$, $\mathfrak{z}\in\mathbb{R}^D$) the requirements in Definition 2.4.7 are then some higher moment conditions (concerning the law of $W$) on functionals of the form

(2.4.13) $$\int_s^T\!\!\int \mathbf{L}[W,\xi](\mathrm{d}r,\mathrm{d}y)\ p(T-r,\mathfrak{z}-y),$$

or even worse with $p^2$ instead of $p$ (note that the integrand becomes singular as $[r,y] \to [T,\mathfrak{z}]$).

Recall that the expected total mass $\Pi_{0,a} A(\mathbb{R}_+)$ of an additive functional $A$ which in the present case equals $\Pi_{0,a} \mathbf{L}[W,\xi](\mathbb{R}_+ \times \mathbb{R}^D)$, as a function of $a$, is usually called the *potential* of the additive functional $A$ (see, for instance, Blumenthal and Getoor (1968), Chapter V). In contrast to this, we interpret the double integral expression in (2.4.13) as the *random $[s,T]$-potential density of the collision local time* $\mathbf{L}[W,\xi]$ *at* $\mathfrak{z}$. (See also the discussion of a special case after condition (4.3.2) below.) Roughly speaking, a branching rate functional $A_\xi$ of the formal structure (2.4.11) is regular at $\mathfrak{z}$ if the random potential density at $\mathfrak{z}$ of the related collision local time $\mathbf{L}[W,\xi]$ has sufficiently good higher moment properties. By the way, $\sup_b \Pi_{s,b} A(s,T) < \infty$ is sufficient for (2.4.8) (but is not always fulfilled in the examples below).

**Remark** 2.4.14. Note that for a $\vartheta$-regular $A$ the assertions (2.4.8)-(2.4.10) also hold for $\varepsilon_m \equiv 0$ or $\varepsilon_n \equiv 0$. In fact, simply apply (2.4.4) and Fatou's lemma. ∎

**Remark** 1.4.15. If $A$ is $\vartheta$-regular with $\vartheta$-admissible sequence $\underline{\varepsilon}$, where $\vartheta$ has the form $\sum_{i=1}^k \lambda_i \delta_{\mathfrak{z}(i)}$ then, for *any* other choice $\lambda'_1,\ldots,\lambda'_k \geq 0$ of the weights $\lambda_1,\ldots,\lambda_k \geq 0$ the branching rate functional $A$ is $\vartheta'$-regular, where $\vartheta' := \sum_{i=1}^k \lambda'_i \delta_{\mathfrak{z}(i)}$, and the *same* zero sequence $\underline{\varepsilon}$ may serve as $\vartheta'$-admissible sequence. Consequently, $\underline{\varepsilon}$ only depends on the finite support of $\vartheta$. ∎

## 2.5. Fundamental Solutions of the Cumulant Equation

Now we are dealing with the question of existence of and convergence to *fundamental solutions* of the cumulant equation (2.2.3) with branching rate functional $A$, i.e. solutions with degenerate "terminal conditions" as $u(T-,\bullet)$

$= \delta_{\mathfrak{z}}$ (instead of $u(T,\bullet) = \varphi \in \Phi_+$). These fundamental solutions will be needed later for the construction of the random density of the measure $\mathcal{X}_T$ at $\mathfrak{z}\in\mathbb{R}^D$ (for Lebesgue almost all $\mathfrak{z}\in\mathbb{R}^D$).

To be more precise, as in the previous subsection set $I=[L,T)$, $0\le L<T$, and for $\vartheta\in\mathcal{M}_f$ consider the *cumulant equation* in the form

$$(2.5.1)\qquad u(s,a) = S^I\vartheta(s,a) - \Pi_{s,a}\int_s^T A(\mathrm{d}r)\, u^2(r,W_r),\qquad s\in I,\ a\in\mathbb{R}^D,$$

that is, now with *measure-valued terminal condition* $u(T-,\bullet) = \vartheta$. For all $\vartheta$-regular branching rate functionals $A$ this is a well-posed problem:

**Theorem** 2.5.2 **(fundamental solutions).** *Let $\vartheta$ belong to $\mathcal{M}_f$ and $A$ be $\vartheta$-regular (recall the Definition* 2.4.7*). There is exactly one measurable non-negative function $U^I[A,\vartheta]$ defined on $I\times\mathbb{R}^D$ which solves equation* (2.5.1). *Moreover,*

$$(2.5.3)\qquad \lambda^{-1}U^I[A,\lambda\vartheta](s,\bullet) \xrightarrow[\lambda\downarrow 0]{bp} S^I\vartheta(s,\bullet),\qquad s\in I,$$

*(first derivative with respect to a small parameter $\lambda$). The solution $U^I[A,\vartheta]$ is continuous with respect to the operation of* **regularization** *of $\vartheta$ (recall* (2.4.2)*) in the following sense: If $\underline{\varepsilon} := \{\varepsilon_n; n\ge 1\}$ is a $\vartheta$-admissible sequence (according to the Definition* 2.4.7*) then*

$$(2.5.4)\qquad U^I[A,\vartheta_{\varepsilon_n}](s,\bullet) \xrightarrow[n\to\infty]{b\,p} U^I[A,\vartheta](s,\bullet),\qquad s\in I.$$

The proof of this key theorem will be provided in Subsection 3.1 below.

## 2.6. Main Result: Absolutely Continuous States

Consider a super-Brownian motion $\mathcal{X}$ with branching rate functional $A$ which starts off at time $s\ge 0$ with any initial state $\mu$ in $\mathcal{M}_f$. (We do not necessarily impose that $A\in\mathfrak{A}$.) The purpose of this subsection is to formulate our main result which states that to a fixed time $T>s$ the random measure $\mathcal{X}_T$ is a.s. *absolutely continuous*, provided that for $I=[L,T)$ with $s\le L<T$ (i.e. in a lower neighborhood of $T$) the branching rate functional $A$ satisfies

conditions as in Definition 2.4.7 but with $\vartheta=\delta_{\mathfrak{z}}$, for Lebesgue almost all $\mathfrak{z}\in\mathbb{R}^D$. To be more precise, we introduce the following definition:

**Definition** 2.6.1 (**a.e.-regular branching rate functionals**). Fix a halfopen interval $I:=[L,T)$, $0\leq L<T$. A branching rate functional $A$ is said to be *a.e.-regular* if there exists a Borel subset $N$ of $\mathbb{R}^D$ of Lebesgue measure $0$, called an *exceptional* set, such that $A$ is $\vartheta$-regular (Definition 2.4.7) for all point measures $\vartheta$ on $\mathbb{R}^D$ with finite support contained in $\mathbb{R}^D\backslash N$. ∎

Roughly speaking, an a.e.-regular branching rate functional is regular at *Lebesgue almost every* $\mathfrak{z}$. In the case of a branching rate functional $A=A_{\xi}$ related to the collision local time $\mathbf{L}[W,\xi]$ as discussed after Definition 2.4.7, the present definition requires, loosely speaking, that for Lebesgue almost all $\mathfrak{z}\in\mathbb{R}^D$ the I-potential density of $\mathbf{L}[W,\xi]$ at $\mathfrak{z}$ has well-behaved higher moments. *Examples* of such a.e.-regular branching rate functionals will be discussed in Subsection 4.4 below.

Now we are in a position to state our *main result:*

**Theorem** 2.6.2 (**absolutely continuous states**). *Let* $\mathcal{X} = [\mathcal{X},\mathbb{P}_{s,\mu},s\geq 0,\mu\in\mathcal{M}_f]$ *be a super-Brownian motion with branching rate functional A (recall our Definition* 2.2.1*). Assume that A is a.e.-regular concerning the interval* $I=[L,T)$ *(recall Definition* 2.6.1*). Then for fixed* $0\leq s\leq L\leq T$ *and* $\mu\in\mathcal{M}_f$, *there exists a random measurable function* $x_T$ *on* $\mathbb{R}^D$ *such that*

$$\mathbb{P}_{s,\mu}\left\{\mathcal{X}_T(\mathrm{d}z) = x_T(z)\mathrm{d}z\right\} = 1.$$

*Moreover, for each finite collection* $\mathfrak{z}_1,\dots,\mathfrak{z}_k$ *of points in* $\mathbb{R}^D\backslash N$ *(where N is a Lebesgue zero set as in Definition* 2.6.1*), the Laplace function of the random vector* $[x_T(\mathfrak{z}_1),\dots,x_T(\mathfrak{z}_k)]$ *with respect to* $\mathbb{P}_{s,\mu}$ *is given by*

$$\text{(2.6.3)}\qquad \mathbb{P}_{s,\mu}\exp\left[-\textstyle\sum_{i=1}^k \lambda_i x_T(\mathfrak{z}_i)\right] = \exp\langle\mu,-u(s,\bullet)\rangle,\qquad \lambda_1,\dots,\lambda_k\geq 0,$$

*where u is the continuation of the fundamental solution* $U^I[A,\vartheta]$ *(according to*

*Theorem* 2.5.2) *to the interval* $[s,T]$, *and* $\vartheta := \sum_{i=1}^{k} \lambda_i \delta_{\mathfrak{z}(i)}$. *In particular, the following expectation and variance formulas hold:*

(2.6.4) $$\mathbb{P}_{s,\mu}\mathfrak{X}_T(\mathfrak{z}) = \int \mu(\mathrm{d}a)\; p(T-s,\mathfrak{z}-a),$$

(2.6.5) $$\mathbb{V}\mathbf{ar}_{s,\mu}\mathfrak{X}_T(\mathfrak{z}) = 2 \int \mu(\mathrm{d}a)\; \Pi_{s,a} \int_s^T A(\mathrm{d}r)\; p(T-r,\mathfrak{z}-W_r).$$

## 2.7. A Basic Lemma

The previous theorem is heavily based on Theorem 2.5.2 in conjunction with the following lemma, taken from [Va]. Actually, there $\delta$-measures are approximated by uniform distributions on small intervals, but our change to Gaussian densities $p(\varepsilon) = p(\varepsilon,\bullet)$ with small variance can easily be justified by a modification of *Lebesgue's density theorem.* Moreover, the vector assertions in the end of the lemma follow from a simple modification of the proof of the other statements given in [Va].

**Lemma** 2.7.1 **(basic lemma).** *Let* $\nu$ *be a random element in the space* $M_f$ *of finite measures over a probability space* $[\Omega,\mathcal{F},\mathbf{P}]$ *satisfying the following two conditions:*

**(i)** *There is a Borel subset* $N$ *of* $\mathbb{R}^D$ *of Lebesgue measure* $0$ *such that for each* $\mathfrak{z}\in\mathbb{R}^D\setminus N$ *there is a zero sequence* $\underline{\varepsilon}_{\mathfrak{z}}$ *such that* $\nu * p(\varepsilon_{\mathfrak{z},n})(\mathfrak{z})$ *converges in law to a random variable* $\eta(\mathfrak{z})$ *in* $\mathbb{R}_+$ *as* $n\to\infty$.

**(ii)** *The expectation of* $\eta(\mathfrak{z})$, *say* $\mathbf{e}(\mathfrak{z})$, *depends on* $\mathfrak{z}\notin N$ *in a measurable way, is locally integrable and*

$$\mathbf{P}(\nu,\varphi) = \int \mathrm{d}z\; \varphi(z)\; \mathbf{e}(z), \qquad \varphi\in\Phi_+.$$

*Then, (over the same probability space) there exists a random measurable function* $f$ *on* $\mathbb{R}^D$ *such that* $\mathbf{P}\{\nu(\mathrm{d}z) = f(z)\mathrm{d}z\} = 1$, *and for each* $\mathfrak{z}\notin N$ *the random variables* $f(\mathfrak{z})$ *and* $\eta(\mathfrak{z})$ *have the same distribution. In particular,* $\nu$ *is an* **absolutely continuous** *measure with* **P**-*probability one.*

*Moreover, if* **(i)** *even holds for vectors, i.e. there is an exceptional* *set* $N$ *such that for each choice of finitely many points* $\mathfrak{z}_1,\dots,\mathfrak{z}_k$ *in* $\mathbb{R}^D\backslash N$ *there is a zero sequence* $\underline{\varepsilon}$ *(depending on* $\mathfrak{z}_1,\dots,\mathfrak{z}_k$*) such that*

$$[\nu*p(\varepsilon_n)(\mathfrak{z}_1),\dots,\nu*p(\varepsilon_n)(\mathfrak{z}_k)] \underset{n\to\infty}{\Rightarrow} \text{ some } [\eta(\mathfrak{z}_1),\dots,\eta(\mathfrak{z}_k)] \quad \text{in law,}$$

*then* $[f(\mathfrak{z}_1),\dots f(\mathfrak{z}_k)] = [\eta(\mathfrak{z}_1),\dots,\eta(\mathfrak{z}_k)]$ *in law.*

Roughly speaking, if the "local densities" $\frac{d\nu}{d\mathfrak{z}} = (\nu,\delta_{\mathfrak{z}}) =: \eta(\mathfrak{z})$ of $\nu$ at $\mathfrak{z}$ exist in law for Lebesgue almost all $\mathfrak{z}$ and their expectations "create" the *full* locally finite intensity measure $\mathbf{P}\nu$, than $\nu$ is a.s. absolutely continuous and the density function is in law given by $\eta$.

The completion of proof of Theorem 2.6.2 is postponed to Subsection 3.2.

# 3. Proofs of the Theorems

## 3.1. Fundamental Solutions: Proof of Theorem 2.5.2

The purpose of this subsection is to prove Theorem 2.5.2. Fix $I=[L,T)$, $0\le L<T$, $\vartheta\in\mathcal{M}_f$, a $\vartheta$-regular branching rate functional $A$, and let $\underline{\varepsilon}$ be a related $\vartheta$-admissible zero sequence according to the Definition 2.4.7. In accordance with Definition 2.2.1, assume that $u_n$ is a (non-negative) solution of the cumulant equation (2.5.1) with $\vartheta$ replaced by $\vartheta_n$ where $\vartheta_n := \vartheta_{\varepsilon(n)}$, $n\ge 1$, (recall the definition (2.4.3); note that each terminal function $\vartheta_n$ is bounded). We want to show that $\{u_n(s,\bullet);\ n\ge 1\}$ is a *Cauchy sequence* in the Banach space $\Phi$, for each fixed $s\in I$.

First of all, for $n\ge 1$ and $s\in I$, we have the following *domination:*

$$0 \le u_n(s,\bullet) \le S^I\vartheta_n(s,\bullet) \le \|\vartheta\|\ p(\varepsilon_n+T-s,0), \tag{3.1.1}$$

where $\|\vartheta\|$ denotes the total mass of $\vartheta$. Moreover, for the $D$-dimensional Brownian density function we have

$$p(\varepsilon+t,0) \le p(t,0) = p(1,0)\ t^{-D/2}, \qquad \varepsilon\ge 0,\ t>0. \tag{3.1.2}$$

Using $|a^2-b^2| = |a+b|\,|a-b|$, for $m,n\ge 1$, $s\in I$ and $a\in\mathbb{R}^D$ we get

$$|u_m-u_n|(s,a) \le |S^I\vartheta_m-S^I\vartheta_n|(s,a) + \Pi_{s,a}\int_s^T A(dr)[S^I\vartheta_m+S^I\vartheta_n](r,W_r)\,|u_m-u_n|(r,W_r).$$

*Iterating* this inequality $K\geq 1$ times yields

$$(3.1.3) \qquad |u_m - u_n|(s,a) \leq \|(S^I\vartheta_m - S^I\vartheta_n)(s,\bullet)\|_\infty + E + F,$$

where $\|\bullet\|_\infty$ denotes the supremum norm, and we set

$$E := \Sigma_{k=1}^K \Pi_{s,a} \int_s^T A(dr_1)\ldots\int_{r(k-1)}^T A(dr_k)$$
$$\Big(\Pi_{i=1}^k [S^I\vartheta_m + S^I\vartheta_n](r_i, W_{r(i)})\Big) \Big|S^I\vartheta_m - S^I\vartheta_n\Big|(r_k, W_{r(k)})$$

and

$$F := \Pi_{s,a} \int_s^T A(dr_1)\ldots\int_{r(K)}^T A(dr_{K+1})$$
$$\Big(\Pi_{i=1}^{K+1} [S^I\vartheta_m + S^I\vartheta_n](r_i, W_{r(i)})\Big) [S^I\vartheta_m + S^I\vartheta_n](r_{K+1}, W_{r(K+1)}).$$

For each $r\in I$,

$$(3.1.4) \qquad q_{m,n}(r) := \|(S^I\vartheta_m - S^I\vartheta_n)(r,\bullet)\|_\infty \leq \|\vartheta\| \; \|p(\varepsilon_m + T - r,\bullet) - p(\varepsilon_n + T - r,\bullet)\|_\infty.$$

Therefore,

$$(3.1.5) \qquad \lim_{m,n\to\infty} \sup_{L\leq r\leq s} q_{m,n}(r) = 0, \qquad s\in I,$$

by the uniform continuity of the Brownian transition density function $p$ on $[c,\infty)\times\mathbb{R}^d$, for $c>0$ fixed. In particular, the first term on the r.h.s. of (3.1.3) is negligible as $m,n\to\infty$ (for fixed $s$).

Concerning the two other terms $E,F$ in (3.1.3), first rearrange the order of integration in all integrals to the reversed order.

In the $k$-th summand of $E$ we introduce the additional indicator $\mathbf{1}\{r_k\leq t\}$, for fixed $t\in[s,T)$. Estimate the absolute value expression from above by $q_{m,n}(r_k)$ defined in (3.1.4), which by (3.1.5) is uniformly small as $m,n\to\infty$. The remaining expectation can be estimated by

$$\leq \text{const}\ \Pi_{s,a}\Big(\int_s^t A(d\tau)\ [S^I\vartheta_m + S^I\vartheta_n](\tau, W_\tau)\Big)^k.$$

Use the simple inequality

$$(3.1.6) \qquad |x+y|^k \leq 2^{k-1}[x^k + y^k], \qquad x,y\geq 0,\ k\geq 1,$$

and (2.4.8) to get a finite limit superior as $m,n\to\infty$. Summarizing, the $k$-th summand of $E$ restricted to $\{r_k\leq t\}$ with a fixed $t$ is negligible as $m,n\to\infty$.

Now we turn our attention to the reverse restriction $\{t<r_k\}$. Recalling that we rearranged the order of integration and passing from the difference

sign to the addition sign, we can turn to the upper estimate

$$\leq \text{const}\ \Pi_{s,a} \int_t^T A(d\tau)\ [S^I\vartheta_m+S^I\vartheta_n]^2(\tau,W_\tau)\ \left(\left[\int_s^\tau A(dr)\ [S^I\vartheta_m+S^I\vartheta_n]\right]\right)^{k-1}.$$

Apply again (3.1.6) and use (2.4.9) to see that the limit superior as $m,n\to\infty$ of this term can be made arbitrarily small by choosing $t$ sufficiently close to $T$. Consequently, we are able to handle $E$, and we are left with $F$.

Write the inner $K$ integrals concerning the variables $r_K,\ldots,r_1$ in a symmetric way getting out a factor $1/K!$ . Apply again (3.1.6) and then (2.4.10) to see that the limit superior of $F$ as $m,n\to\infty$ can be made small by choosing $K$ sufficiently large.

Summarizing, in view of the domination (3.1.1), we established the existence of a non-negative measurable function $u$ on $I\times\mathbb{R}^D$ such that

$$u_n(s,\bullet)\ \xrightarrow[n\to\infty]{b\ p}\ u(s,\bullet), \qquad s\in I.$$

Repeating the procedure from the beginning with this $u$ instead of $u_m$ (relating $u$ to $\varepsilon_m\equiv 0$ but keeping $\varepsilon_n$), and taking into account Remark 2.4.14 we conclude that $u$ solves equation (2.5.1). That is, we *constructed a solution* $u$ with measure-valued terminal condition $\vartheta$.

By even simpler arguments (take $\varepsilon_m\equiv 0\equiv\varepsilon_n$) we conclude that $u$ is *uniquely* determined by the equation. Summarizing, we now have existence, uniqueness and the continuity statement (2.5.4).

It remains to verify the *asymptotic property* (2.5.3). First note that the branching functional $A$ is $\lambda\vartheta$-regular for all $\lambda\geq 0$ (by Remark 1.4.15)). Fix $s\in I$. By the equation (2.5.1) and the domination (3.1.1) (both with $\vartheta$ replaced by $\lambda\vartheta$),

$$(3.1.7)\qquad |U^I[A,\lambda\vartheta]\ -\ \lambda S^I\vartheta|(s,a)\ \leq\ \lambda^2\ \Pi_{s,a}\int_s^T A(dr)\ [S^I\vartheta(r,W_r)]^2.$$

The latter expectation is finite. In fact, take a point $t\in[s,T)$ and split the domain of integration at $t$. For the lower part, estimate one factor of the integrand by a constant and use (2.4.8) with $\varepsilon_n\equiv 0$ and $k=1$. For the other part, apply (2.4.9) with $\varepsilon_m \equiv 0 \equiv \varepsilon_n$ and $k=0$.

From (3.1.7) we conclude that the functions $\lambda^{-1} U^I[A,\lambda\vartheta](s,\bullet)$ pointwise converge to $S^I\vartheta(s,\bullet)$ as $\lambda \downarrow 0$. But they are all dominated by the bounded function $S^I\vartheta(s,\bullet)$. Thus the convergence statement (2.5.3) follows.

This finishes the proof of Theorem 2.5.2. ■

## 3.2. Absolutely Continuous States: Proof of Theorem 2.6.2

Fix a branching rate functional $A$ which is a.e.-regular on the interval $I=[L,T)$ with $0 \le L < T$. Let $N$ be a related exceptional set according to the Definition 2.6.1. Fix a finite sequence $\mathfrak{z}_1,\ldots,\mathfrak{z}_k \notin N$, and let $\underline{\varepsilon}$ be a $\sum_i \delta_{\mathfrak{z}(i)}$-admissible sequence according to the Definition 2.4.7. Finally, choose a starting time point $s \in [0,L]$ and an initial measure $\mu \in \mathcal{M}_f$. In order to show that $\mathcal{X}_T$ fulfills the requirements with respect to $\mathbb{P}_{s,\mu}$ stated in the theorem, by the Markov property, without loss of generality we may assume that $s=L$. The main step in the proof will be to verify the conditions **(i)** and **(ii)** of the basic lemma 2.7.1.

$1^\circ$ *(assumption* **(i)** *of the basic lemma* 2.7.1, *generalized to vectors).* Let $\lambda_1,\ldots,\lambda_k \ge 0$ and set $\vartheta = \sum_i \lambda_i \delta_{\mathfrak{z}(i)}$. Note that $A$ is $\vartheta$-regular and that the zero sequence $\underline{\varepsilon}$ is also $\vartheta$-admissible (Remark 1.4.15). According to the convergence statement (2.5.4),

$$U^I[A,\vartheta_{\varepsilon(n)}](L,\bullet) \xrightarrow[n\to\infty]{bp} U^I[A,\vartheta](L,\bullet). \tag{3.2.1}$$

Hence, we may integrate these functions against the finite measure $\mu$, and by bounded convergence we get:

$$\big(\mu,\ U^I[A,\vartheta_{\varepsilon(n)}](L,\bullet)\big) \xrightarrow[n\to\infty]{} \big(\mu,\ U^I[A,\vartheta](L,\bullet)\big). \tag{3.2.2}$$

Now use the domination (3.1.1) to obtain:

$$\big(\mu,\ U^I[A,\vartheta](L,\bullet)\big) \le \big(\mu,\ S^I\vartheta(T\text{-}L)\big) \le |\lambda|\ \|\mu\|\ p(T\text{-}L,0) \xrightarrow[|\lambda|\downarrow 0]{} 0 \tag{3.2.3}$$

where $|\lambda| := \max_i \lambda_i$. But by assumption (via the Laplace functional (2.2.2) and the cumulant equation (2.2.3)), the l.h.s. of (3.2.2) determines the Laplace transform of the random vector $\left[(\mathcal{X}_T,\delta_{\mathfrak{z}(1)}*p(\varepsilon_n)),\dots,(\mathcal{X}_T,\delta_{\mathfrak{z}(k)}*p(\varepsilon_n))\right]$ with respect to $\mathbb{P}_{L,\mu}$. Therefore, combined with the continuity property (3.2.3), also the r.h.s. of (3.2.2) determines the Laplace transform of a random vector, we denote by $[\eta(\mathfrak{z}_1),\dots,\eta(\mathfrak{z}_k)]$. Consequently,

$$(3.2.4)\qquad \left[(\mathcal{X}_T,\delta_{\mathfrak{z}(1)}*p(\varepsilon_n)),\dots,(\mathcal{X}_T,\delta_{\mathfrak{z}(k)}*p(\varepsilon_n))\right] \underset{n\to\infty}{\Longrightarrow} [\eta(\mathfrak{z}_1),\dots,\eta(\mathfrak{z}_k)]$$

in law, where

$$(3.2.5)\qquad \mathbf{P}\exp\left[-\textstyle\sum_{i=1}^{k}\lambda_i\eta(\mathfrak{z}_i)\right] = \exp\left(\mu,\ -U^{\mathrm{I}}[A,\vartheta](L,\bullet)\right).$$

$2^{\circ}$ *(assumption* **(ii)** *of the basic lemma* 2.7.1*)*. Set $k=1$ and write $\mathfrak{z}$ instead of $\mathfrak{z}_1$. We want to show that the expectation $\mathbf{e}(\mathfrak{z})$ of $\eta(\mathfrak{z})$ is measurable in $\mathfrak{z}\in\mathbb{R}^D\backslash W$ and satisfies

$$\mathbb{P}_{L,\mu}(\mathcal{X}_T,\varphi) = \int dz\ \varphi(z)\ \mathbf{e}(z),\qquad \varphi\in\Phi_+.$$

But by the expectation formula (2.3.5), the l.h.s. coincides with $(\mu S_{T-L},\varphi)$. Thus it suffices to show that

$$(3.2.6)\qquad \left(\mu,\ \delta_{\mathfrak{z}}*p(T-L)\right) = \mathbf{e}(\mathfrak{z}),\qquad \mathfrak{z}\in\mathbb{R}^D\backslash W.$$

Recall that by (3.2.5),

$$\mathbf{P}\exp[-\lambda\eta(\mathfrak{z})] = \exp\left(\mu,\ -U^{\mathrm{I}}[A,\lambda\delta_{\mathfrak{z}}](L,\bullet)\right),\qquad \lambda\geq 0,\ \mathfrak{z}\notin N.$$

But according to (2.5.3) the difference quotient $\lambda^{-1}U^{\mathrm{I}}[A,\lambda\delta_{\mathfrak{z}}](L,\bullet)$ is bounded pointwise convergent to $\delta_{\mathfrak{z}}*p(T-L)$ as $\lambda\downarrow 0$ . Hence (3.2.6) follows by bounded convergence. In particular, $\{\mathbf{e}(\mathfrak{z}),\mathfrak{z}\in\mathbb{R}^D\backslash N\}$ is integrable and the absolute continuity proof is finished.

$3^{\circ}$ *(completion of the Proof of Theorem* 2.6.2*)*. With (3.2.5), (3.2.6) and the coincidence in law as stated in the basic lemma 2.7.1, we already have (2.6.3) and (2.6.4). Finally, the variance formula (2.6.5) can be proved in the same way as the expectation formula, we leave the details to the reader. ∎

# 4. EXAMPLES

## 4.1. Preliminary Remarks

Recall that all of our examples of branching rate functionals $A$ have the formal structure (2.4.11), that is

$$A_\xi(\mathrm{d}r) = \mathrm{d}r \int \xi(r,\mathrm{d}y)\ \delta_y(W_r) \tag{4.1.1}$$

and that via (2.4.12) $A_\xi$ is formally related to the *collision local time* $\mathbf{L}[W,\xi]$ of the D-dimensional Brownian motion $W$ and the (deterministic) branching rate kernel $\xi$.

Our point of view in this section is that the formal $\delta$-function setting can be justified in each of the following examples. In particular, we assume that for the kernels $\xi$ under consideration the continuous additive functionals $A_\xi$ which are formally defined by (4.1.1), really exist and moreover their moments can be estimated from above by the corresponding formal $\delta$-function setting expressions. Of course, here we take advantage of the fact that our examples have irregularities in at most one coordinate.

Recall that to each *time-homogeneous* continuous additive functional $A$ of the Brownian motion $W$ the so-called *Revuz measure* $\xi_A$ on $\mathbb{R}^D$ is associated. It can be defined via

$$(\xi_A,\varphi) = \int \mathrm{d}a\ \Pi_{0,a} \int_0^1 A(\mathrm{d}s)\ \varphi(\mathcal{X}_s), \qquad \varphi\in\Phi_+.$$

If $\xi_A$ is finite and has finite potential in $B_m(0)$, $m\geq 1$, where $B_r(y)$ denotes the open ball in $\mathbb{R}^D$ with center $y$ and radius $r>0$, then

$$\sup_{s\leq t}\left| A(0,s) - \int_0^s \mathrm{d}r \int \xi_A(\mathrm{d}y)\ |B_\varepsilon(y)|^{-1}\ \mathbf{1}\{W_r\in B_\varepsilon(y)\}\right| \xrightarrow[\varepsilon\downarrow 0]{} 0 \tag{4.1.2.}$$

in $\Pi_{0,a}$-probability, $a\in\mathbb{R}^D$, where $|B|$ denotes the volume of $B$; see Theorem 3.12 in Bass (1984). Roughly speaking, the weighted occupation density of W in the vicinity of the support of the Revuz measure $\xi_A$ approaches the additive functional $A$. In other words, $A$ is in fact the collision local time $\mathbf{L}[W,\xi_A](\bullet\times\mathbb{R}^D)$ (recall (2.4.12)), and with (4.1.2) we get a justification for the formal expression (4.1.1) (for time-homogeneous $A$).

Of course, in the *one-dimensional case* the matter drastically simplifies by the existence of the well-behaved family $\{\mathbf{L}^y(\mathrm{d}r); y\in\mathbb{R}\}$ of Brownian local times. Indeed, here we have

$$\mathbf{L}[W,\xi_A](\mathrm{d}[r,y]) = \xi_A(\mathrm{d}y)\ \mathbf{L}^y(\mathrm{d}r)$$

(see, for instance, [RY], Theorem 10.2.9),

$$\Pi_{0,a}\mathbf{L}[W,\xi_A](\mathrm{d}[r,y]) = \xi_A(\mathrm{d}y)\ p(r,y-a)\ \mathrm{d}r,$$

and analogous formulas hold for higher moment measures (see also Proposition 13.2.1 in [RY]).

As in Subsection 2.2, let $I$ denote a finite non-empty subinterval of $\mathbb{R}_+$ with right boundary point $T$. During the discussion of examples, for convenience we will often indicate the dimension $D$ of the $D$-dimensional Brownian transition density function by a lower index: $p =: p_D$.

## 4.2. Examples of Branching Rate Functionals $A$ in $\mathfrak{A}^I$

To warm up, let us consider a few examples of functionals $A_\xi$ satisfying the exponential moment assumptions of Definition 2.3.1.

**Example** 4.2.1 **(bounded regular branching rate).** First of all, the case of a bounded regular branching rate $\xi$ is covered. In fact, assume

$$\text{(4.2.2)}\qquad \xi(r,\mathrm{d}y) = \xi(r,y)\mathrm{d}y, \quad r\in I, \quad \text{with } \xi\in\Phi_+^I$$

(recall Subsection 2.1). Then by (4.1.1), $A_\xi(s,T) \le \|\xi\|_\infty\ |I|$, $s\in I$, (where $|I|$ denotes the length of the interval $I$). Indeed,

$$\text{(4.2.3)}\qquad \int\mathrm{d}y\ \delta_y(z) = \int\mathrm{d}y\ \delta_z(y) = 1, \quad z\in\mathbb{R}^D.$$

Consequently, the requirements (2.3.2) and (2.3.3) are certainly fulfilled, i.e. all functionals $A_\xi$ with a bounded regular $\xi$ belong to $\mathfrak{A}^I$. ∎

**Example** 4.2.4 **(one-dimensional uniformly finite branching rate kernels $\xi$).** Next, in the case $D=1$, a variety of branching rate functionals $A_\xi$ obey the

Definition 2.3.1. In fact, assume that $\xi$ is a kernel satisfying

(4.2.5) $$\|\xi\|_I := \sup_{r\in I} \xi(r,\mathbb{R}) < \infty.$$

Then, for $s\in I$, $a\in\mathbb{R}$, $\lambda>0$, we have

(4.2.6) $$\Pi_{s,a}\exp[\lambda A_\xi(s,T)] \le \Pi_{s,0}\exp\left[\lambda\|\xi\|_I\, \mathbf{L}^0(s,T)\right] \le \Pi_{0,0}\exp\left[\lambda\|\xi\|_I\, \mathbf{L}^0(0,T)\right] < \infty,$$

where $\mathbf{L}^0$ denotes the *Brownian local time* at $0$. Indeed, expand the left hand side in a Taylor series and, for $k\ge 1$, rewrite

(4.2.7) $$\Pi_{s,a}[\lambda A_\xi(s,T)]^k = k!\ \lambda^k\ \Pi_{s,a} \int_s^T A_\xi(\mathrm{d}r_1) \ldots \int_{r(k-1)}^T A_\xi(\mathrm{d}r_k).$$

Use the Markov property at time $r_{k-1}$ and, by (4.1.1),

$$\Pi_{r(k-1),b} \int_{r(k-1)}^T A_\xi(\mathrm{d}r_k) \le \int_{r(k-1)}^T \mathrm{d}r_k \int \xi(r_k,\mathrm{d}y)\ p_1(r_k-r_{k-1},y-b), \qquad b\in\mathbb{R},$$

(with $r_0:=s$ and $b=W_{r(k-1)}$). But the Gaussian densities $p(t,\bullet)$, $t>0$, are maximal at $0$, and by applying the assumption (4.2.5) we find the upper bound

$$\|\xi\|_I \int_{r(k-1)}^T \mathrm{d}r_k\ p_1(r_k-r_{k-1},0) = \|\xi\|_I\ \Pi_{r(k-1),0} \int_{r(k-1)}^T \mathbf{L}^0(\mathrm{d}r_k).$$

Continuing $k$-$1$ times we arrive at the bound

$$k!\ \lambda^k\ \|\xi\|_I^k\ \Pi_{s,0} \int_s^T \mathbf{L}^0(\mathrm{d}r_1) \ldots \Pi_{r(k-1),0} \int_{r(k-1)}^T \mathbf{L}^0(\mathrm{d}r_k) = \lambda^k\ \|\xi\|_I^k\ \Pi_{s,0}[\mathbf{L}^0(s,T)]^k$$

for (4.2.7). Here we used that at all the times $r_i$ "selected by" $\mathbf{L}^0$ the Brownian paths are at $0$, and then the Markov property. Consequently, the first inequality in the claim (4.2.6) is true. By time-homogeneity of the Brownian motion law and the monotonicity of the local time, we may even pass to $s=0$. But all these exponential moments are finite, since the law of $\mathbf{L}^0(0,T)$ with respect to $\Pi_{0,0}$ has Gaussian tails. Hence, the $A_\xi$ under discussion satisfy the conditions (2.3.2) and (2.3.3). Consequently, all branching rate functionals $A_\xi$ attached to uniformly finite branching rate kernels $\xi$ belong to $\mathfrak{A}^I$. ∎

**Example** 4.2.8 **(factored branching rate kernels $\xi$).** To discuss also an irregular $\xi$ in higher dimensions, we consider the following factorization example. Write the dimension $D$ as $D=d+1$, $d\ge 1$, that is we split up an *extra coordinate*. Assume that

(4.2.9) $\qquad \xi(r,\mathrm{d}y) = \xi_d(r,y_d)\mathrm{d}y_d\ \xi_1(r,\mathrm{d}y_1), \qquad r\in I,\ y=[y_d,y_1]\in\mathbb{R}^d\times\mathbb{R},$

where $\xi_1$ is a one-dimensional uniformly finite kernel as in the previous example (satisfying (4.2.5)), whereas $\xi_d$ is a measurable and bounded function on $I\times\mathbb{R}^d$, as in Example 4.2.1 above. In particular, $\xi$ *is irregular in at most one coordinate*. To see that a branching rate functional $A_\xi$ with such $\xi$ belongs to $\mathfrak{A}^I$, we use that $\delta_y(W_r)$ in (4.1.1) also factorizes, namely to $\delta_{y(d)}(W^d_r)\ \delta_{y(1)}(W^1_r)$ where $W=[W^d,W^1]$ and again $y=[y_d,y_1]$. Then we can bound $\xi_d$ by a constant and may apply (4.2.3) to arrive at

$$A_\xi(\mathrm{d}r) \le \text{const}\ \mathrm{d}r \int\xi_1(r,\mathrm{d}y_1)\ \delta_{y(1)}(W^1_r).$$

Then continue as in Example 4.2.4 above. ∎

## 4.3. Examples of $\vartheta$-Regular Branching Rate Functionals

Here we want to discuss the requirements in the Definition 2.4.7 of $\vartheta$-regular branching rate functionals in the case $A=A_\xi$ as indicated in (4.1.1) (regardless whether $A_\xi$ belongs to $\mathfrak{A}^I$ or not). We pay attention only to the most important case $\vartheta = \sum_{j=1}^{l}\delta_{\mathfrak{z}(j)}$ for some fixed $\mathfrak{z}(1),\dots,\mathfrak{z}(l) \in \mathbb{R}^D$ (related to the fundamental solution with terminal condition $\vartheta$). To this end, fix $I=[L,T),\ L<T$.

**Example** 4.3.1 **(one-dimensional kernels $\xi$).** First of all, for a variety of one-dimensional kernels $\xi$, the corresponding branching rate functionals $A_\xi$ fulfill the conditions in Definition 2.4.7. In fact, suppose $D=1$ and that

(4.3.2) $\qquad \limsup_{m\to\infty}\ \sup_{r\in I}\int\xi(r,\mathrm{d}y)\ p_1(\varepsilon_m+T-r,y-\mathfrak{z}(i)) < \infty, \qquad 1\le i\le l,$

for some zero sequence $\underline{\varepsilon}$ (cf. the condition (3.8) in [Va]). Of course, each bounded regular $\xi$ as in (4.2.2) satisfies (4.3.2), and also each single atomic $\xi$, that is $\xi(r,\bullet) \equiv \delta_c$, provided that $c \ne \mathfrak{z}(1),\dots,\mathfrak{z}(l)$. The condition says, roughly speaking, that, as $r$ approaches $T$, the measure $\xi(r,\cdot)$ should have a *finite density of mass "at"* the $\mathfrak{z}(i)$. By the way, this then implies (in the present one-dimensional case) that the super-Brownian motion with branching rate kernel $\xi$ has at time $T$ a finite (random) density of mass at $\mathfrak{z}(1),\dots,\mathfrak{z}(l)$.

To verify the requirements in Definition 2.4.7, start by looking at the expectation expression in the definition 2.4.6 of $H^{k,t}_{m,n}$ with $\vartheta = \Sigma_j \delta_{\mathfrak{z}(j)}$. Write the $k$-th power of the second integral as $k$-fold iterated integrals and interchange the order of all integrations to arrive at

$$(4.3.3)\qquad k!\ \Pi_{s,a} \int_s^T A_\xi(\mathrm{d}r_1)\dots\int_{r(k-1)}^T A_\xi(\mathrm{d}r_k)\ \prod_{i=1}^k \Sigma_{j=1}^l\ p\big(\varepsilon_n+T-r_i, W_{r(i)}-\mathfrak{z}(j)\big)$$
$$\int_{t\vee r(k)}^T A_\xi(d\tau)\ \Big[\Sigma_{j=1}^l\ p\big(\varepsilon_m+T-\tau, W_{\tau-}-\mathfrak{z}(j)\big)\Big]^2.$$

Taking into account the Markov property at time $r_k$, the interior integral can be estimated from above by

$$(4.3.4)\qquad \int_{t\vee r(k)}^T \mathrm{d}\tau\ \int \xi(\tau,\mathrm{d}y)\ p(\tau-r_k, y-W_{r(k)})\ \Big[\Sigma_{j=1}^l\ p\big(\varepsilon_m+T-\tau, y-\mathfrak{z}(j)\big)\Big]^2.$$

By (3.1.2),

$$p(\tau-r_k, y-W_{r(k)})\ \Sigma_{j=1}^l\ p\big(\varepsilon_m+T-\tau, y-\mathfrak{z}(j)\big) \le \text{const}\ (\tau-r_k)^{-1/2}\ (T-\tau)^{-1/2}.$$

This leads to remaining $l$ integrals as in (4.3.2). Pass to their suprema on $\tau \in (r_k,T)$, bring them out of all of the integrals, and note that they have finite limit superiors as $m\to\infty$ by assumption (4.3.2). For (4.3.4) the term

$$(4.3.5)\qquad \text{const} \int_{t\vee r(k)}^T \mathrm{d}\tau\ (\tau-r_k)^{-1/2}\ (T-\tau)^{-1/2}$$

remains. Denote this integral by $I(r_k,t)$. Proceeding step by step similarly with the other integrals in (4.3.3), we obtain the upper estimate

$$(4.3.6)\qquad C^k\ k! \int_s^T \mathrm{d}r_1\dots\int_{r(k-1)}^T \mathrm{d}r_k\ \prod_{i=1}^k\ p_1(r_i-r_{i-1},0)\ I(r_k,t)$$

for $H^{k,t}_{m,n}$, where again we set $r_0=s$, and $C$ is a constant. But $I(r_k,t)$ is uniformly bounded (to see this, switch from the variable $\tau$ to $\sigma$ by the substitution $(\tau-r_k) = (T-r_k)\sigma$). Moreover, $I(r_k,t)$ converges to $0$ as $t\uparrow T$, for each fixed $r_k$. By dominated convergence, for all three properties in Definition 2.4.7 it remains to check that

$$(4.3.7)\qquad k! \int_s^T \mathrm{d}r_1\dots\int_{r(k-1)}^T \mathrm{d}r_k\ \prod_{i=1}^k\ p_1(r_i-r_{i-1},0)$$

is finite and of order $o(k!/\lambda^k)$ as $k\to\infty$, for each $\lambda>0$. But (4.3.7) is simply $\Pi_{s,0}\,[\mathbf{L}^0(s,T)]^k$ with $\mathbf{L}^0$ the Brownian local time at $0$, and the statement follows from the existence of all its exponential moments, recall (4.2.6).

Summarizing, branching rate functionals $A_\xi$ corresponding to one-dimensional kernels $\xi$ with (4.3.2) are $\vartheta$-regular where $\vartheta = \sum_j \delta_{\mathfrak{z}(j)}$. ∎

**Example** 4.3.8 **(factored branching rate kernels** $\xi$**).** Again write the dimension $D$ as $D=d+1,\ d\geq 1.$ Assume the factorization $\xi = \xi_d\times\xi_1$ of (4.2.9) with a bounded regular function $\xi_d$ on $I\times\mathbb{R}^d$ and a one-dimensional kernel $\xi_1$ but now satisfying the condition

(4.3.9) $\quad \limsup_{m\to\infty} \sup_r\, (\varepsilon_m+T-r)^{-d} \int\xi_1(r,\mathrm{d}y_1)\, p_1\big(\varepsilon_m+T-r, y_1-\mathfrak{z}_1(i)\big) < \infty,\quad 1\leq i\leq l,$

for some zero sequence $\underline{\varepsilon}$. Consequently, roughly speaking, for $r$ close to $T$, the measure $\xi_1(r,\bullet)$ should approach a "*potentially small*" *density of mass* at all $\mathfrak{z}_1(j)$. (Compare with the weaker condition (4.3.2) where the singular factor in front of the integral is missing.) Of course, this condition is fulfilled, for instance, for $\xi_1(r,\bullet)\equiv\delta_c$ with $c\neq\mathfrak{z}_1(1),\dots,\mathfrak{z}_1(l)$, but it fails to hold if $\xi_1(r,\bullet)$ is identical to a uniform distribution around some $\mathfrak{z}_1(j)$. Consequently, opposed to the one-dimensional case (4.3.2), the random medium $\xi$ has to be "*sufficiently thin*" at those $\mathfrak{z}(j)$. (Again, this implies for the super-Brownian motion $\mathcal{X}$ with that branching kernel $\xi$ that $\mathcal{X}_T$ has a finite density of mass at $\mathfrak{z}(1),\dots,\mathfrak{z}(l)$.)

To check that in the present case the requirements in Definition 2.4.7 are fulfilled we have only to modify the arguments concerning the previous example. In fact, proceed up to (4.3.4). The latter can now be written as

$$\int_{t\vee r_k}^T \mathrm{d}\tau \int\xi_d(\tau,y_d)\mathrm{d}y_d \int\xi_1(\tau,\mathrm{d}y_1)\, p_D(\tau-r_k, y-W_{r_k}) \Big[\textstyle\sum_j p_D\big(\varepsilon_m+T-\tau, y-\mathfrak{z}(j)\big)\Big]^2.$$

Pass to a bound of $\xi_d$, and factorize the $(d+1)$-dimensional Brownian motion and transition density function as follows:

$$W = [W^d, W^1],\quad p_{d+1}(t,y) = p_d(t,y_d)\, p_1(t,y_1),\qquad t>0,\ y=[y_d,y_1]\in\mathbb{R}^d\times\mathbb{R}.$$

Then estimate

$$p_1(\tau-r_k,y_1-W^1_{r_k})\ p_d\big(\varepsilon_m+T-\tau,y_d-\mathfrak{z}_d(j)\big)\ p_d\big(\varepsilon_m+T-\tau,y_d-\mathfrak{z}_d(j')\big)\ p_1\big(\varepsilon_m+T-\tau,y_1-\mathfrak{z}_1(j)\big)$$
$$\leq \text{const } (\tau-r_k)^{-1/2}\ (\varepsilon_m+T-\tau)^{-d}\ (T-\tau)^{-1/2}, \qquad 1\leq j,j'\leq l,$$

and integrate the density function $p_d(\tau-r_k,\bullet-W^d_{r(k)})$ with respect to $dy_d$. Extract terms as in condition (4.3.9). It remains an expression as in formula line (4.3.5).

In the next integral we use

$$p_1(r_k-r_{k-1},y_1-W^1_{r(k-1)})\ p_d\big(\varepsilon_n+T-r_k,y_d-\mathfrak{z}_d(j)\big)\ p_1\big(\varepsilon_n+T-r_k,y_1-\mathfrak{z}_1(j)\big)$$
$$\leq \text{const } p_1(r_k-r_{k-1},0)\ (\varepsilon_n+T-r_k)^{-d/2}\ p_1\big(\varepsilon_n+T-r_k,y_1-\mathfrak{z}_1(j)\big).$$

We continue now as in the previous example, where the present case is even simpler since here we have only to handle the singular factor $(\varepsilon_n+T-r_k)^{-d/2}$ instead of $(\varepsilon_n+T-r_k)^{-d}$.

Summarizing, branching rate functionals $A_\xi$ with factored branching rate kernels $\xi = \xi_d\times\xi_1$ where the one-dimensional kernels $\xi_1$ are sufficiently irregular at $\mathfrak{z}(1),\dots,\mathfrak{z}(l)$ in the sense of (4.3.9), are $\vartheta$-regular with $\vartheta = \sum_j\delta_{\mathfrak{z}(j)}$. ∎

## 4.4. Examples of a.e.-Regular Branching Rate Functionals

The aim of this subsection is to deal with examples of a.e.-regular branching rate functionals $A_\xi$ according to Definition 2.6.1. For this purpose, fix $I:=[L,T),\ 0\leq L<T,$ set $D=d+1,\ d\geq 0,$ and restrict the attention to a *factored branching rate kernel* $\xi = \xi_d\times\xi_1$ as in (4.2.9) with a bounded regular $\xi_d$ (with the obvious interpretations in the boundary case $d=0$ we include at this place; for instance, read $dy_d$ as $\delta_0$ if $d=0$). The remaining one-dimensional kernel $\xi_1$ is, for simplicity, assumed to be constant in time.

$\xi_1(L,\bullet)$ will be sampled from some probability space $[\Omega,\mathcal{F},\mathcal{P}]$ as described in the two examples below *(random medium)*. In both cases, the realization of the measure $\xi_1(L,\bullet)$ is supported by a countable set $\{x_i;i\geq 1\}$. Hence we may interpret the super-Brownian motion with such (randomly selected) branching rate functional $\xi$ as a model in which branching is allowed only at a *countable collection of hyperplanes.*

**Example** 4.4.1 (**branching restricted to infinitely many isolated hyperplanes**). Suppose that $\xi_1(L,\bullet)$ is sampled from a homogeneous *stochastic point process* $\pi$ on $\mathbb{R}$ of finite intensity. That is, $\pi$ is a random locally finite counting measure on $\mathbb{R}$ whose law is shift invariant, and whose intensity measure $\mathcal{P}\pi(\mathrm{d}y_1)$ is a finite multiple of the Lebesgue measure $\mathrm{d}y_1$.

We need to show that for almost all realizations $\pi$ of the point process there is a Lebesgue zero set $N$ (depending on the realization $\pi$), such that for each choice of $\mathfrak{z}(1),\dots,\mathfrak{z}(l) \notin N$ there is a zero-sequence $\underline{\varepsilon}$ such that the "kernel" $\xi_1$ satisfies the condition (4.3.9) with this $\underline{\varepsilon}$. In fact, by the arguments in the discussion of the examples in Subsection 4.3, then the corresponding branching rate functional $A_\xi$ is a.e.-regular. But in the present time-homogeneous case $\xi_1(r,\bullet)\equiv\xi_1(L,\bullet)=\pi$ the l.h.s. in (4.3.9) can be estimated from above by

$$\sup_{0\le t\le T-L+1} t^{-d} \int \pi(\mathrm{d}y_1)\, p_1\big(t,y_1-\mathfrak{z}_1(i)\big).$$

Therefore it suffices to demonstrate that for all $K>1$

$$\mathcal{P}\int \mathrm{d}z_1\, \mathbf{1}\Big\{\sup_{0<t\le K} t^{-d} \int \pi(\mathrm{d}y_1)\, p_1(t,\, y_1-z_1) = \infty\Big\} = 0$$

holds. By Fubini's theorem, it is even enough to show that

$$\mathcal{P}\Big\{\sup_{0<t\le K} t^{-d} \int \pi(\mathrm{d}y_1)\, p_1(t,\, y_1-z_1) < \infty\Big\} = 1, \qquad z_1\in\mathbb{R},$$

Moreover, by the homogeneity of the point process, we may focus at the case $z_1=0$. Actually, we will even verify that for each constant $K>1$ with $\mathcal{P}$-probability one

$$\int \pi(\mathrm{d}y_1)\, \sup_{0<t\le K} t^{-d}\, p_1(t,y_1) < \infty. \tag{4.4.2}$$

To this end, distinguish between $|y_1|\ge 1$ and the complement. In the first case, take the expectation and note that

(4.4.3) $t^{-d}\, p_1(t,y_1) \le \text{const}\; t^{-d-1/2} \exp[-1/2t] \exp[-y_1^2/2K] \le \text{const} \exp[-y_1^2/2K]$,

which is integrable with respect to the Lebesgue measure $dy_1$. Concerning the second case $|y_1|<1$, we have only to note that the integrand in (4.4.2) is finite as long as $y_1 \neq 0$, and, on the other hand, that, with probability one, $\pi$ has only finitely many points in $\{|y_1|<1\}$, and they are different from $0$. ∎

**Example** 4.4.4 **(branching restricted to densely situated hyperplanes).** We modify the previous example as follows. Replace $\pi$ by a *stable random measure*

$$\Gamma = \Sigma_{i=1}^{\infty} \alpha_i \delta_{x(i)}$$

on $\mathbb{R}$ with index $\gamma \in (0, 1/(2D-1))$, characterized by its Laplace functional

(4.4.5) $\mathcal{P}\exp(\Gamma,-f) = \exp\left[-\int dy_1\, f^{\gamma}(y_1)\right], \quad f\geq 0$ measurable.

Note that the positions $\{x_i; i\geq 0\}$ of the atoms of $\Gamma$ are *densely* situated in $\mathbb{R}$. We stress also the fact that for $D>1$ by assumption the index $\gamma$ cannot be arbitrarily close to $1$. This, of course is only a sufficient condition. On the other hand, for growing $\gamma$ the random measure $\Gamma$ becomes more and more small atoms, that is, it comes closer and closer to the Lebesgue measure (boundary case $\gamma=1$). But under $D>1$ and regular branching kernels $\xi$ the super-Brownian motion has singular states. Summarizing, in the case $D>1$ we presupposed that $\xi_1(L,\bullet)$ is not "too close" to the Lebesgue measure.

To see that a branching rate functional $A_{\xi}$ with such a $\xi$ is a.e.-regular, follow the constructions and arguments in Example 4.4.1 up to (4.4.2), that is, up to the statement

(4.4.6) $\int \Gamma(dy_1) \sup_{0<t\leq K} t^{-d}\, p_1(t,y_1) < \infty \qquad \mathcal{P}\text{-a.s.}$

To verify this, it suffices to show that the Laplace transform of this random variable is $1$ at $0$. But applying the Laplace functional (4.4.5), this will follow if we verify that

(4.4.7) $\int dy_1 \left(\sup_{0<t\leq K} t^{-d}\, p_1(t,y_1)\right)^{\gamma} < \infty.$

Under the additional restriction $|y_1| \geq 1$, we use (4.4.3). But if $|y_1| < 1$ note that by elementary calculus

$$t^{-d-1/2} \exp[-y_1^2/t] \leq \text{const } |y_1|^{-d-D}, \quad t>0,\ y_1 \neq 0. \tag{4.4.8}$$

The latter function of $y_1$ is $\gamma$-fold integrable around $0$ if and only if $\gamma(d+D) < 1$ which we assumed. Consequently, (4.4.7) is true and we are done. ∎

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